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# A distribution free plotting position

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**Abstract.** Many plotting position formulae have been proposed for the past few decades. These formulae are derived or obtained under some specific assumption of probability distribution. Because in practice the data are often plotted in order to determine its probability distribution, it causes difficulty and confusion in selecting the plotting position formula. The objective of this study is to find a plotting position formula which is distribution free. In this study, the plotting position formulae corresponding to the order statistic mean, mode and median are investigated. The order statistic mean, mode and median values are determined by numerical integration and differentiation, and the corresponding plotting position formulae are obtained by regression analysis. The results indicate that both the plotting position formulae for the order statistic mean and mode vary with the distribution of data, but the plotting position formula for the order statistic median is distribution free. The distribution free plotting position formula for the order statistic median is proposed in this study as $(i - 0.326)/(n + 0.348)$.

## 1 Introduction

Graphic methods are important tools used by engineers and scientists in practice. The basic motive of using a probability paper is to modify the scales of the random variable, $X$, and its cumulative probability, $F(x)$, in such a way that the plot against $X$ of its cumulative probability, $F(x)$, appears as a straight line. The probability paper allows one to decide whether a sample of independent observations belongs to a specific family of probability distribution, and extrapolate the curve.

If a sample of $n$ independent observations, $\{x_i\}$, $i = 1, 2, \ldots, n$, is to be plotted on a specified probability paper, it is natural to arrange them in ascending order as $\{x_{(i)} : x_{(1)} < x_{(2)} < \cdots < x_{(n)}\}$. $\{x_{(i)}\}$ vs. $\{P_i\}$ is then plotted. $P_i$ is the sample cumulative probability which is determined by the plotting position.

The plotting position problem has been discussed by many researchers, including Hazen (1930), Weibull (1939), Beard (1943), Gumbel (1943), Benard and Bos-Levenbach (1953), Blom (1958), Harter (1961), Tukey (1962), Gringorten (1963), Cunnane (1978), Adamowski (1981), Hosking (1990), and Hosking and

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Wallis (1995). A general discussion can be found in Cunnane (1978) and Rao and Hamed (2000). The plotting position formula was usually expressed in the form of $(i-\alpha)/(n+1-2\alpha)$. Blom's (1958) formula, $\alpha = 3/8$, gives a good approximation to the probability corresponding to the mean value of the $i$-th order statistic for the random variables of normal distribution. Meanwhile, Tukey (1962) suggested that $\alpha = 1/3$. Harter (1961) published a five-decimal-place table of the $i$-th largest order statistic for $n = 2(1)100(25)250(50)400$. He also tabulated the values of $\alpha$ for $n = 2(2)10(5)25, 50, 100, 200$ and 400. Gringorten (1963) suggested $\alpha = 0.44$ for the extreme value type I distribution. In Beard's (1943) formula, values of $\alpha$ are 0.31 and 1.00 when the median and mode of order statistic of random variables are considered, respectively. Benard and Bos-Levenbach (1953) suggested $\alpha = 0.30$. On the other hand, the well-known Hazen (1930) formula, $\alpha = 1/2$, provides a closer approximation for the random variables of extreme value type I distribution. Weibull's (1939) formula, $\alpha = 0.0$, is shown to be accurate and unbiased for the random variables of uniform distribution. If a compromise formula is considered for all distributions, Cunnane (1978) suggested $\alpha = 0.40$, while Adamowski (1981) suggested $\alpha = 0.25$. A plotting position in the form of $(i-0.35)/(n)$ believed to give acceptable results for some common three-parameter distributions is used in Hosking (1990), Hosking and Wallis (1995), and Rao and Hamed (2000).



Since there is a large number of plotting position formulae, it is difficult to select the optimal plotting position formula for data in practice to determine the distribution of data. The selection of plotting position formula has been discussed in the literature (Chernoff and Lieberman, 1954, 1956; Kimball, 1960; Cunnane, 1978; Adamowski, 1981). Cunnane (1978) rebuked the attitude that the criterion for choice of plotting position is arbitrary, and showed that a worthwhile criterion can be based on desired statistical properties of the plot. These properties are that any quantile estimate made from the plot should be unbiased and should have smallest mean square error among all such estimates. According to graphical comparison of plotting positions, Cunnane (1978) conclude a simple distribution free plotting position with $\alpha = 0.4$ is the best compromise. The objective of this study is to find a plotting formula which is distribution free without considering any specific criteria for choice of plotting position. The probabilities $P_i$ of the mean, mode and median values of $x_{(i)}$ for a given probability distribution are determined by numerical method in this study. Then, the corresponding plotting position formulae are proposed. Seven probability distributions frequently used in practice are used in this study.

The paper is organized as follows. Section 2 of the paper details the method and the computational scheme. The results are presented and discussed in Sect. 3. Finally, conclusions are given in Sect. 4.

## 2 Theoretical aspects

Let $x_1, x_2, \ldots, x_n$ denote a random sample from a distribution of the continuous type having a probability density function (p.d.f.) $f(x)$, provided that $x$ is defined over $(x_l, x_u)$. $x_{(i)}$ is called the $i$-th order statistic of the random sample. The joint p.d.f. of $\{x_{(i)}\}, i = 1, 2, \ldots, n$ is

$$g\left(x_{(1)}, x_{(2)}, \ldots, x_{(n)}\right) = \begin{cases} n!\, f\left(x_{(1)}\right) f\left(x_{(2)}\right) \cdots f\left(x_{(n)}\right), & \text{if } a < x_{(1)} < x_{(2)} < \cdots < x_{(n)} < b \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

The marginal distribution of any order statistic, say $x_{(k)}$, in terms of $F(x)$ and $f(x)$ is given as follows:

$$g\left(x_{(k)}\right) = \begin{cases} \frac{n!}{(k-1)!(n-k)!}\left[F\left(x_{(k)}\right)\right]^{k-1}[1-F(x_k)]^{n-k} f\left(x_{(k)}\right), & \text{if } a < x_{(k)} < b \\ 0, & \text{elsewhere } . \end{cases} \tag{2}$$

Blom (1958) derived a plotting position by approximating the $i$-th order statistic for a Gaussian-distributed sample of size $n$



$$\mathrm{Mean}\left[X_{(i)}\right] = E\left[X_{(i)}\right] = F^{-1}\left(\frac{i-\alpha}{n-2\alpha+1}\right) , \tag{3}$$

where

$$F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}x ,$$

and

$$f(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) .$$

The values of $\alpha$ for $i = 1(1)[n/2]$ when $n = 2(2)10(5)20$ were given by Blom (1958), which increase as $n$ increases. The lowest value is 0.33 for $n = 2$ and $i = 1$. For a given $n, \alpha$ has the smallest value for $i = 1$, rises quickly to a peak for a relatively small value of $i$, and then drops off slowly. He conjectured that $\alpha$ always lies in the interval (0.33, 0.50) and suggested that $\alpha = 3/8$ as a compromise value. For a given $n$ and order $i$, Eq. (3) yields

$$\alpha_{i,n} = \frac{i-(n+1)F\left(\mathrm{Mean}\left[X_{(i)}\right]\right)}{1-2F\left(\mathrm{Mean}\left[X_{(i)}\right]\right)} \tag{4}$$

The values of $\alpha_{i,n}$ for $i = 1(1)[(1/2)n]$ when $n = 25, 50, 100, 200$ and 400 were given by Harter (1961). He proposed a value of $\alpha_n$ for each $n$ and all values of $i$, for which the maximum error of $\mathrm{Mean}[X_{(i)}]$ is less than 0.004.

Since Eq. (3) might not be applicable to all distributions, the general form of plotting position formula in Eq. (5) is used in this study:

$$P_i = \frac{i-b}{n+a} . \tag{5}$$

The parameters $a$ and $b$ are determined by regression on $(i, P_i)$, $i = 1, 2, \ldots, n$. In addition to the mean of the $i$-th order statistic ($\mathrm{Mean}[X_{(i)}]$), the mode of the $i$-th order statistic ($\mathrm{Mode}[X_{(i)}]$) and median of $i$-th order statistic ($\mathrm{Median}[X_{(i)}]$) are also investigated in this study. Thus, the equations to be investigated in this study include:

$$\mathrm{Mean}\left[X_{(i)}\right] = F^{-1}\left(\frac{i-b_1}{n+a_1}\right) \text{ or } F^{-1}\left(\frac{i-\alpha_1}{n-2\alpha_1+1}\right),$$

$$\mathrm{Mode}\left[X_{(i)}\right] = F^{-1}\left(\frac{i-b_2}{n+a_2}\right) \text{ or } F^{-1}\left(\frac{i-\alpha_2}{n-2\alpha_2+1}\right),$$

and

$$\mathrm{Median}\left[X_{(i)}\right] = F^{-1}\left(\frac{i-b_3}{n+a_3}\right) \text{ or } F^{-1}\left(\frac{i-\alpha_3}{n-2\alpha_3+1}\right) .$$



In order to present the computational scheme of this study, the extreme value type I distribution is used as an example. Let $x_1, x_2, \ldots, x_n$ be extreme value type I distributed, the p.d.f. is given in Eq. (6):

$$f(x) = \alpha \exp(-\alpha(x-\beta) - \exp(-\alpha(x-\beta))) \quad (6)$$

The cumulative density function (c.d.f.) of extreme value type I distribution is

$$F(x) = \exp(-\exp(-\alpha(x-\beta))) \quad (7)$$

The parameters $\alpha$ and $\beta$ could be expressed in terms of mean $\mu$ and standard deviation $\sigma$ as follows:

$$\alpha = \frac{1.2825}{\sigma} , \quad (8)$$

$$\beta = \mu - \frac{0.4500}{\sigma} . \quad (9)$$

Thus, if the first moment $\mu$ and second moment $\sigma^2$ (or the coefficient of variation $C_v$) are given, $\alpha$ and $\beta$ can be determined by Eqs. (8) and (9). Then, $f(x)$ and $F(x)$ can be obtained by using Eqs. (6) and (7).

The marginal p.d.f., $g\left(x_{(i)}\right)$, of the order statistic $x_{(i)}$ for $i = 1, 2, \ldots, n$ can be calculated by Eq. (2). Consequently, the mean, mode and median values of each order statistic can be calculated by using Eqs. (10), (11) and (12), respectively.

$$\mathrm{Mean}\left[X_{(i)}\right] = \int_{-\infty}^{\infty} x_{(i)} g\left(x_{(i)}\right) \mathrm{d}x_{(i)} , \quad (10)$$

$$0.5 = \int_{-\infty}^{\mathrm{median}\left[X_{(i)}\right]} g\left(x_{(i)}\right) \mathrm{d}x_{(i)} , \quad (11)$$

and

$$\left.\frac{\mathrm{d}g\left(x_{(i)}\right)}{\mathrm{d}x_{(i)}}\right|_{x_{(i)}=\mathrm{mode}\left[X_{(i)}\right]} = 0 . \quad (12)$$

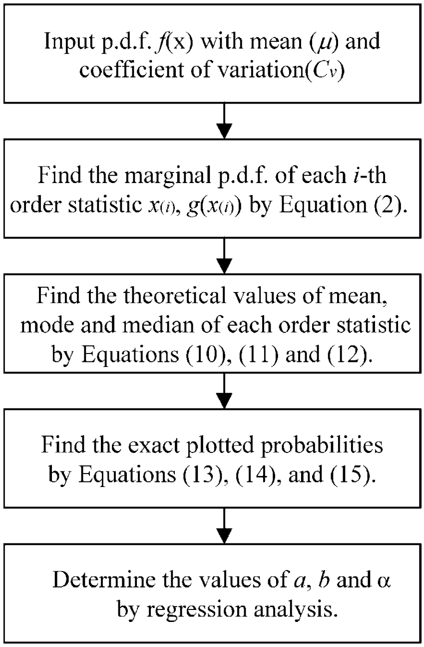


**Fig. 1.** Computation scheme

The numerical integration and differentiation are used to determine the exact values of Mean$[x_{(i)}]$, Mode$[x_{(i)}]$ and Median$[x_{(i)}]$ for any p.d.f. and $i = 1, 2, \ldots, n$. Once the Mean$[x_{(i)}]$, Mode$[x_{(i)}]$ and Median$[x_{(i)}]$ are obtained, their corresponding theoretical probabilities $P_i$ can then be obtained by solving Eqs. (13), (14) and (15), respectively.

$$P_i^{\text{mean}} = \int_{-\infty}^{\text{mean}[X_{(i)}]} f(x)\mathrm{d}x \ , \tag{13}$$

$$P_i^{\text{median}} = \int_{-\infty}^{\text{median}[X_{(i)}]} f(x)\mathrm{d}x \ , \tag{14}$$

$$P_i^{\text{mode}} = \int_{-\infty}^{\text{mode}[X_{(i)}]} f(x)\mathrm{d}x \ . \tag{15}$$

The parameters $a$, $b$ and $\alpha$ of the plotting position formula (Eq. (5)) are then determined by least-square regression on $(i, P_i^{\text{mean}})$, $(i, P_i^{\text{median}})$ and $(i, P_i^{\text{mode}})$, respectively. Consequently, the plotting positions for the mean, mode and median of order statistic for any given distribution with specified parameters can thus be determined. The computational scheme is summarized in Fig. 1.

Seven distributions frequently used in practice, including uniform, normal, Weibull, extreme value type I, Gamma, Rayleigh and 2-parameter log-normal distributions are used in this study. The FORTRAN computer programs are

**Table 1.** Probability distributions used in this study

| Probability distribution | p.d.f. and parameters ($E(x) = \mu, \mathrm{Var}(x) = \sigma^2$) |
|---|---|
| Uniform | $\begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{else} \end{cases}$ <br> $\mu = (a+b)/2, \ \sigma^2 = (b-a)^2/1$ |
| Normal | $\frac{1}{\sqrt{2\pi}\sigma}\exp(-(x-\mu)^2/2\sigma^2), \quad -\infty < x < \infty$ |
| Weibull | $\begin{cases} \frac{\alpha}{\theta^\alpha} x^{\alpha-1}\exp(-(x/\theta)^\alpha), & x > 0; \ \alpha, \theta > 0 \\ 0, & \text{else} \end{cases}$ <br> $\mu = \theta\Gamma(1+1/\alpha), \ \sigma^2 = \theta^2[\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)]$ |
| Extreme value type I | $\begin{cases} \alpha\exp(-\alpha(x-\beta) - \exp(-\alpha(x-\beta))), & -\infty < x < \infty \\ 0, & \text{else} \end{cases}$ <br> $\alpha = 1.2825/\sigma, \ \beta = \mu - 0.4500\sigma$ |
| Gamma | $\begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1}\exp(-x/\theta), & x > 0; \ \alpha, \theta > 0 \\ 0, & \text{else} \end{cases}$ <br> $\mu = \alpha\theta, \ \sigma^2 = \alpha\theta^2$ |
| Rayleigh | $\begin{cases} \frac{x}{\alpha^2}\exp(-(x^2/2\alpha^2)), & x > 0; \\ 0, & \text{else} \end{cases}$ <br> $\mu = \sqrt{\pi/2}\alpha$ |
| 2-parameter log-normal | $\begin{cases} \frac{1}{x\sigma_2\sqrt{2\pi}}\exp(-[\ln(x) - \mu_2]^2/2\alpha_2^2), & x > 0 \\ 0, & \text{else} \end{cases}$ <br> $\mu = e^{\mu_2 + (\sigma_2^2/2)}, \ \sigma^2 = (\exp(\sigma_2^2) - 1)\mu^2$ |

written in double precision. IMSL FORTRAN subroutines such as DQDAG, DBINOM, DGAMMA, DGAMI and DNORDF are used for numerical calculation on PC-486 with installation of IMSL MATH/LIBRARY, STAT/LIBRARY and SFUN/LIBRARY version 1.1. The distributions used in this study and their corresponding relationship between the parameters and lower moments are given in Table 1. There are nine combinations of parameter sets mean $\mu = 20(20)60$ and coefficient of variation $C_v = 0.1(0.2)0.5$ specified for each distribution with sample size $n = 2(1)10(5)50(10)120$. Multiple cases of synthetic data with different $\mu$ and $C_v$ are used to examine the theoretical properties of the plotting positions. For a number of the cases the theoretical and closed form results are available, the results are also compared to the available literature. However, the general forms of theoretical plotting position are not derivable or not available in the literature.

## 3
## Results and discussions

For each distribution and sample size $n$ with parameter sets $\mu = 20(20)60$ and $C_v = 0.1(0.2)0.5$, values of $\alpha$ are determined and rounded to three decimal places.

The maximum value of $\alpha$, the range of $\alpha$, and the maximum absolute error in estimating probability $P_i$ are determined from results of all parameter sets, and summarized for each distribution and sample size $n$. The range of $\alpha$ is the difference between the maximum value and the minimum value of $\alpha$ from nine parameter sets. It should be noted that if $\alpha$ is independent of the parameters $\mu$ and $C_v$ of each distribution, $\alpha$ is the same; therefore, the range of $\alpha$ is zero. On the other hand, the greater the value of the range of $\alpha$ deviates from zero, the more $\alpha$ is dependent on the parameters of probability distribution. The results for the order statistic mean, mode and median are presented and discussed in details as below.



### 3.1
### $\alpha$ determined for the order statistic mean, mode and median

The results for the order statistic means of random variables of uniform, normal and extreme value type I distributions are given in Table 2. The complete results are given in Yu (1992). As expected for the normal distribution, the results indicate that the value of $\alpha$ for each $n$ is invariant with the parameters $\mu$ and $C_v$. The values of $\alpha$ for the normal distribution given in Table 2 are in good agreement with Harter (1961). Furthermore, the maximum absolute errors in estimating probability for all $n$ are less than 0.000733. For uniform distribution, the results indicate that $\alpha = 0$ and $\alpha$ is invariant with the parameters $\mu$ and $C_v$, which is in good agreement with the well-known Weibull formula. For extreme

**Table 2.** Results of the order statistic mean ($\mu$ = 20(20)60 and $C_v$ = 0.1(0.2)0.5)

| $n$ | Uniform | | | Normal | | | Extreme value type I | | |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (1) | (2) | (3) | (1) | (2) | (3) |
| 2 | 0.000 | 0.000 | 0.000000 | 0.330 | 0.000 | 0.000000 | 0.376 | 0.000 | 0.047829 |
| 3 | 0.000 | 0.000 | 0.000000 | 0.341 | 0.000 | 0.000000 | 0.370 | 0.000 | 0.031718 |
| 4 | 0.000 | 0.000 | 0.000000 | 0.348 | 0.000 | 0.000598 | 0.372 | 0.000 | 0.024324 |
| 5 | 0.000 | 0.000 | 0.000000 | 0.354 | 0.000 | 0.000733 | 0.375 | 0.000 | 0.020269 |
| 6 | 0.000 | 0.000 | 0.000000 | 0.359 | 0.000 | 0.000716 | 0.378 | 0.000 | 0.017146 |
| 7 | 0.000 | 0.000 | 0.000000 | 0.363 | 0.000 | 0.000652 | 0.381 | 0.000 | 0.014863 |
| 8 | 0.000 | 0.000 | 0.000000 | 0.366 | 0.000 | 0.000608 | 0.384 | 0.000 | 0.013144 |
| 9 | 0.000 | 0.000 | 0.000000 | 0.369 | 0.000 | 0.000633 | 0.386 | 0.000 | 0.011727 |
| 10 | 0.000 | 0.000 | 0.000000 | 0.371 | 0.000 | 0.000644 | 0.388 | 0.000 | 0.010633 |
| 15 | 0.000 | 0.000 | 0.000000 | 0.380 | 0.000 | 0.000608 | 0.395 | 0.000 | 0.007175 |
| 20 | 0.000 | 0.000 | 0.000000 | 0.085 | 0.000 | 0.000536 | 0.400 | 0.000 | 0.005420 |
| 25 | 0.000 | 0.000 | 0.000000 | 0.389 | 0.000 | 0.000468 | 0.403 | 0.000 | 0.004354 |
| 30 | 0.000 | 0.000 | 0.000000 | 0.392 | 0.000 | 0.000412 | 0.406 | 0.000 | 0.003639 |
| 35 | 0.000 | 0.000 | 0.000000 | 0.394 | 0.000 | 0.000365 | 0.408 | 0.000 | 0.003127 |
| 40 | 0.000 | 0.000 | 0.000000 | 0.396 | 0.000 | 0.000327 | 0.410 | 0.000 | 0.002742 |
| 45 | 0.000 | 0.000 | 0.000000 | 0.397 | 0.000 | 0.000295 | 0.411 | 0.000 | 0.002441 |
| 50 | 0.000 | 0.000 | 0.000000 | 0.398 | 0.000 | 0.000268 | 0.412 | 0.000 | 0.002200 |
| 60 | 0.000 | 0.000 | 0.000000 | 0.400 | 0.000 | 0.000225 | 0.414 | 0.000 | 0.001838 |
| 70 | 0.000 | 0.000 | 0.000000 | 0.402 | 0.000 | 0.000193 | 0.415 | 0.000 | 0.001578 |
| 80 | 0.000 | 0.000 | 0.000000 | 0.403 | 0.000 | 0.000168 | 0.416 | 0.000 | 0.001383 |
| 90 | 0.000 | 0.000 | 0.000000 | 0.404 | 0.000 | 0.000148 | 0.417 | 0.000 | 0.001231 |
| 100 | 0.000 | 0.000 | 0.000000 | 0.434 | 0.000 | 0.000132 | 0.418 | 0.000 | 0.001109 |
| 110 | 0.000 | 0.000 | 0.000000 | 0.405 | 0.000 | 0.000119 | 0.418 | 0.000 | 0.001010 |
| 120 | 0.000 | 0.000 | 0.000000 | 0.405 | 0.000 | 0.000108 | 0.419 | 0.000 | 0.000926 |

(1) – The maximum value of $\alpha$; (2) – the range of $\alpha$; and (3) – the maximum absolute error in estimating probability $P_i$

Table 3. Results of the order statistic mode ($\mu$ = 20(20)60 and $C_v$ = 0.1(0.2)0.5)

| $n$ | Uniform | | | Normal | | | Extreme value type I | | |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (1) | (2) | (3) | (1) | (2) | (3) |
| 2 | 1.000 | 0.000 | 0.000000 | 0.209 | 0.000 | 0.000000 | 0.359 | 0.000 | 0.045338 |
| 3 | 1.000 | 0.000 | 0.000000 | 0.201 | 0.000 | 0.000000 | 0.265 | 0.000 | 0.036456 |
| 4 | 1.000 | 0.000 | 0.000009 | 0.196 | 0.000 | 0.000774 | 0.228 | 0.000 | 0.029329 |
| 5 | 1.000 | 0.000 | 0.000009 | 0.193 | 0.000 | 0.001035 | 0.209 | 0.000 | 0.024106 |
| 6 | 1.000 | 0.000 | 0.000004 | 0.191 | 0.000 | 0.001037 | 0.198 | 0.000 | 0.020233 |
| 7 | 1.000 | 0.000 | 0.000007 | 0.189 | 0.000 | 0.000933 | 0.190 | 0.000 | 0.017290 |
| 8 | 1.000 | 0.000 | 0.000010 | 0.188 | 0.000 | 0.001008 | 0.184 | 0.000 | 0.014999 |
| 9 | 1.000 | 0.000 | 0.000005 | 0.187 | 0.000 | 0.001093 | 0.179 | 0.000 | 0.013174 |
| 10 | 1.000 | 0.000 | 0.000009 | 0.186 | 0.000 | 0.001155 | 0.176 | 0.000 | 0.011695 |
| 15 | 1.000 | 0.000 | 0.000012 | 0.182 | 0.000 | 0.001263 | 0.166 | 0.000 | 0.007213 |
| 20 | 1.000 | 0.000 | 0.000012 | 0.181 | 0.000 | 0.001238 | 0.161 | 0.000 | 0.005014 |
| 25 | 1.000 | 0.000 | 0.000011 | 0.179 | 0.000 | 0.001178 | 0.158 | 0.000 | 0.003741 |
| 30 | 1.000 | 0.000 | 0.000015 | 0.178 | 0.000 | 0.001111 | 0.157 | 0.000 | 0.002926 |
| 35 | 1.000 | 0.000 | 0.000011 | 0.178 | 0.000 | 0.001046 | 0.155 | 0.000 | 0.002366 |
| 40 | 1.000 | 0.000 | 0.000014 | 0.177 | 0.000 | 0.000987 | 0.154 | 0.000 | 0.001962 |
| 45 | 1.000 | 0.000 | 0.000008 | 0.177 | 0.000 | 0.000932 | 0.153 | 0.000 | 0.001659 |
| 50 | 1.001 | 0.000 | 0.000012 | 0.177 | 0.000 | 0.000883 | 0.153 | 0.000 | 0.001425 |
| 60 | 1.001 | 0.001 | 0.000013 | 0.176 | 0.000 | 0.000799 | 0.152 | 0.000 | 0.001090 |
| 70 | 1.001 | 0.001 | 0.000015 | 0.176 | 0.000 | 0.000730 | 0.151 | 0.000 | 0.000864 |
| 80 | 1.001 | 0.001 | 0.000010 | 0.175 | 0.000 | 0.000672 | 0.151 | 0.000 | 0.000704 |
| 90 | 1.001 | 0.001 | 0.000014 | 0.175 | 0.000 | 0.000623 | 0.150 | 0.000 | 0.000586 |
| 100 | 1.001 | 0.001 | 0.000017 | 0.175 | 0.000 | 0.000580 | 0.150 | 0.000 | 0.000495 |
| 110 | 1.001 | 0.001 | 0.000013 | 0.175 | 0.000 | 0.000544 | 0.150 | 0.000 | 0.000424 |
| 120 | 1.001 | 0.001 | 0.000013 | 0.175 | 0.000 | 0.000512 | 0.149 | 0.000 | 0.000368 |

(1) – The maximum value of $\alpha$; (2) – the range of $\alpha$; and (3) – the maximum absolute error in estimating probability $P_i$



value type I and Rayleigh distributions, $\alpha$ is invariant with the parameters $\mu$ and $C_V$, but the corresponding plotting position has larger errors in estimating probability. However, for Weibull, Gamma and 2-parameter log-normal distributions, $\alpha$ varies with the parameters $\mu$ and $C_V$. In fact, $\alpha$ largely depends on $C_V$. It should be noted that the values of the range of $\alpha$ are almost zero when $n \geq 20$ for 2-parameter log-normal distribution, and the maximum values of $\alpha$ for different distributions are quite different. Therefore, it is clear that the plotting position formula varies with the distribution of the random variable when the order statistic mean is considered.

The results for the order statistic mode of random variables of uniform, normal and extreme value type I distributions are given in Table 3. The complete results are given in Yu (1992). For uniform, normal, extreme value type I, and Rayleigh distributions, $\alpha$ is invariant with the parameters of $\mu$ and $C_V$. However, $\alpha$ varies with the parameters of $\mu$ and $C_V$ for Weibull, Gamma and 2-parameter log-normal random variables. It should be noted that the maximum errors in estimating probability for uniform and normal distributions are very small. It is obvious that the plotting position formula varies with different distributions when the order statistic mode is considered.

The results for the order statistic median of random variables of uniform, normal and extreme value type I distributions are given in Table 4. The complete results are given in Yu (1992). The results indicate that the maximum values of $\alpha$ for each $n$ are the same for all distributions. Furthermore, values of the range of $\alpha$

**Table 4.** Results of the order statistic median ($\mu$ = 20(20)60 and $C_v$ = 0.1(0.2)0.5)

| $n$ | Uniform | | | Normal | | | Extreme value type I | | |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (1) | (2) | (3) | (1) | (2) | (3) |
| 2 | 0.293 | 0.000 | 0.000000 | 0.293 | 0.000 | 0.000000 | 0.293 | 0.000 | 0.000021 |
| 3 | 0.298 | 0.000 | 0.000000 | 0.298 | 0.000 | 0.000000 | 0.298 | 0.000 | 0.000016 |
| 4 | 0.301 | 0.000 | 0.000582 | 0.301 | 0.000 | 0.000589 | 0.301 | 0.000 | 0.000566 |
| 5 | 0.304 | 0.000 | 0.000734 | 0.304 | 0.000 | 0.000717 | 0.304 | 0.000 | 0.000764 |
| 6 | 0.306 | 0.000 | 0.000717 | 0.306 | 0.000 | 0.000681 | 0.306 | 0.000 | 0.000748 |
| 7 | 0.308 | 0.000 | 0.000644 | 0.308 | 0.000 | 0.000609 | 0.308 | 0.000 | 0.000629 |
| 8 | 0.310 | 0.000 | 0.000637 | 0.310 | 0.000 | 0.000657 | 0.310 | 0.000 | 0.000641 |
| 9 | 0.311 | 0.000 | 0.000672 | 0.311 | 0.000 | 0.000673 | 0.311 | 0.000 | 0.000660 |
| 10 | 0.312 | 0.000 | 0.000693 | 0.312 | 0.000 | 0.000684 | 0.312 | 0.000 | 0.000698 |
| 15 | 0.317 | 0.000 | 0.000690 | 0.317 | 0.000 | 0.000692 | 0.317 | 0.000 | 0.000684 |
| 20 | 0.319 | 0.000 | 0.000635 | 0.319 | 0.000 | 0.000625 | 0.319 | 0.000 | 0.000634 |
| 25 | 0.321 | 0.000 | 0.000575 | 0.321 | 0.000 | 0.000573 | 0.321 | 0.000 | 0.000566 |
| 30 | 0.323 | 0.000 | 0.000523 | 0.322 | 0.000 | 0.000518 | 0.323 | 0.000 | 0.000522 |
| 35 | 0.324 | 0.000 | 0.000477 | 0.324 | 0.000 | 0.000482 | 0.323 | 0.000 | 0.000476 |
| 40 | 0.324 | 0.000 | 0.000438 | 0.324 | 0.000 | 0.000441 | 0.324 | 0.000 | 0.000441 |
| 45 | 0.325 | 0.000 | 0.000404 | 0.325 | 0.000 | 0.000405 | 0.325 | 0.000 | 0.000406 |
| 50 | 0.326 | 0.000 | 0.000376 | 0.326 | 0.000 | 0.000377 | 0.326 | 0.000 | 0.000373 |
| 60 | 0.327 | 0.000 | 0.000328 | 0.327 | 0.000 | 0.000328 | 0.327 | 0.000 | 0.000328 |
| 70 | 0.327 | 0.000 | 0.000291 | 0.327 | 0.000 | 0.000291 | 0.327 | 0.000 | 0.000289 |
| 80 | 0.328 | 0.000 | 0.000262 | 0.328 | 0.000 | 0.000260 | 0.328 | 0.000 | 0.000261 |
| 90 | 0.328 | 0.000 | 0.000238 | 0.329 | 0.000 | 0.000239 | 0.328 | 0.000 | 0.000238 |
| 100 | 0.329 | 0.000 | 0.000218 | 0.329 | 0.000 | 0.000219 | 0.329 | 0.000 | 0.000218 |
| 110 | 0.329 | 0.000 | 0.000201 | 0.329 | 0.000 | 0.000203 | 0.329 | 0.000 | 0.000202 |
| 120 | 0.329 | 0.000 | 0.000187 | 0.329 | 0.000 | 0.000186 | 0.329 | 0.000 | 0.000188 |

(1) – The maximum value of $\alpha$; (2) – the range of $\alpha$; and (3) – the maximum absolute error in estimating probability $P_i$



are zero for all $n$ and distributions, and the maximum absolute errors for all $n$ and distributions are less than 0.000764. It is shown that the plotting position formula determined from the order statistic median is invariant with distributions, but varies with the sample size $n$.

In order to examine the uniqueness of the plotting position formula for the order statistic median, random variables with $\mu = 20$ and $C_v = 0.4(0.2)2.0$ for each distribution and $n = 2(1)10(10)120$ are also investigated in this study. For 7 (distributions) $*$ 9 (values of $C_v$) $= 63$ sets of parameters, the maximum value of $\alpha$, the range of $\alpha$ and the maximum absolute error in estimating probability for each $n$ are given in Table 5. It is noted that values of the range of $\alpha$ are less than 0.0007 for all $n$ and $C_v$. Furthermore, values of the maximum absolute errors are less than 0.000781 for all $n$ and $C_v$. These results indicate that the plotting position formula for the order statistic median is distribution free.

## 3.2
## The general form of plotting position formula

The existing plotting position formulae are usually expressed in the form of $(i - \alpha)/(n + 1 - 2\alpha)$. Another general form of $(i - b)/(n + a)$ is investigated in this study. Presented in this paper are the results corresponding to $\mu = 40$ and $C_v = 0.3$ for $n = 2, 5, 10, 30, 50, 90$ and 120. The parameters $\alpha, a$ and $b$ for the order statistic mean and median are given in Tables 6 and 8,

**Table 5.** Results of the order statistic median ($\mu$ = 20 and $C_v$ = 0.4(0.2)0.2) for all cases used in this study

| $n$ | (1) | (2) | (3) |
|---|---|---|---|
| 2 | 0.2933 | 0.0007 | 0.000065 |
| 3 | 0.2979 | 0.0006 | 0.000067 |
| 4 | 0.3013 | 0.0004 | 0.000628 |
| 5 | 0.3042 | 0.0004 | 0.000781 |
| 6 | 0.3064 | 0.0004 | 0.000754 |
| 7 | 0.3084 | 0.0004 | 0.000677 |
| 8 | 0.3100 | 0.0004 | 0.000661 |
| 9 | 0.3113 | 0.0004 | 0.000694 |
| 10 | 0.3126 | 0.0004 | 0.000714 |
| 20 | 0.3195 | 0.0003 | 0.000650 |
| 30 | 0.3228 | 0.0005 | 0.000529 |
| 40 | 0.3246 | 0.0003 | 0.000442 |
| 50 | 0.3260 | 0.0005 | 0.000380 |
| 60 | 0.3268 | 0.0003 | 0.000331 |
| 70 | 0.3277 | 0.0004 | 0.000295 |
| 80 | 0.3281 | 0.0003 | 0.000265 |
| 90 | 0.3286 | 0.0003 | 0.000241 |
| 100 | 0.3289 | 0.0003 | 0.000220 |
| 110 | 0.3292 | 0.0006 | 0.000203 |
| 120 | 0.3295 | 0.0003 | 0.000189 |

(1) – The maximum value of $\alpha$; (2) – the range of $\alpha$; and (3) – the maximum absolute error in estimating probability $P_i$



**Table 6.** Determined plotting position parameters for the order statistic mean ($\mu$ = 40 and $C_v$ = 0.3)

| $n$ | Uniform | | | Normal | | | Extreme value type I | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $a$ | $b$ | $\alpha$ | $a$ | $b$ | $\alpha$ | $a$ | $b$ |
| 2 | 0.000 | 1.000 | 0.000 | 0.330 | 0.340 | 0.330 | 0.376 | 0.326 | 0.431 |
| 5 | 0.000 | 1.000 | 0.000 | 0.354 | 0.292 | 0.354 | 0.375 | 0.272 | 0.457 |
| 10 | 0.000 | 1.000 | 0.000 | 0.371 | 0.257 | 0.371 | 0.388 | 0.235 | 0.477 |
| 30 | 0.000 | 1.000 | 0.000 | 0.392 | 0.216 | 0.392 | 0.406 | 0.192 | 0.500 |
| 50 | 0.000 | 1.000 | 0.000 | 0.398 | 0.204 | 0.398 | 0.412 | 0.178 | 0.507 |
| 90 | 0.000 | 1.000 | 0.000 | 0.404 | 0.193 | 0.404 | 0.417 | 0.167 | 0.512 |
| 120 | 0.000 | 1.000 | 0.000 | 0.405 | 0.189 | 0.405 | 0.419 | 0.163 | 0.515 |

respectively. The maximum absolute errors in estimating probability by $(i-\alpha)/(n+1-2\alpha)$ and $(i-b)/(n+a)$ for the order statistic mode and median are given in Tables 7 and 9, respectively. The complete results are given in Yu (1992).

For the order statistic mean and mode of normal or uniform distributed random variables which has a symmetric probability distribution, the values of $\alpha$ and $b$ are the same, and the values of $a$ are very close to $(1-2\alpha)$. However, for the order statistic mean and mode of the random variable with a non-symmetric distribution (extreme value type I, Weibull, Gamma, Rayleigh and 2-parameter log-normal distributions), values of $\alpha$ are quite different from $b$ and values of $a$ are not equal to $(1-2\alpha)$. In Tables 6 and 8, the formula $(i-1)/(n-1)$ for the order statistic mode of uniform random variables, which is perfectly fitted, is in

**Table 7.** The maximum absolute error in estimating probability for the order statistic mode ($\mu = 40$ and $C_v = 0.3$)

| $n$ | Gamma | | Rayleigh | | 2-parameter log-normal | |
|---|---|---|---|---|---|---|
| | (1) | (2) | (1) | (2) | (1) | (2) |
| 2 | 0.059557 | 0.000000 | 0.080358 | 0.000000 | 0.091803 | 0.000000 |
| 5 | 0.023013 | 0.002384 | 0.034160 | 0.002059 | 0.033433 | 0.003347 |
| 10 | 0.012357 | 0.002938 | 0.018130 | 0.002339 | 0.017474 | 0.003920 |
| 30 | 0.004410 | 0.002087 | 0.006724 | 0.001553 | 0.006181 | 0.002675 |
| 50 | 0.002686 | 0.001552 | 0.004224 | 0.001134 | 0.003781 | 0.001975 |
| 90 | 0.001528 | 0.001041 | 0.002460 | 0.000748 | 0.002132 | 0.001312 |
| 120 | 0.001159 | 0.000843 | 0.001884 | 0.000600 | 0.001617 | 0.001060 |

Plotting position formulae: (1) – $(i - \alpha)/(n + 1 - 2\alpha)$; and (2) – $(i - b)/(n + a)$



**Table 8.** Determined plotting position parameters for the order statistic median ($\mu = 40$ and $C_v = 0.3$)

| $n$ | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|
| | $\alpha$ | $a$ | $b$ | $\alpha$ | $a$ | $b$ |
| 2 | 0.293 | 0.414 | 0.293 | 0.293 | 0.414 | 0.293 |
| 5 | 0.304 | 0.392 | 0.304 | 0.304 | 0.392 | 0.304 |
| 10 | 0.312 | 0.375 | 0.312 | 0.312 | 0.376 | 0.312 |
| 30 | 0.323 | 0.355 | 0.323 | 0.322 | 0.355 | 0.322 |
| 50 | 0.326 | 0.349 | 0.326 | 0.326 | 0.348 | 0.326 |
| 90 | 0.328 | 0.343 | 0.328 | 0.329 | 0.343 | 0.329 |
| 120 | 0.329 | 0.341 | 0.329 | 0.329 | 0.341 | 0.329 |
| | Extreme value type I | | | 2-parameter log-normal | | |
| | $\alpha$ | $a$ | $b$ | $\alpha$ | $a$ | $b$ |
| 2 | 0.293 | 0.414 | 0.293 | 0.293 | 0.414 | 0.293 |
| 5 | 0.304 | 0.392 | 0.304 | 0.304 | 0.392 | 0.304 |
| 10 | 0.312 | 0.375 | 0.312 | 0.312 | 0.375 | 0.312 |
| 30 | 0.323 | 0.355 | 0.323 | 0.323 | 0.355 | 0.322 |
| 50 | 0.326 | 0.349 | 0.326 | 0.326 | 0.349 | 0.326 |
| 90 | 0.328 | 0.343 | 0.328 | 0.328 | 0.343 | 0.328 |
| 120 | 0.329 | 0.341 | 0.329 | 0.329 | 0.341 | 0.329 |

**Table 9.** The maximum absolute error in estimating probability for the order statistic median ($\mu = 40$ and $C_v = 0.3$)

| $n$ | Uniform | | Normal | | Extreme value type I | | 2-parameter log-normal | |
|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (1) | (2) | (1) | (2) | (1) | (2) |
| 2 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000021 | 0.000000 | 0.000036 | 0.000000 |
| 5 | 0.000731 | 0.000731 | 0.000717 | 0.000717 | 0.000764 | 0.000758 | 0.000735 | 0.000749 |
| 10 | 0.000689 | 0.000689 | 0.000684 | 0.000684 | 0.000698 | 0.000696 | 0.000686 | 0.000686 |
| 30 | 0.000521 | 0.000521 | 0.000518 | 0.000518 | 0.000522 | 0.000523 | 0.000521 | 0.000521 |
| 50 | 0.000375 | 0.000375 | 0.000377 | 0.000377 | 0.000375 | 0.000375 | 0.000379 | 0.000380 |
| 90 | 0.000238 | 0.000238 | 0.000239 | 0.000239 | 0.000238 | 0.000239 | 0.000240 | 0.000240 |
| 120 | 0.000186 | 0.000186 | 0.000186 | 0.000186 | 0.000188 | 0.000188 | 0.000187 | 0.000186 |

Plotting position formulae: (1) – $(i - \alpha)/(n + 1 - 2\alpha)$; and (2) – $(i - b)/(n + a)$

**Table 10.** Determined plotting position formulae in this study ($x = \log_{10} n$)

| p.d.f. | Order statistic mean | Order statistic mode | Order statistic median |
|---|---|---|---|
| Uniform | $\frac{i}{n+1}$ | $\frac{i-1}{n-1}$ | |
| Normal | $\frac{i-0.399}{n+0.203}$<br>$\alpha = 0.304914 + 0.083175x - 0.016684x^2$ | $\frac{i-0.177}{n+0.647}$<br>$\alpha = 0.220112 - 0.045813x + 0.011712x^2$ | $\frac{i-0.326}{n+0.348}$ |
| Extreme value type I | $\frac{i-0.507}{n+0.176}$<br>$a = 0.380485 - 0.182836x + 0.037633x^2$<br>$b = 0.402264 + 0.093251x - 0.018788x^2$ | $\frac{i-0.047}{n+0.704}$<br>$a = 0.692216 + 0.007157x - 0.000305x^2$<br>$b = 0.065184 - 0.023971x - 0.007414x^2$ | $\alpha = 0.280646 + 0.039468x - 0.007666x^2$ |
| Rayleigh | $\frac{i-0.349}{n+0.401}$<br>$a = 0.503293 - 0.134431x + 0.026155x^2$<br>$b = 0.314307 + 0.077267x - 0.015569x^2$ | $\frac{i-0.173}{n+0.349}$<br>$a = 0.295980 + 0.056978x - 0.014825x^2$<br>$b = 0.210695 - 0.041336x + 0.010899x^2$ | |

good agreement with Beard's (1943) potting position. The results given in Tables 7 and 9 indicate that the maximum absolute errors in estimating probability obtained by using $(i - b)/(n + a)$ for extreme value type I, Gamma, Rayleigh and 2-parameter log-normal distributions are much less than those of obtained by $(i - \alpha)/(n + 1 - 2\alpha)$. Therefore, the form of $(i - b)/(n + a)$ is more appropriate than $(i - \alpha)/(n + 1 - 2\alpha)$ when the order statistic mean and mode of random variables are considered. Furthermore, values of $a$ are not equal to $1 - 2\alpha$ for the order statistic mean and mode of extreme value type I, Gamma, Rayleigh and 2-parameter log-normal distributed random variables. Consequently, these results indicate that the plotting position $(i - 0.44)/(n + 0.12)$ for the order statistic mean of extreme value type I proposed by Gringorten (1963) might not be correct.



However, in Table 8 it is shown in the results determined from the order statistic median that the values of $\alpha$ and $b$ are the same and the values of $a$ are very close to $(1 - 2\alpha)$ for all $n$ and every distribution. The results given in Table 9 show that the maximum absolute errors in estimating probability are less than 0.000764 for all $n$ and distributions. These results indicate that the plotting position formula in the form of $(i - \alpha)/(n + 1 - 2\alpha)$ is still valid when the order statistic median is considered.

### 3.3
### The proposed plotting position formulae

The values of $\alpha$ determined previously vary with sample size $n$ and order $i$. For the convenience of practical use, the compromised plotting position formulae $(i - b)/(n + a)$ given in Table 10 are averaged over all $n$ and order $i$. For the order statistic mean and mode, the values of $a$ and $b$ vary with the parameters of some distributions, there does not exist a compromise plotting position formula for some distributions. Because the compromise formula could cause larger errors in estimating probability for some $n$, regression equations are determined for the values of $a$ and $b$ over $n$. These regression equations are also given in Table 10. Values of $a$ and $b$ obtained by using these regression equations do not differ from the corresponding tabular values of $a$ and $b$ by more than 0.0032 and 0.0022, respectively, for all $n$ and distributions. The maximum absolute errors in estimating probability of the proposed formulae are given in Table 11. For brevity, only the results of $n = 2, 10$ and 30 are presented. These results indicate high accuracy of the proposed plotting position formulae.

## 4
## Conclusions

1. The plotting position formula varies with distributions of random variables when the order statistic mean and mode are considered.
2. For the order statistic mean and mode of random variables, the plotting position formula $(i - b)/(n + a)$ is more appropriate than $(i - \alpha)/(n + 1 - 2\alpha)$. However, these two forms are the same when the order statistic median of random variables is considered.
3. A distribution free plotting position formula is determined for the order statistic median of random variables. A compromise distribution free plotting position formula $(i - 0.326)/(n + 0.348)$ is proposed to be used for all $n$ and order $i$, when the order statistic median of random variables is considered.

**Table 11.** The maximum absolute errors of the determined plotting position formulae in estimating probability

| p.d.f. | Order statistic mean | | | Order statistic mode | | | Order statistic median | | |
|---|---|---|---|---|---|---|---|---|---|
| | Plotting position formula | $n$ | Max. absolute error | Plotting position formula | $n$ | Max. absolute error | Plotting position formula | $n$ | Max. absolute error |
| Uniform | (1) | 2 | 0.000000 | (1) | 2 | 0.000036 | (1) | 2 | 0.005910 |
| | | 10 | 0.000000 | | 10 | 0.000011 | | | |
| | | 30 | 0.000000 | | 30 | 0.000012 | | | |
| Normal | (1) | 2 | 0.013503 | (1) | 2 | 0.004887 | | 10 | 0.001852 |
| | | 10 | 0.003028 | | 10 | 0.001326 | | | |
| | | 30 | 0.000639 | | 30 | 0.001067 | | | |
| | (2) | 2 | 0.000297 | (2) | 2 | 0.000185 | | 30 | 0.000636 |
| | | 10 | 0.000649 | | 10 | 0.001188 | | | |
| | | 30 | 0.000396 | | 30 | 0.001097 | | | |
| Extreme value type I | (1) | 2 | 0.018205 | (1) | 2 | 0.007364 | (2) | 2 | 0.000255 |
| | | 10 | 0.006152 | | 10 | 0.004307 | | | |
| | | 30 | 0.002204 | | 30 | 0.003164 | | | |
| | (2) | 2 | 0.000575 | (2) | 2 | 0.044087 | | 10 | 0.000716 |
| | | 10 | 0.003474 | | 10 | 0.011443 | | | |
| | | 30 | 0.001948 | | 30 | 0.003976 | | | |
| Rayleigh | (1) | 2 | 0.012398 | (1) | 2 | 0.006109 | | 30 | 0.000517 |
| | | 10 | 0.003426 | | 10 | 0.001761 | | | |
| | | 30 | 0.001766 | | 30 | 0.001522 | | | |
| | (2) | 2 | 0.000298 | (2) | 2 | 0.000426 | | | |
| | | 10 | 0.001363 | | 10 | 0.002379 | | | |
| | | 30 | 0.000932 | | 30 | 0.001535 | | | |

Plotting position formulae: (1) – $(i-\alpha)/(n+1-2\alpha)$; and (2) – $(i-b)/(n+a)$



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