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Real gas flow through heterogeneous porous media: theoretical aspects of upscaling

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Abstract. We consider the problem of upscaling transient real gas flow through heterogeneous bounded reservoirs. One of the commonly used methods for deriving effective permeabilities is based on stochastic averaging of nonlinear flow equations. Such an approach, however, would require rather restrictive assumptions about pressure-dependent coefficients. Instead, we use Kirchhoff transformation to linearize the governing stochastic equations prior to their averaging. The linearized problem is similar to that used in stochastic analysis of groundwater flow. We discuss the effects of temporal localization of the nonlocal averaged Darcy's law, as well as boundary effects, on the upscaled gas permeability. Extension of the results obtained by means of small perturbation analysis to highly heterogeneous porous formations is also discussed.

List of symbols

a	kernel of the integral in Eq. (22)
b	time-dependent function
C_Y	covariance function of Y
c_{g}	isothermal gas compressibility
D	space-dependent function
d	dimensionality
f	arbitrary function
H	Kirchhoff transform of P
K	normalized gas permeability defined in Eq. (11)
K_{G}	geometric mean of K
k	gas permeability of the medium
L_1, L_2, L_3	length of the rectangular box in x_1, x_2, x_3 directions
l_Y	correlation length of Y
$\mathbf{J}$	mean pseudo-pressure gradient, $(J_1, J_2, J_3)^{\mathrm{T}}$
$\mathbf{n}$	unit outward normal to the boundary

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P	gas pressure on Dirichlet boundaries Γ_D
p	gas pressure
p_m	low base pressure
p^*	constant pressure defined in Eq. (11)
Q	Kirchhoff transform of V
$\mathbf{q}$	mass flux vector

Subscripts

D	dimensionless
d	dimensionality
eff	effective
in	initial
st	steady-state
p	pressure-dependence

1 Introduction

Thoroughly conducted field tests and theoretical analyses have revealed that permeability of natural reservoirs is spatially variable, with values ranging over a few orders of magnitude. Detailed reviews of the problems associated with realistic representation of reservoir heterogeneities and their impacts on current trends in reservoir simulations are given, among others, by Fayers and Hewett [1], Wen and Gómez-Hernández [2], and Christie [3]. In recent years, upscaling based on stochastic analysis has become a popular tool. Using this approach, one can calculate "an effective (upscaled) permeability" which can be assigned to either the whole reservoir or a cell of the numerical grid.

In the sequel, we define the effective permeability tensor, $\mathbf{k}_{\text{eff}}$, as a coefficient that, when multiplied by an averaged (mean) pressure gradient, produces an averaged (mean) Darcian velocity [4–8]. This definition is useful if $\mathbf{k}_{\text{eff}}$ depends exclusively on the statistical structure of the random field k. It is worthwhile to note here that thus defined effective permeability differs from meanings ascribed to similar terms by King [9], Desbaratas and Dimitrakopolous [10], Morgan and Babuška [11], and Durlofsky [12]. (For a detailed discussion of different definitions of effective or equivalent parameters, see Neuman and Orr [8] and references therein.) In his recent paper, Christie [3] notes that "one of the main limitations of upscaling is that it usually gives an answer with almost no indication of whether the assumptions made in deriving the answer hold. Limited attempts have been made to analyze the upscaling process [13], but so far, no good theory exists that unequivocally states whether an upscaled value provides a good or bad approximation". In terms of stochastic analysis of groundwater flow, these assumptions are (i) ergodicity, i.e. possibility to interchange space and/or time averages by "expected values" or "ensemble averages"; (ii) mild heterogeneity which allows one to use the perturbation analysis; and (iii) slow space and time variation of the mean pressure gradient which is necessary for localization of the averaged Darcy's law [6, 8, 14–16]. Analysis of gas, or multiphase, flow involves averaging of stochastic nonlinear differential equations and requires additional approximations. Some of these approximations will be discussed later in this paper.

Ergodicity is always required to infer field statistics from spatial measurements; it cannot be proven, only hypothesized, unless falling under the purview of well established ergodic theorems such as the law of large numbers. Nevertheless, it can be shown that [17] "in any application, non-ergodicity usually just means that the random function concerned is, in fact, an artificial union of a number of distinct ergodic stationary functions".

The small perturbation analysis has been applied extensively to the study of flow through heterogeneous formations. Although this approach is rigorously valid when the variance of natural log permeability, $Y = \ln k$, is much smaller than unity, it was shown [18] to work reasonably well for the variance up to 2. One can extend the results based on perturbation analysis to larger variances by means of the Landau–Lifshitz conjecture [4, 5, 8, 9, 19].

Strictly speaking, the averaged flow equation can be localized, and the effective permeability can be introduced, only under conditions of the uniform mean pressure gradient. Application of similar results to nonuniform average flow is possible under assumption that the mean gradient varies slowly in space and time. Tartakovsky and Neuman [20] have investigated the validity and applicability of this assumption.

Thus stochastic analysis of single phase groundwater or oil flows provides definite answers to the questions of applicability and limitations of stochastically derived effective permeability. The analysis becomes much more complicated when applied to real gas flow through heterogeneous porous media, since this flow is described by nonlinear differential equations. To the best of author's knowledge, no attempts to obtain stochastically derived effective permeability for gas flow have been made up to date.

In this paper, we use the Kirchhoff transformation [21] to linearize the governing equations. Subsequent stochastic analysis is similar to that performed for groundwater flow. We investigate localization and boundary effects on effective gas permeability.

2
Statement of the problem

Consider laminar and isothermal flow of a real gas of constant composition through heterogeneous porous media, Ω. We start with a local scale (the scale of an in situ experiment), ω, $(\omega < \Omega)$ on which the Darcy law holds

$$\mathbf{v}(\mathbf{x}, t) = -\frac{k(\mathbf{x})}{\mu(p)} \nabla p(\mathbf{x}, t) \quad \mathbf{x} \in \omega \tag{1}$$

where the volume ω is centered about $\mathbf{x}$, $\mathbf{v}$ is the macroscopic (Darcian) velocity, k is the gas permeability of the medium, μ is the gas viscosity, and p is the pressure. Although the gas permeability k is taken to be independent from p, we show later in the paper that the case of $k(\mathbf{x}, p)$ can be handled as well.

The conservation of mass for isothermal flow implies

$$\nabla \cdot [\rho(p)\mathbf{v}(\mathbf{x}, t)] = -\phi(\mathbf{x}) \frac{\partial \rho(p)}{\partial t} \quad \mathbf{x} \in \omega \tag{2}$$

where ρ is the density, and ϕ is the porosity.

One can formally write Eqs. (1) and (2) for a larger domain Ω. We consider two types of boundary conditions for Ω:

$$p(\mathbf{x}, t) = P(\mathbf{x}, t) \quad \mathbf{x} \in \Gamma_{\mathrm{D}} \tag{3}$$

$$-\rho(p)\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = V(\mathbf{x}, t) \quad \mathbf{x} \in \Gamma_{\mathrm{N}} \tag{4}$$

supplemented by the initial condition

$$p(\mathbf{x},0) = p_{\mathrm{in}}(\mathbf{x}) \quad \mathbf{x} \in \Omega \ . \tag{5}$$

Here $P(\mathbf{x},t)$ and $V(\mathbf{x},t)$ are randomly prescribed head and flux on Dirichlet, Γ_{D}, and Neumann, Γ_{N}, boundary segments whose union forms a boundary Γ of the domain Ω; and $\mathbf{n}(\mathbf{x})$ is the unit outward normal to the boundary Γ. The random initial pressure distribution is given by $p_{\mathrm{in}}(\mathbf{x})$.

For real gases under isothermal conditions the equation of state has the form [22]

$$\rho(p) = \frac{1}{\mathrm{RT}} \frac{p}{Z(p)} \tag{6}$$

where R is the gas constant, T is the constant temperature, and $Z(p)$ is the compressibility factor.

Since the permeability $k(\mathbf{x})$ can be measured only at selected points inside Ω on the scale ω, and varies over orders of magnitude within Ω, it can be treated as a random variable. As $k(\mathbf{x})$ is scale-dependent and random, so will be the pressure and fluxes. In other words, Eqs. (1)–(6) written for $\mathbf{x} \in \Omega$ constitute a system of nonlinear stochastic differential equations.

A traditional approach for obtaining effective permeability in the somewhat similar context of unsaturated flow consists of direct stochastic averaging of Eqs. (1)–(6). The resulting averaged equations contain ensemble means of some deterministic functions of random argument, $\langle f(p) \rangle$. It is common in stochastic analyses of unsaturated flow to expand these terms in Taylor series and retain only the leading term in these expansions, i.e. $\langle f(p) \rangle \approx f(\langle p \rangle)$. Such an approach does not seem to be quite satisfactory since it requires a priori knowledge of the p behavior to guarantee convergence of the Taylor series. Instead, we propose to linearize the governing equations prior to their averaging.

3 Linearization

Substituting Eqs. (1) and (6) into Eq. (2) gives

$$\nabla \cdot \left[k(\mathbf{x}) \frac{p(\mathbf{x},t)}{\mu(p)Z(p)} \nabla p(\mathbf{x},t) \right] = \phi(\mathbf{x}) \frac{\partial}{\partial t} \left[\frac{p(\mathbf{x},t)}{Z(p)} \right] \ . \tag{7}$$

Applying the Kirchhoff transformation [21],

$$\Psi(\mathbf{x},t) = \int_{p_m}^{p} \frac{s}{\mu(s)Z(s)} \mathrm{d}s \tag{8}$$

where p_m is the low base pressure, yields

$$\nabla \cdot [k(\mathbf{x}) \nabla \Psi(\mathbf{x},t)] = \phi(\mathbf{x}) \mu(p) c_{\mathrm{g}}(p) \frac{\partial \Psi(\mathbf{x},t)}{\partial t} \ . \tag{9}$$

Here $c_{\mathrm{g}}(p)$ is the isothermal gas compressibility defined as

$$c_g(p) = \frac{1}{\rho(p)} \frac{\partial \rho(p)}{\partial p} \quad . \tag{10}$$

A similar treatment of the real gas flow equation has been first carried out by Al-Hussainy et al. [23] who have called the quantity 2Ψ a real gas pseudo-pressure; Goggin et al. [24] have named the same term a pseudo-potential. For isothermal flow of an ideal gas, $Z(p) \equiv 1$, μ is constant, and one has for the pseudo-pressure: $2\Psi = p^2$.

While quasi-linear Eq. (9) is rigorously valid for arbitrary pressure gradients and variations of $\mu(p)c_g(p)$, its further linearization requires additional assumptions. For a constant rate of gas production, Aronofsky and Jenkins [25] (for ideal gas) and Al-Hussainy et al. [23] (for real gas) have demonstrated that the solution of Eq. (9) with the coefficient on the right hand side evaluated at initial pressure p_{in}, is in a good agreement with experiments. For other field scenarios, Al-Hussainy et al. noted that evaluating $\mu(p)c_g(p)$ "about half way between the extremes might be quite good". We therefore feel comfortable using a linearized version of Eq. (9) with $\mu(p)c_g(p)$ evaluated at some pressure p^*: 113

$$\nabla \cdot [K(\mathbf{x})\nabla\Psi(\mathbf{x}, t)] = S(\mathbf{x}) \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} \quad , \tag{11}$$

where $K(\mathbf{x}) = k(\mathbf{x})/\mathrm{RT}$, $S(\mathbf{x}) = \phi(\mathbf{x})\mu(p^*)c_g(p^*)/\mathrm{RT}$, and p^* is either one of the extreme pressures (initial or final) or their average. We leave the question of how this approximation influences the behavior of effective gas permeability for future studies.

In terms of the pseudo-pressure, a mass flux $\mathbf{q}(\mathbf{x}, t) = \rho(p)\mathbf{v}(\mathbf{x}, t)$ can be expressed as

$$\mathbf{q}(\mathbf{x}, t) = -K(\mathbf{x})\nabla\Psi(\mathbf{x}, t) \quad . \tag{12}$$

By the same token the boundary and initial conditions (3)–(5) take the form

$$\Psi(\mathbf{x}, t) = H(\mathbf{x}, t) \quad \mathbf{x} \in \Gamma_D \tag{13}$$

$$-\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = Q(\mathbf{x}, t) \quad \mathbf{x} \in \Gamma_N \tag{14}$$

$$\Psi(\mathbf{x}, 0) = \Psi_{in}(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{15}$$

where the random functions $H(\mathbf{x}, t)$, $Q(\mathbf{x}, t)$, and $\Psi_{in}(\mathbf{x})$ are the Kirchhoff transforms of $P(\mathbf{x}, t)$, $\rho(p)V(\mathbf{x}, t)$, and p_{in}, respectively.

Thus the nonlinear problem of gas flow reduces to the linear problem of groundwater flow. It is worthwhile to emphasize here that such a linearization is not new; it is its application to the problem of upscaling which is novel.

4
Stochastic averaging of the linearized problem

In a context of groundwater flow a problem similar to Eqs. (11)–(15) was considered by Tartakovsky and Neuman [26]. Decomposing the random functions into their ensemble means and zero-mean fluctuations about these means yields

$$K(\mathbf{x}) = \langle K \rangle + K'(\mathbf{x}) \quad \langle K'(\mathbf{x}) \rangle \equiv 0 \quad , \tag{16}$$

$$\Psi(\mathbf{x},t) = \langle \Psi(\mathbf{x},t)\rangle + \Psi'(\mathbf{x},t) \quad \langle \Psi'(\mathbf{x},t)\rangle \equiv 0 \ , \tag{17}$$

$$\mathbf{q}(\mathbf{x},t) = \langle \mathbf{q}(\mathbf{x},t)\rangle + \mathbf{q}'(\mathbf{x},t) \quad \langle \mathbf{q}'(\mathbf{x},t)\rangle \equiv 0 \ . \tag{18}$$

Substituting Eqs. (16)–(18) into Eq. (12) and taking the ensemble mean give the averaged Darcy's law,

$$\langle \mathbf{q}(\mathbf{x},t)\rangle = -\langle K\rangle \mathbf{J}(\mathbf{x},t) + \mathbf{r}(\mathbf{x},t) \ . \tag{19}$$

Here $\mathbf{J}(\mathbf{x},t) = \nabla\langle\Psi(\mathbf{x},t)\rangle$ is the mean pseudo-pressure gradient, and $\mathbf{r}(\mathbf{x},t) = -\langle K'(\mathbf{x})\nabla\Psi'(\mathbf{x},t)\rangle$ is a so-called "residual" flux [8]. Numerous studies of groundwater flow have demonstrated that $\mathbf{r}(\mathbf{x},t)$ is not necessarily small and therefore cannot be neglected. Hence $\langle K\rangle$ does not act as an upscaled permeability. Moreover, it has been demonstrated that the residual flux is generally nonlocal in time and space. In terms of our analysis this means that $\mathbf{r}(\mathbf{x},t)$ is given by integrals containing pseudo-pressure gradients at points other than $\mathbf{x}$ and times other than t (see Appendix A for more details). The kernels of these integrals reflect a permeability correlation structure and a shape of the domain Ω. As a result, a general effective permeability that depends only on the properties of the medium cannot be defined. Then one can solve numerically a set of recursive approximations [26] for nonlocal Eq. (19). The advantage of using this procedure is that the averaged quantities involved are relatively smooth functions defined on a coarse grid Ω. This makes it possible to use standard numerical techniques, such as finite element methods, more efficiently.

5
Effective permeability

For the sake of simplicity we, following Tartakovsky and Neuman [27], take Ω to be a box-shaped rectangular grid-block with lateral mean no-flow boundaries separated by distances equal to L_2 and L_3, and two constant head boundaries a distance L_1 apart. A spatially uniform mean pseudo-pressure gradient $\mathbf{J}(t) = \nabla\langle\Psi\rangle$ of magnitude $J_1 = [H_2(t) - H_1(t)]/L_1$ acts between the Dirichlet boundaries parallel to the x_1-coordinate. The permeability field is assumed to be log-normally distributed and statistically homogeneous with an exponential covariance function

$$C_Y(\mathbf{x},\mathbf{y}) = \sigma_Y^2 \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|}{l_Y}\right) \tag{20}$$

where σ_Y^2 and l_Y are the variance and the correlation length of the log-permeability $Y = \ln K$.

As shown in Appendix A, a first-order (in σ_Y^2) approximation of Eq. (19) is given by

$$q_1^{[1]}(\mathbf{x},t) = -K_{\mathrm{G}}\left[1 + \frac{\sigma_Y^2}{2}\right] J_1(t) + r_1^{(1)}(\mathbf{x},t) \tag{21}$$

where

$$r_1^{(1)}(\mathbf{x},t) = \int_0^t a(\mathbf{x},t-\tau) J_1(\tau)\,\mathrm{d}\tau \ , \tag{22}$$

and the kernel a is given by Eq. (41). To introduce an effective permeability k_{eff}, a further localization is necessary. Assuming slow time variation of the mean uniform pseudo-pressure gradient, i.e. $J_1(\tau) \approx J_1(t)$, gives

$$q_1^{[1]}(\mathbf{x}, t) = -k_{\text{eff}}^{[1]}(\mathbf{x}, t) J_1(t) \tag{23}$$

where

$$k_{\text{eff}}^{[1]}(\mathbf{x}, t) = K_G \left[1 + \frac{\sigma_Y^2}{2} \right] - \kappa(\mathbf{x}, t) \tag{24}$$

and

$$\kappa(\mathbf{x}, t) = \int_0^t a(\mathbf{x}, t - \tau) \mathrm{d}\tau \ . \tag{25}$$

It follows from Eq. (24) that thus introduced effective permeability is space and time dependent. As we show below, the space dependence results from boundary effects, while the time dependence is a reflection of transient nature of the gas flow.

5.1 Effective permeability for an infinite domain

When boundaries of Ω are situated far away from the point of interest, one can approximate the domain as being infinite. We start by considering steady-state gas flow in an infinite domain. A corresponding expression for k_{eff} is obtained by taking the limits of Eq. (24) as $t \to \infty$ and $L_1, L_2, L_3 \to \infty$. Tartakovsky and Neuman [27] have demonstrated that at these limits Eq. (25) gives (in d dimensions) $\kappa = K_G \sigma_Y^2 / d$, and Eq. (24) leads to a well known expression [6–8]

$$\frac{k_{\text{eff}}^{[1]}}{K_G} = 1 + \sigma_Y^2 \left[\frac{1}{2} - \frac{1}{d} \right] \ . \tag{26}$$

Under conditions of transient flow, the effective permeability becomes time-dependent and Eqs. (24) and (25) yield [27, 14]

$$\frac{k_{\text{eff}}^{[1]}(t_D)}{K_G} = 1 + \sigma_Y^2 \left[\frac{1}{2} - \frac{1}{d} + \frac{1}{d} b_d(t_D) \right] \tag{27}$$

where $t_D = t K_G / S l_Y$ is the dimensionless time, and the behavior of $b_d(t_D)$ is shown in Fig. 1. One can see that the dimensionless relaxation time t_r, required for transient effects to dissipate, is significantly smaller for three-dimensional flow than for one-dimensional flow. Hence under real reservoir conditions, it might be possible to disregard time-dependence of the effective permeability.

5.2 Effect of temporal localization

To investigate the possibility of temporal localization, we compare the fluxes computed via the nonlocal Darcy's law (21) and its localized version (23) for

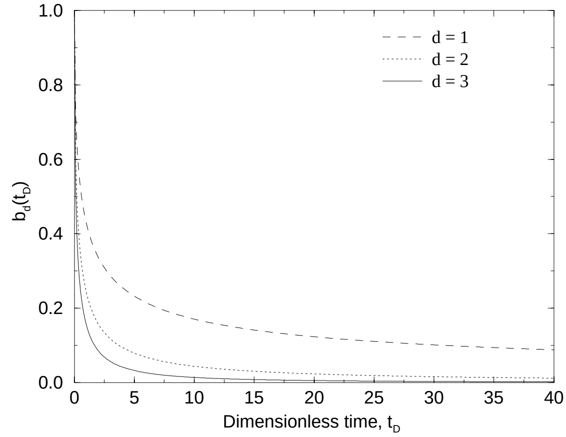

Fig. 1. Time-dependence of the effective permeability for statistically isotropic one-, two-, and three-dimensional heterogeneous media (after Dagan [14])

one- and three-dimensional flows. Tartakovsky and Neuman [20] have evaluated Eqs. (21) and (23) for several functional dependencies $J_1(t_D)$. The authors found that for mildly heterogeneous formations, when $\sigma_Y^2 = 0.1$, there is very little difference between localized and time-nonlocal behaviors. Figure 2 compares mean fluxes, normalized by the geometric mean K_G, computed with $\sigma_Y^2 = 1$ and

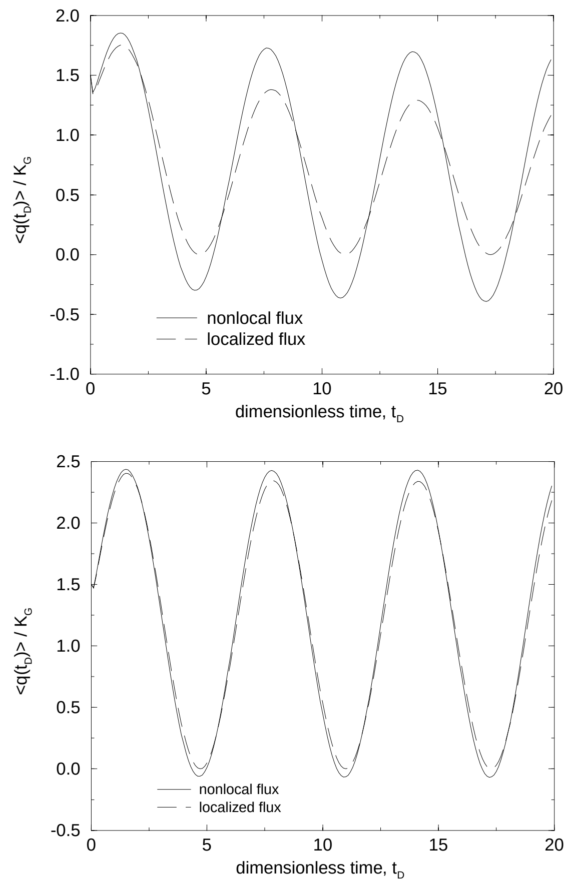

Fig. 2. Normalized nonlocal and localized mean fluxes. (a) One-dimensional flow. (b) Three-dimensional flow. (after Tartakovsky and Neuman [20])

$J_1(t_D) = 1 + \sin(t_D)$. This comparison shows that the difference between local and nonlocal behaviors is more pronounced in one dimension then in three. It also suggests that even for relatively large variances, time-localization of the averaged three-dimensional flow (introduction of the effective permeability) gives satisfactory results.

5.3
Boundary effects

In the regions close to the boundaries of the rectangular box, boundary effects cannot be neglected, and the resulting effective permeability (24) becomes space and time dependent,

$$\frac{k_{\text{eff}}^{[1]}(t_D)}{K_G} = 1 + \sigma_Y^2 \left[\frac{1}{2} - D_{\text{st}}(\chi) + b(\chi, t_D) \right] \tag{28}$$

where χ is a coordinate vector with dimensionless components $\chi_i = x_i/L_i$, D_{st} represents the steady-state component of k_{eff}, and b its transient component. The following analysis is due to Tartakovsky and Neuman [27].

In the region sufficiently away from the boundaries, $D_{\text{st}}(\chi) \equiv 1/3$, $b(\chi, t_D) \equiv b_3(t_D)/3$, and Eq. (28) reduces to Eq. (27). The behavior of the upscaled gas permeability in the boundary layer of the cube $L_1 = L_2 = L_3 \equiv L$ is shown in Figs. 3 and 4. Figure 3 illustrates the dependence of the steady-state component of k_{eff} on a dimensionless cube of size $2\rho = L/l_Y$. One can see that D_{st} reaches 90% of its asymptotic value of $1/3$ at a distance of 10 correlation scales away from the boundaries. Figure 4 shows the behavior of b as function of dimensionless time for several dimensionless sizes of the cube. Note that the presence of the boundaries diminishes the transient component of the upscaled permeability. Similar to the steady-state component, the transient component of the effective permeability reaches its asymptotic value when the box is larger than 10 correlation scales.

6
Generalization to strongly heterogeneous media

The results of the previous section were derived by means of the small perturbation analysis under assumption of mildly heterogeneous porous media, i.e.

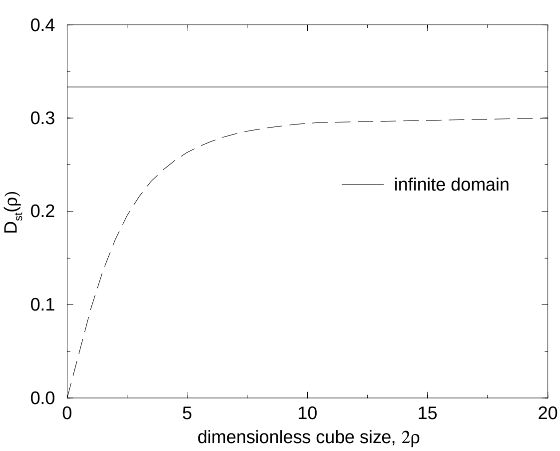

Fig. 3. Dependence of the steady-state component D_{st} of the upscaled gas conductivity k_{eff} on dimensionless box size $2\rho = L/l_Y$ (after Tartakovsky and Neuman [27])

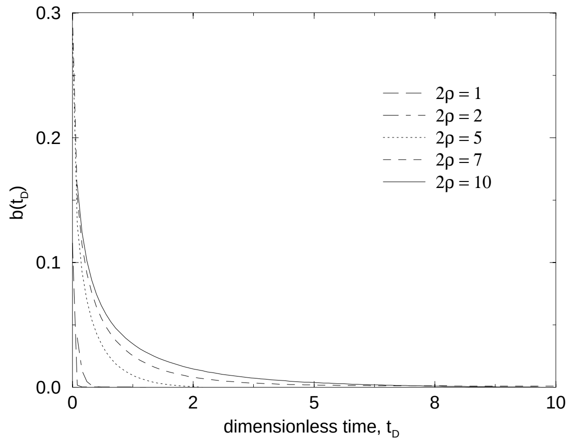

Fig. 4. Time-dependence of the effective permeability for several dimensionless cube sizes

$\sigma_Y^2 \ll 1$. These results can be generalized for the case of strongly heterogeneous media by employing a conjecture that became known in stochastic subsurface hydrology as the Landau–Lifshitz conjecture (LLC) [4, 5, 8, 9, 19]. According to this conjecture, Eqs. (26)–(28) are treated as a Taylor expansion of the corresponding exponents. In particular, Eq. (26) can be thought of as a first-order approximation of the effective gas permeability,

$$\frac{k_{\text{eff}}^{[1]}}{K_{\text{G}}} = \exp\left(\sigma_Y^2\left[\frac{1}{2} - \frac{1}{d}\right]\right) \ . \tag{29}$$

It was proven that LLC is rigorously valid under one-dimensional flow in log-normal fields [28, 29]. It is also rigorously valid under two-dimensional flow in log-normal, statistically isotropic permeability fields [5, 8]. Attempts to prove LLC rigorously for three dimensional flow were reported [9, 19], but De Wit [30] was able to demonstrate that LLC is not rigorously valid for three-dimensional Gaussian isotropic media. Nevertheless, numerical Monte Carlo simulations [31] showed that, for isotropic Gaussian Y with exponential covariance, LLC holds at least up to $\sigma_Y^2 = 7$.

7
Conclusions

We considered transient flow of real gases through bounded heterogeneous porous media. To obtain an effective gas permeability, stochastic analysis of nonlinear differential equations that describe this flow was employed. Standard stochastic averaging was preceded by the linearization based on the Kirchhoff transformation. The linearized stochastic differential equations are similar to those used to describe groundwater flow in heterogeneous formations. Thus, an upscaled gas permeability has the same qualities as an upscaled hydraulic conductivity. In particular,

1. The averaged Darcy's law for gas flow is generally nonlocal, and mean gas permeability $\langle k\rangle$ does not represent an upscaled permeability k_{eff}.
2. Localization of the averaged Darcy's equation requires an assumption of slow space-time variation of the mean pseudo-pressure gradient. For three-dimensional flow, time-localization performs reasonably well.

3. For steady-state gas flow away from the boundaries, the upscaled gas permeability is given by the expression identical to that for the effective conductivity for groundwater flow.
4. Transient flow introduces time-dependence of the upscaled gas permeability. However, under real three-dimensional reservoir conditions, it might be possible to use its steady-state counterpart.
5. Presence of boundaries introduces spatial dependence of k_{eff}. The upscaled gas permeability increases with the size of the domain until it reaches its asymptotic value corresponding to a d-dimensional infinite domain. In box-shaped domains with no-flow and constant head boundaries, boundary effects cease to exist at a distance of 10 correlation scales away from the boundaries.

While this paper dealt with upscaling of the pressure independent gas permeability, the incorporation of pressure-dependence can be handled easily. This formally extends our results to wet condensate gas flow where the effects of Klinkenberg's gas slippage are important. To account for this phenomenon, the pressure-dependent permeability for gas is taken in the form $k_p(\mathbf{x}, p) = k(\mathbf{x})f(p)$ where $f(p)$ is the deterministic pressure-dependent correction. This correction can be easily incorporated in the Kirchhoff transformation (8), and thus all results obtained for upscaling of $k(\mathbf{x})$ hold.

Appendix A: Residual flux

Considering transient groundwater flow through heterogeneous formations, Tartakovsky and Neuman [26] have shown that the residual flux $\mathbf{r}(\mathbf{x}, t)$ can be formally written as the solution of the integral equation

$$\mathbf{r}(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{A}(\mathbf{y}, \mathbf{x}, t - \tau)\mathbf{J}(\mathbf{y}, \tau)\mathrm{d}\mathbf{y}\mathrm{d}\tau + \int_0^t \int_\Omega \mathbf{B}(\mathbf{y}, \mathbf{x}, t - \tau)\mathbf{r}(\mathbf{y}, \tau)\mathrm{d}\mathbf{y}\mathrm{d}\tau \ . \quad (30)$$

The kernels $\mathbf{A}$ and $\mathbf{B}$ are a quadratic form with respect to space variables and a nonsymmetric tensor, respectively. To evaluate these kernels exactly would require calculating the random Green's function for the boundary-value problem (8)–(12). An approximation can be obtained by means of a perturbation analysis. In particular, a first-order (in variance $\sigma_Y^2 = \langle Y'(\mathbf{x})Y'(\mathbf{x})\rangle$ of log-permeability $Y = \ln K$) approximation of Eq. (30) yields [26]

$$\mathbf{r}^{(1)}(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{A}^{(1)}(\mathbf{x}, \mathbf{y}, t - \tau)\mathbf{J}^{(0)}(\mathbf{y}, \tau)\mathrm{d}\mathbf{y}\mathrm{d}\tau \quad (31)$$

where $\mathbf{J}^{(0)}$ is the zeroth-order approximation of $\mathbf{J}$, and $\mathbf{A}^{(1)}$ is the first-order approximation of $\mathbf{A}$ given by

$$\mathbf{A}^{(1)} = K_\mathrm{G} C_Y(\mathbf{x}, \mathbf{y}) \nabla_\mathbf{x} \nabla_\mathbf{y}^\mathrm{T} G_K(\mathbf{x}, \mathbf{y}, t - \tau) \quad (32)$$

Here $K_\mathrm{G} = \exp(\langle Y\rangle)$ is the geometric mean of K, $C_Y = \langle Y'(\mathbf{x})Y'(\mathbf{y})\rangle$ is the covariance function of Y, and the Green's function $G_K(\mathbf{x}, \mathbf{y}, t - \tau)$ is the solution of an equation

$$\nabla^2_{\mathbf{x}} G_K + \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau) = S(\mathbf{x})\frac{\partial G_K}{\partial t} \tag{33}$$

subject to boundary and initial conditions

$$G_K(\mathbf{x},\mathbf{y},t-\tau) = 0 \quad \mathbf{x} \in \Gamma_{\mathrm{D}} \tag{34}$$

$$\nabla_{\mathbf{x}} G_K(\mathbf{x},\mathbf{y},t-\tau)\cdot \mathbf{n}(\mathbf{x}) = 0 \quad \mathbf{x} \in \Gamma_{\mathrm{N}} \tag{35}$$

$$G_K(\mathbf{x},\mathbf{y},0) = 0 \quad \mathbf{x} \in \Omega \ . \tag{36}$$

Expanding $\langle \mathbf{q} \rangle$, $\langle K \rangle$, and $\mathbf{J}$ in Eq. (19) in power series of $Y'(\mathbf{x}) = Y(\mathbf{x}) - \langle Y \rangle$, and neglecting the terms of order higher than σ_Y^2 in these expansions, yields

$$\langle \mathbf{q}^{[1]}(\mathbf{x},t)\rangle = \mathbf{q}^{(0)}(\mathbf{x},t) + \langle \mathbf{q}^{(1)}(\mathbf{x},t)\rangle \tag{37}$$

where

$$\mathbf{q}^{(0)}(\mathbf{x},t) = -K_{\mathrm{G}}\mathbf{J}^{(0)}(\mathbf{x},t) \tag{38}$$

and

$$\langle \mathbf{q}^{(1)}(\mathbf{x},t)\rangle = -K_{\mathrm{G}}\left[\mathbf{J}^{(1)}(\mathbf{x},t) + \frac{\sigma_Y^2}{2}\mathbf{J}^{(0)}(\mathbf{x},t)\right] + \mathbf{r}^{(1)}(\mathbf{x},t) \ . \tag{39}$$

For the mean spatially uniform flow, $\mathbf{J}(\mathbf{x},t) \equiv \mathbf{J}(t) \equiv \mathbf{J}^{(0)}(t)$. Assuming that the mean flow is parallel to the x_1-direction, i.e. $\mathbf{J} = (J_1, 0, 0)^{\mathrm{T}}$, leads directly to Eq. (21). It follows from Eq. (31) that

$$r_1^{(1)}(\mathbf{x},t) = \int_0^t a(\mathbf{x},t-\tau)\mathbf{J}(\mathbf{y},\tau)\mathrm{d}\tau \tag{40}$$

where

$$a(\mathbf{x},t-\tau) = \int_{\Omega} A_{11}^{(1)}(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\mathbf{y} \tag{41}$$

and

$$A_{11}^{(1)} = K_{\mathrm{G}} C_Y(\mathbf{x},\mathbf{y})\frac{\partial^2 G_K(\mathbf{x},\mathbf{y},t-\tau)}{\partial x_1 \partial y_1} \ . \tag{42}$$

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