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ORIGINAL PAPER

Models for the difference of continuous covariance functions

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Abstract

A linear combination, with negative weights, of two continuous covariance functions has been analyzed by a few authors just for special cases and only in the real domain. However, a covariance is a complex valued function: for this reason, the general problem concerning the difference of two covariance functions in the complex domain needs to be analyzed, while it does not yet seem to have been addressed in the literature; hence, exploring the conditions such that the difference of two covariance functions is again a covariance function can be considered a further property. Therefore, this paper yields a contribution to the theory of correlation, hence the results cannot be restricted to the particular field of application. Starting from the difference of two complex covariance functions defined in one dimensional Euclidean space, wide families of models for the difference of two complex covariance functions can be built in any dimensional space, utilizing some well known properties. In particular, the difference of two real covariance functions has been considered; moreover, the difference between some special isotropic covariance functions has also been analyzed. A detailed analysis of the parameters of the models involved has been proposed; this kind of analysis opens a gate for modeling, in any dimensional space, the correlation structure of a peculiar class of complex valued random fields, as well as the subset of real valued random fields. Some relevant hints about how to construct the subset of real covariance functions characterized by negative values have also been given.

Keywords Complex covariance functions · Spectral distribution functions · Spectral density functions

1 Introduction

The class of continuous covariance functions is completely characterized by Bochner's theorem: this important result shows that a covariance function (c.f.) is, in general, a complex valued function; for this reason, the whole family of real c.f.s, which are often utilized in several applications (Cressie and Huang 1999; De Iaco et al. 2001, 2002; Gneiting 2002b), only represents a subset of the wider set of complex c.f.s and are, very often, positive in the whole set in which they are defined.

Properties of c.f.s for second order stationary random fields are well known in the literature (Yaglom 1987; Christakos 2017); in particular, it is well known that, in general, the difference of two c.f.s is not a c.f.

In the literature, a few authors have addressed the problem of investigating the difference (or linear combinations with negative weights) of two c.f.s; moreover, this kind of analysis has been performed only in the real domain. At this purpose, a linear combination, with negative weights, of two real valued spatial or spatio-temporal c.f.s has been analyzed by Ma (2005): however, the two real c.f.s involved in the analysis are isotropic and belong to the same family. In addition, a class of c.f.s which allows negative values has also been proposed by Gregori et al. (2008): although it is not required that the c.f.s involved belong to the same class, the applications are however restricted to the case of Gaussian and Matern families (Matern 1980).

In the very last years some efforts have been made in geostatistics utilizing complex formalism. In particular, an application to predict a wind field has been proposed by De Iaco and Posa (2016), and a first attempt of complex

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formalism in the spatio-temporal context has been provided by Cappello et al. (2020).

As far as we know, there are no contributions in the literature concerning the difference of two c.f.s in the complex domain. Hence, this paper can be considered one of the first attempts to explore the conditions under which the difference of two c.f.s in the complex domain is again a c.f.: these results can be added to the classical properties of c.f.s, well known in the literature. This paper thus yields a general contribution to the theory of correlation, hence the results cannot be restricted to the particular field of application (i.e., geostatistics, time series analysis). In particular, valid models for the difference of c.f.s in the complex domain as well as in the real domain will be proposed; moreover, in order to provide a complete scenario of the subject to be utilized in several and different situations, some examples of the difference of two isotropic c.f.s are also explored for the two and three dimensional Euclidean spaces.

As will be shown, if the conditions for which the difference of two c.f.s provides a new c.f., in $\mathbb{R}$, are satisfied, the models obtained through this difference are characterized by different features: for example, they could be non negative in the whole domain, or could be characterized by negative values in a subset of their domain. Indeed, in several applications concerning biology, hydrology and spatio-temporal turbolence, c.f.s with negative values are often needed (Shkarofsky 1968; Pomeroy et al. 2003).

The various examples, concerning the difference of two c.f.s and proposed in this paper, are flexible enough to be utilized in several applications pertaining to the complex domain, as well as to the subset of the real domain. In particular, it will be shown how some special families of c.f.s, obtained through the difference of two c.f.s, present some relevant and flexible characteristics which the standard parametric families of c.f.s and their linear combinations with non negative coefficients, as well as their products, are not able to satisfy.

This paper is organized as follows: in Sect. 2 some characteristics of continuous c.f.s in the complex domain are summarized. In Sect. 3 conditions for which the difference of two c.f.s is again a c.f. have been analyzed in the complex domain, as well as in the special case of the real domain through a detailed analysis involving the parameters of these models; in particular, starting from separable models, the c.f.s obtained through the difference are non separable. In the case of real c.f.s, some well known models, often utilized in the applications, have been considered; some examples of the difference of two isotropic c.f.s are also explored. The results can be utilized in a flexible way in any dimensional domain. An overview and some relevant hints about how to construct c.f.s with negative values have been given in Sect. 4, whereas the relevance of the results has been pointed out in Sect. 5.

2 A brief overview

In this section, a brief outline of continuous c.f.s, which are completely characterized by Bochner's theorem (Bochner 1959), is provided. Let's denote with $\mathbb{R}^n$ the Euclidean n-dimensional space, with $\mathbb{C}$ the set of the complex numbers and i the imaginary unit.

Theorem 1 Bochner's theorem. *Let $C : \mathbb{R}^n \to \mathbb{C}$, be a Hermitian function, then C is a continuous c.f. if and only if it is of the form*

$$C(\mathbf{x}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})dF(\boldsymbol{\omega}), \tag{1}$$

where F is a non decreasing and non negative bounded measure on $\mathbb{R}^n$.

The c.f. C in (1) can also be written as: $C(\mathbf{x}) = C_{re} + iC_{im}$, where

$$C_{re} = \int_{\mathbb{R}^n} \cos(\boldsymbol{\omega}^T\mathbf{x})dF(\boldsymbol{\omega})$$

and

$$C_{im} = \int_{\mathbb{R}^n} sen(\boldsymbol{\omega}^T\mathbf{x})dF(\boldsymbol{\omega});$$

in particular, C_{re} is a real c.f., whereas C_{im} is an odd function and it is not a c.f..

According to the previous theorem, any continuous c.f. can be represented as in (1) and the converse is also true. Please note that as it is the Fourier transform of the finite measure F, the function C must be continuous. From (1) it follows that:

1. $C(\mathbf{0}) \geq 0$;
2. $C(-\mathbf{x}) = \overline{C}(\mathbf{x})$;
3. $|C(\mathbf{x})| \leq C(\mathbf{0})$,

where the bar denotes complex conjugate. If the non negative spectral distribution function F is absolutely continuous, then there exists: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that:

$$C(\mathbf{x}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})f(\boldsymbol{\omega})d\boldsymbol{\omega}, \tag{2}$$

where f is called *spectral density function*, it is non negative and integrable. In general, there is no guarantee that C is integrable. If C is integrable, then the spectral distribution function F is absolutely continuous and the spectral density function f can be expressed in terms of the c.f. C, i.e.,

$$f(\boldsymbol{\omega}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-i\boldsymbol{\omega}^T\mathbf{x})C(\mathbf{x})d\mathbf{x}. \tag{3}$$

If F is an absolutely continuous spectral distribution function, then

$$\tilde{F}(\boldsymbol{\omega}) = C(\mathbf{0}) - F(-\boldsymbol{\omega})$$

is also a an absolutely continuous spectral distribution function and it is called the *conjugate spectral distribution* of F.

Corollary 1 The c.f. of the conjugate spectral distribution $\tilde{F}$ is the conjugate c.f. of C, i.e.,

$$\overline{C}(\mathbf{x}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x}) d\tilde{F}(\boldsymbol{\omega}).$$

Hence, if C is a c.f., then $\overline{C}$ is also a c.f..

Corollary 2 If the spectral distribution function F is absolutely continuous, then the c.f. of the spectral density function $\tilde{f}(\boldsymbol{\omega}) = f(-\boldsymbol{\omega})$, corresponding to the conjugate spectral distribution $\tilde{F}$, is the conjugate c.f. of C of the spectral density function $\tilde{f}$, i.e.,

$$\overline{C}(\mathbf{x}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x}) \tilde{f}(\boldsymbol{\omega}) d(\boldsymbol{\omega}).$$

An absolutely continuous spectral distribution function is said to be *symmetric* if it is equal to its conjugate, i.e., $\tilde{F}(\boldsymbol{\omega}) = F(\boldsymbol{\omega})$. In this last case, the spectral density function is an even function and the corresponding covariance is a real function. Equivalently, a c.f. is a real function if and only if it is equal to its conjugate, i.e., $\overline{C}(\mathbf{x}) = C(\mathbf{x})$. In order to construct models for the difference of two c.f.s, it is worth underlining the following aspects:

- if the spectral distribution function is not symmetric or, alternatively, the spectral density function is not an even function, the c.f. will necessarily be a complex c.f..
- On the other hand, if the spectral distribution function is symmetric or, alternatively, the spectral density function is an even function, the c.f. will necessarily be a real c.f.. In this case, the two spectral density functions involved in the difference of two c.f.s, are both even functions, or, as will be shown in the next section, the two spectral density functions couldn't both be even functions, however their difference could be, in some cases, an even function.
- Complex c.f.s can be constructed in any dimensional space; at this purpose, it is important to provide some useful details about the various techniques which have been proposed in the literature. In particular,
  - utilizing the well known closure properties of c.f.s;
  - through the construction of separable models, i.e.

$$C(\mathbf{x}) = \prod_{i=1}^{n} C_i(x_i), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{4}$$

where each component in the product is a c.f. in $\mathbb{R}$; in particular, if F_i is the spectral distribution function of C_i in the representation (1), $i = 1, \ldots, n$, then C also has a representation as in (1), where

$$F(\boldsymbol{\omega}) = \prod_{i=1}^{n} F_i(\omega_i), \quad \boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n; \tag{5}$$

  - by considering the subclass of isotropic c.f.s; in this case, different results based on the spectral representation through Bessel functions (Matern 1980; Yaglom 1987), completely monotone functions (Schoenberg 1938), sufficient conditions on positive definiteness for isotropic c.f.s (Polya 1949), have been utilized.

3 Difference of two covariance functions

As already underlined, the results given by Ma (2005) and Gregori et al. (2008), concerning the difference between two c.f.s, are only valid in the real domain. On the other hand, the corollaries presented in this section are valid in the complex domain, i.e., for any c.f. defined as in (1); hence, these results can be considered further properties for the difference of two c.f.s in the complex domain. Valid models for the difference of c.f.s in the complex domain, as well as in the real domain, will be constructed by utilizing the results of the same corollaries. The dependence of a c.f. from a vector of parameters will be properly specified, because all the results depend on these parameters.

Corollary 3 Let $C_i : \mathbb{R}^n \to \mathbb{C}, i = 1, 2$ be c.f.s and define

$$C(\mathbf{x}; \boldsymbol{\Lambda}) = AC_1(\mathbf{x}; \boldsymbol{\alpha}) - BC_2(\mathbf{x}; \boldsymbol{\beta}), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{6}$$

where $A > 0, B > 0, \boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors of parameters and $\boldsymbol{\Lambda} = (A, B, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Because of Bochner's theorem, two different cases need to be analyzed:

- the spectral distribution functions $F_i, i = 1, 2,$ which define $C_i, i = 1, 2,$ respectively, are not absolutely continuous. In this case the corresponding c.f.s can be written as follows:

$$C_1(\mathbf{x};\boldsymbol{\alpha}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})dF_1(\boldsymbol{\omega};\boldsymbol{\alpha}),$$
$$C_2(\mathbf{x};\boldsymbol{\beta}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})dF_2(\boldsymbol{\omega};\boldsymbol{\beta});$$

then

$$C(\mathbf{x};\boldsymbol{\Lambda}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})d(AF_1(\boldsymbol{\omega};\boldsymbol{\alpha}) - BF_2(\boldsymbol{\omega};\boldsymbol{\beta})) \quad (7)$$

is a c.f. if and only if, for some suitable values of the vector of parameters $\boldsymbol{\Lambda}$ and $\boldsymbol{\omega} \in \mathbb{R}^n$, $(AF_1 - BF_2)$ is a non negative bounded function on $\mathbb{R}^n$;

- the spectral distribution functions $F_i, i = 1, 2,$ which define $C_i, i = 1, 2$, respectively, are absolutely continuous. In this case, the corresponding c.f.s can be written as follows:

$$C_1(\mathbf{x};\boldsymbol{\alpha}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})f_1(\boldsymbol{\omega};\boldsymbol{\alpha})d\boldsymbol{\omega},$$
$$C_2(\mathbf{x};\boldsymbol{\beta}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})f_2(\boldsymbol{\omega};\boldsymbol{\beta})d\boldsymbol{\omega},$$

with $f_i(\boldsymbol{\omega};\cdot) \geq 0, \boldsymbol{\omega} \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} f_i(\boldsymbol{\omega};\cdot)d\boldsymbol{\omega} < \infty$, $i = 1, 2$; then

$$C(\mathbf{x};\boldsymbol{\Lambda}) = \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T\mathbf{x})(Af_1(\boldsymbol{\omega};\boldsymbol{\alpha}) - Bf_2(\boldsymbol{\omega};\boldsymbol{\beta}))d\boldsymbol{\omega} \quad (8)$$

is a c.f. if and only if, for some suitable values of the vector of parameters $\boldsymbol{\Lambda}$ and $\boldsymbol{\omega} \in \mathbb{R}^n$,

$$Af_1(\boldsymbol{\omega};\boldsymbol{\alpha}) - Bf_2(\boldsymbol{\omega};\boldsymbol{\beta}) \geq 0,$$
$$\int_{\mathbb{R}^n} (Af_1(\boldsymbol{\omega};\boldsymbol{\alpha}) - Bf_2(\boldsymbol{\omega};\boldsymbol{\beta}))d\boldsymbol{\omega} < +\infty. \quad (9)$$

The second property of (9) is always satisfied because $(Af_1 - Bf_2)$ is a difference of integrable functions, hence only the first property of (9) need to be verified.

Two special cases of Corollary 3 are given hereafter. In particular, in the next Corollary 4 it is assumed that the function C in (6) is defined as the product of the difference of c.f.s, where each c.f. in the difference is defined in $\mathbb{R}$, whereas, in the next Corollary 5 it is assumed that the function C in (6) is defined as the difference of the product of c.f.s, where each c.f. is defined in $\mathbb{R}$.

Of course, the above two models are completely different; in particular, the model in the Corollary 4 has been built by considering the following aspects:

1. it is a special case of Corollary 3 in $\mathbb{R}$,
2. it is an application of Eq. (4).

In the Corollary 5, starting from separable models in $\mathbb{C}$, the components of the real and imaginary part of the resulting c.f. are non separable and anisotropic.

Corollary 4 Let $C_j : \mathbb{R} \to \mathbb{C}, j = 1, 2$ be c.f.s and define

$$C_{12(k)}(x_k;\alpha_k,\beta_k) = A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k), A_k > 0, B_k > 0, k = 1,\ldots,n, \quad (10)$$

and

$$C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\Big[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\Big], \quad (11)$$

where $\mathbf{A} = (A_1,\ldots,A_n), \mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\boldsymbol{\Theta} = (\mathbf{A}, \mathbf{B}, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Because of Bochner's theorem, two different cases need to be analyzed:

- the spectral distribution functions $F_1(x_k;\alpha_k), F_2(x_k;\beta_k)$, which define $C_1(x_k;\alpha_k)$ and $C_2(x_k;\beta_k), k = 1,\ldots,n$, respectively, are not absolutely continuous. In this case the c.f.s can be written as follows:

$$C_1(x_k;\alpha_k) = \int_{\mathbb{R}} \exp(i\omega_kx_k)dF_1(\omega_k;\alpha_k),$$
$$C_2(x_k;\beta_k) = \int_{\mathbb{R}} \exp(i\omega_kx_k)dF_2(\omega_k;\beta_k);$$

then, each

$$C_{12(k)}(x_k;\alpha_k,\beta_k) = \int_{\mathbb{R}} \exp(i\omega_kx_k)d(A_kF_k(\omega_k;\alpha_k) - B_kF_2(\omega_k;\beta_k)), \quad (12)$$

is a c.f. if and only if, for some suitable values of the vector of parameters $\boldsymbol{\Theta}$ and $\omega_k \in \mathbb{R}, k = 1,\ldots,n$

$$A_kF_1(\omega_k;\alpha_k) - B_kF_2(\omega_k;\beta_k) \geq 0, \quad k = 1,\ldots,n, \quad (13)$$

and each $(A_kF_1 - B_kF_2), k = 1,\ldots,n,$ is a non decreasing function.

As a consequence, C defined in Eq. (11) is a c.f..

- The spectral distribution functions $F_1(x_k;\alpha_k), F_2(x_k;\beta_k)$, which define $C_1(x_k;\alpha_k)$ and $C_2(x_k;\beta_k), k = 1,\ldots,n$, respectively, are absolutely continuous. In this case, these c.f.s can be written as follows:

$$C_1(x_k;\alpha_k) = \int_{\mathbb{R}} \exp(i\omega_kx_k)f_1(\omega_k;\alpha_k)d\omega_k,$$
$$C_2(x_k;\beta_k) = \int_{\mathbb{R}} \exp(i\omega_kx_k)f_2(\omega_k;\beta_k)d\omega_k,$$

with $f_j(\omega_k;\cdot) > 0, \omega_k \in \mathbb{R}$ and $\int_{\mathbb{R}} f_j(\omega_k;\cdot)d\omega_k < \infty$, $j = 1, 2; k = 1,\ldots,n$;

then, each

$$C_{12(k)}(x_k;\alpha_k,\beta_k) = \int_{\mathbb{R}} \exp(i\omega_k x_k) d(A_k F_k(\omega_k;\alpha_k) - B_k F_2(\omega_k;\beta_k)) \quad (14)$$

is a c.f. if and only if, for some suitable values of the vector of parameters $\mathbf{\Theta}$ and $\omega_k \in \mathbb{R}, k = 1,\ldots,n$,

$$A_k f_1(\omega_k;\alpha_k) - B_k f_2(\omega_k;\beta_k) \geq 0,$$
$$\int_{\mathbb{R}} (A_k f_1(\omega_k;\alpha_k) - B_k f_2(\omega_k;\beta_k)) d\omega_k < +\infty. \quad (15)$$

Note that the second condition in (15) is always satisfied; as a consequence of Eq. (4), C defined in Eq. (11) is a c.f..

Corollary 5 Let $C_j : \mathbb{R} \to \mathbb{C}, j = 1,2$ be c.f.s and define

$$C(\mathbf{x};\mathbf{\Theta}) = \prod_{k=1}^{n} A_k C_1(x_k;\alpha_k) - \prod_{k=1}^{n} B_k C_2(x_k;\beta_k),$$
$$A_k > 0, B_k > 0, k = 1,\ldots,n, \quad (16)$$

where $\mathbf{A} = (A_1,\ldots,A_n), \mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\mathbf{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$. Let the spectral distribution functions which define $C_1(x_k;\alpha_k)$ and $C_2(x_k;\beta_k)$, $k = 1,\ldots,n$, respectively, be absolutely continuous, with

$$f_j(\omega_k;\cdot) > 0, \omega_k \in \mathbb{R}, \quad \int_{\mathbb{R}} f_j(\omega_k;\cdot) d\omega_k < \infty,$$
$$j = 1,2, \quad k = 1,\ldots,n.$$

In this case, the expression (16) can be written as follows:

$$C(\mathbf{x};\mathbf{\Theta}) = \prod_{k=1}^{n}\left[A_k \int_{\mathbb{R}} \exp(i\omega_k x_k) f_1(\omega_k;\alpha_k) d\omega_k\right] - \prod_{k=1}^{n}\left[B_k \int_{\mathbb{R}} \exp(i\omega_k x_k) f_2(\omega_k;\beta_k) d\omega_k\right]$$
$$= \int_{\mathbb{R}^n} \exp(i\boldsymbol{\omega}^T \mathbf{x}) \left[\prod_{k=1}^{n} A_k f_1(\omega_k;\alpha_k) - \prod_{k=1}^{n} B_k f_2(\omega_k;\beta_k)\right] d\boldsymbol{\omega}. \quad (17)$$

Then, (17) is a c.f. if and only if, for some suitable values of the vector of parameters $\mathbf{\Theta}$ and $\boldsymbol{\omega} \in \mathbb{R}^n$,

$$\prod_{k=1}^{n}\left[\frac{A_k f_1(\omega_k;\alpha_k)}{B_k f_2(\omega_k;\beta_k)}\right] \geq 1, \quad \omega_k \in \mathbb{R}, k = 1,\ldots,n. \quad (18)$$

Remarks

- Note that if (15) is satisfied, then inequality (18) is verified; hence, the second part of Corollary 4 implies Corollary 5.
- No hypothesis on the spectral distribution function (i.e. to be symmetric or not), as well as on the spectral density function (i.e. to be even or not) are made in the Corollaries 3, 4 and 5; hence, these corollaries provide a general result on the c.f. C in (7), (8), (11) and (16), which could be, in general, a complex c.f..
- The complex c.f. C defined in (11) is the product of complex c.f.s $C_{12(k)}, k = 1,\ldots,n$; then C is a separable c.f., where separability is referred to the complex domain $\mathbb{C}$. However, the real and the imaginary components outcoming from the above product are non separable in $\mathbb{R}^n$ and are characterized by non geometric anisotropy.
- The complex c.f. C defined in (16) is the difference of two products of complex c.f.s, where each product is separable in the complex domain $\mathbb{C}$. However, the real and the imaginary components of the resulting complex c.f. are non separable in $\mathbb{R}^n$ and are characterized by non geometric anisotropy.
- If the two spectral density functions f_1 and f_2 in (8) are even functions and conditions (9) are satisfied, then $(AC_1 - BC_2)$ is a real c.f..
- If just one of the two spectral density functions f_1 or f_2 in (8) is an even function, then $(Af_1 - Bf_2)$ cannot be an even function, as a consequence, if conditions (9) are satisfied, $(AC_1 - BC_2)$ is a complex c.f..
- It may happen that if the two spectral density functions f_1 and f_2 in (8) are not even functions, then $(Af_1 - Bf_2)$ could be an even function, as a consequence, if conditions (9) are satisfied, $(AC_1 - BC_2)$ is a real c.f., as shown in the following example.

Example 1 Consider, in $\mathbb{R}$, the following spectral density functions:

$$f_1(x) = e^{-|x|}(x^2 + x + 3), \qquad f_2(x) = e^{-|x|}(x^2 + x + 2),$$

which are not even functions, hence the corresponding c.f.s are complex functions. It is easy to verify that: $f_1(x) - f_2(x) = e^{-|x|}$, is an even spectral density function, as a consequence, the corresponding c.f. is a real function. A generalization of this example will be given in Corollary 7.

- If C_1 and C_2 are real and continuous c.f.s and $C_1(\mathbf{x}) \leq C_2(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, then $(C_2 - C_1)$ is not, in general, a c.f., as shown in the following example.

Example 2 Consider the following c.f.s: $C_1(x) = e^{-2|x|}$, $C_2(x) = e^{-|x|}$,with $C_1(x) \leq C_2(x), x \in \mathbb{R}$. Note that $C(x) = C_2(x) - C_1(x)$ is not a c.f., because $C(0) = 0$, and $C(x) > C(0), x \neq 0$, which contradicts the third property of a c.f., given in Section 2.

- If C_1 and C_2 are real correlation functions, i.e., $C_1(\mathbf{0}) = C_2(\mathbf{0}) = 1$, then $C(\mathbf{x}) = C_1(\mathbf{x}) - C_2(\mathbf{x})$ cannot ever be a c.f. (except for the trivial case $C_1(\mathbf{x}) = C_2(\mathbf{x}) = 0, \mathbf{x} \in \mathbb{R}^n$); indeed, in this case $C(\mathbf{0}) = 0$, however, $|C(\mathbf{x})| \geq C(\mathbf{0})$, which contradicts one of the main properties of c.f.s.

- If the spectral distribution functions $F_i, i = 1, 2$, which define $C_i, i = 1, 2$, respectively, in Corollary 3 are absolutely continuous, then the resulting c.f. C is strictly positive definite (De Iaco et al. 2011; De Iaco and Posa 2018).

Some significant examples, utilizing the results of the previous corollaries, will be given hereafter. In particular,

1. in the Sect. 3.1, some examples for the difference of c.f.s in the complex domain will be provided.
2. In the Sect. 3.2, some examples for the difference of c.f.s in the real domain will be also given; in this last case, differently from the models described in Ma (2005) which are isotropic and they belong to the same class, the c.f.s proposed in this paper, obtained as the difference of two real c.f.s, are anisotropic and non separable, although they have been built starting from separable models and they do not necessarily belong to the same class. Some well known models, utilized in the applications, have been considered.
3. In the Sect. 3.3 some examples for the difference of two isotropic c.f.s are provided for the two and three dimensional Euclidean space; in particular, for some of these examples the c.f.s involved in the difference could not belong to the same family.

3.1 Difference of two complex covariance functions

The following result shows a special case in which the difference of two spectral distribution functions, which are not absolutely continuous, is again a spectral distribution function. As a consequence, the Fourier transform of this last spectral distribution function, corresponding to the difference of two c.f.s, will be a c.f..

Corollary 6 Let F_1 and F_2 be spectral distribution functions on $\mathbb{R}$, i.e.,:

$$F_1(\omega; \boldsymbol{\alpha}) = \begin{cases} 0 & \omega < t_1 \\ \alpha_k & t_k \leq \omega < t_{k+1} \quad i = 1, \ldots, n-1 \\ \alpha_n & \omega \geq t_n, \end{cases} \tag{19}$$

$$F_2(\omega; \boldsymbol{\beta}) = \begin{cases} 0 & \omega < t_1 \\ \beta_k & t_k \leq \omega < t_{k+1} \quad k = 1, \ldots, n-1 \\ \beta_n & \omega \geq t_n, \end{cases} \tag{20}$$

where

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n),$$
$$\beta_k < \alpha_k, \alpha_k > 0, \beta_k > 0, k = 1, \ldots, n$$

with $\alpha_k < \alpha_{k+1}, \beta_k < \beta_{k+1}$, and $(\alpha_k - \beta_k) < (\alpha_{k+1} - \beta_{k+1})$, $k = 1, \ldots, n-1$; let

$$C_1(x; \boldsymbol{\alpha}) = \int_{\mathbb{R}} \exp(i\omega x) dF_1(\omega; \boldsymbol{\alpha}),$$
$$C_2(x; \boldsymbol{\beta}) = \int_{\mathbb{R}} \exp(i\omega x) dF_2(\omega; \boldsymbol{\beta}).$$

Then: $C_1(x; \boldsymbol{\alpha}) - C_2(x; \boldsymbol{\beta}) = \int_{\mathbb{R}} \exp(i\omega x) d(F_1(\omega; \boldsymbol{\alpha}) - F_2(\omega; \boldsymbol{\beta}))$, is a c.f., because $(F_1 - F_2)$ is a spectral distribution function, i.e., it is non negative and a non decreasing function. Moreover, if $(F_1 - F_2)$ is a non symmetric spectral distribution function, then $(C_1 - C_2)$ is a complex c.f..

Corollary 6 is valid in $\mathbb{R}$, however, utilizing Eqs. (4) and (5), it can be extended to any dimensional space $\mathbb{R}^n$. In the next examples, as well as in the examples shown in the Sect. 3.2, the same formalism utilized for the Corollaries 3, 4 and 5 will be retained: the generic coordinate of the one dimensional space $\mathbb{R}$ will be denoted with $x_k, k = 1, \ldots, n$. In this way, starting from the one dimensional space, covariance models in $\mathbb{R}^n$ will be easily constructed applying (11) and (16). These aspects will be underlined at the end of the examples shown hereafter and at the end of the examples discussed in the Sect. 3.2.

An application of Corollary 3 for absolutely continuous spectral distribution functions is given in the following examples, where the difference of two spectral density functions is considered: such a difference is a spectral density function if conditions (9) are satisfied; as a consequence, the difference of the corresponding c.f.s is again a c.f.. In particular, in the Examples 3, 4 and 5 the spectral density functions f_1 and f_2 are not both even functions, whereas in the Example 6 f_1 is an even function while f_2 is not an even function.

Example 3 Let's consider, in $\mathbb{R}$, the following spectral density functions, with $0 < \beta_k < \alpha_k, 0 < B_k < A_k, \omega_k \in \mathbb{R}$, $k = 1, \ldots, n$:

$$f_1(\omega_k; \alpha_k) = \begin{cases} 0 & \omega_k < 0 \\ 1 & 0 \leq \omega_k < \alpha_k \\ 0 & \omega_k \geq \alpha_k \end{cases} \tag{21}$$

$$f_2(\omega_k; \beta_k) = \begin{cases} 0 & \omega_k < 0 \\ 1 & 0 \leq \omega_k < \beta_k \\ 0 & \omega_k \geq \beta_k \end{cases} \tag{22}$$

which are not even functions; since: $A_k f_1(\omega_k; \alpha_k) - B_k f_2(\omega_k; \beta_k) \geq 0, \omega_k \in \mathbb{R}$, then the following difference:

$$
\begin{aligned}
&A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\\
&\quad = \int_{\mathbb{R}} \exp(i\omega_k x_k)(A_k f_1(\omega_k;\alpha_k) - B_k f_2(\omega_k;\beta_k))d\omega_k\\
&\quad = A_k\frac{\sin(\alpha_k x_k)}{x_k} - B_k\frac{\sin(\beta_k x_k)}{x_k}\\
&\qquad + i\left[A_k\frac{2}{x_k}\sin^2\left(\frac{\alpha_k x_k}{2}\right) - B_k\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right)\right],
\end{aligned}
$$

with $A_kC_1(0;\alpha_k) - B_kC_2(0;\beta_k) = A_k\alpha_k - B_k\beta_k$, is a complex c.f., $x_k \in \mathbb{R}, k = 1,\ldots,n$.

According to Corollaries 4 and 5, the following functions:

$$
C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\left[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\right],
$$

$$
C^*(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}A_kC_1(x_k;\alpha_k) - \prod_{k=1}^{n}B_kC_2(x_k;\beta_k),
$$

are complex c.f.s in $\mathbb{R}^n$, where $\mathbf{A} = (A_1,\ldots,A_n)$, $\mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n), \boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ and $\boldsymbol{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, and

$$
C_1(x_k;\alpha_k) = \frac{\sin(\alpha_k x_k)}{x_k} + i\frac{2}{x_k}\sin^2\left(\frac{\alpha_k x_k}{2}\right),
$$

$$
C_2(x_k;\beta_k) = \frac{\sin(\beta_k x_k)}{x_k} + i\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right).
$$

Example 4 Let's consider, in $\mathbb{R}$, the following spectral density functions, with $0 < \alpha_k < \beta_k$, $0 < B_k < A_k, \omega_k \in \mathbb{R}, k = 1,\ldots,n$

$$
f_1(\omega_k;\alpha_k) = \begin{cases} 0 & \omega_k < 0\\ \exp(-\alpha_k\omega_k) & \omega_k \geq 0\end{cases} \tag{23}
$$

$$
f_2(\omega_k;\beta_k) = \begin{cases} 0 & \omega_k < 0\\ \exp(-\beta_k\omega_k) & \omega_k \geq 0\end{cases} \tag{24}
$$

which are not even functions and let C_1 and C_2 be the corresponding c.f.s of f_1 and f_2, respectively; since: $A_kf_1(\omega_k;\alpha_k) - B_kf_2(\omega_k;\beta_k) \geq 0$, $\omega_k \in \mathbb{R}$, then the following difference:

$$
\begin{aligned}
&A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\\
&\quad = \int_{\mathbb{R}} \exp(i\omega_k x_k)(A_k f_1(\omega_k;\alpha_k) - B_k f_2(\omega_k;\beta_k))d\omega_k\\
&\quad = \frac{A_k\alpha_k}{(x_k^2+\alpha_k^2)} - \frac{B_k\beta_k}{(x_k^2+\beta_k^2)} + i\left[\frac{A_kx_k}{(x_k^2+\alpha_k^2)} - \frac{B_kx_k}{(x_k^2+\beta_k^2)}\right],\\
&\qquad x_k \in \mathbb{R}, k = 1,\ldots,n,
\end{aligned} \tag{25}
$$

is a complex c.f..

According to Corollaries 4 and 5, the following functions:

$$
C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\left[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\right],
$$

$$
C^*(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}A_kC_1(x_k;\alpha_k) - \prod_{k=1}^{n}B_kC_2(x_k;\beta_k),
$$

are complex c.f.s, both defined in $\mathbb{R}^n$, where $\mathbf{A} = (A_1,\ldots,A_n), \mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ and $\boldsymbol{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, and

$$
C_1(x_k;\alpha_k) = \frac{\alpha_k}{(x_k^2+\alpha_k^2)} + i\frac{x_k}{(x_k^2+\alpha_k^2)},
$$

$$
C_2(x_k;\beta_k) = \frac{\beta_k}{(x_k^2+\beta_k^2)} + i\frac{x_k}{(x_k^2+\beta_k^2)}.
$$

Example 5 Let's consider, in $\mathbb{R}$, the spectral density functions defined in the Eqs. (23) and (22) respectively, with $\alpha_k > 0, \beta_k > 0, A_k > 0, B_k > 0, k = 1,\ldots,n$; note that f_1 and f_2 are not even functions and define:

$$
\begin{aligned}
&f(\omega_k;\alpha_k,\beta_k,A_k,B_k) = A_kf_1(\omega_k;\alpha_k) - B_kf_2(\omega_k;\beta_k),\\
&\omega_k \in \mathbb{R}, k = 1,\ldots,n;
\end{aligned} \tag{26}
$$

this last function is a spectral density function if $0 < B_k < A_k$ and

$$
\beta_k = \frac{\ln\left(\frac{A_k}{B_k}\right)}{\alpha_k}, \quad k = 1,\ldots,n.
$$

If C_1 and C_2 are the corresponding c.f.s of f_1 and f_2, respectively, then the following difference:

$$
\begin{aligned}
&A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\\
&\quad = \int_{\mathbb{R}} \exp(i\omega_k x_k)(A_k f_1(\omega_k;\alpha_k) - B_k f_2(\omega_k;\beta_k))d\omega_k\\
&\quad = \frac{A_k\alpha_k}{(x_k^2+\alpha_k^2)} - B_k\frac{\sin(\beta_k x_k)}{x_k}\\
&\qquad + i\left[\frac{A_kx_k}{(x_k^2+\alpha_k^2)} - B_k\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right)\right],\\
&\qquad x_k \in \mathbb{R}, k = 1,\ldots,n
\end{aligned} \tag{27}
$$

with $A_kC_1(0;\alpha_k) - B_kC_2(0;\beta_k) = \frac{A_k}{\alpha_k} - B_k\beta_k$, is a complex c.f..

According to Corollaries 4 and 5, the following functions:

$$C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\left[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\right],$$
$$C^*(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}A_kC_1(x_k;\alpha_k) - \prod_{k=1}^{n}B_kC_2(x_k;\beta_k),$$

are complex c.f.s, both defined in $\mathbb{R}^n$, where $\mathbf{A} = (A_1,\ldots,A_n), \mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ and $\boldsymbol{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, and

$$C_1(x_k;\alpha_k) = \frac{\alpha_k}{(x_k^2+\alpha_k^2)} + i\frac{x_k}{(x_k^2+\alpha_k^2)},$$
$$C_2(x_k;\beta_k) = \frac{\sin(\beta_k x_k)}{x_k} + i\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right).$$

Example 6 Let's consider, in $\mathbb{R}$, the spectral density functions

$$f_1(\omega_k) = \exp(-\alpha_k|\omega_k|), \quad \alpha_k > 0,$$

and f_2 defined in Eq. (22), where f_1 is an even function and f_2 is not an even function and define, with $\alpha_k > 0, \beta_k > 0, A_k > 0, B_k > 0, k = 1,\ldots,n$:

$$f(\omega_k;\alpha_k,\beta_k,A_k,B_k) = A_kf_1(\omega_k;\alpha_k) - B_kf_2(\omega_k;\beta_k), \quad \omega_k \in \mathbb{R}, k = 1,\ldots,n; \tag{28}$$

this last function is a spectral density function if $0<B_k<A_k$ and $\beta_k = \frac{\ln\left(\frac{A_k}{B_k}\right)}{\alpha_k}$, $k = 1,\ldots,n$. If C_1 and C_2 are the corresponding c.f.s of f_1 and f_2, respectively, then the following difference:

$$\begin{aligned}A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k) &= \int_{\mathbb{R}}\exp(i\omega_k x_k)(A_kf_1(\omega_k;\alpha_k) - B_kf_2(\omega_k;\beta_k))d\omega_k \\ &= \frac{2A_k\alpha_k}{(x_k^2+\alpha_k^2)} - B_k\frac{\sin(\beta_k x_k)}{x_k} - iB_k\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right), \\ & x_k \in \mathbb{R}, k = 1,\ldots,n,\end{aligned} \tag{29}$$

is a complex c.f.. According to Corollaries 4 and 5, the following functions:

$$C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\left[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\right],$$
$$C^*(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}A_kC_1(x_k;\alpha_k) - \prod_{k=1}^{n}B_kC_2(x_k;\beta_k),$$

are complex c.f.s, where $\mathbf{A} = (A_1,\ldots,A_n)$, $\mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n), \boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ and $\boldsymbol{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, and $C_1(x_k;\alpha_k) = \frac{2\alpha_k}{(x_k^2+\alpha_k^2)}$, $C_2(x_k;\beta_k) = \frac{\sin(\beta_k x_k)}{x_k} + i\frac{2}{x_k}\sin^2\left(\frac{\beta_k x_k}{2}\right)$.

3.2 Difference of two real covariance functions

In the following, some applications of Corollary 3, 4 and 5 are given for the subset of the real c.f.s: hence, C_1 and C_2 defined in the same corollaries are real c.f.s which depend on some parameters. In particular, starting from some well known c.f.s (exponential, Gaussian and rational models), the difference of the various combinations of these last models has been analyzed: it will be shown that the difference of two selected c.f.s could be a c.f. for some values of the parameters on which these models depend. On the other hand, the same difference could not be a c.f. for other values of these parameters. The models obtained as the difference of two c.f.s could be negative in a subset of the domain on which the c.f.s are defined. The proposed models $C_i, i = 1,2$, are integrable, hence, according to (3), the spectral density functions can be expressed as the Fourier transform of these covariance models.

Example 7 Let

$$C_1(x_k;\alpha_k) = \exp\left(-\alpha_k x_k^2\right), \quad x_k \in \mathbb{R}, \quad \alpha_k > 0, k = 1,\ldots,n,$$
$$C_2(x_k;\beta_k) = \exp\left(-\beta_k x_k^2\right), \quad x_k \in \mathbb{R} \quad \beta_k > 0, k = 1,\ldots,n,$$

be Gaussian c.f.s; these models are integrable, then their Fourier transforms f_1 and f_2 are, respectively:

$$\begin{aligned}f_1(\omega_k;\alpha_k) &= \frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(-ix_k\omega_k\right)\exp\left(-\alpha_k x_k^2\right)dx_k \\ &= \frac{1}{2\sqrt{\pi\alpha_k}}\exp\left[-\left(\frac{\omega_k^2}{4\alpha_k}\right)\right], \\ f_2(\omega_k;\beta_k) &= \frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(-ix_k\omega_k\right)\exp\left(-\beta_k x_k^2\right)dx_k \\ &= \frac{1}{2\sqrt{\pi\beta_k}}\exp\left[-\left(\frac{\omega_k^2}{4\beta_k}\right)\right],\end{aligned}$$

$\omega_k \in \mathbb{R}, \quad k = 1,\ldots,n$. Then

$$C(x_k;A_k,B_k,\alpha_k,\beta_k) = A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k), \quad x_k \in \mathbb{R}, A_k > 0, B_k > 0, \tag{30}$$

according to Corollaries 3 and 4, is a c.f., for each $k = 1,\ldots,n$, if its Fourier transform is a spectral density function, i.e., $A_kf_1(\omega_k;\alpha_k) - B_kf_2(\omega_k;\beta_k) \geq 0, \quad \omega_k \in \mathbb{R}$; the previous inequality is satisfied if:

$$\frac{\omega_k^2}{4}\left(\frac{1}{\beta_k} - \frac{1}{\alpha_k}\right) > \ln\left(\frac{B_k}{A_k}\sqrt{\frac{\alpha_k}{\beta_k}}\right); \tag{31}$$

then, (31) is always satisfied if $\alpha_k > \beta_k$ and $\frac{B_k}{A_k} < \sqrt{\frac{\beta_k}{\alpha_k}}, k = 1, \ldots, n.$

Example 8 Given the following exponential c.f.s:

$$C_1(x_k;\alpha_k) = \exp\left(-\alpha_k|x_k|\right), \quad x_k \in \mathbb{R}, \alpha_k > 0, k = 1,\ldots,n,$$
$$C_2(x_k;\beta_k) = \exp\left(-\beta_k|x_k|\right), \quad x_k \in \mathbb{R}, \beta_k > 0, k = 1,\ldots,n;$$

then, C_1 and C_2 are integrable and

$$C(x_k;A_k,B_k,\alpha_k,\beta_k) = A_k \exp\left(-\alpha_k|x_k|\right) - B_k \exp\left(-\beta_k|x_k|\right), \quad A_k > 0, B_k > 0, \tag{32}$$

is a c.f. if its Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k;A_k,B_k,\alpha_k,\beta_k) = \frac{A_k}{\pi}\frac{\alpha_k}{(\omega_k^2+\alpha_k^2)} - \frac{B_k}{\pi}\frac{\beta_k}{(\omega_k^2+\beta_k^2)} \geq 0, \omega_k \in \mathbb{R}; \tag{33}$$

in particular, (33) is satisfied if:

$$\frac{(\omega_k^2+\beta_k^2)}{(\omega_k^2+\alpha_k^2)} - \frac{B_k}{A_k}\frac{\beta_k}{\alpha_k} \geq 0, \quad \omega_k \in \mathbb{R}.$$

According to the values of the parameters α_k and $\beta_k, k = 1, \ldots, n$, some special situations can be analyzed:

1. $\alpha_k > \beta_k, k = 1, \ldots, n$; in this case, it is easy to verify that the minimum value of the function:

$$\zeta(\omega_k;\alpha_k,\beta_k) = \frac{(\omega_k^2+\beta_k^2)}{(\omega_k^2+\alpha_k^2)}$$

   is obtained for $\omega_k^* = 0$; hence $\zeta(0;\alpha_k,\beta_k) = \frac{\beta_k^2}{\alpha_k^2}$; as a consequence, (33) is a spectral density function if: $\frac{B_k}{A_k} < \frac{\beta_k}{\alpha_k} < 1.$

2. $\alpha_k < \beta_k, k = 1, \ldots, n$; in this case, $\zeta(\omega_k;\alpha_k,\beta_k) = \frac{(\omega_k^2+\beta_k^2)}{(\omega_k^2+\alpha_k^2)} > 1, \quad \omega_k \in \mathbb{R}$, then (33) is a spectral density function if:

$$\frac{B_k}{A_k} < \frac{\alpha_k}{\beta_k}. \tag{34}$$

Example 9 Given the following c.f.s,

$$C_1(x_k;\alpha_k) = \frac{1}{(x_k^2+\alpha_k^2)}, \quad C_2(x_k;\beta_k) = \frac{1}{(x_k^2+\beta_k^2)},$$
$$x_k \in \mathbb{R}, \alpha_k > 0, \beta_k > 0,$$

then C_1 and C_2 are integrable and

$$C(x_k;A_k,B_k,\alpha_k,\beta_k) = \frac{A_k}{(x_k^2+\alpha_k^2)} - \frac{B_k}{(x_k^2+\beta_k^2)}, \quad A_k > 0, B_k > 0, \tag{35}$$

is a c.f. if its Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k;A_k,B_k,\alpha_k,\beta_k) = \frac{A_k}{2\alpha_k}\exp\left(-\alpha_k|\omega_k|\right) - \frac{B_k}{2\beta_k}\exp\left(-\beta_k|\omega_k|\right) \geq 0, \quad \omega_k \in \mathbb{R}; \tag{36}$$

in particular, (36) is satisfied if $\beta_k > \alpha_k$, and $\frac{B_k}{A_k} < \frac{\beta_k}{\alpha_k}, k = 1, \ldots, n.$

Example 10 Given the following c.f.s:

$$C_1(x_k;\alpha_k) = \exp\left(-\alpha_k|x_k|\right),$$
$$C_2(x_k;\alpha_k) = \frac{\pi}{2\alpha_k^3}(1+\alpha_k|x_k|)\left[\exp\left(-\alpha_k|x_k|\right)\right],$$

with $x_k \in \mathbb{R}, \alpha_k > 0, k = 1, \ldots, n$; then, C_1 and C_2 are integrable and

$$C(x_k;A_k,B_k,\alpha_k) = A_k C_1(x_k;\alpha_k) - B_k C_2(x_k;\alpha_k) = \exp\left(-\alpha_k|x_k|\right)\left[A_k - \frac{\pi B_k}{2\alpha_k^3}(1+\alpha_k|x_k|)\right], A_k > 0, B_k > 0, \tag{37}$$

is a c.f. if its Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k;A_k,B_k,\alpha_k) = \frac{A_k}{\pi}\frac{\alpha_k}{(\omega_k^2+\alpha_k^2)}\left[1 - \frac{\pi B_k}{A_k\alpha_k(\omega_k^2+\alpha_k^2)}\right] \geq 0, \quad \omega_k \in \mathbb{R}.$$

Note that the maximum value of the function

$$\epsilon(\omega_k;\alpha_k) = \frac{1}{(\omega_k^2+\alpha_k^2)}$$

is obtained for $\omega_k = 0$, then f is a spectral density function, hence C is a c.f. if:

$$\frac{B_k}{A_k} < \frac{\alpha_k^3}{\pi}.$$

Note that:

$$C^*(x_k; A_k, B_k, \alpha_k) = B_k C_2(x_k; \alpha_k) - A_k C_1(x_k; \alpha_k),$$
$$A_k > 0, B_k > 0,$$

cannot be a c.f. because its Fourier transform:

$$f(\omega_k; A_k, B_k, \alpha_k) = \frac{A_k}{\pi} \frac{\alpha_k}{(\omega_k^2 + \alpha_k^2)} \left[\frac{\pi B_k}{A_k \alpha_k (\omega_k^2 + \alpha_k^2)} - 1 \right], \quad \omega_k \in \mathbb{R},$$

cannot be a spectral density function, since the function: $\epsilon(\omega_k; \alpha_k) = \frac{1}{(\omega_k^2 + \alpha_k^2)}$, does not present a minimum value.

Example 11 Given the following c.f.s

$$C_1(x_k; \alpha_k) = \exp\left(-\alpha_k |x_k| \right), \quad C_2(x_k; \beta_k) = \exp\left(-\beta_k x_k^2 \right),$$
$$x_k \in \mathbb{R}, \alpha_k > 0, \beta_k > 0,$$

$k = 1, \ldots, n$, then C_1 and C_2 are integrable and

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = A_k \exp\left(-\alpha_k |x_k| \right) - B_k \exp\left(-\beta_k x_k^2 \right), \quad A_k > 0, B_k > 0, \tag{38}$$

is a c.f., for each $k = 1, \ldots, n$, if the corresponding Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k; A_k, B_k, \alpha_k, \beta_k) = \left[\frac{A_k}{\pi} \frac{\alpha_k \exp(\omega_k^2 / 4\beta_k)}{(\omega_k^2 + \alpha_k^2)} - \frac{B_k}{2\sqrt{\pi \beta_k}} \right] \exp\left(-\frac{\omega_k^2}{4\beta_k} \right) \geq 0, \quad \omega_k \in \mathbb{R}.$$

According to the values of the parameters α_k and $\beta_k, k = 1, \ldots, n$, some special situations can be analyzed:

1. $\alpha_k^2 - 4\beta_k > 0, k = 1, \ldots, n$; in this case, it is easy to verify that the minimum value of the function: $\kappa(\omega_k, \alpha_k, \beta_k) = \dfrac{\alpha_k \exp(\omega_k^2 / 4\beta_k)}{(\omega_k^2 + \alpha_k^2)}$ is obtained for $\omega_k^* = 0$; then, taking into account the inequalities $\alpha_k^2 - 4\beta_k > 0, k = 1, \ldots, n$, f is a spectral density function, as a consequence C defined in (38) is a c.f., if
$$0 < \frac{B_k}{A_k} < \frac{2\sqrt{\pi}}{\pi} \frac{\sqrt{\beta_k}}{\alpha_k} < \frac{1}{\sqrt{\pi}}, \quad hence: 0 < \frac{B_k}{A_k} < \frac{1}{\sqrt{\pi}} \approx 0.564.$$
2. $\alpha_k^2 - 4\beta_k < 0, k = 1, \ldots, n$; in this case, the minimum value of the function: $\kappa(\omega_k, \alpha_k, \beta_k) = \frac{\alpha_k \exp(\omega_k^2 / 4\beta_k)}{(\omega_k^2 + \alpha_k^2)}$ is obtained for $\omega_k^* = \sqrt{4\beta_k - \alpha_k^2}$; then, taking into account the inequalities $\alpha_k^2 - 4\beta_k < 0, k = 1, \ldots, n, f$ is a spectral density function, as a consequence C defined in (38) is a c.f., if $0 < \dfrac{B_k}{A_k} < \dfrac{e}{\sqrt{\pi}} \approx 1.53$.

Note that:

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = B_k C_2(x_k; \beta_k) - A_k C_1(x_k; \alpha_k),$$
$$A_k > 0, B_k > 0,$$

cannot be a c.f. because its Fourier transform:

$$f(\omega_k; A_k, B_k, \alpha_k, \beta_k) = \left[\frac{B_k}{2\sqrt{\pi \beta_k}} - \frac{A_k}{\pi} \frac{\alpha_k \exp(\omega_k^2 / 4\beta_k)}{(\omega_k^2 + \alpha_k^2)} \right] \exp\left(-\frac{\omega_k^2}{4\beta_k} \right), \quad \omega_k \in \mathbb{R},$$

cannot be a spectral density, since the function: $\kappa(\omega_k; \alpha_k, \beta_k) = \dfrac{\alpha_k \exp(\omega_k^2 / 4\beta_k)}{(\omega_k^2 + \alpha_k^2)}$ does not present an absolute maximum value.

Example 12 Given the c.f.s

$$C_1(x_k; \alpha_k) = \exp\left(-\alpha_k |x_k| \right), \quad C_2(x_k; \beta_k) = \frac{1}{(x_k^2 + \beta_k^2)},$$
$$x_k \in \mathbb{R}, \alpha_k > 0, \beta_k > 0,$$

$k = 1, \ldots, n$; then C_1 and C_2 are integrable and

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = A_k \exp\left(-\alpha_k |x_k| \right) - \frac{B_k}{(x_k^2 + \beta_k^2)}, \quad A_k > 0, B_k > 0, \tag{39}$$

is a c.f. if the corresponding Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k; A_k, B_k, \alpha_k, \beta_k) = \left[\frac{A_k}{\pi} \frac{\alpha_k \exp(\beta_k |\omega_k|)}{(\omega_k^2 + \alpha_k^2)} - \frac{B_k}{2\beta_k} \right] \frac{1}{\exp\left[(\beta_k |\omega_k|) \right]} \geq 0, \quad \omega_k \in \mathbb{R}.$$

According to the values of the parameters α_k and $\beta_k, k = 1, \ldots, n$, some special situations can be analyzed:

1. $\alpha_k \beta_k > 1, k = 1, \ldots, n$; in this case, the minimum value of the function
$$\chi(\omega_k; \alpha_k, \beta_k) = \frac{\alpha_k \exp(\beta_k |\omega_k|)}{(\omega_k^2 + \alpha_k^2)}, \tag{40}$$
is obtained for $\omega_k^* = 0$; hence, f is a spectral density function, as a consequence C defined in (39) is a c.f. if:
$$0 < \frac{B_k}{A_k} < \frac{2}{\pi} \frac{\beta_k}{\alpha_k} < \frac{2}{\pi} \beta_k^2.$$
2. $\alpha_k \beta_k < 1, k = 1, \ldots, n$; in this case, there exist two minimum values for the function (40): $\omega_k^* = 0$ and $\omega_k^{**} = \frac{1 + \sqrt{1 - \alpha_k^2 \beta_k^2}}{\beta_k}$. By comparing the values of the function χ, defined in (40), computed in ω_k^* and ω_k^{**}, i.e.,
$$\chi(\omega_k^{**}; \alpha_k, \beta_k) - \chi(\omega_k^*; \alpha_k, \beta_k) > 0, \tag{41}$$
it results that (41) is satisfied if:

$$(\alpha_k\beta_k)^2 \exp\left(1+\sqrt{1-(\alpha_k\beta_k)^2}\right) > 2\left(1+\sqrt{1-(\alpha_k\beta_k)^2}\right), \quad k=1,\ldots,n. \tag{42}$$

Inequality (42) is satisfied for $0.8 < \alpha_k\beta_k < 1$, where as for $0 < \alpha_k\beta_k < 0.8$ it results $\chi(\omega_k^{**};\alpha_k,\beta_k) - \chi(\omega_k^{*};\alpha_k,\beta_k) < 0$. Hence, according to the previous results:

- $0.8 < \alpha_k\beta_k < 1$, expression (39) is a c.f. if: $0 < \frac{B_k}{A_k} < \frac{2}{\pi}\frac{\beta_k}{\alpha_k} < \frac{2}{\pi}\frac{1}{\alpha_k^2}$;
- $0 < \alpha_k\beta_k < 0.8$, expression (39) is a c.f. if: $0 < \frac{B_k}{A_k} < \frac{2}{\pi}\chi(\omega_k^{**};\alpha_k,\beta_k)\beta_k$.

Note that: $C(x;A_k,B_k,\alpha_k,\beta_k) = B_kC_2(x_k;\beta_k) - A_kC_1(x_k;\alpha_k)$, $A_k > 0, B_k > 0$, cannot be a c.f. because its Fourier transform:

$$f(\omega_k;A_k,B_k,\alpha_k,\beta_k) = \left[\frac{B_k}{2\beta_k} - \frac{A_k}{\pi}\frac{\alpha_k\exp(\beta_k|\omega_k|)}{(\omega_k^2+\alpha_k^2)}\right]\frac{1}{\exp(\beta_k|\omega_k|)}, \quad \omega_k\in\mathbb{R},$$

cannot be a spectral density function, since the function

$$\chi(\omega_k;\alpha_k,\beta_k) = \frac{\alpha_k\exp(\beta_k|\omega_k|)}{(\omega_k^2+\alpha_k^2)},$$

does not present an absolute maximum value.

Example 13 Given the following c.f.s,

$$C_1(x_k;\alpha_k) = \frac{1}{(x_k^2+\alpha_k^2)}, \quad C_2(x_k;\beta_k) = \exp\left(-\beta_k x_k^2\right),$$
$$x_k\in\mathbb{R}, \alpha_k>0, \beta_k>0,$$

$k=1,\ldots,n$, then C_1 and C_2 are integrable and

$$C(x_k;A_k,B_k,\alpha_k,\beta_k) = \frac{A_k}{(x_k^2+\alpha_k^2)} - B_k\exp\left(-\beta_k x_k^2\right), A_k>0, B_k>0, \tag{43}$$

is a c.f. if its Fourier transform f is a spectral density function, i.e.,

$$f(\omega_k;A_k,B_k,\alpha_k,\beta_k) = \frac{A_k}{2\alpha_k}\exp\left(-\alpha_k|\omega_k|\right) - \frac{B_k}{2\sqrt{\pi\beta_k}}\exp\left[-\left(\frac{\omega_k^2}{4\beta_k}\right)\right] \geq 0, \omega_k\in\mathbb{R}, \tag{44}$$

which is equivalent to the following inequality:

$$\exp\left(\frac{\omega_k^2}{4\beta_k} - \alpha_k|\omega_k|\right) - \frac{B_k\alpha_k}{A_k\sqrt{\pi\beta_k}} \geq 0, \quad \omega_k\in\mathbb{R}; \tag{45}$$

the minimum value of the function $\tau(\omega_k;\alpha_k,\beta_k) = \exp\left(\frac{\omega_k^2}{4\beta_k} - \alpha_k|\omega_k|\right)$ is obtained for $\omega_k^* = 2\alpha_k\beta_k$; hence, (44) is a spectral density function if: $\exp(\alpha_k^2\beta_k) < \frac{A_k\sqrt{\pi\beta_k}}{B_k\alpha_k}$, $\quad k=1,\ldots,n$.

Note that:

$$C(x_k;A_k,B_k,\alpha_k,\beta_k) = B_kC_2(x_k;\beta_k) - A_kC_1(x_k;\alpha_k),$$
$$A_k>0, B_k>0,$$

cannot be a c.f. because its Fourier transform:

$$f(\omega_k;A_k,B_k,\alpha_k,\beta_k) = \frac{B_k}{2\sqrt{\pi\beta_k}}\exp\left[-\left(\frac{\omega_k^2}{4\beta_k}\right)\right] - \frac{A_k}{2\alpha_k}exp\left(-\alpha_k|\omega_k|\right), \omega_k\in\mathbb{R},$$

cannot be a spectral density function, since the following inequality

$$\frac{B_k\alpha_k}{A_k\sqrt{\pi\beta_k}} - \exp\left(\frac{\omega_k^2}{4\beta_k} - \alpha_k|\omega_k|\right) \geq 0, \quad \omega_k\in\mathbb{R},$$

is not satisfied, because the function τ does not present an absolute maximum value.

In the same way as all that proposed in all the examples shown in the Sect. 3.1 and according to Corollaries 4 and 5, the following functions:

$$C(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}\left[A_kC_1(x_k;\alpha_k) - B_kC_2(x_k;\beta_k)\right],$$
$$C^*(\mathbf{x};\boldsymbol{\Theta}) = \prod_{k=1}^{n}A_kC_1(x_k;\alpha_k) - \prod_{k=1}^{n}B_kC_2(x_k;\beta_k),$$

are real c.f.s, both defined in $\mathbb{R}^n$, where

$\mathbf{A} = (A_1,\ldots,A_n), \mathbf{B} = (B_1,\ldots,B_n), \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n), \boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$

and $\boldsymbol{\Theta} = (\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta})$ are vectors of parameters, $\mathbf{x} = (x_1,\ldots,x_n)\in\mathbb{R}^n$, where C_1 and C_2 are the c.f.s defined in the Examples 7, 8, 9, 10, 11, 12 and 13.

3.3 Difference of isotropic covariance functions

As already underlined, an isotropic c.f. is necessarily a real valued c.f.. Although there exist several examples of real c.f.s for stationary random fields in one dimensional space, however some of the same c.f.s cannot be examples of isotropic c.f.s in any dimensional space (see for example, the triangular c.f.s). Nevertheless, there exist functions which can be isotropic c.f.s in any dimensional space $\mathbb{R}^n, n\in\mathbb{N}$. A linear combination, with negative weights, of two continuous spatial isotropic or spatio-temporal c.f.s has been analyzed by Ma (2005) in all dimensions: however, the two spatial or spatio-temporal c.f.s involved in the analysis belong to the same family. In this subsection some examples for the difference of two isotropic c.f.s are provided for the two and three dimensional spaces; in particular, for some of these examples the c.f.s involved in the difference could not belong to the same family.

The following equations express the multidimensional spectral density f in terms of the one-dimensional spectral density f_1, for the two and three dimensional spaces, i.e., $n=2$ and $n=3$, which are the most relevant cases in the applications:

$$f(\omega) = -\frac{1}{\pi}\int_{\omega}^{\infty}\frac{df_1(\omega_1)}{d\omega_1}\frac{d\omega_1}{(\omega_1^2-\omega^2)^{1/2}}, \quad n=2; \tag{46}$$

$$f(\omega) = -\frac{1}{2\pi\omega}\frac{df_1(\omega)}{d\omega}, \quad n=3. \tag{47}$$

To check whether or not a given function C is an n-dimensional isotropic c.f. (for $n=2$ and $n=3$) it is important to find, first of all, the one dimensional Fourier transform f_1 of the function C, then computing the corresponding two and three dimensional spectral density f from f_1, utilizing Eqs. (46) and (47) and finally verifying whether or not the function f is everywhere non negative.

The exponential and Gaussian c.f.s are isotropic c.f.s in the space $\mathbb{R}^n$, for any $n \in \mathbb{N}$. In the following examples, starting from the one dimensional spectral density functions given in the Examples 7 and 8 and applying Eqs. (46) and (47), the difference, considering various combinations, between the isotropic exponential and Gaussian c.f.s have been analyzed for the two and three dimensional spaces.

Example 14 In the following example, the difference between two isotropic Gaussian c.f.s has been considered in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

- $n=2$. In the Example 7, given the one dimensional c.f.s $C_1(x_k)$ and $C_2(x_k)$ and the corresponding one dimensional spectral density functions $f_1(\omega_k)$ and $f_2(\omega_k)$, by utilizing Eq. (46), the two dimensional spectral density functions $f_1(\omega)$ and $f_2(\omega)$ can be obtained:

$$f_1(\omega;\alpha) = \frac{1}{4\pi\alpha}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right], \qquad f_2(\omega;\beta) = \frac{1}{4\pi\beta}\exp\left[-\left(\frac{\omega^2}{4\beta}\right)\right], \tag{48}$$

where $\alpha>0$, $\beta>0$, $\boldsymbol{\omega}=(\omega_1,\omega_2)$ and $\omega=\sqrt{\omega_1^2+\omega_2^2}$. Then

$$C(x;A,B,\alpha,\beta) = A\exp(-\alpha x^2) - B\exp(-\beta x^2), A>0, B>0, \tag{49}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e.,

$$Af_1(\omega;\alpha) - Bf_2(\omega;\beta) = \frac{A}{4\pi\alpha}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right] - \frac{B}{4\pi\beta}\exp\left[-\left(\frac{\omega^2}{4\beta}\right)\right] \geq 0, \quad \omega\in\mathbb{R};$$

the previous inequality is satisfied if:

$$\frac{\omega^2}{4}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) > \ln\left(\frac{B}{A}\frac{\alpha}{\beta}\right); \tag{50}$$

then, (50) is always satisfied if $\alpha>\beta$ and $\frac{B}{A}\frac{\alpha}{\beta}<1$, i.e., $1<\frac{\alpha}{\beta}<\frac{A}{B}$.

- $n=3$. In this case, utilizing Eq. (47), the following three dimensional spectral density functions $f_1(\omega)$ and $f_2(\omega)$ can be obtained:

$$f_1(\omega;\alpha) = \frac{1}{8(\pi\alpha)^{2/3}}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right]; \qquad f_2(\omega;\beta) = \frac{1}{8(\pi\beta)^{2/3}}\exp\left[-\left(\frac{\omega^2}{4\beta}\right)\right], \tag{51}$$

where $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3), \omega=\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}, \alpha>0$, $\beta>0$. Then

$$C(x;A,B,\alpha,\beta) = A\exp(-\alpha x^2) - B\exp(-\beta x^2), A>0, B>0, \tag{52}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e.,

$$Af_1(\omega;\alpha) - Bf_2(\omega\beta) = \frac{A}{8(\pi\alpha)^{2/3}}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right] - \frac{B}{8(\pi\beta)^{2/3}}\exp\left[-\left(\frac{\omega^2}{4\beta}\right)\right] \geq 0, \quad \omega\in\mathbb{R};$$

the previous inequality is satisfied if:

$$1<\frac{\alpha}{\beta}<\left(\frac{A}{B}\right)^{3/2}.$$

Example 15 In the following example, the difference between two isotropic exponential c.f.s has been considered in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

- $n=2$. In the Example 8, given the c.f.s $C_1(x_k)$ and $C_2(x_k)$ and the corresponding one dimensional spectral density functions $f_1(\omega_k)$ and $f_2(\omega_k)$, by utilizing

Eq. (46), the two dimensional spectral density functions $f_1(\omega)$ and $f_2(\omega)$ can be obtained:

$$f_1(\omega;\alpha) = \frac{\alpha}{2\pi(\omega^2+\alpha^2)^{3/2}}, \quad f_2(\omega;\beta) = \frac{\beta}{2\pi(\omega^2+\beta^2)^{3/2}},$$

where $\alpha > 0$, $\beta > 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and $\omega = \sqrt{\omega_1^2+\omega_2^2}$. Then

$$C(x;A,B,\alpha,\beta) = A\exp\big(-\alpha|x|\big) - B\exp\big(-\beta|x|\big),\quad A>0, B>0, \tag{53}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e.,

$$\frac{A\alpha}{2\pi(\omega^2+\alpha^2)^{3/2}} - \frac{B\beta}{2\pi(\omega^2+\beta^2)^{3/2}} \geq 0, \quad \omega \in \mathbb{R}; \tag{54}$$

in particular, (54) is satisfied if: $\left(\frac{\omega^2+\beta^2}{\omega^2+\alpha^2}\right)^{3/2} - \frac{B}{A}\frac{\beta}{\alpha} \geq 0, \quad \omega \in \mathbb{R}.$

If $\alpha < \beta$, then (54) is satisfied if: $1 < \frac{\beta}{\alpha} < \frac{A}{B}$; on the other hand, if $\alpha > \beta$, then (54) is satisfied if: $1 < \frac{\alpha}{\beta} < \sqrt{\frac{A}{B}}$.

- $n = 3$. In this case, utilizing Eq. (47), the following three dimensional spectral density functions f_1 and f_2 can be obtained:

$$f_1(\omega;\alpha) = \frac{\alpha}{\pi^2(\omega^2+\alpha^2)^2}, \quad f_2(\omega;\beta) = \frac{\beta}{\pi^2(\omega^2+\beta^2)^2},$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3), \omega = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2}, \alpha > 0$, $\beta > 0$. Then

$$C(x;A,B,\alpha,\beta) = A\exp\big(-\alpha|x|\big) - B\exp\big(-\beta|x|\big),\quad A>0, B>0, \tag{55}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e.,

$$\frac{A\alpha}{\pi^2(\omega^2+\alpha^2)^2} - \frac{B\beta}{\pi^2(\omega^2+\beta^2)^2} \geq 0, \quad \omega \in \mathbb{R}; \tag{56}$$

in particular, (56) is satisfied if: $\left(\frac{\omega^2+\beta^2}{\omega^2+\alpha^2}\right)^{2} - \frac{B}{A}\frac{\beta}{\alpha} \geq 0, \quad \omega \in \mathbb{R}.$

If $\alpha < \beta$, then (56) is satisfied if: $1 < \frac{\beta}{\alpha} < \frac{A}{B}$; on the other hand, if $\alpha > \beta$, then (56) is satisfied if: $1 < \frac{\alpha}{\beta} < \left(\frac{A}{B}\right)^{1/3}$.

Example 16 In the following example, the difference between an isotropic exponential and an isotropic Gaussian c.f.s has been considered in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

- $n = 2$. In the Examples 8 and 7, given the one dimensional c.f.s $C_1(x_k)$ and $C_2(x_k)$ for the exponential and the Gaussian models, respectively, with the corresponding one dimensional spectral density functions $f_1(\omega_k)$ and $f_2(\omega_k)$, by utilizing Eq. (46), the two dimensional spectral density functions $f_1(\omega)$ and $f_2(\omega)$, with $\alpha = \beta$, can be obtained:

$$f_1(\omega;\alpha) = \frac{\alpha}{2\pi(\omega^2+\alpha^2)^{3/2}},$$
$$f_2(\omega;\alpha) = \frac{1}{4\pi\alpha}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right], \alpha > 0,$$

where $\alpha > 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and $\omega = \sqrt{\omega_1^2+\omega_2^2}$. Then

$$C(x;A,B,\alpha) = A\exp\big(-\alpha|x|\big) - B\exp\big(-\alpha x^2\big), A>0, B>0, \tag{57}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e., $\frac{A\alpha}{2\pi(\omega^2+\alpha^2)^{3/2}} - \frac{B}{4\pi\alpha}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right] \geq 0, \quad \omega \in \mathbb{R};$ the previous inequality is satisfied if: $\left[\exp\left(\omega^2/4\alpha\right)\right]/\left(\omega^2+\alpha^2\right)^{3/2} \geq B/(2A\alpha^2)$.

The minimum value of the function $g(\omega;\alpha) = \left[\exp\left(\omega^2/4\alpha\right)\right]/(\omega^2+\alpha^2)^{3/2}$ is obtained for $\omega = 0$, if $\alpha > 6$: in this case, (57) is a c.f. if $\alpha \leq (2A)/B$; on the other hand, the minimum value of the function g is obtained for $\omega = \sqrt{6\alpha - \alpha^2}$ if: $0 < \alpha < 6$. In this last case, (57) is a c.f. if: $e^{(6-\alpha)/4}\sqrt{\alpha} > (3B\sqrt{6})/A$.

- $n = 3$. In this case, utilizing Eq. (47), the following three dimensional spectral density functions f_1 and f_2 can be obtained:

$$f_1(\omega;\alpha) = \frac{\alpha}{\pi^2(\omega^2+\alpha^2)^2},$$
$$f_2(\omega;\alpha) = \frac{1}{8(\pi\alpha)^{3/2}}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right],$$

where $\alpha > 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3), \omega = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$. Then

$$C(x;A,B,\alpha) = A\exp\big(-\alpha|x|\big) - B\exp\big(-\alpha x^2\big), A>0, B>0, \tag{58}$$

according to Corollary 3, is a c.f. if its Fourier transform is a spectral density function, i.e., $\frac{A\alpha}{\pi^2(\omega^2+\alpha^2)^2} - \frac{B}{8(\pi\alpha)^{3/2}}\exp\left[-\left(\frac{\omega^2}{4\alpha}\right)\right] \geq 0;$ the previous inequality is satisfied if: $\left[\exp\left(\omega^2/4\alpha\right)\right]/(\omega^2+\alpha^2)^2 \geq B\sqrt{\pi}/\ (8A\alpha^2\sqrt{\alpha})$.

The minimum value of the function $\delta(\omega;\alpha) = \left[\exp\left(\omega^2/4\alpha\right)\right]/(\omega^2+\alpha^2)^2$ is obtained for $\omega = 0$, if $\alpha > 8$: in this case, (58) is a c.f. if $\alpha\sqrt{\alpha} \leq (8A)/(B\sqrt{\pi})$; on the other hand, the minimum value of the function δ is obtained for $\omega = \sqrt{8\alpha - \alpha^2}$ if: $0<\alpha<8$. In this last case, (58) is a c.f. if: $e^{(8-\alpha)/4}\sqrt{\alpha} > (8B\sqrt{\pi})/A$.

3.4 Further properties of real covariance functions

Some properties related to isotropic/anisotropic and separable/nonseparable c.f.s are given hereafter.

- If C_1 and C_2 in (6) are isotropic c.f.s, then C is an isotropic c.f.;
- if C_1 or C_2 in (6) is an anisotropic c.f., then C is an anisotropic c.f.;
- the sum of two anisotropic c.f.s could be an isotropic c.f.: let $C_3(\mathbf{x}) = C_1(\|\mathbf{x}\|) - C_2(\mathbf{x})$, where C_3 satisfies Corollary 3, C_1 is an isotropic c.f. and C_2 is an anisotropic c.f.; then $C(\mathbf{x}) = C_3(\mathbf{x}) + C_2(\mathbf{x})$ is an isotropic c.f..
- although the c.f. C in Eq. (11) is separable in $\mathbb{R}^n$, the c.f. C in (16) is non separable; anyway both models are characterized by non geometric anisotropy.

Real c.f.s can also be obtained as the Fourier transform of the difference of two spectral density functions, which are not both even functions. The following result is a generalization of Example 1.

Corollary 7 Let: $f_1(\omega;\alpha) = \exp(-\alpha|\omega|)P_n(\omega)$, $f_2(\omega;\alpha) = \exp(-\alpha|\omega|)Q_m(\omega)$, be spectral density functions, $\omega \in \mathbb{R}$, and let

$$f(\omega;\alpha) = A_1 f_1(\omega;\alpha) - A_2 f_2(\omega;\alpha) = \exp(-\alpha|\omega|)\big(A_1 P_n(\omega) - A_2 Q_m(\omega)\big),$$

where P_n and Q_m are polynomials of degree n and m, respectively. Let C_1 and C_2 be the c.f.s corresponding, through Bochner's theorem, to f_1 and f_2, respectively. If:

$$A_1 P_n(\omega) - A_2 Q_m(\omega) \geq 0, \quad \omega \in \mathbb{R}, \quad \textit{and}$$
$$A_1 P_n(\omega) - A_2 Q_m(\omega) = A_1 P_n(-\omega) - A_2 Q_m(-\omega), \quad \omega \in \mathbb{R},$$

then $(A_1 f_1 - A_2 f_2)$ is an even spectral density function and

$$C(x;\alpha) = A_1 C_1(x;\alpha) - A_2 C_2(x;\alpha) = \int_{\mathbb{R}} \exp(i\omega x)(A_1 f_1(\omega;\alpha) - A_2 f_2(\omega;\alpha))d\omega,$$

is a real c.f., although C_1 and C_2 could be complex c.f.s if $P_n(\omega) \neq P_n(-\omega)$, for some $\omega \in \mathbb{R}$, and $Q_m(\omega) \neq Q_m(-\omega)$, for some $\omega \in \mathbb{R}$.

Note that it is necessary that the integers n and m in the previous corollary must be even numbers, otherwise f_1 and f_2 cannot be spectral density functions.

4 Covariance functions with negative values

The set of c.f.s with negative values is, of course, a subset of the real c.f.s. Apart from some particular cases which have been published in the literature, some examples of c.f.s characterized by negative values, are given hereafter. In general, for a complex c.f., only its real part is a c.f., hence it is possible to analyze the sign of the real part: some complex c.f.s whose real part is a covariance with negative values have been proposed. Some families regarding the difference of two real c.f.s, already described in subsections, characterized by negative values, have also been selected.

1. Yaglom (1987) furnished oscillatory c.f.s, based on the Bessel function of the first kind; on the other hand, Gneiting (2002a) proposed some families of c.f.s characterized by negative values through the turning bands technique.
2. C.f.s which are constructed as a linear combinations of c.f.s, with some negative weights, as described in Ma (2005) and in Gregori et al. (2008).

   In particular, in Ma (2005) linear combinations of two isotropic c.f.s of the same type were considered with further generalizations in the space-time domain.

   In Gregori et al. (2008) the following spatio-temporal class of the generalized sum of product models was analyzed, i.e.,

   $$C_{st}(\mathbf{x},u) = \sum_{j=1}^{m} k_j C_{sj}(\mathbf{x}) C_{tj}(u), \quad (\mathbf{x},u) \in \mathbb{R}^n \times \mathbb{R}, \quad (59)$$

   where C_{sj} and $C_{tj}, j = 1,\ldots,m$, are continuous and integrable spatial and temporal c.f.s, respectively. Through a suitable analysis on the inf and sup of the ratios of the spectral density functions, the authors pointed out that the class (59) is still a class of c.f.s for some negative weights k_j.
3. The real part of a complex c.f. is a c.f.; there exist several classes of complex c.f.s whose real part is characterized by negative values in a subset of their domain, as pointed out in the following examples.

Example 17 Let's consider, in $\mathbb{R}$, with $\lambda > 0$, $p > 0$, the following spectral density function (gamma distribution):

$$f(\omega; \lambda, p) = \begin{cases} \frac{\lambda^p}{\Gamma(p)} \omega^{p-1} e^{-\lambda\omega} & \omega > 0 \\ 0 & \omega \leq 0, \end{cases} \quad (60)$$

and the corresponding c.f.:

$$C(x) = \left(\frac{\lambda}{\lambda - ix} \right)^p. \quad (61)$$

Please note that (60) is not an even spectral density function, hence (61) is a complex c.f.; a special case of (61), for $p = 2$ and $\lambda = 1$, is the following c.f.:

$$C(x) = \frac{1 - x^2}{(x^2 + 1)^2} + i \frac{2x}{(x^2 + 1)^2}.$$

Example 18 Let's consider, in $\mathbb{R}$, with $n \in \mathbb{N}_+$, the following spectral density function (χ^2 distribution):

$$f(\omega; n) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} \omega^{n/2-1} e^{-\omega/2} & \omega > 0 \\ 0 & \omega \leq 0, \end{cases} \quad (62)$$

and the corresponding c.f.:

$$C(x) = \left(\frac{1}{1 - 2ix} \right)^{n/2}, \quad (63)$$

which is very similar to (61). Even in this case, (62) is not an even spectral density function, hence (63) is a complex c.f.. A special case of (63), for $n = 4$, is the following c.f.:

$$C(x) = \frac{1 - 4x^2}{(4x^2 + 1)^2} + i \frac{4x}{(4x^2 + 1)^2}.$$

On the other hand, the difference of two real c.f.s can be also characterized by negative values, for suitable values of the vector of parameters, in a subset of their domain, as shown in the following example.

Example 19 Let's consider, in $\mathbb{R}$, the following spectral density functions:

$f_1(\omega) = e^{-|\omega|}(1 + |\omega|)$, $f_2(\omega) = e^{-|\omega|}$, which are even functions; then,

$$C(x; A_1, A_2) = \int_{\mathbb{R}} \exp(i\omega x)[A_1 f_1(\omega) - A_2 f_2(\omega)] d\omega \quad (64)$$

is a real c.f. if and only if: $A_1 f_1(\omega) - A_2 f_2(\omega) \geq 0$, $\omega \in \mathbb{R}$; hence, (64) is a real c.f. if $A_2 < A_1$. Then, the c.f. $C(x; A_1, A_2) = \frac{4A_1}{(x^2 + 1)^2} - \frac{2A_2}{(x^2 + 1)}$ is negative in a subset of its domain, i.e., $x < -\sqrt{\frac{2A_1}{A_2} - 1}$, $x > \sqrt{\frac{2A_1}{A_2} - 1}$.

4. C.f.s characterized by negative values can be selected among most of the families described in the examples proposed in Sect. 3.2 and the same formalism will be retained: the generic coordinate of the one dimensional space $\mathbb{R}$ will be denoted with x_k.

In particular, in Example 7, it is easy to verify that the c.f. C in (30) is negative for all the values $x_k \in \mathbb{R}$ which satisfy: $(\alpha_k - \beta_k)x_k^2 > \ln\left(\frac{A_k}{B_k}\right)$, $k = 1, \ldots, n$;

In Example 8, if $\alpha_k > \beta_k$, it is easy to verify that the c.f. (32) is negative for the values $x_k \in \mathbb{R}$ which satisfy: $(\alpha_k - \beta_k)|x_k| > \ln\left(\frac{A_k}{B_k}\right), k = 1, \ldots, n$; on the other hand, if $\alpha_k < \beta_k$, the c.f. (32) should be negative if: $(\beta_k - \alpha_k)|x_k| < \ln\left(\frac{B_k}{A_k}\right)$; however, because of (34), $\frac{B_k}{A_k} < \frac{\alpha_k}{\beta_k} < 1$; then, if $\alpha_k < \beta_k, k = 1, \ldots, n$, the c.f. (32) cannot ever be negative. In Example 9, the parametric family (35), in $\mathbb{R}$, becomes: $C(x_k; A_k, B_k, \alpha_k, \beta_k) = \frac{A_k}{(x_k^2 + \alpha_k^2)} - \frac{B_k}{(x_k^2 + \beta_k^2)}$;

in particular, this last family is characterized just by positive values in the whole domain if $B_k < A_k$ and $\beta_k > \alpha_k$, and by negative values in a subset of its domain if: $1 < \frac{B_k}{A_k} < \frac{\beta_k}{\alpha_k}$. In Example 10, the parametric family (37), in $\mathbb{R}$, becomes:

$$C(x_k; A_k, B_k, \alpha_k) = \exp(-\alpha|x_k|)\left[A_k - \frac{\pi^3 B_k}{8\alpha_k^3}(1 + \alpha_k|x_k|)\right],$$

which is always characterized by negative values in a subset of its domain.

In Example 11, the parametric family (38), in $\mathbb{R}$, becomes:

$$C(x_k; A_k, A_k, \alpha_k, \beta_k) = A_k \exp(-\alpha_k|x_k|) - B_k \exp(-\beta_k x_k^2).$$

This last family is negative for the values $x_k \in \mathbb{R}$ which satisfy: $(\beta_k x_k^2 - \alpha_k|x_k|) < \ln\left(\frac{B_k}{A_k}\right)$; taking into account the minimum value of the function: $G(x_k; \alpha_k, \beta_k) = (\beta_k x_k^2 - \alpha_k|x_k|)$, the following condition must be satisfied for the c.f., defined in (38), to be negative, i.e., $\left(\frac{\alpha_k^2}{4\beta_k}\right) > \ln\left(\frac{A_k}{B_k}\right)$; hence, if the above condition is not satisfied, the c.f., defined in (38), cannot ever be negative. Moreover, if $\alpha_k^2 < 4\beta_k$ and $0 < \frac{B_k}{A_k} < \frac{\sqrt{\pi}}{\pi^2} e^3$, $\ln\left(\frac{A_k}{B_k}\right) < \frac{\alpha_k^2}{4\beta_k} < 1$,

the same family of c.f.s could assume negative values in a subset of its domain.

In Example 12, if $\alpha_k \beta_k > 1$, the c.f.

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = A_k \exp\left(-\alpha_k |x_k|\right) - \frac{B_k}{(x_k^2 + \beta_k^2)}, \quad A_k > 0, B_k > 0,$$

is negative for the values $x_k \in \mathbb{R}$ which satisfy: $(x_k^2 + \beta_k^2)\exp(-\alpha_k|x_k|) < \frac{B_k}{A_k}$;

this last inequality can be easily satisfied for sufficiently high values of x_k, since

$$\lim_{x_k \to \infty} \left[(x_k^2 + \beta_k^2) \exp(-\alpha_k |x_k|) \right] = 0.$$

In Example 14 the difference between two isotropic Gaussian c.f.s has been considered in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively: it is easy to verify that the c.f.s C in (49) and (52) are negative for all the values $x \in \mathbb{R}$ which satisfy: $(\alpha - \beta)x^2 > \ln\left(\frac{A}{B}\right)$.

In Example 15 the difference between two isotropic exponential c.f.s has been considered in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. In particular, if $\alpha < \beta$, the c.f.s C in (53) and (55) should be negative for the values $x \in \mathbb{R}$ which satisfy: $(\beta - \alpha)|x| < \ln\left(\frac{B}{A}\right)$; however, this last inequality cannot ever be satisfied because $1 < \frac{\beta}{\alpha} < \frac{A}{B}$; on the other hand, if $\alpha > \beta$, the c.f.s C in (53) and (55) are negative for the values $x \in \mathbb{R}$ which satisfy: $(\alpha - \beta)|x| > \ln\left(\frac{A}{B}\right)$.

5 Relevance and impact of the results

The motivation for this paper arises from the need to address various problems, whose importance concerns theoretical and practical aspects as specified hereafter:

- a covariance is a complex valued function and its properties are well known and described in the literature. Moreover, it is also well known that, in general, the difference of two c.f.s is not a c.f. and this issue, in the literature, does not yet seem to have been addressed in detail. For this reason, the general problem concerning the difference of two c.f.s in the complex domain has been analyzed in this paper; hence, exploring the conditions such that the difference of two c.f.s is again a c.f. can be considered a further property. The results of this paper thus yield a general contribution to the theory of correlation, hence they cannot be restricted to the particular field of application. The above issue has been explored in the complex domain as well as in the subset of the real domain; in this latter case, the difference of isotropic c.f.s has also been discussed for the two and the three dimensional Euclidean space. In particular, a detailed analysis involving the parameters of these models has been proposed. Although the results provided in this paper represent a natural consequence of Bochner's theorem, the significant aspects of the same results derive from properly specifying the characteristics of the spectral distribution function or of the spectral density function in Bochner's representation.
- All the families of c.f.s, which are constructed utilizing the standard models (Gaussian, general exponential and rational, which are non negative and non decreasing c.f.s) and by applying the usual and well known properties (such as linear combinations with non negative coefficients or products), preserve the main typical characteristics of the standard models, regardless of the values of their parameters, as formally shown hereafter.

Let C_j be c.f.s such that $C_j(x) \geq 0, x \in \mathbb{R}$; $C_j'(x) < 0, x > 0, j = 1, \ldots, n$ and

$$C_S(x) = \sum_{j=1}^{n} \lambda_j C_j(x), \quad C_P(x) = \prod_{j=1}^{n} C_j(x), \quad \lambda_j \geq 0, j = 1, \ldots, n; \tag{65}$$

then:

$$C_S(x) \geq 0, x \in \mathbb{R}, \quad \textit{and} \quad C_S'(x) < 0, x > 0,$$
$$C_P(x) \geq 0, x \in \mathbb{R} \quad \textit{and} \quad C_P'(x) < 0, x > 0;$$

in fact, a linear combination of these standard models (C_j in the above expressions) with non negative coefficients or their product, generate c.f.s which present the same behaviour of the same standard models: they are characterized by a systematic decreasing behaviour and are always non negative in their domain.

On the other hand, there exist models constructed through the difference of standard models which behave differently by properly changing the values of the non negative parameters from which they depend. Hence, some special families of c.f.s obtained through the difference of two standard c.f.s provide a significant contribution to modeling wide classes of correlation structures. For example, the class of models defined in (32), i.e.,

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = A_k \exp\left(-\alpha_k|x_k|\right) - B_k \exp\left(-\beta_k|x_k|\right), \quad A_k > 0, B_k > 0,$$

is flexible enough to describe several situations with reespect to the standard families of c.f.s as the ones defined in Eq. (65); indeed, the class (32) is always characterized by a linear behavior near the origin, it is always positive if: $\frac{B_k}{A_k} < \frac{\alpha_k}{\beta_k} < 1$; moreover, in this case, this class presents a systematically decreasing behavior for $x_k > 0$. On the other hand, if: $\frac{B_k}{A_k} < \frac{\beta_k}{\alpha_k} < 1$, the same class is negative for all the values $x_k \in \mathbb{R}$ which satisfy: $(\alpha_k - \beta_k)|x_k| > \ln\left(\frac{A_k}{B_k}\right)$. Hence, the same class behaves differently by properly changing the values of the parameters on which it depends.

Likewise, the class of models defined in (35), i.e.,

$$C(x_k; A_k, B_k, \alpha_k, \beta_k) = \frac{A_k}{(x_k^2 + \alpha_k^2)} - \frac{B_k}{(x_k^2 + \beta_k^2)},$$
$$A_k > 0, B_k > 0,$$

is always characterized by a parabolic behavior near the origin and it is always positive if: $\beta_k > \alpha_k$ and $B_k < A_k$; moreover, in this case, this class presents a systematically decreasing behavior for $x_k > 0$. On the other hand, if: $B_k > A_k$, the same class is negative for $x_k \in \mathbb{R}$ which satisfy: $(B_k - A_k)x_k^2 > A_k\beta_k^2 - B_k\alpha_k^2$.

Finally, some traditional families of c.f.s, characterized by damped oscillations and by negative values in subsets of their domain, have been constructed by multiplying standard models of c.f.s with a cosinusoidal c.f.; however, these last families of c.f.s are unable to behave differently by changing the values of the parameters on which they depend, as explained hereafter.

Let $C_D(x) = C(x)\cos(x)$, where C is a c.f. such that $C(x) \geq 0, x \in \mathbb{R}$ and $C'(x) < 0, x > 0$. Then $C'_D(x) = 0$ if $\tan(x) = \frac{C'(x)}{C(x)}$; hence, there always exists $x \in \mathbb{R}_+$ such that C_D changes its sign.

- From a practical point of view, the various examples, concerning the difference of two c.f.s proposed in this paper, are flexible enough to be implemented and utilized in several applications pertaining to the complex domain, as well as to the subset of the real domain. In particular, the models obtained through this difference are characterized by different features: for example, they could be non negative in the whole domain, or could be characterized by negative values in a subset of their domain. Indeed, in several applications in biology, hydrology and spatio-temporal turbolence, c.f.s with negative values are often needed, as pointed out by various authors (Shkarofsky 1968; Levinson et al. 1984; Pomeroy et al. 2003; Xu et al. 2003a, b; Yakhot et al. 1989). The theory of complex random fields can give a significant contribution to several phenomena which often occur in ecological and environmental science: in particular, in meteorology, oceanography, signal analysis and geophysics. For this reason, in the very last years some efforts have been made in geostatistics utilizing complex formalism. In particular, an application to predict a wind field has been proposed by De Iaco and Posa (2016); moreover, a first attempt to utilize complex formalism in the spatio-temporal context has also been provided by Cappello et al. (2020): the complex c.f., indexed in time, is estimated and modeled for data, derived from high frequency radar systems, collected from 207 stations along the US East and Gulf Coast. In addition, in other particular scientific applications, flexible c.f.s models like the ones defined in Eqs. (32) and (35) are often required: in particular, in genetics and molecular biology the Price equation (Martini et al. 2016) utilizes c.f.s to provide a mathematical description of evolution and natural selection; in micrometeorology, the Eddy c.f. (Moncrieff et al. 2004) is a key atmospherics measurement technique; finally, c.f.s play a key role in feature extraction (Sahidullah and Kinnunen 2016), to capture the spectral variability of a signal.

6 Conclusions

In this paper, conditions for which the difference of two c.f.s in the complex domain, as well as in the subset of the real domain, is again a c.f. have been analyzed. This can be considered a further property which addresses some relevant issues: (a) it provides wider families of c.f.s which can be utilized to model further correlation structures; (b) it supplies families of c.f.s also characterized by negative values, utilized in several applications; (c) some special families of c.f.s obtained through the difference of two c.f.s provide a significant contribution to modeling wide classes of correlation structures because they are flexible enough to describe various behaviours as compared to the standard families. The c.f.s proposed in this paper can be utilized to model complex random fields, but also the subset of real random fields; in this latter case, some well known models have been considered. Moreover, in order to provide a complete analysis of this subject, some examples of the difference of two isotropic c.f.s have been also explored; in particular, for some of these examples the c.f.s involved in the difference could not belong to the same family. The results can be utilized in a flexible way in any dimensional domain.

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