

Estimates of effective permeability for non-Newtonian fluid flow in randomly heterogeneous porous media

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Published online: 27 April 2010
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Abstract An understanding of the interplay between non-Newtonian effects in porous media flow and field-scale domain heterogeneity is of great importance in several engineering and geological applications. Here we present a simplified approach to the derivation of an effective permeability for flow of a purely viscous power-law fluid with flow behavior index n in a randomly heterogeneous porous domain subject to a uniform pressure gradient. A standard form of the flow law generalizing the Darcy's law to non-Newtonian fluids is adopted, with the permeability coefficient being the only source of randomness. The natural logarithm of the permeability is considered a spatially homogeneous and correlated Gaussian random field. Under the ergodic hypothesis, an effective permeability is first derived for two limit 1-D flow geometries: flow parallel to permeability variation (serial-type layers), and flow transverse to permeability variation (parallel-type layers). The effective permeability of a 2-D or 3-D isotropic domain is conjectured to be a power average of 1-D results, generalizing results valid for Newtonian fluids under the validity of Darcy's law; the conjecture is validated comparing our results with previous literature findings. The conjecture is then extended, allowing the exponents of the power averaging to be functions of the flow behavior index. For Newtonian flow, novel expressions for the effective permeability reduce to those derived in the past. The effective permeability is shown to be a function of flow dimensionality, domain heterogeneity, and flow behavior index. The impact of heterogeneity is significant, especially for

shear-thinning fluids with a low flow behavior index, which tend to exhibit channeling behavior.

Keywords Non-Newtonian · Power-law fluid · Porous media · Heterogeneity

1 Introduction

Non-Newtonian fluid flow through natural porous media is of great interest in many petroleum and environmental engineering applications.

In the oil industry, many fluids used to enhance oil recovery from underground reservoirs are significantly non-Newtonian, exhibiting shear-dependence of viscosity and other distinctly non-linear effects. During water flooding operations, chemical additives, polymeric solutions or foams are often added to the injected water to minimize the instability effects, thus improving the overall sweeping efficiency; surfactants are also added to the water phase to decrease the surface tension between the aqueous and oil phases. Heavy and waxy oils displaying non-Newtonian behavior are also commonly found in underground reservoirs. Fracturing agents and drilling muds with complex rheologic behavior are routinely used in low-permeability formations.

In environmental applications, liquid pollutants and wastes such as suspensions, solutions and emulsions of various substances, certain asphalts and bitumen, greases, sludges, and slurries may migrate in the subsurface and penetrate underground reservoirs, leading to groundwater contamination. Non-Newtonian fluid flow in porous media may also be relevant in soil remediation processes involving the removal of liquid pollutants via chemical (often polymerization) reactions.

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A large body of literature exists on flow of complex fluids in porous media: for exhaustive reviews see Savins (1969), Goldstein and Entov (1994), and Chhabra et al. (2001). Several studies are specifically concerned with evaluation of macroscopic models relating the Darcy velocity of the fluid to the applied pressure gradient; these are necessary for the prediction of flowrates and/or coupling with the continuity equation for large-scale modeling. Different formulations of a modified Darcy law for non-Newtonian fluids have been proposed (e.g. Bird et al. 1960; Christopher and Middleman 1965; Kozicki et al. 1967; Teeuw and Hesselink 1980; Pearson and Tardy 2002), particularly for pseudoplastic fluids described by a power-law model. Most of these models have been derived using a capillary representation of the porous medium, and include a constant related to its tortuosity. Recently, the lattice Boltzmann technique has been successfully applied to direct simulation of power-law fluid flow in porous media (Sullivan et al. 2006).

In field-scale modeling of porous media flow at scales ranging from local to regional, domain heterogeneity plays a crucial role (Dagan 1989); in the hydrologic literature, the ubiquitous effect of heterogeneity in model parameters, most notably the hydraulic conductivity, has prompted the development of stochastic models aimed at deriving representative properties as functions of statistical parameters describing heterogeneity (for a review see Sanchez-Vila et al. 2006). Broadly speaking, a representative parameter controls the average behavior of groundwater flow within an aquifer at a given scale, and is frequently needed in numerical modeling and field applications to simulate fluid flow in heterogeneous domains at a scale larger than that of the heterogeneities. This is usually termed the porous media stochastic upscaling problem: for a critical look at its epistemic foundations, see Christakos (2003).

Clearly, when non-Newtonian fluids are involved, the development of stochastic subsurface models is hampered by the nonlinearity of the flow; generally speaking, the effect of field-scale domain heterogeneity on non-Newtonian fluid flow in porous media has received but scant attention in the literature. As a consequence, very few papers are specifically concerned with the derivation of upscaled representative parameters for such nonlinear flows; a notable exception is the work by Fadili et al. (2002), who derived the effective conductivity tensor for power-law fluids flowing in an heterogeneous medium via stochastic homogenization. For flow of such fluids in spatially variable fractures, both a deterministic (Di Federico 1997) and stochastic (Di Federico 1998) aperture variation was considered to deduce an equivalent fracture aperture; tortuosity effects were subsequently incorporated into the deterministic model (Di Federico 2001). In a related study involving nonlinear flow, Fourar et al. (2005) derived effective parameters (hydraulic

conductivity and inertial coefficient) for Forchheimer-type flow in heterogeneous porous media.

In this paper, a simplified approach to the derivation of an effective permeability for flow of a non-Newtonian fluid in a randomly heterogeneous porous medium subject to a uniform pressure gradient is presented. First, one-dimensional flow in two limit cases is considered: in the first case, flow is parallel to permeability variation (serial-type layers); in the second, flow is transversal to permeability variation (parallel-type layers). The permeability is taken to vary as a spatially homogeneous and correlated random field, with a given probability density function; two different distributions are considered.

The effective permeability of a 2-D or 3-D isotropic permeability field is then derived by a suitable averaging procedure, generalizing results valid for Newtonian fluids under the validity of Darcy's law (Sanchez-Vila et al. 2006); results are then compared with those by Fadili et al. (2002); based on the outcome of the comparison, an extension of the conjecture is proposed.

The plan of the paper is as follows: first, the most common formulations of the modified Darcy's law for flow of power-law fluids in porous media are reviewed; then, analytical relationships for effective conductivity for limit 1-D cases (serial and parallel layers) are derived; finally, a conjecture for the effective permeability of 2-D and 3-D isotropic media is formulated, and results are discussed as functions of model parameters.

2 Modified Darcy's law for a power-law fluid

In the following we focus our attention on the simplest rheological model available for a complex fluid; we therefore consider the flow of a non-Newtonian viscoelastic fluid described by the power-law (Ostwald-deWaele) model, given in simple shear flow conditions by

$$\tau = H\dot{\gamma}^n, \quad (1)$$

in which τ is the shear stress, $\dot{\gamma}$ the shear rate, H [$M L^{-1} T^{n-2}$] the consistency index and n the flow behavior index (a positive real number): $n < 1$ describes pseudoplastic (shear-thinning) behavior, $n > 1$ dilatant (shear-thickening) behavior, and $n = 1$ means Newtonian fluid; in the latter case, H reduces to Newtonian viscosity μ .

In his monumental review of non-Newtonian flow in porous media, Savins (1969) noticed that the bulk of the published investigations pertain to aqueous solutions of polymers; in turn, most, but not all, of these exhibit shear-thinning behavior. For such fluids, the power-law model is often viewed as a convenient approximation, within a given shear stress interval, of a more complex stress-shear rate relationship, adequately described by the three-parameter

Ellis model (Park et al. 1975). Savins summarized a large number of data from earlier laboratory experiments; when interpreted with the power-law model, these data yielded values of n in the range $0.5 \div 1$. Christopher and Middleman (1965) reported laboratory experimental values of n ranging from 0.50 to 0.96 for polymeric solutions; for similar fluids, Kozicki et al. (1967) obtained $n = 0.92 \div 0.97$. Gogarty (1967) derived via laboratory experiments $n = 0.61 \div 0.68$ for surfactant-stabilized dispersions of water in hydrocarbon; for the same fluids Van Poolen and Jargon (1969) obtained $n = 0.67$ in an injection field test. The experimental work of Park et al. (1975) yielded $n = 0.41 \div 0.46$ for commercial aqueous polymer solutions. Odeh and Yang (1979) obtained $n = 0.14$ in well injection tests with biopolymers. Ikoku and Ramey (1979) and Wu et al. (1991) adopted values of the flow behavior index in the range $0.2 \div 1$ in the sensitivity analysis of their analytical solutions, respectively for transient radial flow and immiscible non-Newtonian displacement in porous media.

A large number of later papers, of an experimental or theoretical nature, reported or adopted values of the flow behavior index n between 0.2 and 1; it is virtually impossible to list even a fraction of these, but a partial list of references may be found in Orgéas et al. (2006) and Sochi (2009).

However, the case of fluids showing dilatant behavior ($n > 1$) is not entirely academic: starch suspensions, gum solutions, and aqueous solutions of titanium dioxide tend to exhibit dilatant behavior (Shenoy 1995), as well as polymeric solutions in certain ranges of shear stress (Savins 1969). Therefore, for the sake of generality, in the sequel both shear-thinning and shear thickening behavior will be considered.

The flow law for the motion of such fluids in porous media is a modified Darcy’s law taking into account the nonlinearity of the rheological equation, which can be expressed in the following multi-dimensional form (Shenoy 1995)

$$\nabla P = -\frac{H}{k^*} |\mathbf{q}|^{n-1} \mathbf{q}, \tag{2}$$

where $P = p + \rho g z$ is the generalized pressure, p the pressure, z the vertical coordinate, ρ the fluid density, g the specific gravity, \mathbf{q} the Darcy flux [$L T^{-1}$], and k^* a modified permeability [L^{n+1}], defined as

$$k^* = \frac{1}{2C} \left(\frac{50k}{3\phi} \right)^{(n+1)/2} \left(\frac{n\phi}{3n+1} \right)^n \tag{3}$$

where k is the intrinsic permeability coefficient [L^2], ϕ the porosity, and C the tortuosity factor. Equation 2 can be recast equivalently as (Pascal 1983)

$$\nabla P = -\frac{\mu_{eff}}{k} |\mathbf{q}|^{n-1} \mathbf{q}, \tag{4}$$

in which μ_{eff} [$M L^{-n} T^{n-2}$] is the effective viscosity, reducing for $n = 1$ to Newtonian viscosity μ ; since obviously $\mu_{eff} k = H/k^*$, it follows that μ_{eff} is a function of H , n , ϕ and k according to

$$\mu_{eff} = 2C \left(\frac{3}{50} \right)^{(n+1)/2} \left(\frac{3n+1}{n} \right)^n H \phi^{(1-n)/2} k^{(1-n)/2} \tag{5}$$

For $C = 1$, Eq. 5 coincides with the expression first presented by Bird et al. (1960); when $C = 25/12$, one recovers the widely used “bed factor” of Christopher and Middleman (1965) and Savins (1969)

$$\mu_{eff}^I = \frac{1}{12} \left(\frac{9n+3}{n} \right)^n (150)^{(1-n)/2} H \phi^{(1-n)/2} k^{(1-n)/2}, \tag{6}$$

while for $C = (25/12)^{(n+1)/2}$, one has

$$\mu_{eff}^{II} = \frac{2}{(8)^{(1+n)/2}} \left(\frac{3n+1}{n} \right)^n H \phi^{(1-n)/2} k^{(1-n)/2}, \tag{7}$$

first proposed by Teeuw and Hesselink (1980) and adopted in a number of papers by Pascal and coworkers (for a partial list of references see Pascal and Pascal 1997). Additional formulations of the effective viscosity, adopting slightly different expressions for the tortuosity factor C , are available in Bird et al. (1987) and Shenoy (1995).

Auriault et al. (2002) derived via homogenization techniques the following flow law, valid for a locally isotropic medium

$$\mathbf{q} = -C' |\nabla P|^{(1-n)/n} \nabla P, \tag{8}$$

with C' being a function of H and n ; an analogous expression was earlier derived by Slattery (1967), and a similar one was adopted for one-dimensional flow by Sullivan et al. (2006).

Finally, the formulation of the flow law proposed by Fadili et al. (2002) for local-scale 3-D flow in a porous medium reads

$$\mathbf{q} = -K |\mathbf{q}|^{1-n} \nabla P, \tag{9}$$

where K [$M^{-1} L^{(2+n)} T^{(2-n)}$] is a function of n .

Our succinct literature review shows that in general the Darcy flux and the gradient of the generalized pressure (hereinafter referred to simply as pressure) are related in a nonlinear fashion via a power-law relationship, which includes a constant of proportionality (conductivity parameter) having disparate dimensions and labeled with different symbols by the authors; the chemical engineering literature provides several additional models (see for example Liu and Masliyah 1998), in which the conductivity parameter is an explicit function of pore-scale variables.

However, the focus of this paper is on the interplay between non-Newtonian effects and domain heterogeneity at the field scale (Dagan 1989). In the hydrologic literature, spatial heterogeneity is commonly modeled via the stochastic variability of the hydraulic conductivity tensor \mathbf{K} [LT^{-1}], related to the intrinsic permeability tensor \mathbf{k} [L^2] via $\mathbf{K} = (\rho g/\mu)\mathbf{k}$; dealing with a non-Newtonian fluid, it is appropriate to view directly \mathbf{k} as spatially variable. Hence, we find it convenient to adopt a local (or Darcy-scale) flow law where the intrinsic permeability appears explicitly, without further discussing its dependency on pore-scale variables. We then adopt for the local flow law Eq. 4, which has the additional advantage of incorporating the dependence from the fluid consistency index. To elucidate in a preliminary fashion the impact of the adoption of either expression (6) or (7), most commonly adopted in the literature for the effective viscosity, in Fig. 1 the ratio $\mu_{eff}^I/\mu_{eff}^{II}$ is shown as a function of n in the range [0,2]; it is seen that the two expressions are of the same order in the entire range; when $0.3 < n < 1.5$, the difference between the two expressions is $<20\%$.

Finally, to explore the dependency of representative parameters from the permeability k we find it convenient to re-formulate the expression of k/μ_{eff} appearing in Eq. 4 as follows

$$\begin{aligned} \frac{k}{\mu_{eff}} &= \Lambda k^{(1+n)/2}; \quad \Lambda = \Lambda(\phi, H, n) \\ &= \frac{1}{2C} \left(\frac{50}{3}\right)^{(n+1)/2} \left(\frac{n}{3n+1}\right)^n \frac{\phi^{(n-1)/2}}{H}. \end{aligned} \quad (10)$$

3 Representative permeability for non-Newtonian flow in randomly heterogeneous porous media

In this section we wish to derive a representative intrinsic permeability (hereinafter referred to simply as

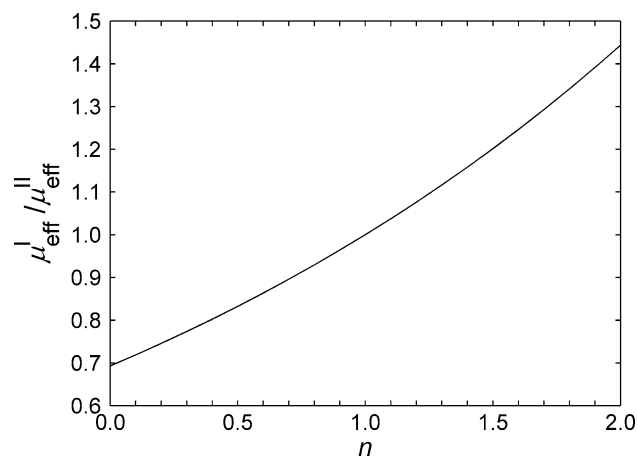


Fig. 1 Ratio $\mu_{eff}^I/\mu_{eff}^{II}$ as a function of n in the range [0,2]

permeability) for non-Newtonian fluid flow in randomly heterogeneous porous media subject to a uniform pressure gradient. By representative permeability we mean a parameter controlling the average behavior of fluid flow within a porous medium at a given scale. In their recent review on representative hydraulic conductivities in saturated groundwater flow, Sanchez-Vila et al. (2006) define different representative hydraulic conductivities, namely effective, pseudo-effective, equivalent and intrinsic ones. The effective conductivity relates the ensemble averages of flux and head gradient, and is constant throughout the entire flow domain; if it varies with the location within the domain, it is termed pseudo-effective or apparent. The equivalent conductivity relates the spatial averages of flux and head gradient within a given domain, although the term is sometimes used in place of pseudo-effective or apparent (Severino et al. 2008). Under ergodicity, ensemble and spatial averages are interchangeable, and so are effective and equivalent parameters. Finally, the interpreted conductivity derives from interpretation of field data. While the above definitions may be readily applied to the relationship between specific flux and the pressure gradient in a linear context, their application to non-Darcy flow is not straightforward, given the nonlinearity of the flow. In fact, when gas flow in randomly heterogeneous media is examined (Tartakovsky 2000; Tartakovsky and Guadagnini 2000), effective parameters according to the above definition are shown to exist only in special cases, namely under a uniform pressure gradient. A similar result was shown to hold for unsaturated groundwater flow (Tartakovsky et al. 2003), in which a necessary condition for the existence of effective relative permeability is the existence of pressure gradients that vary slowly in space over significant parts of the flow domain.

In the context of nonlinear seepage flow due to inertial effects, commonly known as Forchheimer flow, Fourar et al. (2005) defined an effective hydraulic conductivity and an effective Forchheimer coefficient for flow in a randomly heterogeneous porous medium subject to a uniform pressure gradient and under the assumption of a constant local Forchheimer coefficient. In their paper, the effective parameters were derived analytically in the case of serial and parallel layers; for two-dimensional correlated media, numerical simulations were performed, allowing to determine the large-scale Forchheimer coefficient as a function of the Reynolds number and the correlation length.

To our knowledge, the determination of effective parameters for non-Newtonian flow in randomly heterogeneous porous media at the field scale has been tackled only by Fadili et al. (2002). Starting from a local scale flow law given by Eq. 9, they modeled the conductivity K appearing there as a second-order homogeneous random field characterized by its mean and autocovariance

function, under the assumption of a lognormal conductivity distribution. They derived a macro-scale relationship between flux and pressure gradient through an homogenization technique, adopting two complementary formulations (macro-conductivity and macro-resistivity approaches) that were then matched via an exponential extrapolation. Their analytical formula for the effective conductivity, valid to second-order in the log-conductivity standard deviation for one, two, and three dimensions, was then validated in 2-D via numerical simulations; their analytical results showed good agreement with their numerical simulations for $0.2 \leq n \leq 1$, while for lower values of n (of limited practical value) the theoretical results significantly overestimate the numerical simulations.

Following the same approach, here the permeability k is taken to vary as a spatially homogeneous and correlated random field, with an assigned probability density function (pdf) $f(k)$.

The domain dimensions are assumed to be much larger than the integral scale of the permeability autocovariance function; then, under ergodicity, spatial averages and ensemble averages are interchangeable, and a single realization can be examined. To derive an expression for the effective permeability k_{eff} , one-dimensional flow is first considered in two limiting cases: (i) serial-type layers, when the pressure gradient is parallel to the permeability variation; (ii) parallel-type layers, when the pressure gradient is transverse to the permeability variation.

3.1 Serial-type layers

Consider an heterogeneous porous domain of length L , subject to an external uniform pressure difference $\Delta P = P(0) - P(L)$; the domain is made of N homogeneous layers of equal thickness Δx and with an area A perpendicular to the flow direction x ; the i th layer has a permeability k_i . By virtue of mass conservation, volumetric flux Q through each layer is the same; assuming that the 1-D version of the flow law given by Eqs. 4 and 10 holds locally, the pressure difference in each layer is given by

$$\Delta P_i = \frac{1}{\Lambda} \left(\frac{Q}{A}\right)^n k_i^{-(n+1)/2} \Delta x. \tag{11}$$

Summing over the layers yields, since $\Delta x = L/N$

$$\Delta P = \frac{1}{\Lambda} \left(\frac{Q}{A}\right)^n L \frac{1}{N} \sum_{i=1}^N k_i^{-(n+1)/2}. \tag{12}$$

Taking the limit as $N \rightarrow \infty$, the length of each layer tends to zero and the discrete permeability variation to a continuous one; then under ergodicity

$$\begin{aligned} \Delta P &= \frac{1}{\Lambda} \left(\frac{Q}{A}\right)^n L \int_0^\infty k^{-(n+1)/2} f(k) dk \\ &= \frac{1}{\Lambda} \left(\frac{Q}{A}\right)^n L \langle k^{-(n+1)/2} \rangle, \end{aligned} \tag{13}$$

where $\langle \cdot \rangle$ is expected value. The effective permeability k_{eff}^s for serial-type layers is then defined by means of the relationship

$$\Delta P = \frac{1}{\Lambda} \left(\frac{Q}{A}\right)^n L \left(k_{eff}^s\right)^{-(n+1)/2}. \tag{14}$$

Comparison between Eqs. 13 and 14 gives

$$k_{eff}^s = \langle k^{-(n+1)/2} \rangle^{-2/(n+1)}. \tag{15}$$

For a Newtonian fluid ($n = 1$), Eq. 15 reduces to $k_{eff}^s = \langle k^{-1} \rangle^{-1} = k_H$, i.e. the harmonic mean, as first shown by Matheron (1967).

3.2 Parallel-type layers

Consider now the same porous domain as above, but made of N homogeneous layers of length L and area ΔA , perpendicular to the flow direction x ; the i th layer has a permeability k_i . The volumetric flux Q_i in the i th layer is then

$$Q_i = \Lambda^{1/n} \left(\frac{\Delta P}{L}\right)^{1/n} k_i^{(n+1)/2n} \Delta A. \tag{16}$$

Summing over the layers yields, since $\Delta A = A/N$

$$Q = \Lambda^{1/n} \left(\frac{\Delta P}{L}\right)^{1/n} A \frac{1}{N} \sum_{i=1}^N k_i^{(n+1)/2n}. \tag{17}$$

Taking the limit as $N \rightarrow \infty$, the transverse area of each layer tends to zero, and under ergodicity

$$\begin{aligned} Q &= \Lambda^{1/n} \left(\frac{\Delta P}{L}\right)^{1/n} A \int_0^\infty k^{(n+1)/2n} f(k) dk \\ &= \Lambda^{1/n} \left(\frac{\Delta P}{L}\right)^{1/n} A \langle k^{(n+1)/2n} \rangle. \end{aligned} \tag{18}$$

The effective permeability k_{eff}^p for parallel-type layers is then defined by means of the relationship

$$Q = \Lambda^{1/n} \left(\frac{\Delta P}{L}\right)^{1/n} A \left(k_{eff}^p\right)^{(n+1)/2n}. \tag{19}$$

Comparison between Eqs. 18 and 19 gives

$$k_{eff}^p = \langle k^{(n+1)/2n} \rangle^{2n/(n+1)}. \tag{20}$$

For $n = 1$, Eq. 20 reduces to $k_{eff}^p = \langle k \rangle = k_A$, i.e. the arithmetic mean (Matheron 1967).

3.3 Effective permeability estimates for serial-type and parallel-type layers

To evaluate the flow rate and effective permeability in the above expressions, the lognormal distribution is adopted for the permeability; its probability density function is given by

$$f(k) = \frac{1}{\sqrt{2\pi}k\sigma_Y} e^{-(\ln k - \ln k_G)^2 / 2\sigma_Y^2} \quad (21)$$

where $Y = \ln k$, σ_Y^2 is its variance, $k_G = \langle k \rangle e^{-\sigma_Y^2/2}$ is the geometric mean, and $\langle k \rangle = k_A$ is the expected value (arithmetic mean). Utilizing Eqs. 15 and 20 with Eq. 21 provides the following expressions for the effective permeability, respectively in serial-type and parallel-type layers (Gradshteyn and Ryzhik 1994, Eq. 3.323.2, p. 355):

$$k_{eff}^s = k_G e^{-(1+n)\sigma_Y^2/4} = \langle k \rangle e^{-(3+n)\sigma_Y^2/4}, \quad (22)$$

$$k_{eff}^p = k_G e^{(1+n)\sigma_Y^2/4n} = \langle k \rangle e^{(1-n)\sigma_Y^2/4n}. \quad (23)$$

Figure 2a and b depict the ratio $k_{eff}/\langle k \rangle$ as a function of σ_Y^2 for various values of the flow behavior index n , respectively for flow in serial-type and parallel-type layers.

For 1-D flow in serial-type layers, $k_{eff}^s < k_G < \langle k \rangle$; Fig. 2a shows that the ratio $k_{eff}^s/\langle k \rangle$ decreases markedly with increasing log-permeability variance σ_Y^2 , more so for $n > 1$ than for $n < 1$; this is so because the flow rate is controlled mainly by the lower permeability cells along the 1-D domain.

For 1-D flow in parallel-type layers, $k_{eff}^p > k_G$ in any case, while $k_{eff}^p > \langle k \rangle$ for $n < 1$, and $k_{eff}^p < \langle k \rangle$ for $n > 1$. Figure 2b shows that $k_{eff}^p/\langle k \rangle$ increases significantly with increasing σ_Y^2 for $n < 1$; in this range of n the value of $k_{eff}^p/\langle k \rangle$ is very sensitive to the n value. For $n > 1$, $k_{eff}^p/\langle k \rangle$ decreases moderately with increasing σ_Y^2 , with n having very little influence on the result.

To explore the influence of the adoption of a different probability density function for the permeability, we next consider the case of the gamma distribution, recently advocated as an alternative to the lognormal distribution on the basis of field data analysis (Loaiciga et al. 2006); assuming a zero shape parameter, its pdf is given by

$$f(k) = \frac{k^{\alpha-1} e^{-k/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad (24)$$

where α and β are the distribution parameters and $\Gamma(\cdot)$ is the gamma function; the mean and variance of k are given by $\langle k \rangle = \alpha\beta$ and $\sigma_k^2 = \alpha\beta^2$. Performing the same steps provides the following expressions for the effective permeability, respectively in serial-type and parallel-type layers (Gradshteyn and Ryzhik 1994, Eq. 3.381.4, p. 364):

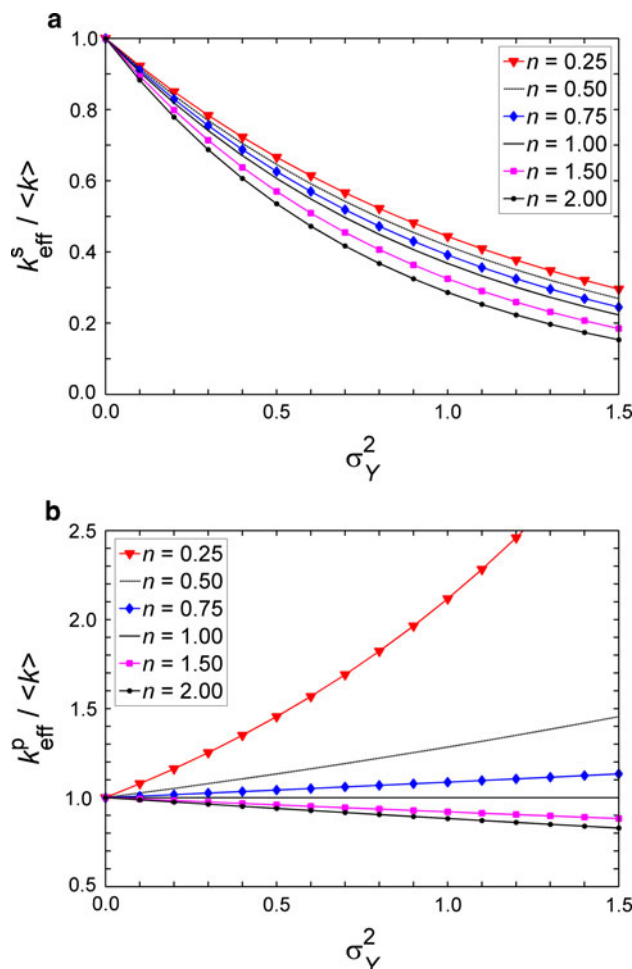


Fig. 2 Ratio $k_{eff}/\langle k \rangle$ as a function of log-permeability variance σ_Y^2 for various values of n , a lognormal pdf of k and flow in: **a** 1-D serial-type domain; **b** 1-D parallel-type domain

$$k_{eff}^s = \frac{\langle k \rangle}{\alpha} \left[\frac{\Gamma(\alpha - \frac{1}{2} - \frac{n}{2})}{\Gamma(\alpha)} \right]^{2/(n+1)}; \quad (\alpha > (n + 1)/2), \quad (25)$$

$$k_{eff}^p = \frac{\langle k \rangle}{\alpha} \left[\frac{\Gamma(\alpha + \frac{1}{2} + \frac{1}{2n})}{\Gamma(\alpha)} \right]^{2n/(n+1)}. \quad (26)$$

For $n = 1$, due to the properties of the gamma function, Eq. 25 reduces to $k_{eff}^s = \langle k \rangle (\alpha - 1) / \alpha = k_H$, while Eq. 26 reduces to $k_{eff}^p = \langle k \rangle = k_A$ as expected. Figure 3a and b depicts the ratio $k_{eff}/\langle k \rangle$ as a function of the coefficient of variation $C_v = \sigma_k/\langle k \rangle$ for various values of the flow behavior index n , respectively for flow in serial-type and parallel-type layers.

For flow in serial-type layers, $k_{eff}^s < \langle k \rangle$; Fig. 3a shows that the ratio $k_{eff}^s/\langle k \rangle$ again decreases significantly with increasing permeability heterogeneity (higher C_v), more so for $n > 1$ than for $n < 1$.

For flow in parallel-type layers, $k_{eff}^p > \langle k \rangle$ for $n < 1$, and $k_{eff}^p < \langle k \rangle$ for $n > 1$. Figure 3b shows that $k_{eff}^p/\langle k \rangle$ increases moderately with increasing C_v for $n < 1$, while for $n > 1$,

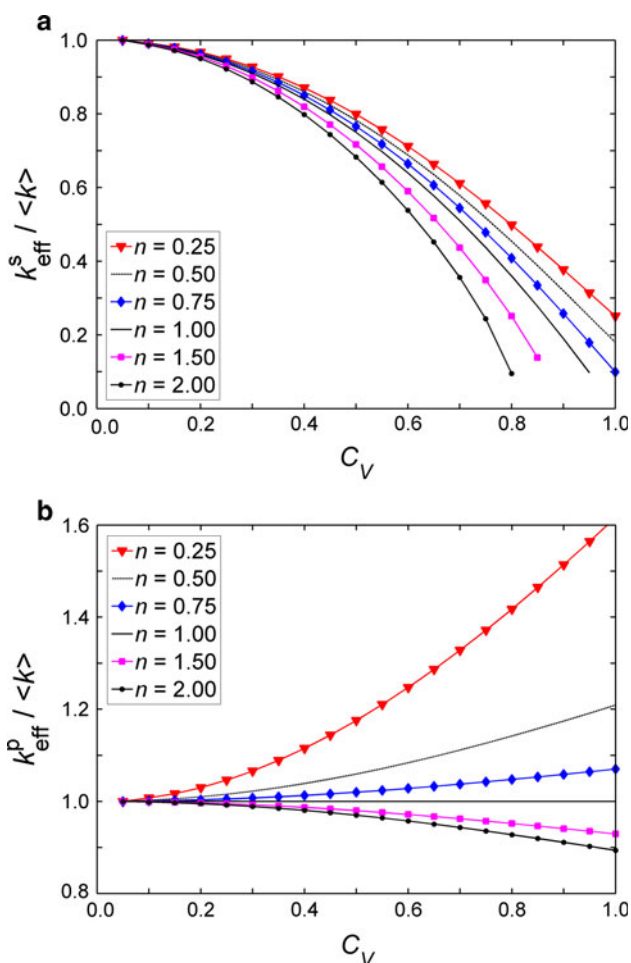


Fig. 3 Ratio $k_{eff}/\langle k \rangle$ as a function of log-permeability variance σ_Y^2 for various values of n , a gamma pdf of k and flow in: **a** 1-D serial-type domain; **b** 1-D parallel-type domain

$k_{eff}^p/\langle k \rangle$ decreases moderately with increasing C_V , with the value of n having a limited impact on the result.

A comparison of the results obtained for 1-D effective permeability in serial-type and parallel-type layers shows that the adoption of different functions for the permeability distribution yields results that are qualitatively comparable.

3.4 Isotropic d -dimensional porous medium

Consider now non-Newtonian power law fluid flow in a d -dimensional porous domain ($d = 1, 2, 3$) subject to an external constant pressure difference $\Delta P = P(0) - P(L)$ over a length L ; the domain is characterized by an isotropic permeability variation with a lognormal permeability distribution. Clearly, Eqs. 15 and 20 constitute respectively the upper and lower bounds for the effective permeability, in analogy with the Newtonian case discussed for hydraulic conductivity by Matheron (1967), in which the upper and lower bounds for the effective conductivity are respectively the arithmetic and harmonic means.

If the domain is seen as a random mixture of elements in which the fluid flows either transversal or parallel to permeability variation, the effective permeability is determined by the spatial configuration (connectivity) of the local permeabilities (Hristopoulos and Christakos 1999), and the effective permeability can be approximated by a suitable average of one-dimensional results. Considering again Newtonian (Darcy) flow as a reference case, the effective permeability in a d -dimensional heterogeneous porous domain with a lognormal permeability distribution is given by (Sanchez-Vila et al. 2006)

$$k_{eff}(\text{Newtonian}) = k_A^{(d-1)/d} k_H^d \tag{27}$$

For $d = 1$, the above equation restitutes k_H . To extend its validity to non-Newtonian flow, it is natural to substitute in place of k_A and k_H their non-Newtonian counterparts k_{eff}^p and k_{eff}^s thus conjecturing that

$$k_{eff} = \left(k_{eff}^p\right)^{(d-1)/d} \left(k_{eff}^s\right)^d, \tag{28}$$

which reduces to Eq. 27 for Newtonian fluid flow. In one dimension ($d = 1$) the results valid for serial-type layers are recovered, while in two and three dimensions one has respectively:

$$k_{eff}^{2D} = k_G e^{(1-n^2)\sigma_Y^2/8n} = \langle k \rangle e^{(1-4n-n^2)\sigma_Y^2/8n}, \tag{29}$$

$$k_{eff}^{3D} = k_G e^{(2+n-n^2)\sigma_Y^2/12n} = \langle k \rangle e^{(2-5n-n^2)\sigma_Y^2/12n}. \tag{30}$$

For a 2-D isotropic domain, inspection of Eq. 29 shows that when $n < \sqrt{5} - 2 \cong 0.24$, $k_{eff}^{2D} > \langle k \rangle > k_G$; when $0.24 < n < 1$, $\langle k \rangle > k_{eff}^{2D} > k_G$; for $n > 1$, $k_{eff}^{2D} < k_G < \langle k \rangle$. Figure 4a shows that $k_{eff}^{2D}/\langle k \rangle$ decreases significantly with increasing σ_Y^2 for all values of n , with the exception of very pseudoplastic fluids ($n < 0.24$); for these, $k_{eff}^{2D}/\langle k \rangle$ increases with increasing σ_Y^2 , with small variations in the value n strongly influencing the result.

Results for a 3-D isotropic domain are somewhat analogous to the 2-D case, with $k_{eff}^{3D} > \langle k \rangle > k_G$ for $n < (\sqrt{33} - 5)/2 \cong 0.37$, $k_G < k_{eff}^{3D} < \langle k \rangle$ when $0.37 < n < 2$; $k_{eff}^{3D} < k_G < \langle k \rangle$ for $n > 2$. Again (Fig. 4b), $k_{eff}^{3D}/\langle k \rangle$ decreases with increasing σ_Y^2 , with the exception of very pseudoplastic fluids ($n < 0.37$).

Upon comparing results for non-Newtonian flow in 1-D (serial-type layers), 2-D and 3-D isotropic fields, it is noted that, for given n and σ_Y^2 , k_{eff} increases with dimensionality, since a larger number of paths are available to the fluid to bypass low-permeability areas; the increase is marked moving from 1-D to 2-D, and moderate from 2-D to 3-D.

The conjecture (28) is validated comparing the present results with the findings of Fadili et al. (2002) (hereinafter FA). For a mean pressure gradient ∇P_0 aligned with the x_1 axis, their effective conductivity in the mean flow direction $(K_e)_{11}$, relating mean flux and gradient of mean pressure in

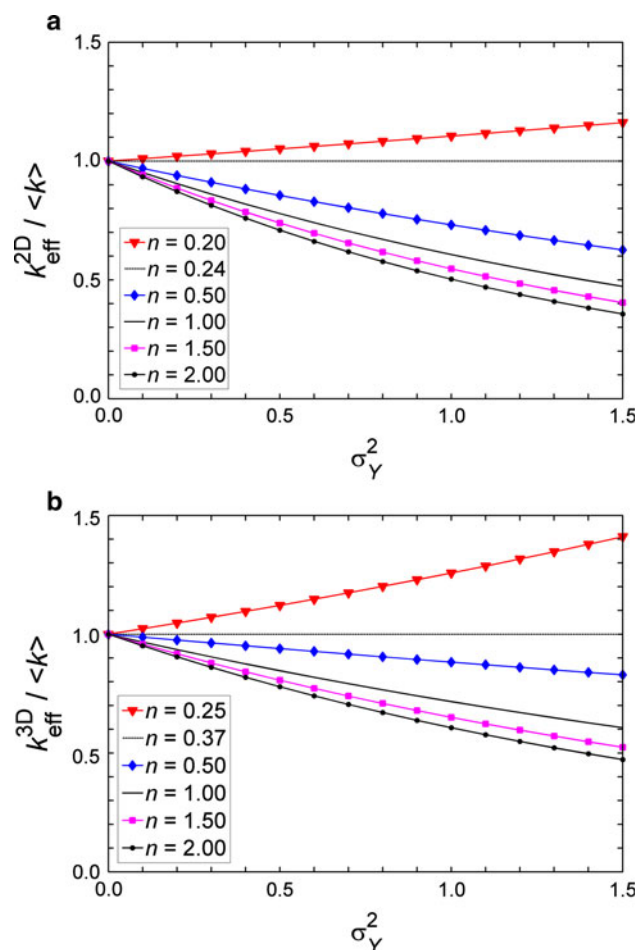


Fig. 4 Ratio $k_{eff} / \langle k \rangle$ according to Eq. 28 as a function of log-permeability variance σ_Y^2 for various values of n , a lognormal pdf of k , and flow in **a** 2-D isotropic domain; **b** 3-D isotropic domain

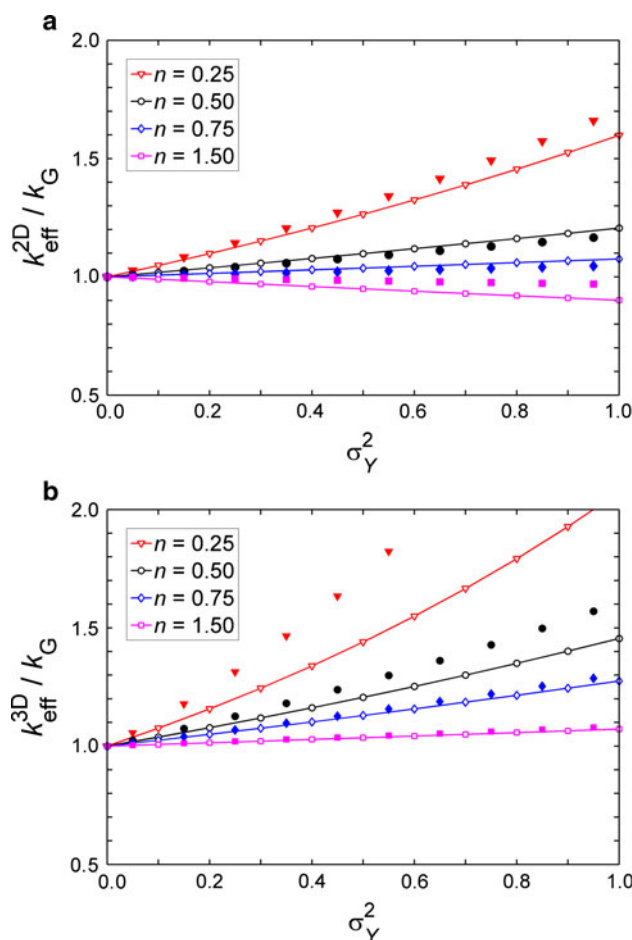


Fig. 5 Comparison between k_{eff} / k_G derived according to Eq. 28 (lines with open symbols) and analytical result by Fadili et al. (2002) (filled symbols) as a function of log-permeability variance σ_Y^2 for various values of n , a lognormal pdf of k and flow in: **a** 2-D isotropic domain; **b** 3-D isotropic domain

the upscaled version of Eq. 9, takes in d dimensions the form

$$(K_e)_{11}(n) = K_G \exp \left(\frac{\sigma^2}{2} \int_{R^d} \frac{\mathbf{w} \cdot \mathbf{w} - 2w_1^2}{[\mathbf{n}\mathbf{w} \cdot \mathbf{w} + (1-n)w_1^2]} S_{aa}(\mathbf{w}) d\mathbf{w} \right), \tag{31}$$

where $K_G = \exp(A)$, $A = \ln(K)$, σ^2 and S_{aa} are the log-conductivity variance and spectral density, and \mathbf{w} is the wave number vector.

In two dimensions, Eq. 31 reduces to the following formula, irrespective of the isotropic correlation function adopted:

$$(K_e)_{11} = K_G \exp \left(\frac{\sigma^2}{2} \frac{1 - \sqrt{n}}{(\sqrt{n}(1 + \sqrt{n}))} \right), \tag{32}$$

In Fig. 5a, 2-D results based on our conjecture (Eq. 28) are compared with FA analytical results given by Eq. 32

for selected values of n in the range [0.25–1.50], and σ_Y^2 varying from 0 to 2. The comparison takes into account that, according to Eqs. 4, 9 and 10, $k_{eff} \propto K_e^{2/(n+1)}$.

Figure 5b provides the same comparison between our conjecture and FA analytical results for the 3-D case; since a closed-form counterpart to Eq. 32 was not available for 3-D flow, Eq. 31 was integrated numerically. Upon inspection of Fig. 5a and b, it is noted that our conjecture overestimates or underestimates results by FA depending on the value of n and dimensionality; the discrepancy between the two approaches increases with domain heterogeneity. In 2-D, our results moderately underestimate FA for low values of n up to 0.38, slightly overestimate them for n between 0.38 and 1, and slightly underestimate FA for dilatant fluids ($n > 1$). In 3-D, our conjecture underestimates FA results in the entire range $0.25 \leq n \leq 1.50$; the discrepancy is significant for low values of n in the range 0.25–0.50, moderate for $0.50 \leq n \leq 1$, and quite insignificant for dilatant fluids.

In general, the agreement between our conjecture and FA results is decidedly better for 2-D than for 3-D.

Equation 28, despite model simplifications, captures the dependence of the effective permeability from the rheological parameter (n) and the porous medium heterogeneity (σ_Y^2), including its marked increase with increasing σ_Y^2 for pseudoplastic fluids with a low flow behavior index, due to the fluid preference for flowpaths of least tortuosity among those with high permeability, a phenomenon already noted in FA.

To achieve full consistency between our conjecture and FA results, retaining the structure of Eq. 28, we propose to generalize it as

$$k_{eff} = \left(k_{eff}^p\right)^{c_p} \left(k_{eff}^s\right)^{c_s}; \left(c_p + c_s = 1\right), \tag{33}$$

where the exponents c_p and c_s are functions of n ; their sum amounts to one as in Eq. 28. They are easily determined for the 2-D and 3-D case by fitting Eq. 33 to FA results. In the interval $0.25 \leq n \leq 1.50$, the fitting yields the values reported in Table 1.

To obtain the exponents c_p and c_s for any value of n between 0.25 and 1.50, the following interpolation curves may be derived for the 2-D case (with a coefficient of determination $R^2 = 0.998$)

$$c_p^{2D}(n) = 0.681 - 0.747n + 0.807n^2 - 0.239n^3, \tag{34}$$

$$c_s^{2D}(n) = 0.319 + 0.747n - 0.807n^2 + 0.239n^3, \tag{35}$$

and for the 3-D case ($R^2 = 0.999$)

$$c_p^{3D}(n) = 1.137 - 1.181n + 0.968n^2 - 0.257n^3, \tag{36}$$

$$c_s^{3D}(n) = -0.137 + 1.181n - 0.968n^2 + 0.257n^3. \tag{37}$$

The behavior of the exponents c_p and c_s as functions of n allows grasping the relative weight of the parallel and serial configurations in the non-Newtonian case and compare it to the Newtonian one for the 2-D ($c_p^{2D} = 0.5, c_s^{2D} = 0.5$) and the 3-D case ($c_p^{3D} = 0.67, c_s^{3D} = 0.33$). It is seen that for non-Newtonian flow in two dimensions, the weight of the parallel configuration is higher than in the Newtonian case for $n < 0.38$; the same is true in three dimensions in the whole shear-thinning range (most common in field

applications), with a more pronounced effect for very low values of n . Since the parallel configuration intuitively resembles the channeling phenomenon, we find our observation consistent with that of FA, who demonstrate that, in randomly heterogeneous correlated media, as n decreases, most of the flowrate occurs through a limited number of paths, and with those of Sullivan et al. (2006), who concluded that shear-thinning fluids show an increased preference for flow down higher permeability channels.

4 Summary and conclusions

We have investigated non-Newtonian power-law fluid flow through randomly heterogeneous porous media subject to a uniform pressure gradient, with the aim of deriving an expression for the effective permeability analogous with known relationships valid for Newtonian fluids. Its estimate was obtained representing the domain permeability as a spatially homogeneous random field with a known probability distribution function. Under ergodicity, analytical expressions were first derived for serial-type and parallel-type one-dimensional porous media; results for 2-D or 3-D isotropic domains were derived via power averaging of the one-dimensional expressions. Our conjecture was validated comparing our results with previous literature findings. An extension of the conjecture was proposed, in which the exponents of the power averaging are functions of the flow behavior index. The final expression may be used to evaluate the effective permeability under mean uniform flow for a broad range of values of the flow behavior index. When Newtonian flow is considered, all expressions reduce to those derived in the past.

Our work leads to the following conclusions:

1. In 1-D serial-type domains, the ratio between effective and mean permeability decreases with increasing heterogeneity, with a moderate impact of the flow behavior index value; in 1-D parallel-type domains, the effective permeability increases or decreases with increasing heterogeneity, depending whether the fluid is pseudoplastic or dilatant.
2. In 2-D and 3-D flows, the ratio between effective and mean permeability decreases with increasing log-permeability variance, except for very pseudoplastic fluids with a flow behavior index smaller than a limit value depending on flow dimensionality; the effective permeability value is highly sensitive to the value of n for pseudoplastic fluids with $n \leq 0.50$, less so for dilatant fluids. Moreover, the ratio increases to some extent with flow dimensionality.
3. The proposed expression for the effective permeability captures, albeit in a simplified fashion, the high sensitivity of power-law fluids to heterogeneity, and

Table 1 Exponents c_p and c_s for values of n between 0.25 and 1.50

n	2-D		3-D	
	c_p	c_s	c_p	c_s
0.25	0.541	0.459	0.898	0.102
0.50	0.477	0.523	0.756	0.244
0.75	0.476	0.524	0.689	0.311
1.00	0.500	0.500	0.667	0.333
1.50	0.568	0.432	0.677	0.323

confirms the tendency of non-Newtonian shear-thinning fluids to exhibit channeling behavior.

4. Several questions remain unanswered, such as the effect of finite domain size, anisotropy and non-uniform pressure gradients on representative permeability, in analogy with the Newtonian case. These issues need to be investigated via a combination of analytical approaches, numerical simulations, and field and laboratory experiments.

Acknowledgements This work was supported by the Università di Bologna 'RFO (Ricerca Fondamentale Orientata)' 2007 and 2008 funds. We thank the reviewers for their comments, which improved the quality of the final manuscript.

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