Stoch Environ Res Risk Assess (2009) 23:451–462
DOI 10.1007/s00477-008-0232-8

ORIGINAL PAPER

Fuzzy process capability indices for quality control of irrigation water

Cengiz Kahraman · İhsan Kaya

Published online: 29 April 2008

Abstract Water covers over 70% of the Earth surface and is a very important resource to people and the environment. Water pollution affects drinking water, rivers, lakes and oceans all over the world. This consequently harms human health and the natural environment. Water pollution can also affect the crops. So, water pollution is an important issue for humanity. Therefore, the control of irrigation water is a necessity. In this paper, a methodology based on process capability indices (PCIs) has been presented to control the levels of pH, dissolved oxygen (DO) and temperature (*T*) in dam's water for irrigation. Fuzzy PCIs have been proposed for this aim. The fuzzy estimates of $\hat{C}_p$ and $\hat{C}_{pk}$ are obtained for pH, DO, and *T* based on Buckley's interval estimation approach and based on fuzzy specification limits. An application has been made for Kesikköprü Dam in Ankara, Turkey. In this paper, Buckley's approach is re-arranged to obtain a triangular fuzzy membership function because it cannot be obtained from Buckley's approach in some situation.

Keywords Fuzzy process capability · Fuzzy estimate · Fuzzy specification limits · Water pollution · Quality · Irrigation

C. Kahraman · İ. Kaya (✉)
Department of Industrial Engineering, Faculty of Management, Istanbul Technical University, 34367 Istanbul, Turkey
e-mail: kayai@itu.edu.tr

C. Kahraman
e-mail: kahramanc@itu.edu.tr

1 Introduction

Water pollution is a large set of adverse effects upon water bodies (lakes, rivers, oceans, groundwater) caused by human activities. Although natural phenomena such as volcanoes, storms, earthquakes, etc., also cause major changes in water quality and the ecological status of water, these are not deemed to be pollution. Water pollution has many causes and characteristics. Increases in nutrient loading may lead to eutrophication. Organic wastes such as sewage impose high oxygen demands on the receiving water leading to oxygen depletion with potentially severe impacts on the whole eco-system. Industries discharge a variety of pollutants in their wastewater including heavy metals, organic toxins, oils, nutrients, and solids. Even many of the municipal water supplies in developed countries can present health risks. Water pollution is a serious problem in the global context. It has been suggested that it is the leading worldwide cause of deaths and diseases and that it accounts for the deaths of more than 14,000 people daily (Anonymous 2006).

Principal sources of water pollution are: industrial discharge of chemical wastes and byproducts, discharge of poorly treated or untreated sewage, surface runoff containing pesticides, slash and burn farming practice, which is often an element within shifting cultivation agricultural systems, surface runoff containing spilled petroleum products, surface runoff from construction sites, farms, or paved and other impervious surfaces, e.g. silt, discharge of contaminated and/or heated water used for industrial processes, acid rain caused by industrial discharge of sulfur dioxide (by burning high-sulfur fossil fuels), excess nutrients added by runoff containing detergents or fertilizers, underground storage tank leakage, leading to soil contamination, thence aquifer contamination (Anonymous 2006).

Aim of this study is control of water quality for irrigation. Water quality describes the condition of a water body and its related suitability for different purposes (also known as environmental values). In a healthy water body, the water quality supports a rich and varied community of organisms, sustains public health and/or agricultural applications. Measurement of water quality is a very important issue for the decision if the water is suitable for aim. These measurements include (from simple and basic to more complex): Conductivity (also see salinity), dissolved oxygen (DO), pH, color of water, taste and odor, turbidity, total suspended solids (TSS), chemical oxygen demand (COD), biochemical oxygen demand (BOD), microorganisms such as fecal coliform bacteria, dissolved metals and metalloids (lead, mercury, arsenic, etc.), dissolved organics, temperature, pesticides, and heavy metals (Anonymous 2003). Quality classes for physical and inorganic chemical parameters of inside water resources are presented in Table 1. The water, which is aimed at using for agriculture or other aims, must satisfy these conditions. In this paper the usability of water for irrigation is investigated. Especially crops are affected by pH values. This parameter must be especially controlled. Table 1 shows the quality classes of inside water resources. If pH values change between 6.5 and 8.5, the water can be suitable for irrigation (Anonymous 1991).

A dam is a barrier across flowing water that obstructs, directs or retards the flow, often creating a reservoir, lake or impoundment. Dams can be formed by human agency, natural causes, or by the intervention of wildlife such as beavers. Man-made dams are typically classified according to their structure, intended purpose or height. According to International Commission on Large Dams (ICOLD) standards, Turkey has 555 large dam reservoirs. The names and surface areas (km^2) of the large ones are Atatürk (817), Keban (675), Karakaya (268), Hirfanlı (263), Altınkaya (118), and Kurtboğazı (6). Turkey is rich in terms of streams and rivers. (Anonymous 2006).

The water, which is stored in dam, can be used for different aims. In this study, the water of Kesikköprü Dam is investigated if it is suitable for irrigation or not. In Table 2, some values of this dam are presented.

In this study, the parameters which affect the quality of water are investigated and measured. In Table 3, the values of these parameters are summarized. The observation has been collected for 36 weeks in every 0, 2, 4, 6,…, 20 m. In Table 3, they are showed as weekly means.

The rest of this paper is organized as follows: Traditional PCIs and the main characteristics of PCIs are explained in Sect. 2. Potential process capability index (C_p) and actual process capability index (C_{pk}) are analyzed and applied for the water stored in a dam in this section. In Sect. 3, PCIs are investigated under fuzzy environment and fuzzy process capability indices (FPCIs) have been proposed based on Buckley's approach and triangular fuzzy number (TFN). At the same time, fuzzy estimate of PCIs (Parchami and Mashinchi 2007) is re-arranged to obtain triangular fuzzy membership functions. Buckley's approach and recent publications on FPCIs are summarized in this section. Fuzzy estimates of C_p and C_{pk} are also illustrated in this section. Also, the case of fuzzy specification limits is investigated in this section. FPCIs are applied for the water stored in Kesikköprü Dam, Turkey and they are compared with traditional process capability indices (PCIs) in Sect. 4. Section 5 includes conclusions and suggestions for future research.

Table 1 Quality classes for physical and inorganic chemical parameters of inside water resources (Anonymous 1998; Icaga et al. 2006, Icaga 2007)

Limits of quality classes	I	II	III	IV
Temperature (T) (°C)	25	25	30	>30
pH	6.5–8.5	6.5–8.5	6–9	<6–9>
Dissolved oxygen (DO) (g/m^3)	8	6	3	<3
Oxygen saturate (OS) (g/m^3)	90	70	40	<40
Chloride (Cl) (g/m^3)	25	200	400	>400
Sulphate (SO_4) (g/m^3)	200	200	400	>400
Ammonia (NH_3) (g/m^3)	0.2	1	2	>2
Nitrite (NO_2) (g/m^3)	0.002	0.01	0.05	>0.05
Nitrate (NO_3) (g/m^3)	5	10	20	>20
Total phosphors (g/m^3)	0.02	0.16	0.65	>0.65
Total dissolved solid (TDS) (g/m^3)	500	1,500	5,000	>5,000
Color (Pt-co unit)	5	50	300	>300
Sodium (Na) (g/m^3)	125	125	250	>250

2 Process capability analysis

Process capability is broadly defined as the ability of a process to satisfy customer expectations. Some processes do a good job of meeting customer requirements and therefore are considered "capable", while others do not

Table 2 Kesikköprü Dam

Location	Ankara
River	Kızılırmak
Construction (starting and completion) year	1959–1966
Dam volume	900 hm^3
Height (from river bed)	49.1 m
Reservoir volume at normal water surface elevation	95 hm^3
Reservoir area at normal water surface elevation	6.5 km^2
Irrigation area	11,860 ha.
Capacity	76 MW
Annual generation	250 G,Wh

Table 3 The observed values of Kesikköprü Dam (weekly mean)

Week	Temperature	SpCond (uS/cm)	TDS (g/L)	Salinity (ppt)	DO (%)	DO (mg/L)	pH
1	12.597	1,348.600	0.877	0.680	81.425	8.593	7.670
2	13.303	1,347.000	0.876	0.680	67.955	7.041	7.765
3	13.782	1,345.300	0.876	0.676	85.725	8.798	7.468
4	14.620	1,348.400	0.876	0.677	90.290	9.078	7.601
5	15.843	1,353.200	0.878	0.683	84.200	8.224	7.741
6	16.467	1,353.800	0.880	0.682	91.080	8.767	7.942
7	17.162	1,359.800	0.883	0.685	84.580	8.009	8.018
8	17.455	1,357.200	0.881	0.682	78.470	7.355	7.973
9	18.364	1,362.000	0.887	0.687	80.910	7.449	7.621
10	18.183	1,364.600	0.890	0.688	64.050	5.924	7.289
11	18.599	1,366.600	0.890	0.690	64.405	5.866	7.428
12	19.266	1,367.400	0.890	0.690	60.430	5.451	7.560
13	19.948	1,370.500	0.891	0.690	69.550	6.107	7.678
14	19.660	1,370.700	0.890	0.690	73.005	6.504	7.738
15	20.189	1,374.500	0.893	0.690	69.550	5.451	8.065
16	20.143	1,374.500	0.893	0.690	49.980	4.352	7.407
17	20.026	1,378.700	0.898	0.690	62.435	5.495	7.707
18	19.703	1,382.200	0.900	0.696	43.965	3.901	7.565
19	19.722	1,383.500	0.900	0.697	43.640	3.822	7.661
20	19.380	1,385.100	0.900	0.698	44.945	4.019	7.420
21	19.300	1,385.700	0.900	0.698	72.825	6.591	7.483
22	19.078	1,385.300	0.900	0.700	55.465	5.082	6.992
23	18.496	1,387.900	0.900	0.700	53.420	4.976	6.863
24	18.188	1,390.000	0.902	0.700	58.520	5.484	7.000
25	14.833	1,394.700	0.910	0.701	72.370	7.295	7.678
26	13.676	1,397.600	0.910	0.706	70.790	7.317	7.513
27	9.308	1,401.200	0.910	0.710	86.960	9.935	7.278
28	3.616	1,419.700	0.920	0.710	82.615	10.893	7.593
29	3.514	1,426.800	0.930	0.710	99.170	13.109	7.810
30	7.273	1,426.300	0.928	0.719	110.360	13.204	7.993
31	7.383	1,433.900	0.930	0.720	99.330	11.891	7.041
32	7.832	1,433.500	0.931	0.720	91.905	10.882	8.497
33	8.632	1,432.600	0.931	0.720	92.600	10.742	8.316
34	10.509	1,420.000	0.923	0.717	92.310	10.208	8.108
35	10.716	1,424.200	0.927	0.720	98.275	10.818	8.027
36	11.907	1,417.700	0.923	0.717	98.230	10.456	8.313
Mean	14.9631	1,385.2972	0.9007	0.6975	75.7149	7.7525	7.6617

and are designated “not capable” (Bothe 1997). Measure of process capability summarizes some aspects of a process’s ability to satisfy customer requirements. Some PCIs are used to measure the ability of process. A PCI is a number which summarizes the behavior of a product or process characteristic relative to specifications. Generally, this comparison is made by forming the ratio of the width between the process specification limits to the width of the natural tolerance limits. These indices help us to decide how well the process meets the specification limits (Montgomery 2005). Several PCIs such as C_p, C_{pk}, and C_{pm} are used to estimate the capability of process (Kotz and Johnson 2002).

2.1 Potential process capability index (C_p)

C_p is defined as the ratio of specification width over the process spread. The specification width represents customer and/or product requirements. The process variations are represented by the specification width. If the process variation is very large, the C_p value is small and it represents a low process capability.

$$C_p = \frac{\text{Specification width}}{\text{Process spread}} = \frac{\text{Allowable process spread}}{\text{Actual process spread}} = \frac{\text{USL} - \text{LSL}}{6\sigma} = \frac{w}{6\sigma} \quad (1)$$

where σ is the standard deviation of the process, USL and LSL are upper and lower specification limits, respectively.

C_p is usually estimated by the following equation:

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6S} = \frac{w}{6S} \quad (2)$$

where S denotes the sample standard deviation.

C_p indicates how well the process fits within the two specification limits. It never considers any process shift as indicated by Eq. (1), and presented in Fig. 1. C_p simply measures the spread of the specifications relative to the six-sigma spread in the process. If the process average is not centered near the midpoint of specifications limits, the C_p index gives misleading results (see Kane 1986; Bothe 1997; Kotz and Johnson 2002; Montgomery 2005, for more details).

The small values of C_p indicate that the natural range of variation of the process does not fit within the tolerance band since these values are not acceptable. Obviously, it is desirable to have C_p as large as possible. For a process with two-sided specification limits, the percentage of nonconforming items (NC %) can be solved $\Phi\left(\frac{\text{LSL}-\mu}{\sigma}\right) + \left[1 - \Phi\left(\frac{\text{USL}-\mu}{\sigma}\right)\right]$. where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution and μ is the process mean.

Table 4 lists various values of C_p, the corresponding values of standard normal variable Z, and the fractions of nonconformity (defect rate) in parts per million (ppm).

The six quality conditions and the corresponding C_p values are summarized in Table 5 (Tsai and Chen 2006).

2.2 Actual process capability index (C_{pk})

The process capability ratio C_p does not take into account where the process mean is located relative to specifications (Montgomery 2005). C_p focuses the dispersion of the studied process and does not take into account the centering of the process. To overcome this problem, Kane (1986) introduced C_{pk}. The C_{pk} index is used to provide an indication of the variability associated with a process. It shows how a process confirms to its specification. The index is usually used to relate the "*natural tolerance* (3σ)" to the specification limits. C_{pk} describes how well the process fits within the specification limits, taking into account the location of the process mean. Process target is a point within the specification width. It is reflects the best value of the customer satisfaction as shown in Fig. 2. Generally, $T = \frac{\text{USL}+\text{LSL}}{2}$. If the mean of the process is equal to target value, customers gain the best satisfaction.

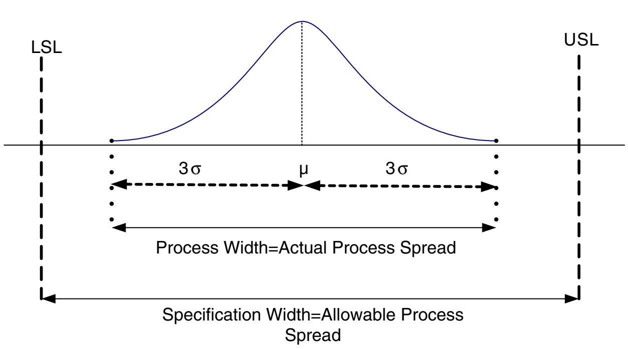

Fig. 1 Centered process

Table 4 Cp values, Z values and the corresponding ppm

C_p	0.67	1.00	1.33	1.67	2.00
Z	2	3	4	5	6
ppm	45,500	2,700	66	0.54	0.002

Table 5 Quality conditions and C_p values

Quality condition	C_p value
Super excellent	$2.00 \leq C_p$
Excellent	$1.67 \leq C_p \leq 2.00$
Satisfactory	$1.33 \leq C_p \leq 1.67$
Capable	$1.00 \leq C_p \leq 1.33$
Inadequate	$0.67 \leq C_p \leq 1.00$
Poor	$C_P \prec 0.67$

C_{pk} measures this real capability when the process is the off-center. The variation factor k is defined as

$$k = \frac{|T - \mu|}{0.5(\text{USL} - \text{LSL})} \quad (3)$$

$$C_{pk} = C_p(1 - k) \quad (4)$$

If the process centered, $k = 0$ and $C_{pk} = C_p$.

If the process target is not determined, C_{pk} should be calculated differently based on Eqs. (5–7) (see Kane 1986; Bothe 1997; Kotz and Johnson 2002; Montgomery 2005, for more details).

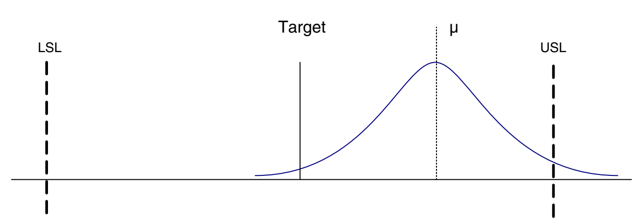

Fig. 2 Off-center process

$$C_{pk} = \min\{C_{pl}, C_{pu}\} = \frac{\min\{\mathrm{USL} - \mu, \mu - \mathrm{LSL}\}}{3\sigma} \quad (5)$$

$$C_{pl} = \frac{(\mu - \mathrm{LSL})}{3\sigma} \quad (6)$$

$$C_{pu} = \frac{(\mathrm{USL} - \mu)}{3\sigma}$$

C_{pk} is usually estimated by the following equation:

$$C_{pk} = \frac{\min\{\mathrm{USL} - \bar{x}, \bar{x} - \mathrm{LSL}\}}{3S} \quad (7)$$

The index C_{pk} has been regarded as a yield-based index since it provides bounds on the process yield, $2\Phi(3C_{pk}) - 1 \leq \text{yield} \leq \Phi(3C_{pk})$, for a normally distributed process (Boyles 1991). For a C_{pk} at level 1, statistically, one would expect that the product's fraction of defectives, is at most 2,700 parts per million (ppm), falling outside the specification limits. At a C_{pk} level of 1.33, the defect rate drops to 66 ppm. To achieve less than 0.544 ppm defect rate, a C_{pk} level of 1.67 is needed. At a C_{pk} level of 2.0, the likelihood of a defective part drops to two parts per billion (ppb).

In this paper traditional C_p and C_{pk} values have been determined by aid of Minitab 14.0 for pH value. These statistics are shown in Fig. 3. The minimum values of PCI for a critical parameter for two-sided and one-sided specifications are 1.50 and 1.45, respectively (Montgomery 2005). For the pH parameter, we can say that C_p is not suitable since $C_p = 1.25 \leq 1.50$. And the percentage of the specification band is $P = \left(\frac{1}{C_P}\right) \times 100 = \left(\frac{1}{1.25}\right) \times 100 = 80$. The water only uses the 80% of the specification band. Generally if $C_p = C_{pk}$, the process is centered at the midpoint of specification, and when $C_{pk} < C_p$ the process is off-center. C_{pk} is 1.04 for pH parameter and it is smaller than $C_p = 1.25$. Therefore it can be said that the process is off-center.

3 Fuzzy process capability analysis

In some works, specification limits are crisp but a fuzzy estimation of PCI is made by defining the significance level as α-cut level. And in some other works the specification limits are not precise numbers and have uncertainty but are expressed in fuzzy terms and hence classical capability indices cannot be applied.

In the literature, there are some papers on fuzzy PCIs. They are summarized in the following briefly. Chen and Chen (2007) presented a method to incorporate fuzzy inference with process capability. They proposed a fuzzy inference approach that employed the maximum–minimum product composition to operate fuzzy *if–then* rules to evaluate the multi-process capability based on distance values of a confidence box. They illustrated an example of color STN display demonstrated that the presented method was effective for assessment of multi-process capability.

Hsu and Shu (2007) presented a method combining the vector of fuzzy numbers to produce the membership function of fuzzy estimator of Taguchi index, the loss-based process capability index C_{pm}, for further testing process capability. This approach allowed the consideration of imprecise output data resulting from the measurements of the products quality. They proposed two useful fuzzy inference criteria, the critical value and the fuzzy P value, to assess the manufacturing process capability based on C_{pm}.

Parchami and Mashinchi (2007) applied Buckley's estimation approach to find fuzzy estimates of several PCIs. They proposed an algorithm for fuzzy estimation of PCIs based on predefined α-cuts using Buckley's approach. They created triangular fuzzy membership functions of PCIs using this approach. They also presented a method for comparing estimated PCIs. They illustrated some numerical

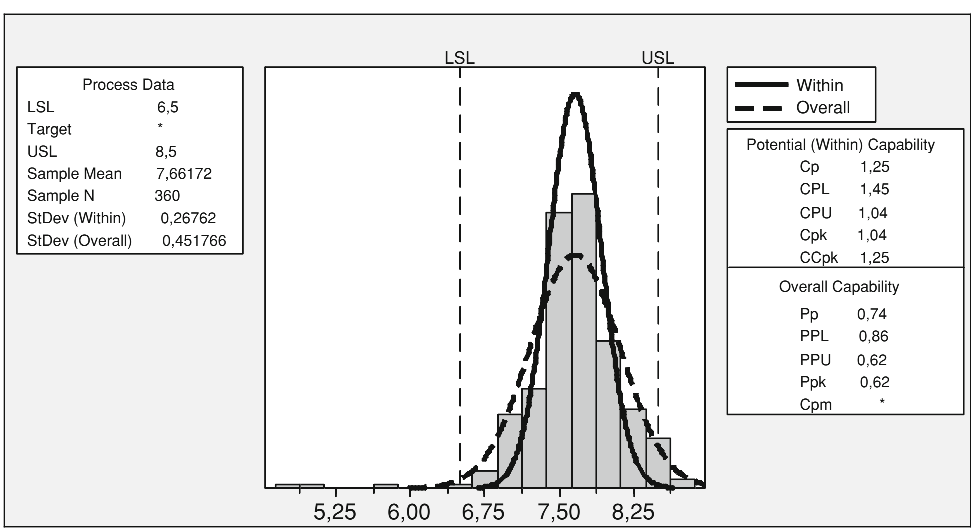

Fig. 3 Process capability analysis of water for pH value

examples to test the performance of the method. Parchami et al. (2006) obtained a $(1-\alpha)100\%$ fuzzy confidence interval for fuzzy PCIs. They defined the specification limits as fuzzy numbers. They also presented some interpretations for the fuzzy confidence interval.

Tsai and Chen (2006) extended the application of the process capability index C_p in the manufacturing industry to a fuzzy environment. They proposed a methodology for testing the C_p of fuzzy numbers. They formulated a pair of nonlinear functions to find the α-cuts of $\tilde{C}_p$. The membership functions of $\tilde{C}_p$ are constructed from various values of α. They calculated the probability of rejecting the null hypothesis based on this membership function. Their methodology shows a grade of acceptability of the null hypothesis and the alternative hypothesis, respectively. They illustrated an example for testing the performance of the proposed methodology.

Parchami et al. (2005) discussed the fuzzy quality. They analyzed fuzzy PCIs. They introduced new PCIs as TFNs, where the engineering specification limits are also TFNs. They determined the relations between the fuzzy process capabilities indices. They also presented a methodology based on a binary relation which was used for the comparison of fuzzy processes. They also applied two examples to clarify this methodology.

Chen et al. (2003a) proposed a method to incorporate the fuzzy inference with the process capability index in the bigger-the-best type quality characteristics assessments. They used a concise score concept to represent the grade of the process capability. They also developed an evaluation procedure to use the method efficiently. They illustrated an example to demonstrate the effectiveness and feasibility of the presented method.

Chen et al. (2003b) proposed a fuzzy inference method to select the best among the competing suppliers based on an estimated capability index of C_{pm} calculated from sampled data. Both input and output are described by linguistic variables to account for the uncertain information associated with them. Triangular and trapezoid membership functions are used to represent uncertain information about process variables. They also illustrated an example of color STN displays demonstrated that the proposed method was effective and feasible for the evaluation of competing process capability.

Gao and Huang (2003) emphasized that process tolerances had influences not only on manufacturing costs, but also on the achievement of the required specifications of a product. They dealt with the more complex nonlinear situations of manufacturing processes. They proposed a nonlinear optimal process tolerance allocation approach which was to optimize process tolerances based on manufacturing capability indices. The results of the testing and the results of a comparison with the existing methods showed that the proposed approach was quite stable and was able to provide improvements in acceptable process probability, as the scrap rates were reduced.

Lee (2001) proposed a model to calculate the fuzzy process capability index when observations were fuzzy numbers. This approach could mitigate the effect when the normal assumption was inappropriate. Lee (2001) not only concentrated the construction of the membership function of the fuzzy process capability index, but also, the complexity of constructing the membership function of a type other than triangular was much more difficult.

Lee et al. (1999) presented a model for designing process tolerances to maximize the process capability index. This model is consolidated into a single objective fuzzy programming. The proposed model simultaneously optimized the process capability of each operation. They determined the lower and upper bounds of the process capability index via fuzzy membership function. They noted that that low manufacturing cost resulted from wide process tolerances, whereas large process tolerances contributed to good process capability. Therefore, they constructed a multi-objective problem into a single objective formulation as fuzzy model. Then they proposed a fuzzy approach to maximize the process capability index of each operation by obtaining a maximum value for the fuzzy number.

Yongting (1996) defined a formula of process capability index C_{pk} to measure fuzzy quality. He determined the value of the fuzzy process capability index C_{pk} changed between 0 and 1, which was different from the classical range of $[-\infty, \infty]$.

In the rest of this paper, Buckley's (2004) approach is applied to find the fuzzy estimates of C_p and C_{pk}. Later, the case of fuzzy specification limits is examined to construct the membership functions of C_p and C_{pk}. They are applied to control of irrigation water stored in Kesikköprü Dam, Turkey.

3.1 Buckley's approach

In this paper, Buckley's approach (Buckley 2004, 2005a, b) for fuzzy estimation is used to produce triangular membership functions of PCIs. In this section, this approach is summarized briefly (Parchami and Mashinchi 2007).

Before the explanation of this approach, we should explain the notation. A triangular shaped fuzzy number "N" is a fuzzy subset of the real numbers "R" satisfying:

- $N(x) = 1$ for exactly one $x \in R$.
- For $\alpha \in (0, 1]$, the α-*cut of* N is a closed and bounded interval, which is denoted by $N_\alpha = [n_1(\alpha), n_2(\alpha)]$, where $n_1(\alpha)$ is increasing, $n_2(\alpha)$ is decreasing continuous functions.

In this paper, triangular shaped fuzzy numbers are used to parameter estimation. Let X is a random variable with probability density function (p.d.f.) $f(x;\theta)$ for a single parameter θ. Assume that θ is unknown and must be estimated from a random sample $X_1,\ldots,X_n$. Let $Y=u(X_1,\ldots,X_n)$ is a statistic used to estimate θ. According to the values of these random variables, e.g. $X_i=x_i$, $1\le i\le n$, we obtain a point estimate $\hat{\theta}=y=u(x_1,\ldots,x_n)$ for θ. There is no any expectation that this point estimate be exactly equal to θ, so a $(1-\beta)100\%$ confidence interval for θ is often also computed.

The $(1-\beta)100\%$ confidence interval for θ is denoted as $[\theta_1(\beta),\theta_2(\beta)]$, for $0<\beta<10$. Thus the interval $\theta_1=\left[\hat{\theta},\hat{\theta}\right]$ is the 0% confidence interval for θ and $\theta_0=\Theta$ is a 100% confidence interval for θ, where Θ is the whole parameter space. Consequently, it is obtained that a family of $(1-\beta)100\%$ confidence intervals for θ, where $0\le\beta\le1$. β is used here since α, usually employed for confidence intervals, is reserved for α-cuts of fuzzy numbers. If these confidence intervals are placed, one on top of the other, a triangular shaped fuzzy number θ whose α-*cut* are the following confidence intervals is obtained:

$$\begin{aligned}&\theta_\alpha=[\theta_1(\alpha),\theta_2(\alpha)]\quad \text{for } 0<\alpha<1\\ &\theta_0=\Theta\\ &\theta_1=\left[\hat{\theta},\hat{\theta},\right]\end{aligned}\tag{8}$$

3.1.1 Fuzzy estimate of C_p

The standard deviation, σ, of X in the traditional process capability formula which is showed Eq. (1) can be estimated. We know that s is the natural estimator of σ. If $X_1,X_2,\ldots,X_n$ are independent, and they are distributed as random variables with p.d.f. $N(\mu,\sigma^2)$, then the sum of squared deviation from the mean is distributed by Chi-square distribution (χ^2). Therefore, s^2 is distributed as

$$\sigma^2\times\frac{\chi^2_{n-1}}{(n-1)}.\tag{9}$$

From Eq. (5)

$$\Pr\left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\beta}{2}}}\le\sigma^2\le\frac{(n-1)s^2}{\chi^2_{n-1,\frac{\beta}{2}}}\right]=1-\beta$$

where $\Pr\left[\chi^2_{n-1}\le\chi^2_{n-1,\varepsilon}\right]=\varepsilon$.

A random sample X_1, X_2,…, X_n from $N(\mu,\sigma^2)$ to estimate C_p is taken. Then $(1-\beta)$ 100% confidence interval for σ^2 is (Buckley 2004, 2005a, b);

$$\left[\sigma_1^2(\beta),\sigma_2^2(\beta)\right]=\left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\beta}{2}}},\frac{(n-1)s^2}{\chi^2_{n-1,\frac{\beta}{2}}}\right]\tag{10}$$

where s is the natural estimator of σ, and $(n-1)$ is the degree of freedom for Chi-square distribution. According to Buckley's approach;

$$\left(S^2\right)_\alpha=\left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}},\frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right]\tag{11}$$

Let $b\in\left(S^2\right)_\alpha$; $\alpha\in(0,1)$. Let us define

$$C_p(b)=\frac{U-L}{6\sqrt{b}},\tag{12}$$

then where (Parchami and Mashinchi 2007)

$$\left(C_p\right)_\alpha=\left[\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\frac{\alpha}{2}}}{n-1}},\hat{C}_p\sqrt{\frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{n-1}}\right]\quad\text{for } 0<\alpha<1$$
$$\text{and}\quad\hat{C}_p=\frac{\text{USL}-\text{LSL}}{6s}\tag{13}$$

In some situations, a triangular membership function cannot be obtained from Eq. (13). To obtain a triangular fuzzy membership function from Eq. (13), it is re-arranged as in Eq. (14). The main idea of this arrangement is that the membership value of this function when $\alpha=1$ should be equal to the value found in the crisp case. To do this we add the difference between $\hat{C}_p$ and $\left(\hat{C}_p\right)_{\alpha=1.0}$ to the both sides of Eq. (13):

$$\begin{aligned}\left(C_p\right)_\alpha=\Bigg[&\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\frac{\alpha}{2}}}{n-1}}+\left(\hat{C}_p-\left(\hat{C}_p\sqrt{\frac{\chi^2_{n-1,0.5}}{n-1}}\right)\right),\\ &\hat{C}_p\sqrt{\frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{n-1}}+\left(\hat{C}_p-\left(\hat{C}_p\sqrt{\frac{\chi^2_{n-1,0.5}}{n-1}}\right)\right)\Bigg]\end{aligned}\tag{14}$$

3.1.2 Fuzzy estimate of C_{pk}

In the previous section, we suppose that the fuzzy estimate of σ^2 is a TFN S^2 and its α-*cuts* are shown in Eq. (10). If we suppose that the fuzzy estimate of μ is a TFN $(\bar{X})$, its α-*cuts* are as follows (Parchami and Mashinchi 2007):

$$\left(\bar{X}\right)_\alpha=\left[\bar{x}-t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}},\bar{x}+t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}}\right]\tag{15}$$

where t_{n-1} has t distribution with $n-1$ degree of freedom and $\Pr\left[t_{n-1}\ge t_{n-1,\varepsilon}\right]=\varepsilon$.

Let $a\in\left(\bar{X}\right)_\alpha$ and $b\in\left(S^2\right)_\alpha$; $\alpha\in(0,1)$. C_{pk} is defined as

$$C_{pk}(a,b)=\frac{\text{USL}-\text{LSL}-2|a-M|}{6\sqrt{b}}.\tag{16}$$

Where $M=\frac{\text{USL}+\text{LSL}}{2}$. The α-cuts for C_{pk} are defined as follows (Parchami and Mashinchi 2007):

$$(C_{pk})_{\alpha}=\left[\frac{\text{USL}-\text{LSL}-2|\bar{x}-M|-2\times t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,\frac{\alpha}{2}}}}},\frac{\text{USL}-\text{LSL}-2|\bar{x}-M|+2\times t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,1-\frac{\alpha}{2}}}}}\right] \tag{17}$$

In some situations, a triangular membership function cannot be obtained from Eq. (17) as seen in Fig. 4. To obtain a triangular fuzzy membership function from Eq. (17), it is re-arranged as in Eq. (18).

$$(C_{pk})_{\alpha}=\left[\begin{array}{l}\frac{\text{USL}-\text{LSL}-2|\bar{x}-M|-2\times t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,\frac{\alpha}{2}}}}}+\left(\hat{C}_{pk}-\left(\frac{\text{USL}-\text{LSL}-2|\bar{x}-M|-2\times t_{n-1,0.5}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,0.5}}}}\right)\right),\\ \frac{\text{USL}-\text{LSL}-2|\bar{x}-M|+2\times t_{n-1,\frac{\alpha}{2}}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,1-\frac{\alpha}{2}}}}}+\left(\hat{C}_{pk}-\left(\frac{\text{USL}-\text{LSL}-2|\bar{x}-M|-2\times t_{n-1,0.5}\times\frac{s}{\sqrt{n}}}{6\times s\times\sqrt{\frac{n-1}{\chi^2_{n-1,0.5}}}}\right)\right)\end{array}\right] \tag{18}$$

3.2 Fuzzy specification limits

In this subsection, PCIs are analyzed as TFNs when the specifications are expressed in fuzzy numbers. Any $A \in \mathrm{F}(\Re)$ is called a fuzzy quantity on $\Re$ and any $T_{a,b,c} \in F_T(\Re)$ is called a TFN. In this paper we use the notation $T(a,b,c)$ for a TFN. Let $T(a,b,c)$ and $T(d,e,f) \in \mathrm{F_T}(\Re)$, $\mathrm{k} \in \Re$, $\mathrm{k} > 0$, and $\mathrm{a} \geq \mathrm{f}$.

Definitions of the operations subtraction (Θ) and division ($\varnothing$) on $F_T(\Re)$ are as follows:

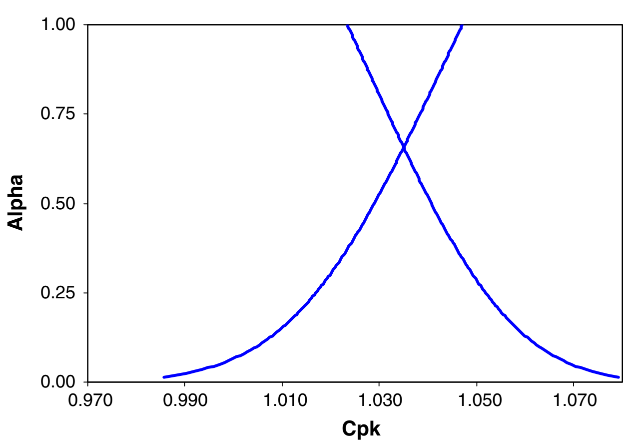

Fig. 4 Fuzzy estimates of PCIs

$$T(a,b,c) \ominus T(d,e,f) = T(a-f, b-e, c-d) \tag{19}$$

$$T(a,b,c) \varnothing \mathrm{k} = T(a/k, b/k, c/k) \tag{20}$$

Suppose we have a fuzzy process with fixed σ, for which the upper and lower specification limits are TFNs, $\text{USL} = (a_u, b_u, c_u)$, $\text{LSL} = (a_l, b_l, c_l) \in \mathrm{F_T}(\Re)$.

The fuzzy process capability index is a TFN, $\tilde{C}_p = \left(U\tilde{S}L - L\tilde{S}L\right) \varnothing 6\sigma$. Therefore, the membership function of $\tilde{C}_p$ was as follows (Parchami et al. 2005):

$$\tilde{C}_p = T\left(\frac{a_u - c_l}{6\sigma}, \frac{b_u - b_l}{6\sigma}, \frac{c_u - a_l}{6\sigma}\right) \tag{21}$$

The traditional $\tilde{C}_{pk}$ formula can be written as follows:

$$C_{pk} = \frac{\text{USL} - \text{LSL} - 2|\mu - M|}{6\sigma} \tag{22}$$

where $M = \frac{(\text{USL}+\text{LSL})}{2}$

The membership function of $\tilde{C}_{pk}$ was determined as follows (Parchami et al. 2005):

$$\tilde{C}_{pk} = T\left(\frac{a_u - c_l - 2|\mu - m|}{6\sigma}, \frac{b_u - b_l - 2|\mu - m|}{6\sigma}, \frac{c_u - a_l - 2|\mu - m|}{6\sigma}\right)$$

$$\text{where } m = \frac{(b_u + b_l)}{2} \tag{23}$$

4 Application

In this study, the water of Kesikköprü Dam is investigated if it is suitable for irrigation or not based on FPCIs. The FPCIs approach is used to construct the membership functions of C_p and C_{pk} for pH, DO, and T.

4.1 Application of fuzzy estimate of PCIs

Specification limits for pH are determined as $L = 6.5$ and $U = 8.5$, respectively. Process mean and process standard deviation are calculated from the observed data for 36 weeks as $\mu = 7.66$, and $\sigma = 0.267$. Totally 360 samples have been taken for 36 weeks to calculate the mean and standard deviation, therefore $n = 360$. The α-cuts of $\hat{C}_p$ for pH are computed as in Eq. (24):

$$(C_p)_\alpha = \left[\begin{array}{l} \frac{8.5-6.5}{6\times 0.267}\sqrt{\frac{\chi^2_{359,\frac{\alpha}{2}}}{359}} + \left(\frac{8.5-6.5}{6\times 0.267} - \left(\frac{8.5-6.5}{6\times 0.267}\sqrt{\frac{\chi^2_{359,0.5}}{359}}\right)\right); \\ \frac{8.5-6.5}{6\times 0.267}\sqrt{\frac{\chi^2_{359,1-\frac{\alpha}{2}}}{359}} + \left(\frac{8.5-6.5}{6\times 0.267} - \left(\frac{8.5-6.5}{6\times 0.267}\sqrt{\frac{\chi^2_{359,0.5}}{359}}\right)\right) \end{array} \right] \qquad (24)$$

The graph created by Ms Excel of the membership function $\hat{C}_p$ is shown in Fig. 5. Note that in the classical method, substituting the standard deviation in Eq. (1), one can find the estimate $\hat{C}_p = 1.25$.

The fuzzy estimate contains more information than a point or interval estimate. We can see that, $\hat{C}_p = 1.25$ belongs to fuzzy estimate $\hat{C}_p$ with a $\hat{C}_p(\alpha = 1.0) = 1.25$. We can also say that $\hat{C}_p(\alpha = 0.31) = 1.20$ and $\hat{C}_p(\alpha = 0.485) = 1.20$ from Fig. 5. In Fig. 6, the distributions of the pH values with respect to extreme C_p and C_{pk} values are illustrated. The best and the worst situations of the water (according to pH) are seen from Fig. 6.

The α-cuts of $\hat{C}_{pk}$ for pH are computed as in Eq. (25):

$$(C_{pk})_\alpha = \left[\begin{array}{l} \frac{8.5-6.5-2|7.67-7.5| - 2\times t_{359,\frac{\alpha}{2}} \times \frac{0.260}{\sqrt{360}}}{6\times 0.267 \times \sqrt{\frac{359}{\chi^2_{359,\frac{\alpha}{2}}}}} + \left(1.03 - \left(\frac{8.5-6.5-2|7.67-7.5| - 2\times t_{359,0.5} \times \frac{0.260}{\sqrt{360}}}{6\times 0.267 \times \sqrt{\frac{359}{\chi^2_{359,0.5}}}}\right)\right), \\ \frac{8.5-6.5-2|7.67-7.5| + 2\times t_{359,\frac{\alpha}{2}} \times \frac{0.260}{\sqrt{360}}}{6\times 0.267 \times \sqrt{\frac{359}{\chi^2_{359,1-\frac{\alpha}{2}}}}} + \left(1.03 - \left(\frac{8.5-6.5-2|7.67-7.5| - 2\times t_{359,0.5} \times \frac{0.260}{\sqrt{360}}}{6\times 0.267 \times \sqrt{\frac{359}{\chi^2_{359,0.5}}}}\right)\right) \end{array} \right] \qquad (25)$$

The graph created by Ms Excel of the membership function $\hat{C}_{pk}$ is shown in Fig. 7. We can see that, $\hat{C}_{pk} = 1.04$ belongs to fuzzy estimate $\hat{C}_{pk}$ with a $\hat{C}_{pk}(\alpha = 1) = 1.04$. We can also say that $\hat{C}_{pk}(\alpha = 0.23) = 1.00$ and $\hat{C}_{pk}(\alpha = 0.34) = 1.06$ from Fig. 7.

Dissolved oxygen and temperature (T) are other two critical parameters which are analyzed by FPCIs in this paper. The membership functions of fuzzy estimate of these PCI are also created. For DO, LSL and USL are determined as 6.0 and 8.0, respectively, according to Table 1. The membership functions of this parameter are illustrated in Figs. 8 and 9 for C_p and C_{pk}.

For DO, we can see that, $\hat{C}_p = 0.76$ belongs to fuzzy estimate $\hat{C}_p$ with a $\hat{C}_p(\alpha = 1) = 0.76$. We can also say that $\hat{C}_p(\alpha = 0.46) = 0.739$ and $\hat{C}_p(\alpha = 0.17) = 0.80$ from Fig. 8. For DO, we can see that, $\hat{C}_{pk} = 0.189$ belongs to fuzzy estimate $\hat{C}_p$ with a $\hat{C}_{pk}(\alpha = 1) = 0.189$. We can also say that $\hat{C}_{pk}(\alpha = 0.17) = 0.1785$ and $\hat{C}_{pk}(\alpha = 0.045) = 0.200$ from Fig. 9.

For T, USL, and LSL are determined as 30 and 25, respectively, according to Table 1. The membership functions of this parameter are illustrated in Figs. 10 and 11 for C_p and C_{pk}. We can see that, $\hat{C}_p = 0.99$ belongs to fuzzy estimate $\hat{C}_p$ with a $\hat{C}_p(\alpha = 1.0) = 0.99$. We can also say that $\hat{C}_p(\alpha = 0.135) = 1.054$ and $\hat{C}_p(\alpha = 0.19) = 0.95$ from Fig. 10. We can see that, $\hat{C}_{pk} = -4.01$ belongs to fuzzy estimate $\hat{C}_{pk}$ with a $\hat{C}_{pk}(\alpha = 1) = -4.01$. We can also say that $\hat{C}_{pk}(\alpha = 0.15) = -3.78$ and $\hat{C}_p(\alpha = 0.47) = -4.13$ from Fig. 11.

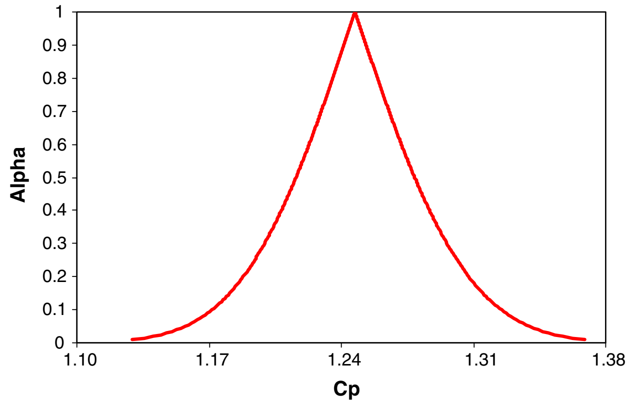

Fig. 5 The membership function of fuzzy estimate for $\hat{C}_p$

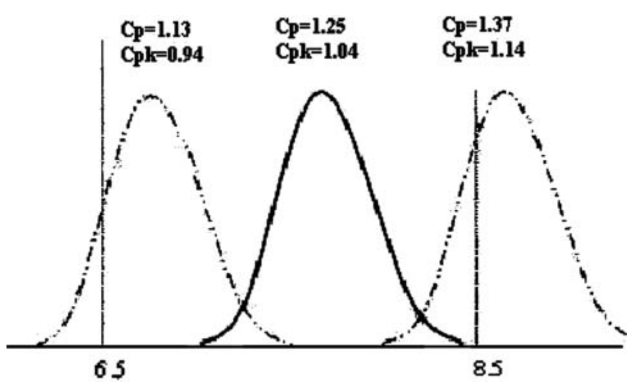

Fig. 6 The distribution of the pH values

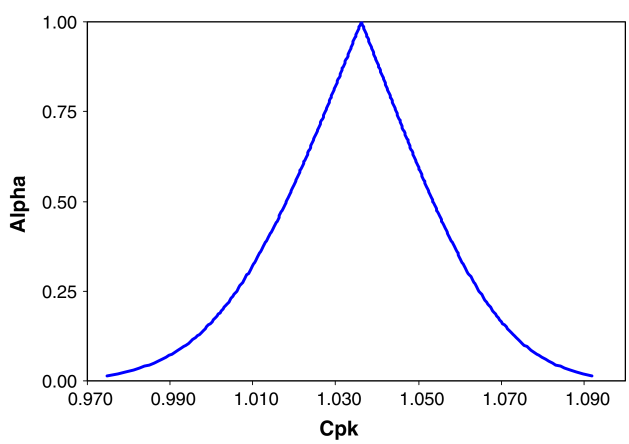

Fig. 7 The membership function of fuzzy estimate for $\hat{C}_{pk}$

4.2 Applications of fuzzy specification limits

In this sub-section, the fuzzy specification limits approach is used to construct the membership functions of C_p and C_{pk} for pH. The fuzzy specification limits are defined as follows:

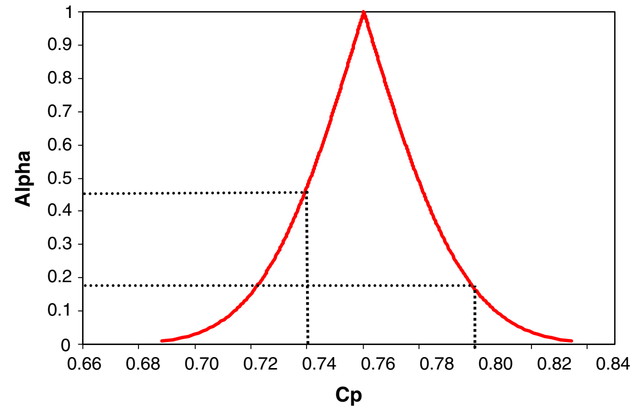

Fig. 8 Membership functions of DO for $\hat{C}_p$

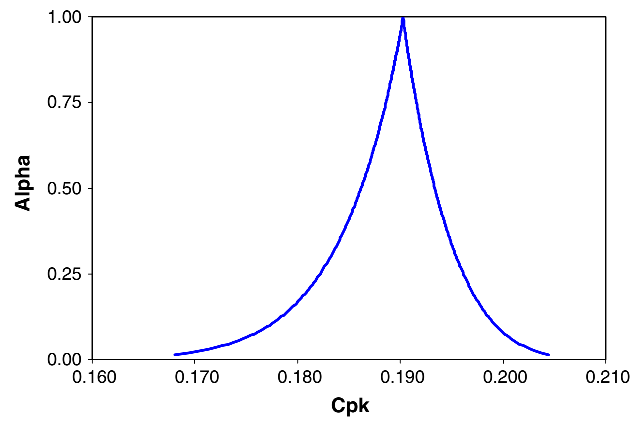

Fig. 9 Membership functions of DO for $\hat{C}_{pk}$

USL = approximately 8.5 = (8.25, 8.5, 8.75);
LSL = approximately 6.5 = (6.25, 6.5, 6.75).

According to Eq. (21) $\tilde{C}_p$ can be evaluated as follows:

$$\tilde{C}_p = T\left(\frac{8.25 - 6.75}{6 \times 0.267}, \frac{8.5 - 6.5_l}{6 \times 0.267}, \frac{8.75 - 6.25}{6 \times 0.267}\right)$$

$$\tilde{C}_p = T\left(\frac{1.500}{1.602}, \frac{2.000_l}{1.602}, \frac{2.500}{1.602}\right)$$

$$\tilde{C}_p = T(0.936, 1.248, 1.561)$$

The membership function of $\tilde{C}_p$ for pH is illustrated in Fig. 12. For pH, we can see that, $\hat{C}_p = 1.25$ belongs to fuzzy estimate $\hat{C}_p$ with a $\hat{C}_p(\alpha = 1) = 1.25$. We can also say that $\hat{C}_p(\alpha = 0.364) = 1.05$ and $\hat{C}_p(\alpha = 0.675) = 1.35$ from Fig. 12.

According to Eq. (23) $\tilde{C}_{pk}$ can be evaluated as follows:

$$\tilde{C}_{pk} = T(0.724, 1.036, 1.348)$$

The membership function of $\tilde{C}_{pk}$ for pH is illustrated in Fig. 13. For pH, we can see that, $\hat{C}_{pk} = 1.04$ belongs to fuzzy estimate $\hat{C}_{pk}$ with a $\hat{C}_p(\alpha = 1) = 1.25$. We can also

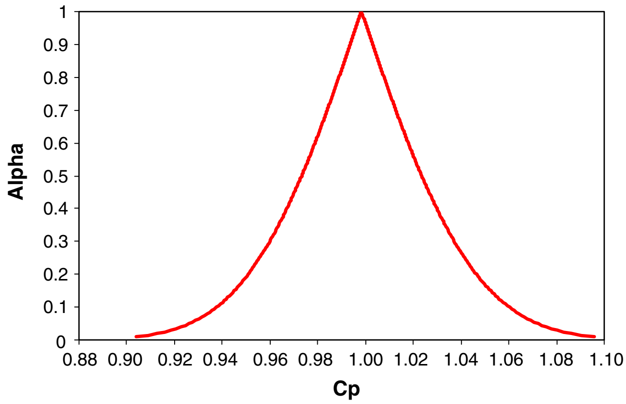

Fig. 10 Membership functions of T for $\hat{C}_p$

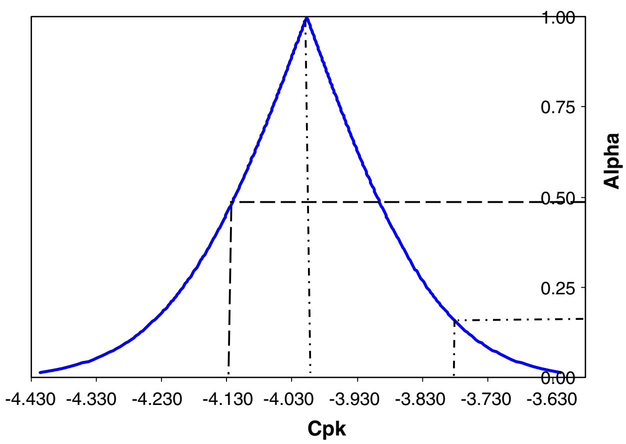

Fig. 11 Membership functions of T for $\hat{C}_{pk}$

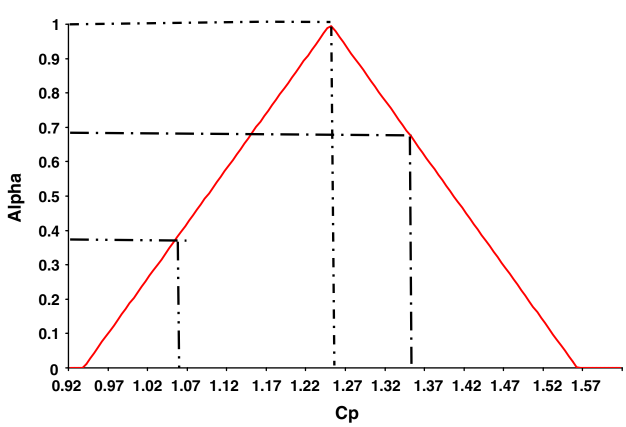

Fig. 12 The membership function of $\tilde{C}_p$

say that $\hat{C}_{pk}(\alpha = 0.275) = 0.81$ and $\hat{C}_{pk}(\alpha = 0.123) = 1.31$ from Fig. 12.

For the other water pollutants (DO and T) the same operations are followed when the specification limits are fuzzy.

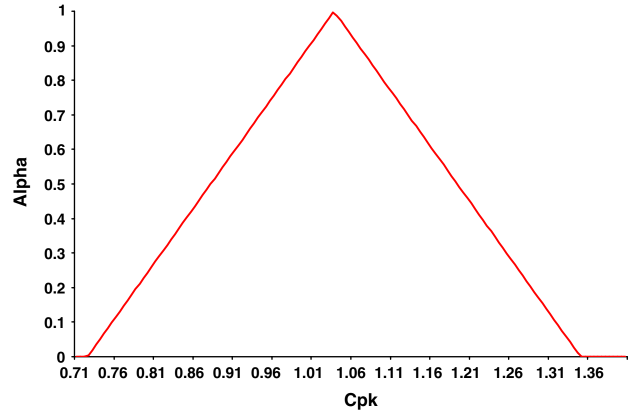

Fig. 13 The membership function of $\tilde{C}_{pk}$

5 Conclusion

Water pollution is one of the most important subjects for all people. It has many dangerous effects on lives. In this paper, we suggest process capability analysis to measure the capability of water for irrigation. Process capability analysis is a conceivable technique to control water pollution. Dam's water must have suitable conditions for irrigation especially for pH, DO, and T factors. This study analyzes the capability of water for pH, DO, and T by PCIs. However, fuzzy estimate of PCIs are analyzed and the membership functions of fuzzy estimate for $\hat{C}_p$ and $\hat{C}_{pk}$ are obtained. The case that specification limits are fuzzy has also been analyzed. Crisp PCIs represent only a single position of the process while FPCIs represent all possible positions of the process. This advantage can be used for the possible estimation of the process capability and it is a very useful advantage to control water pollutants. The corresponding people can interpret the results of FPCIs to solve water pollution problem. In the future researches, FPCIs can also be analyzed according to six-sigma approach. Six-sigma approach tries to reduce the variation of the process by letting the USL–LSL interval be 12 σ. It is more tightened than the traditional PCI approach to control water pollution.

References

Anonymous (1991) Water pollution control, The Official Gazette, whose number is 20748, Ankara

Anonymous (1998) Water pollution control regulations, no. 19919, 13–74 (in Turkish)

Anonymous (2003) Environment protection (Water Quality) policy 2003 and explanatory report. http://www.epa.sa.gov.au/pdfs/epwq_report.pdf

Anonymous (2006) Land and water resources of Turkey, Reports of DSİ (The General Directorate of State Hydraulic Works). http://www.dsi.gov.tr/english/topraksue.htm

Bothe DR (1997) Measuring process capability: techniques and calculations for quality and manufacturing engineers. McGraw-Hill, New York

Boyles RA (1991) The Taguchi capability index. J Qual Technol 23:17–26
Buckley JJ (2004) Fuzzy statistics, studies in fuzziness and soft computing. Springer Book, New York
Buckley JJ (2005a) Fuzzy statistics: regression and prediction. Soft Comput 9:769–775
Buckley JJ (2005b) Fuzzy statistics: hypothesis testing. Soft Comput 9:512–518
Chen TW, Chen KS, Lin JY (2003a) Fuzzy evaluation of process capability for bigger-the-best type products. Int J Adv Manuf Technol 21:820–826
Chen TW, Lin JY, Chen KS (2003b) Selecting a supplier by fuzzy evaluation of capability indices Cpm. Int J Adv Manuf Technol 22:534–540
Chen KS, Chen TW (2007) Multi-process capability plot and fuzzy inference evaluation. Int J Prod Econ. doi:10.1016/j.ijpe.2006.12.056
Gao Y, Huang M (2003) Optimal process tolerance balancing based on process capabilities. Int J Adv Manuf Technol 21:501–507
Hsu BM, Shu MH (2007) Fuzzy inference to assess manufacturing process capability with imprecise data. doi:10.1016/j.ejor.2007.02.023
Icaga Y, Bostanoglu Y, Kahraman E (2006) Water quality statistics of Akarçay Basin. Technol Res EJCT 2(1):43–50
Icaga Y (2007) Fuzzy evaluation of water quality classification. Ecol Ind 7(3):710–718
Kane VE (1986) Process capability indices. J Qual Technol 18(1):41–52
Kotz S, Johnson N (2002) Process capability indices-a review 1992–2000. J Qual Technol 34:2–19
Lee YH, Wei CC, Chang CL (1999) Fuzzy design of process tolerances to maximise process capability. Int J Adv Manuf Technol 15:655–659
Lee HT (2001) Cpk index estimation using fuzzy numbers. Eur J Oper Res 129:683–688
Montgomery DC (2005) Introduction to statistical quality control. Wiley, New York
Parchami A, Mashinchi M, Yavari AR, Maleki HR (2005) Process capability indices as fuzzy numbers. Aust J Stat 34(4):391–402
Parchami A, Mashinchi M, Maleki HR (2006) Fuzzy confidence interval for fuzzy process capability index. J Intell Fuzzy Syst 17:287–295
Parchami A, Mashinchi M (2007) Fuzzy estimation for process capability indices. Inf Sci 177:1452–1462
Tsai CC, Chen CC (2006) Making decision to evaluate process capability index C_p with fuzzy numbers. Int J Adv Manuf Technol 30:334–339
Yongting C (1996) Fuzzy quality and analysis on fuzzy probability. Fuzzy Sets Syst 83:283–290