

New classes of covariance and spectral density functions for spatio-temporal modelling

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Published online: 5 June 2007
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Abstract In the nonseparable spatio-temporal context, several efforts have been made in order to obtain general classes of spatio-temporal covariances. Our aim in this paper is to join several approaches coming from different authors and provide some ideas for the construction of new models of spatio-temporal covariance and spectral density functions. On one hand, we build new covariance families while removing some undesirable features of the previously proposed models, particularly following Stein's (in *J Am Stat Assoc* 100:310–321, 2005) remark about Gneiting's (in *J Am Stat Assoc* 97:590–600, 2002) approach and about some tensorial product covariance models. We show some of the theoretical results and examples obtained with the product or the sum of spatio-temporal covariance functions or even better with the mixed forms. On the other hand, we define new models for spectral densities through the product of two other spectral densities. We give some characterizations and properties as well as several examples. Finally, we present a practical modelling of Irish wind speed data based on some of the space-time covariance models presented in this paper.

Keywords Irish wind speed data · Matérn model · Mixed-form covariance · Nonseparability · Product-sum covariance · Space-time covariance function · Spectral density function

Work partially funded by grant MTM2004-06231 from the Spanish Ministry of Science and Culture.

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1 Introduction

Stochastic models describing how processes vary across space and time are essential to the application of statistics to a wide range of environmental applications. However, the structural analysis of such processes is more difficult than for merely spatial or for temporal ones.

Estimating and modelling the correlation of a space-time process are principal objectives in geostatistical analysis. The extension of geostatistical techniques to the space-time domain in order to provide tools for joint analysis of the space and time components, is only apparent because of certain characteristics of spatio-temporal phenomena (Rouhani and Hall 1989).

Because it is often difficult to think about spatial and temporal variations simultaneously, it is tempting to focus on the analysis of how the covariances at a single place vary across time, and how the covariances at a single time vary across space. If these were the only characteristics that mattered, then separable models would suffice. Allowing the merely spatial and temporal covariances to define the space-time dependence structure is a severe restriction, however.

In modern literature we can find several approaches to the construction of nonseparable covariances. In the first contributions the focus is on extending spatial or temporal methods to spatio-temporal ones, considering the spatio-temporal dependence separately in most of the cases. These extension approaches could be classified into two main categories: the former considers a pure extension of multivariate spatial or temporal models, and the latter is based on considering an univariate spatio-temporal process. Even if there are many points in common between the two approaches, some distinctions are necessary in order to see the opportunity of applying one approach despite to the other.

Following Stein (2005), separable covariance functions generally imply that small changes in the locations of observations can lead to large changes in the correlations between certain linear combinations of observations. The source of this discontinuity can be traced to a lack of smoothness away from the origin of separable functions. Furthermore, many of the nonseparable space-time covariance functions proposed in recent works have a similar lack of differentiability along certain axis and thus similar properties with their implied correlations. One is thus led to seek space-time covariance functions that are smooth everywhere except possibly at the origin. However, there are other properties, besides smoothness away from the origin, to consider in developing models for space-time covariance functions. One desirable such practical property is to have an explicit expression for the space-time covariance function or, if that is not available, at least a fast and accurate algorithm for computing it numerically.

Before continuing with history of spatio-temporal modelling, let us introduce some notations necessary to understand the sequel. Assume that a spatio-temporal process admits a decomposition of the form

$$Z(\mathbf{s}, t) = \mu(\mathbf{s}, t) + \delta(\mathbf{s}, t), \quad (1)$$

where \mathbf{s} is a point lying on \mathbb{R}^d Euclidean space ($d = 1, 2, \dots$), t is a time parameter which is assumed in this case to vary continuously on \mathbb{R} . Furthermore $\mu(\mathbf{s}, t)$ is some deterministic trend and $\delta(\mathbf{s}, t)$ is a spatio-temporal process with zero-mean and variogram structure not necessarily stationary

$$2\gamma(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2) = \text{Var}(Z(\mathbf{s}_1, t_1) - Z(\mathbf{s}_2, t_2)). \quad (2)$$

In the sequel we assume weak stationarity, in order to have a covariance structure depending only on the spatial and temporal lags. Besides, nonstationary models are usually obtained as an extension from stationary forms. Let us focus now on the previously mentioned categories of approach to spatio-temporal modelling, strictly depending on data configuration: whenever the spatial data set is more dense than the temporal one, usually a multivariate spatial process is considered (Egbert and Lettenmaier 1986). In the other case, a multivariate temporal process is considered (Solow and Gorelik 1986). As previously stated, we consider from now on weakly stationary spatio-temporal processes, which is equivalent to assume respectively that the variogram, or the covariogram, only depends on the spatial and temporal lag, indeed,

$$\begin{aligned} \gamma(\mathbf{s} + \mathbf{h}, t + u) &\equiv \gamma(\mathbf{h}, u), \\ C(\mathbf{s} + \mathbf{h}, t + u) &\equiv C(\mathbf{h}, u), \end{aligned} \quad (3)$$

where $C(\dots)$ indicates the spatio-temporal covariance function. Starting from these assumptions several authors tried to extend basic methodologies to the problems arising when modelling a spatio-temporal process. For instance, Dimitrakopoulos and Luo (1994) proposed some geometrically anisotropic models for spatio-temporal data. Guttorp et al. (1992) proposed some separable covariance structures, obtained with the tensor product of a spatial and a temporal covariance (which, in the Gaussian case, is equivalent to state that the process $\delta(\mathbf{s}, t)$ can be factorized into $\delta_1(\mathbf{s})\delta_2(t)$, where $\delta_1(\mathbf{s})$ and $\delta_2(t)$ are mutually independent). In the same context of separability, Rohuani and Hall (1989) proposed the so called sum-variogram. Other approaches are based on the fact that the drift could entirely catch the temporal variability, in order to obtain a purely spatial random process as $\delta(\mathbf{s}, t) = \delta(\mathbf{s})$ (Sampson and Guttorp 1992). In the nonseparable context, several efforts have been made in order to obtain classes (as general as possible) of spatio-temporal covariances. It is worth citing Jones and Zhang (1997), Cressie and Huang (1999), Christakos (2000), De Cesare et al. (2001), Gneiting (2002), Ma (2003), Stein (2005) and Fernández-Casal (2003). The majority of contributions regard stationary spatio-temporal covariances assuming isotropy on space and time. Particularly, Cressie and Huang (1999) proposed a spectral approach to obtain spatio-temporal covariances, and Gneiting's (2002) work represents the natural generalization to this approach, obtained using completely monotone functions and functions whose first derivative is completely monotone. Stein (2005) puts emphasis to the spectral approach and to the fact that a spatio-temporal spectral density must be sufficiently smooth away from the origin, as well as to the problems of differentiability away from the origin, introducing some important criteria allowing to show the differentiability of the covariance function obtained as Fourier transform of a certain class of spectral densities. Finally, Fernández-Casal (2003) extends the Shapiro-Botha approach to flexible variograms in the spatio-temporal context. In the nonstationary context, important contributions come from Christakos (2000), Fuentes and Smith (2001), Fuentes (2002), Christakos (2002) and Kolovos et al. (2004).

In this paper our aim is to join several approaches coming from some of these authors and provide key ideas for the construction of spatio-temporal spectral densities and covariance functions. In particular, we try to remove some undesirable features of the previously proposed models, particularly following Stein's remark about Gneiting's approach and about some tensorial product covariance models. Specifically, Stein (2005) observes that models of the type $\exp(-|\mathbf{x}| - |t|)$, obtained with a tensorial product of two exponential covariance functions on space

and time, are not differentiable at the origin and denote a lack of differentiability along certain axis, which in turn implies discontinuities of the autocorrelation function away from the origin. Furthermore, while emphasizing the need for spatio-temporal covariance functions which are sufficiently smooth away from the origin, Stein (2005) observes that Gneiting’s approach leads to some undesirable features, such as the fact that whatever the lack of smoothness of $C(\mathbf{x},0)$ for \mathbf{x} near zero, it will be shared by $C(\mathbf{x},t)$ for $t \neq 0$ and \mathbf{x} near zero, since $C(\mathbf{x},t)$ is just a rescaling of $C(\mathbf{x},0)$. Starting from Stein’s recommendation, we propose here the symmetric version of Gneiting’s assumption, in which a spatio-temporal covariance function can be obtained in the following way (generalized version):

$$C(\mathbf{h}, \mathbf{u}) = \frac{\sigma^2}{\psi(\|\mathbf{u}\|^2)^{d/2}} \varphi\left(\frac{\|\mathbf{h}\|^2}{\psi(\|\mathbf{u}\|^2)}\right), \quad (\mathbf{h}, \mathbf{u}) \in \mathbb{R}^d \times \mathbb{R}^l, \tag{4}$$

with d, l nonnegative integers. Here $\varphi(\cdot)$ is a completely monotone function defined on $[0, \infty)$ and $\psi(\cdot)$ a positive function with completely monotone derivative. A valid spatio-temporal covariance function can also be obtained, following Gneiting (2002), with the form

$$C(\mathbf{h}, \mathbf{u}) = \frac{\sigma^2}{\psi(\|\mathbf{h}\|^2)^{l/2}} \varphi\left(\frac{\|\mathbf{u}\|^2}{\psi(\|\mathbf{h}\|^2)}\right), \quad (\mathbf{h}, \mathbf{u}) \in \mathbb{R}^d \times \mathbb{R}^l, \tag{5}$$

Starting with this very simple result, we try to obtain some nice models simply by using the tensorial product, the sum or the sum-product (De Cesare’s model 2001) of Eqs. 5, and 6. Particularly, we show that whenever choosing the two completely monotone functions to be proportional to a Matérn–Whittle form, we can obtain some interesting particular cases allowing to show some covariance functions that do not depend on negative exponential structures, which are undesirably not differentiable at the origin.

The plan of the paper is the following. Section 2 describes some of the new theoretical results about the construction of spatio-temporal covariance functions. We show some particular examples obtained with the product or the sum of spatio-temporal covariance functions or better with the mixed forms, while analyzing some particular cases. Section 3 defines new models for spectral densities through the product of two other spectral densities. We give some characterizations and properties as well as several examples. Section 4 illustrates practical strategies in order to construct and apply one of the parametric families presented in this paper to the Irish wind data of

Haslett and Raftery (1989). The paper ends with some conclusions and discussion.

2 Building spatio-temporal covariance functions

We introduce some theoretical results to build new models of readily interpretable spatio-temporal covariance functions. Particular examples based on selected models and parameter settings are presented.

Proposition 1 (Mixed forms) *Let $C_1(\mathbf{h}, u)$ and $C_2(\mathbf{h}, u)$ be valid nonseparable spatio-temporal covariance functions with $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$. Let $C_3(\mathbf{h})$ and $C_4(u)$ be respectively valid spatial and temporal covariance functions. Finally, let $\lambda_{12}, \lambda_i, i = 1, \dots, 4$ be nonnegative constants, and ξ a natural number. Then*

$$C(\mathbf{h}, u) = C_i(\mathbf{h}, u) + C_3(\mathbf{h}) + C_4(u), \quad i = 1, 2 \tag{6}$$

$$C(\mathbf{h}, u) = C_1(\mathbf{h}, u)C_2(\mathbf{h}, u) + C_3(\mathbf{h}) + C_4(u) \tag{7}$$

$$C(\mathbf{h}, u) = C_1(\mathbf{h}, u) + C_2(\mathbf{h}, u) + C_3(\mathbf{h}) + C_4(u) \tag{8}$$

$$C(\mathbf{h}, u) = (\lambda_{12}C_1(\mathbf{h}, u)C_2(\mathbf{h}, u) + \lambda_3C_3(\mathbf{h}) + \lambda_4C_4(u))^\xi \tag{9}$$

and

$$C(\mathbf{h}, u) = (\lambda_1C_1(\mathbf{h}, u) + \lambda_2C_2(\mathbf{h}, u) + \lambda_3C_3(\mathbf{h}) + \lambda_4C_4(u))^\xi \tag{10}$$

are valid spatio-temporal covariance functions.

Proof The proof is direct application of the properties of covariance functions. Specifically, for the sum and product it is sufficient to recall that covariance functions are closed under the sum and tensor product operations. Observe that permissibility of Eqs. 9 and 10 is preserved for ξ natural number, but in general they are not valid for ξ real and positive.

It is interesting to note that following Gneiting, expressions 6, 7, 8, 9 and 10 would take the forms

$$C(\mathbf{h}, u) = \frac{\sigma^2}{\psi(\|\mathbf{h}\|^2)^{l/2}} \varphi_i\left(\frac{|u|^2}{\psi(\|\mathbf{h}\|^2)}\right) + \varphi_3(\|\mathbf{h}\|^2) + \varphi_4(|u|^2), \quad i = 1, 2 \tag{11}$$

$$C(\mathbf{h}, u) = \frac{\sigma^2}{\psi_2(\|\mathbf{h}\|^2)^{l/2}\psi_1(|u|^2)^{d/2}} \varphi_1\left(\frac{\|\mathbf{h}\|^2}{\psi_1(|u|^2)}\right) \varphi_2 \times \left(\frac{|u|^2}{\psi_2(\|\mathbf{h}\|^2)}\right) + \varphi_3(\|\mathbf{h}\|^2) + \varphi_4(|u|^2), \tag{12}$$

$$C(\mathbf{h}, u) = \frac{\sigma^2}{\psi_1(|u|^2)^{d/2}} \varphi_1\left(\frac{\|\mathbf{h}\|^2}{\psi_1(|u|^2)}\right) + \frac{\sigma^2}{\psi_2(\|\mathbf{h}\|^2)^{l/2}} \varphi_2\left(\frac{|u|^2}{\psi_2(\|\mathbf{h}\|^2)}\right) + \varphi_3(\|\mathbf{h}\|^2) + \varphi_4(|u|^2), \tag{13}$$

$$C(\mathbf{h}, u) = \left(\lambda_{12} \frac{\sigma^2}{\psi_2(\|\mathbf{h}\|^2)^{l/2} \psi_1(|u|^2)^{d/2}} \varphi_1\left(\frac{\|\mathbf{h}\|^2}{\psi_1(|u|^2)}\right) \times \varphi_2\left(\frac{|u|^2}{\psi_2(\|\mathbf{h}\|^2)}\right) + \lambda_3 \varphi_3(\|\mathbf{h}\|^2) + \lambda_4 \varphi_4(|u|^2)\right)^\xi, \tag{14}$$

$$C(\mathbf{h}, u) = \left(\lambda_1 \frac{\sigma^2}{\psi_1(|u|^2)^{d/2}} \varphi_1\left(\frac{\|\mathbf{h}\|^2}{\psi_1(|u|^2)}\right) + \lambda_2 \frac{\sigma^2}{\psi_2(\|\mathbf{h}\|^2)^{l/2}} \varphi_2\left(\frac{|u|^2}{\psi_2(\|\mathbf{h}\|^2)}\right) + \lambda_3 \varphi_3(\|\mathbf{h}\|^2) + \lambda_4 \varphi_4(|u|^2)\right)^\xi, \tag{15}$$

where $\varphi_i(t), t \geq 0, i = 1, \dots, 4$ are completely monotone functions and $\psi_j(t), t \geq 0, j = 1, 2$ are positive functions with completely monotone derivative.

Some comments are necessary at this point. Equations 6 and 7 of Proposition 1 represent the generalization of De Cesare’s product-sum model (2001), with the great advantage that we can choose nonseparable spatio-temporal structures in the sense of Gneiting. Besides, the marginal structures act separately as a rescaling of the nonseparable structure on space and time. The completely monotone functions and functions whose derivatives are completely monotone can be selected from Tables 1 and 2 in Gneiting (2002). Furthermore other completely monotone functions can be obtained as a Gaussian Scale Mixture (Gneiting 1997). Finally, the functions φ_3 and φ_4 must be respectively coherent with φ_1 and φ_2 , as they are respectively chosen to be ordinary functions of space and time before creating the nonseparable space-time structures 4 and 5. In summary, in practical terms the procedure would be the following:

1. Choose $\varphi_1, \varphi_2, \psi_1$ and ψ_2 . Use these functions to create nonseparable space-time structures of the type 4 and 5.
2. Use one of the models 11, 12, 13, 14 and 15 implementing separate structures which must be of the same type of φ_1 and φ_2 , respectively, for space and time.

2.1 Particular examples

2.1.1 Example 1

Consider the example (16) in Gneiting, of the form

$$C_1(\mathbf{h}, u) = \frac{\sigma_1^2}{2^{v_1-1} \Gamma(v_1) (a_1 |u|^{2\alpha_1} + 1)^{\delta_1 + \beta d/2}} \times \left(\frac{c_1 \|\mathbf{h}\|}{(a_1 |u|^{2\alpha_1} + 1)^{\beta/2}}\right)^{v_1} \times K_{v_1}\left(\frac{c_1 \|\mathbf{h}\|}{(a_1 |u|^{2\alpha_1} + 1)^{\beta/2}}\right), \tag{16}$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$. Using the same mixture of monotone functions and completely monotone ones, applying 6 we obtain the analogous expression

$$C_2(\mathbf{h}, u) = \frac{\sigma_2^2}{2^{v_2-1} \Gamma(v_2) (a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\delta_2 + \beta l/2}} \times \left(\frac{c_2 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}}\right)^{v_2} \times K_{v_2}\left(\frac{c_2 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}}\right), \tag{17}$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$ and $a_i, c_i, i = 1, 2$, are nonnegative scaling parameters on space and time, respectively, $\alpha_i \in (0, 1], i = 1, 2$ is a smoothing parameter on time, $v_i > 0, i = 1, 2$, is a smoothing parameter on space, $\beta \in [0, 1], \delta_i \geq 0, \sigma_i^2 > 0$, and $K_{v_i}(\cdot)$ is the modified Bessel function of the third kind of order $v_i, i = 1, 2$.

Imposing $v_1 = v_2 = v$ and $\delta_1 + \beta d/2 = \delta_2 + \beta l/2 = \varepsilon$ (indeed, imposing $\beta = \frac{2(\delta_2 - \delta_1)}{d-l}$ and $\delta_2 \geq \delta_1, d > l, 2(\delta_2 - \delta_1) \leq d-l$), and

$$C(\mathbf{h}, u) = C_1(\mathbf{h}, u) C_2(\mathbf{h}, u), \tag{18}$$

we obtain the following expression, based on expressions 16 and 17,

$$C(\mathbf{h}, u) = \frac{k}{B_{u, \mathbf{h}}^{v\beta/2 + \varepsilon}} (a_1 a_2 \|\mathbf{h}\| |u|)^v K_v\left(\frac{a_2 \|\mathbf{h}\|}{(a_1 |u|^{2\alpha_1} + 1)^{\beta/2}}\right) \times K_v\left(\frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}}\right), \tag{19}$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$, $v = \varepsilon + \beta/2$ and $k = \frac{\sigma^4}{2^{2(v-1)}(\Gamma(v))^2}$ with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Moreover,

$$B_{u,\mathbf{h}} = \left(1 + a_1|u|^{2\alpha_1} + a_2\|\mathbf{h}\|^{2\alpha_2} + a_1a_2|u|^{2\alpha_1}\|\mathbf{h}\|^{2\alpha_2}\right).$$

Here we have that a_1, a_2 are nonnegative scaling parameters ($a_1 = c_2, a_2 = c_1$). The smoothing parameters on time and space are, respectively, α_1 and α_2 , both belonging to the interval $(0,1]$. Finally, the parameters $v > 0$ and $\varepsilon > 0$ globally control the process, independently of the other ones, which act locally. Applying Proposition 1, we can add two separate structures on space and time. For instance, we could choose a Matérn covariance function for both space and time, so let us call

$$M_s(\mathbf{h}) = \frac{\sigma_2^2}{2^{2\alpha_2-1}\Gamma(\alpha_2)} (a_2\|\mathbf{h}\|)^{\alpha_2} K_{\alpha_2}(a_2\|\mathbf{h}\|), \tag{20}$$

and

$$M_t(u) = \frac{\sigma_1^2}{2^{2\alpha_1-1}\Gamma(\alpha_1)} (a_1|u|)^{\alpha_1} K_{\alpha_1}(a_1|u|), \tag{21}$$

respectively, the purely spatial and temporal covariance functions. Note that the smoothing parameters here are called, respectively, α_2 and α_1 , in order to be related to the local smoothing parameters in Eq. 19. The same is imposed on the local scale parameters. The aim is to integrate the local smoothing parameter’s behavior into a nonseparable context with a separate contribution on space and time. Besides, relating the global smoothing parameters of the separate structure to the nonseparable local ones in Eq. 19 ensures avoiding problems of monotonicity of the resulting covariance function, obtained as in Eq. 12. Following this approach, a possible model for $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$ could be

$$C(\mathbf{h}, u) = \frac{k}{B_{u,\mathbf{h}}^{v\beta/2+\varepsilon}} (a_1a_2\|\mathbf{h}\||u|)^v K_v\left(\frac{a_2\|\mathbf{h}\|}{(a_1|u|^{2\alpha_1}+1)^{\beta/2}}\right) \times K_v\left(\frac{a_1|u|}{(a_2\|\mathbf{h}\|^{2\alpha_2}+1)^{\beta/2}}\right) + M_s(\mathbf{h}) + M_t(u). \tag{22}$$

Some interesting particular cases of Eq. 19 can be obtained by setting $v = 1/2$ and $v = 3/2$, obtaining the following expressions for $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$

$$C(\mathbf{h}, u \mid v = 1/2) \propto \frac{k}{B_{u,\mathbf{h}}^\varepsilon} \exp\left(-\frac{a_1|u|}{(a_2\|\mathbf{h}\|^{2\alpha_2}+1)^{\beta/2}} - \frac{a_2\|\mathbf{h}\|}{(a_1|u|^{2\alpha_1}+1)^{\beta/2}}\right), \tag{23}$$

$$C(\mathbf{h}, u \mid v = 3/2) \propto \frac{k}{B_{u,\mathbf{h}}^\varepsilon} \exp\left(-\frac{a_1|u|}{(a_2\|\mathbf{h}\|^{2\alpha_2}+1)^{\beta/2}} - \frac{a_2\|\mathbf{h}\|}{(a_1|u|^{2\alpha_1}+1)^{\beta/2}}\right) \chi_{u,\mathbf{h}}, \tag{24}$$

where $\chi_{u,\mathbf{h}} = (1 + a_2\|\mathbf{h}\|(a_1|u|^{2\alpha_1} + 1)^{-\beta/2} + a_1|u|(a_2\|\mathbf{h}\|^{2\alpha_2} + 1)^{-\beta/2})$. Figure 1 shows contour and perspective plots coming from the spatio-temporal model of Example 1. In particular, the first column represents the covariance equation 22 whereas the second column represents covariance equation 19.

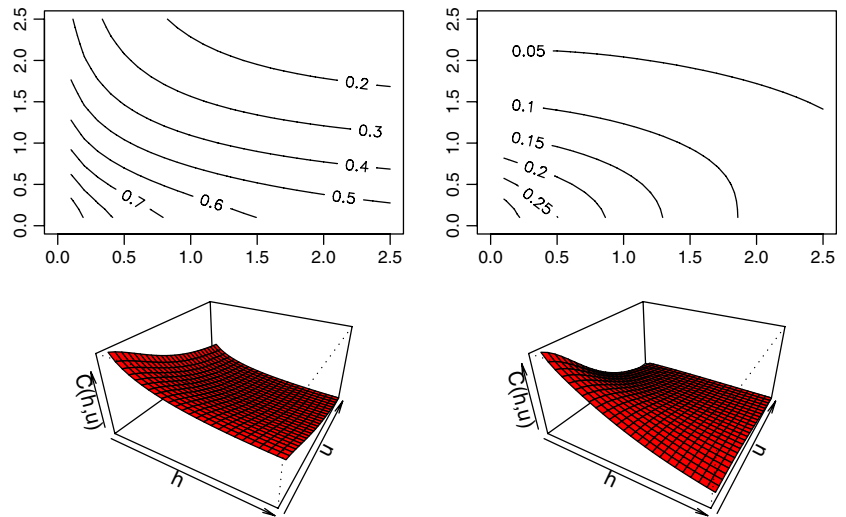
2.1.2 Example 2

Consider the examples (16) and (12) in Gneiting (2002), taking now one of the covariances in Gneiting’s style and the other in the class of the symmetric version. The factorization of the two forms gives the following model

$$C(\mathbf{h}, u) = \frac{k}{B_{u,\mathbf{h}}^\varepsilon} \left(\frac{a_1|u|}{(a_2\|\mathbf{h}\|^{2\alpha_2}+1)^{\beta/2}}\right)^v \times \exp\left(\frac{-a_2\|\mathbf{h}\|^{2\alpha_2}}{(a_1|u|^{2\alpha_1}+1)^{\beta\alpha_2}}\right) \times K_v\left(\frac{a_1|u|}{(a_2\|\mathbf{h}\|^{2\alpha_2}+1)^{\beta/2}}\right), \tag{25}$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$ and $k = \frac{\sigma^4}{2^{(v-1)}(\Gamma(v))}$ and a_1 and a_2 are nonnegative scaling parameters, α_1 and α_2 belong to the interval $(0,1]$ and v and ε are the positive global smoothing parameters. It is interesting to observe that setting $v = 1/2$ and $v = 3/2$ we obtain two particular cases, respectively

Fig. 1 Contour and perspective plots of the spatio-temporal covariance functions given by Eqs. 22 (first column) and 19 (second column) with spatial lag (\mathbf{h}) represented against the temporal one (u). The common parameters are the following: $a_1 = 1, a_2 = 1, a_3 = 0.5, \beta = 0.9, \alpha_1 = 0.9, \alpha_2 = 0.2, \nu = 1.5$ and $\varepsilon = 2$



$$C(\mathbf{h}, u) \propto \frac{\sigma^4}{B_{u,\mathbf{h}}^\varepsilon} \exp \left(- \frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}} - \frac{a_2 \|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta\alpha_2}} \right), \quad (26)$$

$$C(\mathbf{h}, u) \propto \frac{\sigma^4}{B_{u,\mathbf{h}}^\varepsilon} \exp \left(- \frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}} - \frac{a_2 \|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta/2}} \right) \times \left(1 + \frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}} \right), \quad (27)$$

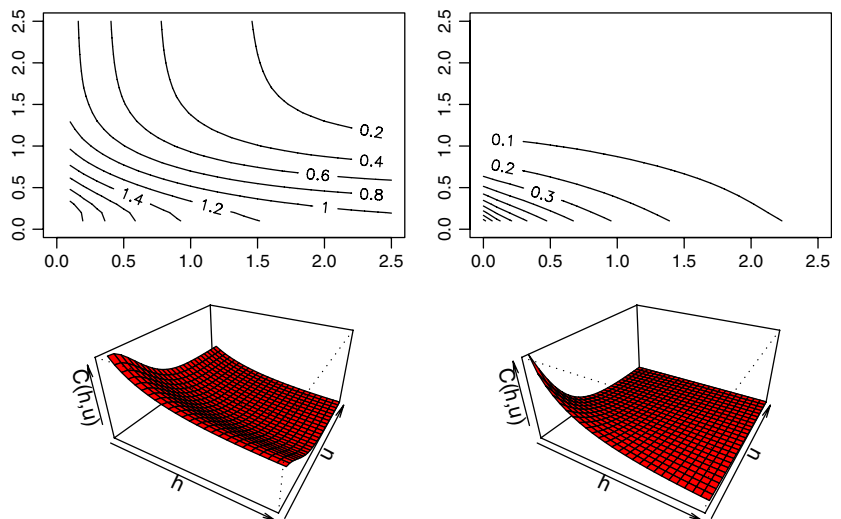
The corresponding mixed form can be obtained taking into account that the temporal structure was initially modelled with a Matérn covariance function and the spatial one with a negative exponential one, thus

$$C(\mathbf{h}, u) = \frac{k}{B_{u,\mathbf{h}}^\varepsilon} \left(\frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}} \right)^\nu \exp \left(\frac{-a_2 \|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta\alpha_2}} \right) \times K_\nu \left(\frac{a_1 |u|}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}} \right) + M_t(u) + \exp(-a_2 \|\mathbf{h}\|^{\alpha_2}). \quad (28)$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$.

Figure 2 shows contour and perspective plots coming from the spatio-temporal model of Example 2. In particular, the

Fig. 2 Contour and perspective plots of the spatio-temporal covariance functions given by Eqs. 25 (first column) and 28 (second column) with spatial lag (\mathbf{h}) represented against the temporal one (u). The common parameters are the following: $a_1 = 1.8, a_2 = 1, a_3 = 0.5, \beta = 0.9, \alpha_1 = 0.9, \alpha_2 = 0.4, \nu = 0.5$ and $\varepsilon = 2$



first column represents the covariance equation 25 whereas the second column represents covariance equation 28.

2.1.3 Example 3

We now choose the following completely monotone function and function with completely monotone derivative

$$\varphi_i(t) = (1 + c_i t^{\alpha_i})^{-\nu}, \quad t \geq 0, \quad i = 1, 2 \tag{29}$$

with $\alpha_i \in (0, 1]$, c_i and ν strictly positive, and

$$\psi_i(t) = (a_i t^{\alpha_i} + 1)^{\beta_i}, \quad t \geq 0, \quad i = 1, 2 \tag{30}$$

with $\alpha_i \in (0, 1]$, $\beta_i \in [0, 1]$, a_i strictly positive. With straightforward calculations we obtain

$$C_1(\mathbf{h}, u) = \frac{\sigma^2}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta_1 d/2}} \left(1 + a_1 \frac{|u|^{2\alpha_1}}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta_1 \alpha_1}} \right)^{-\nu}, \tag{31}$$

where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$, and

$$C_2(\mathbf{h}, u) = \frac{\sigma^2}{(a_1 |u|^{2\alpha_1} + 1)^{\beta_2 d/2}} \left(1 + a_2 \frac{\|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta_2 \alpha_2}} \right)^{-\nu}, \tag{32}$$

with $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$.

The tensorial product gives the following structure, for $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$,

$$C(\mathbf{h}, u) = \frac{\sigma^2}{B_{u, \mathbf{h}}^\varepsilon} \left(1 + a_1 \frac{|u|^{2\alpha_1}}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta_{\alpha_1/2}}} + a_2 \frac{\|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta_{\alpha_2/2}}} + a_1 a_2 \frac{|u|^{2\alpha_1} \|\mathbf{h}\|^{2\alpha_2}}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta_{\alpha_1/2}} (a_1 |u|^{2\alpha_1} + 1)^{\beta_{\alpha_2/2}}} \right)^{-\nu}, \tag{33}$$

where the interpretation and restriction on parameters is the same as in the previous Example 1.

2.1.4 Example 4

Consider the examples (16) and (12) in Gneiting (2002). We propose a closed form of the type

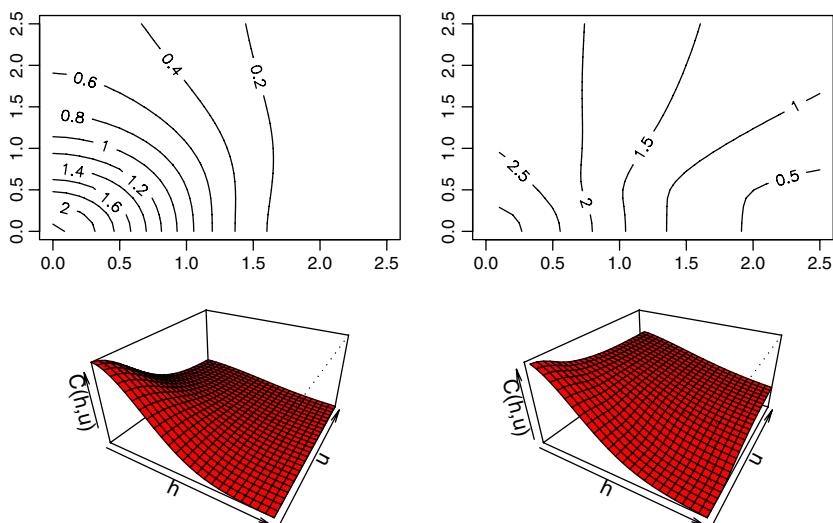
$$C(\mathbf{h}, u) = (\lambda_1 C_1(\mathbf{h}, u) + \lambda_2 C_2(\mathbf{h}, u))^\xi, \tag{34}$$

where $\lambda_1, \lambda_2 \in (0, 1]$, ξ a natural number.

Considering directly the case $\nu = 1/2$, it is possible to obtain the following expression

$$C(\mathbf{h}, u) = \frac{1}{B_{u, \mathbf{h}}^{1/2}} \left\{ \lambda_1 \exp\left(\frac{-a_2 \|\mathbf{h}\|^{2\alpha_2}}{(a_1 |u|^{2\alpha_1} + 1)^{\beta/2}}\right) (a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{1/2} + \lambda_2 \exp\left(-\frac{a_1 |u|^{2\alpha_1}}{(a_2 \|\mathbf{h}\|^{2\alpha_2} + 1)^{\beta/2}}\right) (a_1 |u|^{2\alpha_1} + 1)^{1/2} \right\}^\xi, \tag{35}$$

Fig. 3 Contour and perspective plots of the spatio-temporal covariance functions given by Eqs. 36 (first column) and 35 (second column) with spatial lag (\mathbf{h}) represented against the temporal one (u) . The common parameters are the following: $a_1 = 1, a_2 = 0.7, a_3 = 0.5, \beta = 0.9, \alpha_1 = 0.9, \alpha_2 = 0.9, \nu = 1$ and $\varepsilon = 0.8$



where $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$ and $v = \delta_1 + \beta d/2 = \delta_2 + \beta l/2$ (imposing $\beta = \frac{2(\delta_2 - \delta_1)}{d-l}$ and $\delta_2 \geq \delta_1, d > l, 2(\delta_2 - \delta_1) \leq d-l$).

Finally, a mixed form of the type 22 can be easily obtained

$$C(\mathbf{h}, u) = \frac{1}{B_{u, \mathbf{h}}^{1/2}} \left\{ \begin{aligned} &\lambda_1 \exp\left(\frac{-a_2 \|\mathbf{h}\|^{2x_2}}{(a_1 |u|^{2x_1} + 1)^{\beta/2}}\right) (a_2 \|\mathbf{h}\|^{2x_2} + 1)^{1/2} \\ &+ \lambda_2 \exp\left(\frac{-a_1 |u|^{2x_1}}{(a_2 \|\mathbf{h}\|^{2x_2} + 1)^{\beta/2}}\right) (a_1 |u|^{2x_1} + 1)^{1/2} \end{aligned} \right\}^\zeta + M_s(\mathbf{h}) + \exp(-a_1 |u|^{\alpha_1}), (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R} \tag{36}$$

Figure 3 shows contour and perspective plots coming from the spatio-temporal model of Example 4. In particular, the first column represents the covariance equation 36 whereas the second column represents covariance equation 35.

3 Building spatio-temporal models through the spectral density function

There are different ways to define a model for the covariance function of a random field. Under several conditions, there exists a spectral density function associated to the spectral distribution function. In this section, we are giving a particular structure to this density.

Recall that a continuous function C on $\mathbb{R}^d \times \mathbb{R}, d \in \mathbb{N}$, is positive definite if and only if it is of the form

$$C(\mathbf{h}, u) = \int e^{i\omega \mathbf{h} + i\tau u} dF(\omega, \tau), (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R} \tag{37}$$

where F is the distribution function of a nonnegative and finite measure on $\mathbb{R}^d \times \mathbb{R}$. If F is absolutely continuous (when C is integrable), Bochner’s representation is expressed as

$$C(\mathbf{h}, u) = \int e^{i\omega \mathbf{h} + i\tau u} f(\omega, \tau) d(\omega, \tau), (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R} \tag{38}$$

where f is a continuous, nonnegative and integrable function.

Our proposal is based on defining a new spectral density as a product of two spectral densities, as shown in the following results.

Proposition 2 *Let us consider $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ two spectral density functions, not necessarily different. Consider $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$, and $\alpha, \beta \in \mathbb{N}$ even numbers. If the following function*

$$f(\omega, \tau) = f_1(a_1 |\tau|^\alpha + b_1 \|\omega\|^\beta) f_2(a_2 |\tau|^\alpha + b_2 \|\omega\|^\beta) \tag{39}$$

is integrable, then it is a spectral density.

Proof The proof is simple, as it is based on the properties of norm functions:

- The functions $\tau \rightarrow a_1 |\tau|^\alpha$ and $\omega \rightarrow b_1 \|\omega\|^\beta$ are continuous for construction, as the powers α, β are even numbers. So, it is their sum.
- The composition $(\omega, \tau) \rightarrow a_1 |\tau|^\alpha + b_1 \|\omega\|^\beta \rightarrow f_1(a_1 |\tau|^\alpha + b_1 \|\omega\|^\beta)$ is continuous (because both the first function and f_1 are continuous), and is nonnegative (because f_1 is nonnegative). The same conclusion follows replacing a_1, b_1 by a_2, b_2 , respectively.
- Finally, the function f is continuous and nonnegative because it is the product of two functions with these properties, and it is integrable by assumption.

The most important characteristic of this model is that it includes both the separable and nonseparable cases in the spectral domain, and defines a very general class of models.

One aspect this model should meet is the differentiability of the covariance function obtained as Fourier transform of Eq. 39. This feature can be studied by means of some properties of its associated spectral density function, when it exists. This aspect is studied in Stein (2005), and here we adapt this analysis to the spectral densities defining f . Consider a spectral density $f(\omega) = g_1(\omega)g_2(\omega)$, where $g_1, g_2 : \mathbb{R}^3 \rightarrow \mathbb{R}, \omega = (w_1, w_2, w_3)', x = (x_1, x_2, x_3)' \in \mathbb{R}^3$ and denote by $D^{\mathbf{m}}$ the partial derivative $\frac{\partial^{\mathbf{m}}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}$, being $m = m_1 + m_2 + m_3$ and $\mathbf{m} = (m_1, m_2, m_3)$. Then, parts of Propositions 4 and 5 of Stein (2005) are adapted here to give:

Proposition 3 *Suppose $D^l g_1, D^l g_2$ exist and are integrable for $l \leq k$, and the functions $w_1^{m_1} w_2^{m_2} w_3^{m_3} D^l g_1(\omega)$ and $w_1^{m_1} w_2^{m_2} w_3^{m_3} D^l g_2(\omega)$ are also integrable for $l \leq k$, then $D^{\mathbf{m}} C$ exists, where C is the covariance function associated to f .*

Proof The proof is a direct consequence of Proposition 4 in Stein (2005). In order to show what we state, it is only necessary to see that $D^l g_1 g_2$ exists for $l \leq k$ and $\omega^{\mathbf{m}} D^{\mathbf{k}} g_1 g_2$ are all of them integrable. Observe that $D^l g_1 g_2$ is a linear combination of products $D^l g_1 D^s g_2$, which are integrable by assumption. Same thing happens for $\omega^{\mathbf{m}} D^{\mathbf{k}} g_1 g_2$, being a linear combination of products $\omega_1^{m_1} \omega_2^{m_2} \omega_3^{m_3} D^l g_1(\omega) D^s g_2(\omega)$, integrable also by assumption. Then, $g_1 g_2$ follows assumptions of Proposition 4 of Stein (2005), and the result is obtained.

Proposition 4 *For $j = 1, 2, 3$, assume $\frac{\partial^l}{\partial w_j^l} g_1$ and $\frac{\partial^l}{\partial w_j^l} g_2$ exist and are integrable for $l \leq k$ and $|\omega|^n \frac{\partial^{l_j}}{\partial w_j^{l_j}} g_1, |\omega|^n \frac{\partial^{l_j}}{\partial w_j^{l_j}} g_2$ are also integrable. We have that for $x \neq 0, D^{\mathbf{m}} C(x)$ exists for $\mathbf{m} = (m_1, m_2, m_3)$ such that $m_1 + m_2 + m_3 \leq n$.*

Proof A similar proof follows for this proposition, based on Proposition 5 of Stein (2005), together with the facts that $\frac{\partial^l}{\partial \omega_j^l} g_1 g_2$ is a linear combination of $\frac{\partial^l}{\partial \omega_j^l} g_1 \frac{\partial^s}{\partial \omega_j^s} g_2$, and that $|\omega|^n \frac{\partial^l}{\partial \omega_j^l} g_1 g_2$ is a linear combination of $|\omega|^n \frac{\partial^l}{\partial \omega_j^l} g_1 \frac{\partial^s}{\partial \omega_j^s} g_2$.

Then, $\frac{\partial^l}{\partial \omega_j^l} g_1 g_2$ and $|\omega|^n \frac{\partial^l}{\partial \omega_j^l} g_1 g_2$ are integrable and Proposition 5 of Stein (2005) can be used, getting the final result.

Note that in our model $g_1 = f_1 \circ h_1, g_2 = f_2 \circ h_2$, where $h_1(\omega) = a_1|w_3|^\alpha + b_1\|(w_1, w_2)\|^\beta, h_2(\omega) = a_2|w_3|^\alpha + b_2\|(w_1, w_2)\|^\beta$. The differentiability of g_1, g_2 is indicated by means of these four functions. In particular, if the above indicated partial derivatives exist for f_1, f_2, h_1, h_2 the previous Propositions 3 and 4 can be applied. One simple case could be using $\alpha, \beta \in \dot{2}$, and then, only conditions for f_1, f_2 should be imposed.

In the following subsection we introduce some examples built from some of the most widely used models of spectral densities in geostatistics.

3.1 Particular examples

3.1.1 Example 1

Consider now two different densities associated with two different covariance functions, but both with a Matérn structure, with $a = b = 1$, that is

$$f(\omega, \tau) = \phi_1 \phi_2 (a_1^2 + (a_{11}|\tau| + b_{11}\|\omega\|)^2)^{-\nu_1 - \frac{d_1}{2}} \times (a_2^2 + (a_{21}|\tau| + b_{21}\|\omega\|)^2)^{-\nu_2 - \frac{d_2}{2}} \tag{40}$$

with $d_1, d_2 = 1, 2, 3$, and with ϕ_1, ϕ_2 strictly positive parameters, while the scaling parameters $a_1, a_{11}, a_{21}, b_1, b_{11}$, and b_{21} must be nonnegative. Finally, the smoothing parameters ν_1, ν_2 are nonnegative. With this structure we are able to get a separable density function, just by setting $a_{11} = b_{21} = 0$ or $b_{11} = a_{21} = 0$, as follows:

$$f(\omega, \tau) = \phi_1 \phi_2 (a_1^2 + b_{11}^2 \|\omega\|^2)^{-\nu_1 - \frac{d_1}{2}} (a_2^2 + a_{21}^2 |\tau|^2)^{-\nu_2 - \frac{d_2}{2}} \tag{41}$$

$$f(\omega, \tau) = \phi_1 \phi_2 (a_1^2 + a_{11}^2 |\tau|^2)^{-\nu_1 - \frac{d_1}{2}} (a_2^2 + (b_{21} \|\omega\|)^2)^{-\nu_2 - \frac{d_2}{2}} \tag{42}$$

Figures 4 and 5 show some of these examples. Note that when we work with a linear combination of the norm of the frequencies, the levels in the contour plots are linear functions. Other possibility could be to substitute this linear combination by a quadratic function.

3.1.2 Example 2

Note that most of the cases in the above examples showed approximately linear functions for the levels of the contour

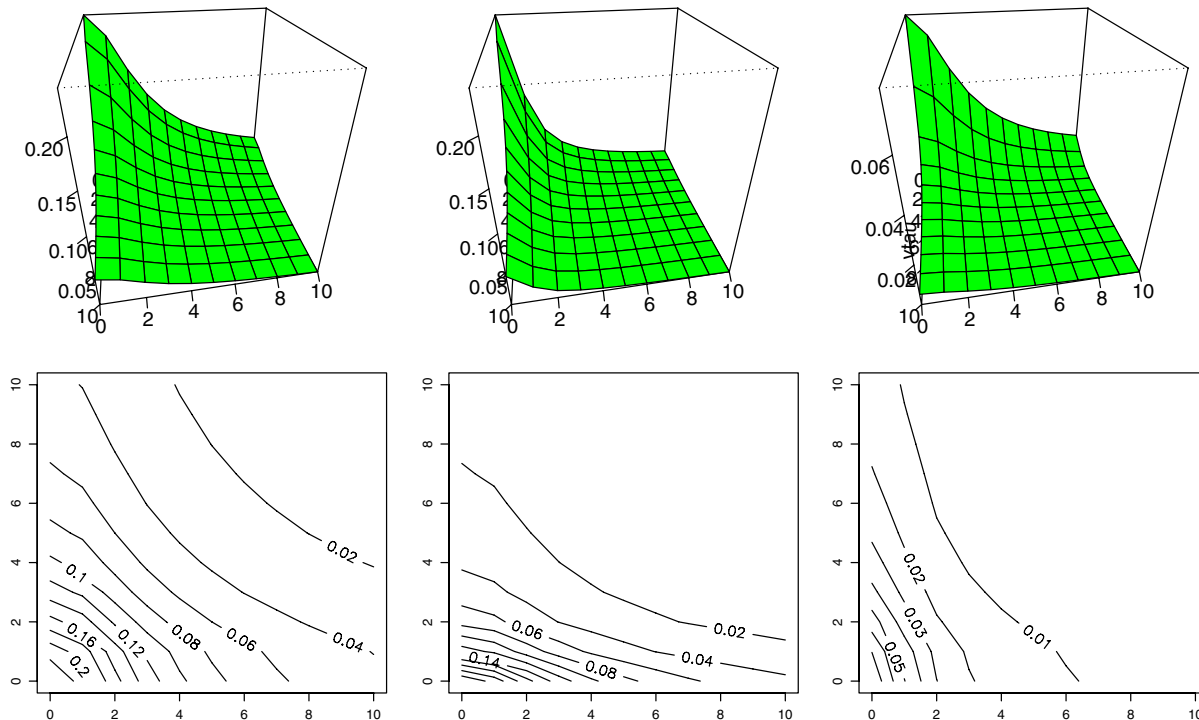


Fig. 4 Perspective and contour plots of the spectral density function given by Eq. 40 corresponding to Example 1 in Sect. 3.1. Horizontal axis of the contour plots corresponds to $|\tau|$, and vertical axis to $\|\omega\|$. Common parameters are $\phi_1 = \phi_2 = 1, b_{11} = a_{21} = 0$. The other

parameters are: *first column:* $a_1 = 2, a_{11} = 1, a_2 = 2, b_{21} = 1, \nu_1 = \nu_2 = 1/60$; *second column:* $a_1 = 2, a_{11} = 3, a_2 = 2, b_{21} = 1, \nu_1 = 1/60, \nu_2 = 1/60$; *third column:* $a_1 = 2, a_{11} = 1, a_2 = 6, b_{21} = 7, \nu_1 = 1/60, \nu_2 = 1/90, d_1 = d_2 = 1$

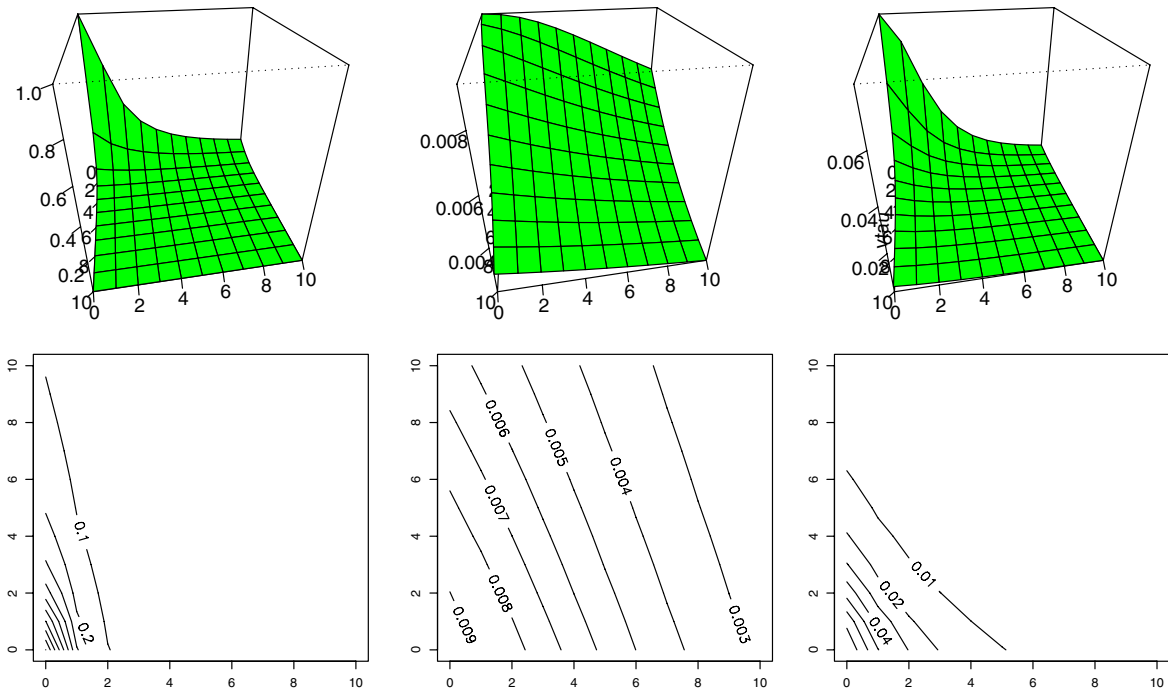


Fig. 5 Perspective and contour plots of the spectral density function given by Eq. 40 corresponding to Example 1 in Sect. 3.1. Horizontal axis of the contour plots corresponds to $|\tau|$, and vertical axis to $\|\omega\|$. Common parameters are $\phi_1 = \phi_2 = 1$. The other parameters are: *first column*: $a_{11} = 1, b_{11} = 2, a_1 = 1, v_1 = 1/120, a_{21} = 0, b_{21} = 1, a_2 = 1$,

$v_2 = 1/100$; *second column*: $a_{11} = 1, b_{11} = 2, a_1 = 10, v_1 = 1/120, a_{21} = 0, b_{21} = 1, a_2 = 10, v_2 = 1/100$; *third column*: $a_{11} = 1, b_{11} = 0.3, a_1 = 2, v_1 = 1/100, a_{21} = 2, b_{21} = 7, a_2 = 6, v_2 = 1/100, d_1 = d_2 = 1$

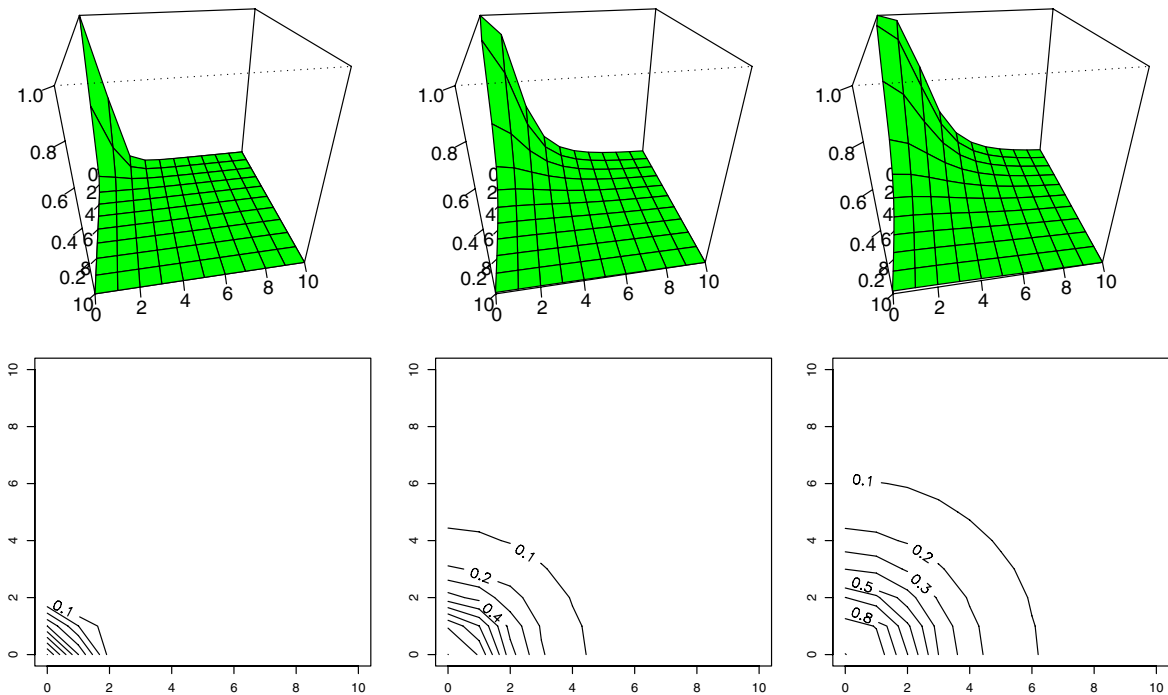


Fig. 6 Perspective and contour plots of the spectral density function given by Eq. 40 corresponding to Examples 1 and 2 in Sect. 3.1. Horizontal axis of the contour plots corresponds to $|\tau|$, and vertical axis to $\|\omega\|$. Common parameters are $\alpha = \beta = 2, a = 1$ and $\phi = 1$.

The other parameters are: *first column*: $a_1 = 1, b_1 = 1, v = 1/2$; *second column*: $a_1 = 0.5, b_1 = 0.5, v = 1/100$; *third column*: $a_1 = 0.25, b_1 = 0.25, v = 1/100$

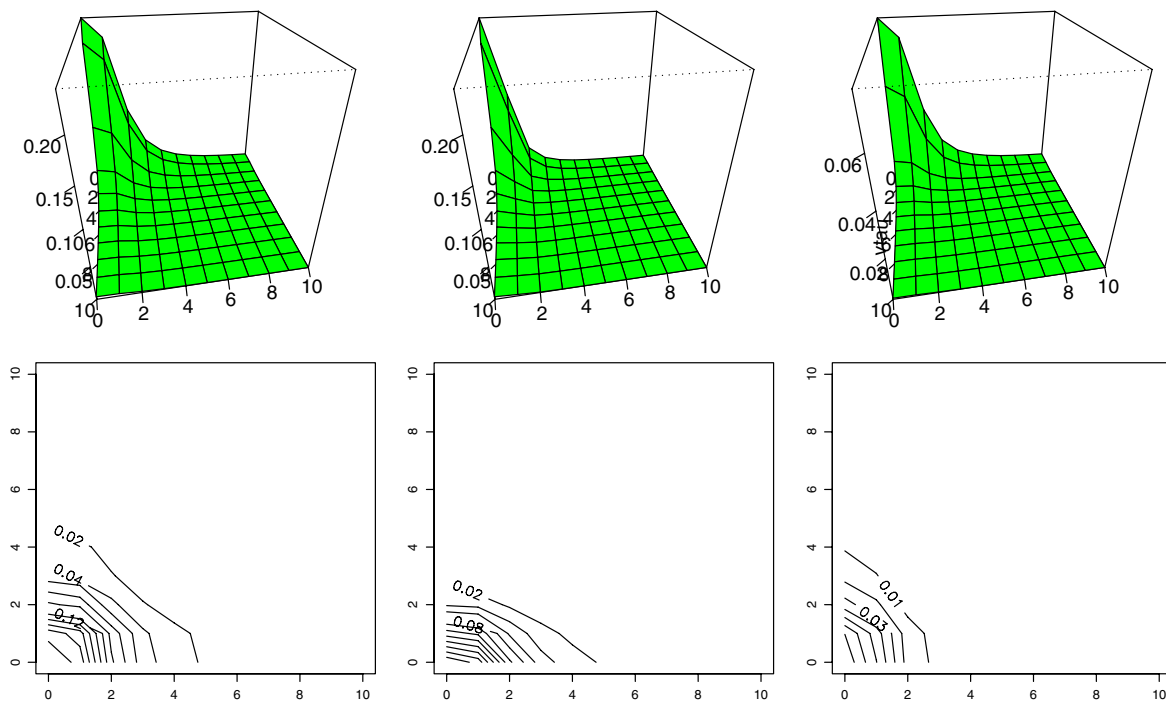


Fig. 7 Perspective and contour plots of the spectral density function given by Eq. 40 corresponding to Examples 1 and 2 in Sect. 3.1. Horizontal axis of the contour plots corresponds to $|\tau|$, and vertical axis to $\|\omega\|$. Common parameters are $\phi_1 = \phi_2 = 1, b_{11} = a_{21} = 0$.

The other parameters are: *first column:* $a_1 = 2, a_{11} = 1, a_2 = 2, b_{21} = 1, v_1 = v_2 = 1/60$; *second column:* $a_1 = 2, a_{11} = 3, a_2 = 2, b_{21} = 1, v_1 = 1/60, v_2 = 1/60$; *third column:* $a_1 = 2, a_{11} = 1, a_2 = 6, b_{21} = 7, v_1 = 1/60, v_2 = 1/90$

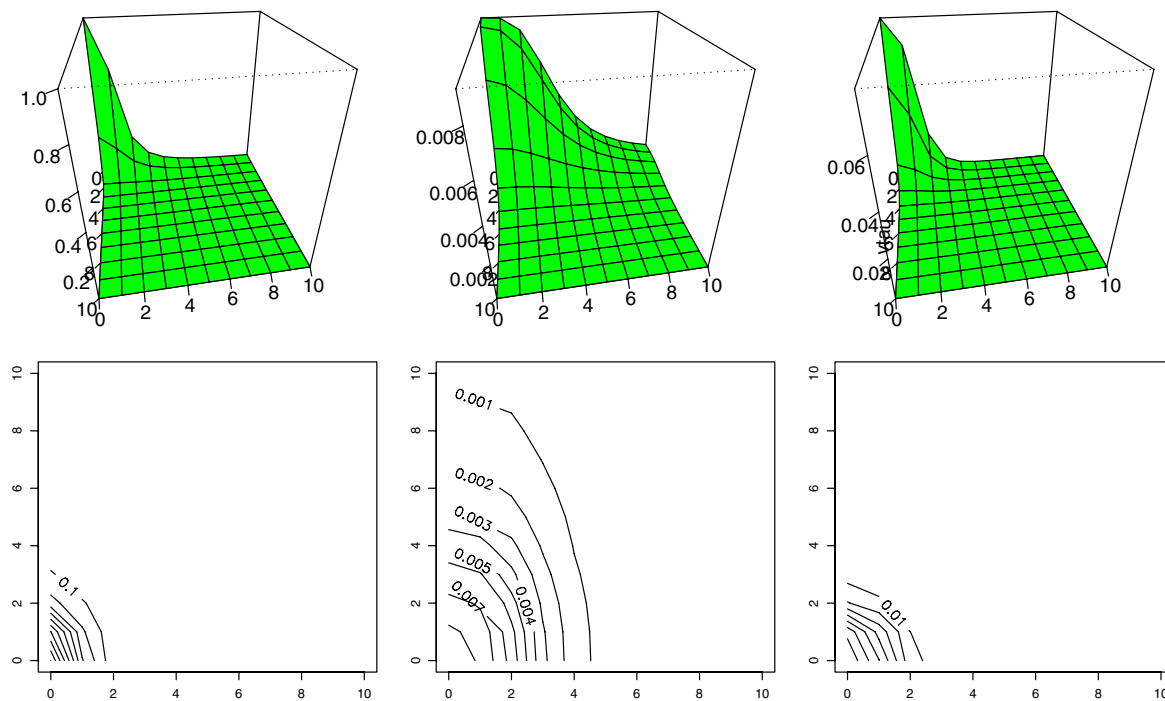


Fig. 8 Perspective and contour plots of the spectral density function given by Eq. 40 corresponding to Examples 1 and 2 in Sect. 3.1. Horizontal axis of the contour plots corresponds to $|\tau|$, and vertical axis to $\|\omega\|$. Common parameters are $\alpha = \beta = 2, \phi_1 = \phi_2 = 1$. The other parameters are: *first column:* $a_{11} = 1, b_{11} = 2, a_1 = 1, v_1 = 1/120, a_{21} = 0, b_{21} = 1, a_2 = 1, v_2 = 1/100$; *second column:* $a_{11} = 1, b_{11} = 2, a_1 = 10, v_1 = 1/120, a_{21} = 0, b_{21} = 1, a_2 = 10, v_2 = 1/100$; *third column:* $a_{11} = 1, b_{11} = 0.3, a_1 = 2, v_1 = 1/100, a_{21} = 2, b_{21} = 7, a_2 = 6, v_2 = 1/100$

120, $a_{21} = 0, b_{21} = 1, a_2 = 1, v_2 = 1/100$; *second column:* $a_{11} = 1, b_{11} = 2, a_1 = 10, v_1 = 1/120, a_{21} = 0, b_{21} = 1, a_2 = 10, v_2 = 1/100$; *third column:* $a_{11} = 1, b_{11} = 0.3, a_1 = 2, v_1 = 1/100, a_{21} = 2, b_{21} = 7, a_2 = 6, v_2 = 1/100$

plots, which could not be a desired feature for a general model. Here we propose, through Figures 6, 7 and 8, some further examples based on the tensorial product of two Matérn structures with $\alpha = \beta = 2$.

4 An application to wind speed data

In this section we consider a brief application in order to show some of the possible strategies that can be chosen for the implementation of a nonseparable structure.

Data refer to Irish wind speed and have already been used in other works, particularly in Haslett and Raftery (1989), Cressie and Huang (1999), Gneiting (2002) and Stein (2005).

The spatio-temporal data set includes 12 synoptic meteorological stations during the period 1961–1978. For every day of the year we have daily means for every station. Data are available at Statlib. As far as the data preliminary analysis is concerned, we followed Haslett and Raftery (1989), so we proceeded with a square root transformation of the original data in order to stabilize the variance and obtain marginal distributions which were approximately normal. Then, we deseasonalized the data calculating the average of the square roots of the daily means over all years and stations for each day of the year. Subtraction of that average to the square roots of the daily means yielded deseasonalized data. After considering the spatial correlation between stations, we omitted Rosslare station, as its correlation with the other stations

was considerably lower than correlation between the other ones. Note that the spatial correlation in this case was actually a conditional one, as we need to fix a certain temporal lag. Once the data was deseasonalized, we considered a subset of the original data, indeed we considered the eleven stations only during the year 1978. Nevertheless, the methodology we propose can be extended to the whole data set, although it can be highly time computing.

We considered a short-term autocorrelation structure for the temporal component, as the 11 stations exhibited a short-term temporal autocorrelation structure. So, we decided to consider a spatio-temporal structure with temporal lags less than or equal to 3 days. In fact, as it can be seen in Fig. 9, although the spatial correlation seemed to be almost decreasing with respect to the spatial distance, while increasing the temporal lag, we obtained an almost null covariance from the first spatial lags.

As far as estimation and fitting are concerned, we decided to follow two procedures in parallel. In the former, the following steps were included: (a) estimation of the temporal and spatial structures separately; (b) factorization of the obtained structures in order to have a separable spatio-temporal covariance; (c) estimation, via weighted nonlinear least squares (WNLS), of the interaction parameter. In the second procedure, we estimated the spatio-temporal structure globally, using the WNLS. For the sake of simplicity, from now on we call the first method *SEM*, as acronym for *Separate Estimation Method*, and the second one as *GEM*, as acronym for *Global*

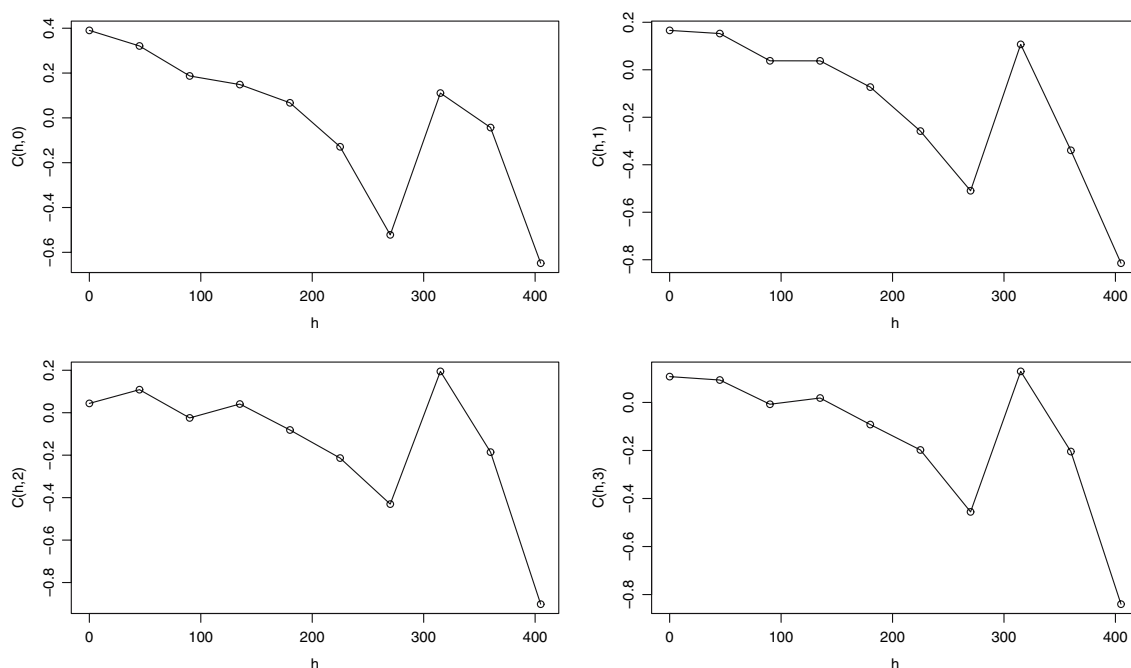


Fig. 9 Conditional spatial covariances. From *up-left* to *down-right*: temporal lags respectively equal to 0, 1, 2, 3

Estimation Method. The *SEM* procedure has been followed by Gneiting (2002), although in our opinion this procedure is not coherent with the classes he proposes. In fact, Gneiting’s classes are completely nonseparable, so it is not possible to start from two separate temporal and spatial structures and then estimate the spatio-temporal interaction parameter.

Let us begin with *SEM*. For the temporal structure we fitted the following model

$$C(u) = 0.03974613 + \frac{0.4}{(1 + 0.0945329|u|)^{0.0000001}} \times \exp(-0.0945329|u|), \tag{43}$$

where the constant 0.4 is the estimated variance for the year 1978. For the spatial structure, we obtained

$$C(\mathbf{h}) = 0.03974613 + \frac{0.4}{(1 + 0.00325\|\mathbf{h}\|^{0.596})^{0.00001}} \times \exp\left(-0.003251164\|\mathbf{h}\|^{2 \times 0.596}\right). \tag{44}$$

Note that, if we consider the class in Eq. 26, the marginal structures can be easily calculated

$$C(\mathbf{0}, u) \propto \frac{\sigma^2}{(1 + a_1|u|^{\alpha_1})^\varepsilon} \exp(-a_1|u|), \tag{45}$$

$$C(\mathbf{h}, 0) \propto \frac{\sigma^2}{(1 + a_2\|\mathbf{h}\|^{\alpha_2})^\varepsilon} \exp(-a_2\|\mathbf{h}\|^{2\alpha_2}). \tag{46}$$

Estimations obtained in Eqs. 43 and 44 can be, respectively, considered as particular cases of Eqs. 45 and 46. It is important to observe that the estimated smoothing parameter is equal in both marginal structures.

The tensorial product of the marginal structures offers the following separable structure

$$C(\mathbf{h}, u) \propto \frac{\sigma^4}{B_{u,\mathbf{h}}^\varepsilon} \exp\left(-a_1|u| - a_2\|\mathbf{h}\|^{2\alpha_2}\right), \tag{47}$$

which is a particular case of Eq. 26 setting the interaction parameter β identically equal to zero.

Thus, we followed the same procedure for the estimated structures, obtaining a separable one which can be considered a particular case of Eq. 26 with no interaction. A final step regarded the estimation of the interaction parameter β . For this estimation, we needed to calculate the empirical spatio-temporal covariances. As we could consider data to be weakly stationary, we used an extension of the Hawkins and Cressie (1984) robust variogram estimator to the empirical spatio-temporal covariance. Indeed, we defined the following expression

Table 1 Fitted values of the parameters for the two analyzed spatio-temporal models: *SEM* and *GEM*

Parameters	<i>SEM</i>	<i>GEM</i>
a_1	0.09453	0.13258
a_2	0.00325	0.00265
α_1	1.00000	0.49905
α_2	0.59628	0.59840
β	0.0000001	0.09227
ε	0.0000001	0.0000001
C_0	0.03974	0.03616

$$\hat{C}(\mathbf{h}, u) = \hat{\sigma}^2 - \left(\frac{1}{|N_u(\mathbf{h})|} \sum_{|N_u(\mathbf{h})|} |\hat{\delta}_{t_i}(\mathbf{s}_i) - \hat{\delta}_{t_j}(\mathbf{s}_j)|^{1/2} \right)^4 / \left(0.457 + \frac{0.494}{|N_u(\mathbf{h})|} \right), \tag{48}$$

where $\hat{\sigma}^2$ is the variance of the process calculated from data. The problem of estimation of the interaction parameter could be solved with a WNLS algorithm, which gave an interaction parameter β equal to $1 \times e^{-6}$. Thus, the *SEM*

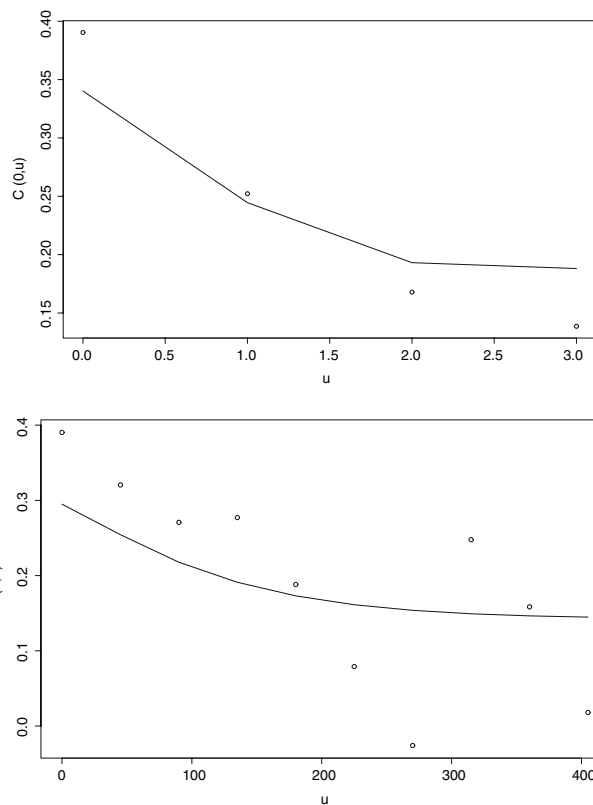
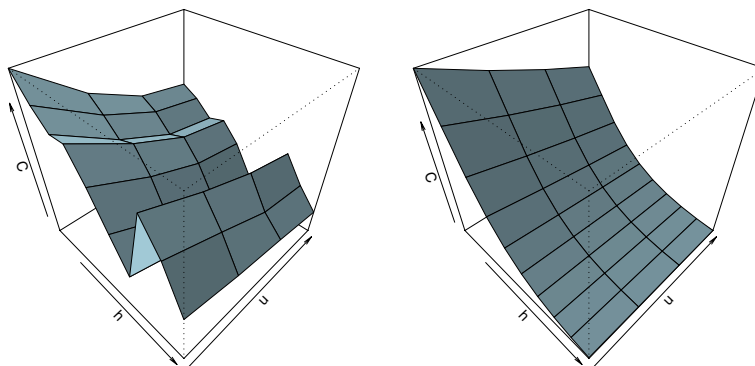


Fig. 10 Purely temporal (left) and spatial (right) covariance estimation (points) and fitted model (solid line) for the *SEM*

Fig. 11 Estimation via *GEM*. From left to right: estimated and fitted surfaces



method showed no interaction between space and time. We believe that this no-interaction model was partially due to the fact that we used a subset of data, while detrendization and deseasonalization were performed with respect to the whole data set. This fact could have caused an oversmoothing of the data.

As the product of the estimated marginal structures gave a spatio-temporal form which was compatible with the class 26, another possibility was to estimate globally all the parameters of the spatio-temporal structure with a unique algorithm, in order to see if in some ways the hypothesis of no interaction between space and time could be accepted. Thus, we performed the *GEM* method.

As far as estimation of the empirical covariance was concerned, we used formula 48 as in the previous procedure. We implemented the WNLS algorithm with respect to the parametric class in Eq. 26. Finally, we got an estimation of the seven parameters which is summarized, together with the results regarding the *SEM* procedure, in Table 1.

As we can see from Table 1, both *SEM* and *GEM* still provided a non-smoothed model, as ε was practically zero. The nugget effect was almost 1/10 of the total variance, and the local spatial smoothing parameters were almost equal. As far as differences were concerned, the *SEM* local temporal smoothing parameter α_1 was twice the corresponding parameter calculated with *GEM*. The most important difference was given by the fact that *GEM* provided a non-null interaction parameter β , while *SEM* highlighted a non-interaction model.

In Fig. 10 the temporal and spatial covariance estimation and corresponding fitted functions are shown for the *SEM* model. Although fitting was quite good, it was highlighted a poor fitting of the short range dependence, as we could see in the first lags. Estimation and fitting for the *GEM* method are shown in Fig. 11. In this case, the fitting seemed quite good as far as the microscale component was concerned.

In order to further compare the two methods, we performed a cross-validation procedure and calculated the CRV_3 quantity (Huang and Cressie 1996) as a measure of goodness of prediction, resulting in 0.752 for *SEM* and

0.602 for *GEM*. So, cross-validation selected *GEM* as a better spatio-temporal model based on a nonseparable structure. Thus, it seems that the *SEM*, which corresponds to the one used in Gneiting (2002), could hide the interaction between the merely spatial and temporal structures, as highlighted by the use of our *GEM*. This confirms our initial feeling at the beginning of our analysis, in which a model taking into account the interaction between the marginal structures should outperform other types of models [as those used by Gneiting (2002) and Stein (2005)]. In this sense, our strategy by using the *GEM* model provides a more refined solution to this problem.

5 Conclusions and discussion

In this paper we have presented some new classes of covariance functions which can be implemented with simple procedures. The former proposal regards covariance functions obtained as sum-product of nonseparable space-time covariances, to which a merely spatial and/or temporal component can be added in order to modify the level of smoothness of the associated process. Observe that with this procedure we can obtain closed forms which are very flexible and easy-to-build. On the other hand, we cannot determinate analytically the level of differentiability associated to these new covariances. This drawback is covered by the second proposal, in which we build spectral densities whose Fourier pair, following Stein (2005), is infinitely differentiable away from the origin.

One of the covariance functions presented is used to model the nonseparable spatio-temporal structure of the Irish wind speed data. The analysis highlights that preferably the parameters characterizing the spatial and temporal dependence has to be estimated jointly, in order to avoid the loss of information about the interaction between space and time.

Acknowledgments The Editor and referees are acknowledged with thanks. Their precise comments and suggestions have clearly improved an earlier version of the manuscript.

References

- Christakos G (2000) Modern spatiotemporal geostatistics. Oxford University Press, New York
- Christakos G (2002) On a deductive logic-based spatiotemporal random field theory. *Probab Theory Math Stat* 66:54–65
- Cressie N, Huang HC (1999) Classes of nonseparable, spatiotemporal stationary covariance functions. *J Am Stat Assoc* 94:1330–1340
- De Cesare L, Myers D, Posa D (2001) Product-sum covariance for space-time modeling: an environmental application. *Environmetrics* 12:11–23
- Dimitrakopoulos R, Lou X (1994) Spatiotemporal modeling: covariances and ordinary kriging system. In: Dimitrakopoulos R (ed) *Geostatistics for the next century*. Kluwer, Dordrecht, pp 88–93
- Egbert GD, Lettenmaier DP (1986) Stochastic modeling of the space-time structure of atmospheric chemical deposition. *Water Resour Res* 22:165–179
- Fernández-Casal R (2003) *Geostatística Espacio-Temporal. Modelos Flexibles de Variogramas Anisotropicos no Separables*. Doctoral Thesis, Santiago de Compostela
- Fuentes M (2002) Spectral methods for nonstationary spatial processes. *Biometrika* 89:197–210
- Fuentes M, Smith R (2001) A new class of nonstationary spatial models. Research report of the Statistics Department at North Carolina State University
- Gneiting T (1997) Normal scale mixtures and dual probability densities. *J Stat Comput Simul* 9:375–384
- Gneiting T (2002) Stationary covariance functions for space-time data. *J Am Stat Assoc* 97:590–600
- Guttorp P, Sampson PD, Newman K (1992) Non-parametric estimation of spatial covariance with application to monitoring network evaluation. In: *Statistics in the environmental and earth sciences*. Edward Arnold, London
- Haslett J, Raftery AE (1989) Space-time modelling with long-memory dependence: assessing ireland's wind power resource. *Appl Stat* 38:1–50
- Hawkins DM, Cressie N (1984) Robust kriging—a proposal. *Math Geol* 16:3–18
- Huang HC, Cressie N (1996) Spatio-temporal prediction of snow water equivalent using the Kalman filter. *Comput Stat Data Anal* 22:159–175
- Jones RH, Zhang Y (1997) Models for continuous stationary space-time processes. In: Gregoire TG, Brillinger DR, Diggle PJ, Russek-Cohen E, Warren WG, Wolfinger RD (eds) *Modelling longitudinal and spatially correlated data*. Springer, New York, pp 289–29
- Kolovos A, Christakos G, Hristopoulos DT, Serre ML (2004) Methods for generating non-separable spatiotemporal covariance models with potential environmental applications. *Adv Water Resour* 27:815–830
- Ma C (2003) Families of spatio-temporal stationary covariance models. *J Stat Plan Inference* 116:489–501
- Rohuani S, Hall TJ (1989) Space-time kriging of groundwater data. In: Armstrong M (ed) *Geostatistics*. Kluwer, Dordrecht, vol 2, pp 639–651
- Sampson PD, Guttorp P (1992) Nonparametric estimation of nonstationary spatial covariance structures. *J Am Stat Assoc* 87:108–119
- Solow AR, Gorelik SM (1986) Estimating missing streamflow values by cokriging. *Math Geol* 18:785–809
- Stein M (2005) Space time covariance functions. *J Am Stat Assoc* 100:310–321