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Green's function for two-and-a-half dimensional elastodynamic problems in a half-space

A. Tadeu, J. António, L. Godinho

Abstract This paper presents an analytical solution, together with explicit expressions, for the steady state response of a homogeneous three-dimensional half-space subjected to a spatially sinusoidal, harmonic line load. These equations are of great importance in the formulation of three-dimensional elastodynamic problems in a half-space by means of integral transform methods and/or boundary elements. The final expressions are validated here by comparing the results with those obtained with the boundary element method (BEM) solution, for which the free surface of the ground is discretized with boundary elements.

Introduction

The aim of this study is to provide Green's functions for calculating the wavefield radiated by a spatially sinusoidal, harmonic line load buried in a half-space. These functions, or fundamental solutions, relate the field variables (stresses or displacements) at some location in the half-space domain caused by a dynamic source placed elsewhere in the medium.

The expressions are developed using the displacement potentials defined by the methodology used by the authors (Tadeu and Kausel, 2000) to evaluate the Green's functions for a harmonic (steady state) line load whose amplitude varies sinusoidally in the third dimension. These displacement potentials are written as a superposition of plane waves following the approach used first by Lamb (1904) for the two-dimensional case and then by Bouchon (1979) and Kim et al. (1993) to compute the three-dimensional field using a discrete wave number representation. The Green's functions for the half-space are then derived, assuming free stress boundary conditions at the surface. The Green's function for a half-space is first written as the sum of the Green's function for a full-space with surface terms, using a technique similar to that described by Kawase (1988). Then the surface term is decomposed into two parts, one of which corresponds to the image-source solution, following a methodology closely related to the work of Kawase and Aki (1989). The expressions presented here make it possible to compute the wavefield inside a half space, without a full discretization of the interior domain, by means of numerical techniques such as the finite differences, or even by discretizing the free surface using boundary elements techniques.

The authors believe that the fundamental solution presented here can be of great value in the formulation of 3D elastodynamic problems, via boundary elements together with integral transforms. The fundamental solutions are expressed in an explicit form, and represent the Green's functions for a harmonic (steady state) line load buried in a half-space whose amplitude varies sinusoidally in the third dimension. Such problems are referred to in the literature as 21/2D problems. The present equations are very important in themselves because not only can they relate the displacements at some point produced by a point load somewhere in the three dimensional space, but they also can be incorporated into a numerical boundary element approach in order to avoid the full discretization of the free surface of the half-space. Notice that this full discretization is only possible using different simplified approaches, such as the use of damping, because it will lead to an system of equations which, due to its size, will be unsolvable.

Fundamental solution

Consider an infinite, homogeneous space subjected at the origin of coordinates to a spatially varying line load in the z direction of the form $p(x,y,z,t) = \delta(x)\delta(y)e^{i(\omega t - k_z z)}$ and acting in one of the three coordinate directions. In this expression, $\delta(x)$ and $\delta(y)$ are Dirac-delta functions, ω is the frequency of the load and k_z is the wavenumber in z (see Fig. 1a). The response to this load can be obtained by applying a spatial Fourier transform in the z direction to the Helmholtz equations for a point load (see e.g. Gradshteyn and Ryzhik, p. 1151, Eq. 17.34.4). The z transformed equations are then

$$\left(\frac{\partial^2 \hat{A}_p}{\partial x^2} + \frac{\partial^2 \hat{A}_p}{\partial y^2} + k_\alpha^2 \hat{A}_p\right) = \frac{-iH_0^{(2)}(-ik_z r)}{4\rho\alpha^2}$$
$$\left(\frac{\partial^2 \hat{A}_s}{\partial x^2} + \frac{\partial^2 \hat{A}_s}{\partial y^2} + k_\beta^2 \hat{A}_s\right) = \frac{-iH_0^{(2)}(-ik_z r)}{4\rho\beta^2} \qquad (1)$$

where $k_\alpha = \sqrt{k_p^2 - k_z^2}$ with $(\mathrm{Imag}(k_\alpha) \le 0)$ and $k_p = \omega/\alpha$, $k_\beta = \sqrt{k_s^2 - k_z^2}$ with $(\mathrm{Imag}(k_\beta) \le 0)$ and $k_s = \omega/\beta$, $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ and $\beta = \sqrt{\mu/\rho}$ are the velocities for P (pressure) waves and S (shear) waves, respectively, λ and μ are the Lamé constants, ρ is the mass density, $\hat{A}_p(x,y,k_z,\omega)$ and $\hat{A}_s(x,y,k_z,\omega)$ are the Fourier transforms of the two potentials $A_p(x,y,z,\omega)$ and $A_s(x,y,z,\omega)$

A. Tadeu (✉), J. António, L. Godinho
University of Coimbra, Department of Civil Engineering, 3030-290 Coimbra, Portugal

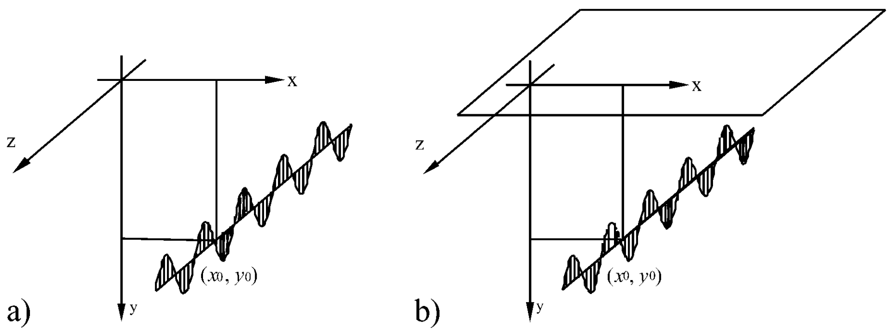

Fig. 1a, b. Geometry of the problem: **a** full-space; **b** half-space

for the irrotational and equivoluminal parts of the displacement vector, $H_n^{(2)}(\ )$ are Hankel functions of the second kind and n-th order, $r = \sqrt{x^2 + y^2}$ and $\mathrm{i} = \sqrt{-1}$. From equilibrium conditions we find $\hat{A}_{\mathrm{p}}$ and $\hat{A}_{\mathrm{s}}$,

$$\begin{aligned} \hat{A}_{\mathrm{p}} &= \frac{\mathrm{i}}{4\rho\omega^2}\left[H_0^{(2)}(k_\alpha r) - H_0^{(2)}(-\mathrm{i}k_z r)\right] \\ \hat{A}_{\mathrm{s}} &= \frac{\mathrm{i}}{4\rho\omega^2}\left[H_0^{(2)}(k_\beta r) - H_0^{(2)}(-\mathrm{i}k_z r)\right] \end{aligned} \tag{2}$$

The displacements G_{ij} in direction i due to a load applied in direction j can now be obtained from the relation

$$G_{ij} = \frac{\partial^2\left(\hat{A}_{\mathrm{p}} - \hat{A}_{\mathrm{s}}\right)}{\partial x_i \partial x_j} + \delta_{ij}\hat{\nabla}^2\hat{A}_{\mathrm{s}} \tag{3}$$

in which δ_{ij} is the Kronecker delta, $x_j = x, y, z$ for $j = 1, 2, 3$, and $\partial/\partial z = -\mathrm{i}k_z$. We may observe that

$$\begin{aligned} \hat{A}_{\mathrm{p}} - \hat{A}_{\mathrm{s}} &= \frac{1}{4\mathrm{i}\rho\omega^2}\left[H_0^{(2)}(k_\beta r) - H_0^{(2)}(k_\alpha r)\right] \\ \hat{\nabla}^2\hat{A}_{\mathrm{s}} &= \frac{1}{4\mathrm{i}\rho\beta^2}H_0^{(2)}(k_\beta r) \end{aligned} \tag{4}$$

A full set of Green's functions for the strains and stresses, are given in Tadeu and Kausel (2000), and these fully agree with a solution for moving loads given earlier by Pedersen et al. (1994) and Papageorgiou et al. (1998).

These same equations can be represented as a continuous superposition of homogeneous and inhomogeneous plane waves.

Load acting in the direction of the x-axis

The displacement potentials resulting from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the x direction, are then given by the expressions,

$$\begin{aligned} \phi^x &= \frac{1}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(\frac{k}{\nu}\mathrm{e}^{-\mathrm{i}\nu|y-y_0|}\right)\mathrm{e}^{-\mathrm{i}k(x-x_0)}\mathrm{d}k \\ \psi_x^x &= 0 \\ \psi_y^x &= \frac{\mathrm{i}(-\mathrm{i}k_z)}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(\frac{\mathrm{e}^{-\mathrm{i}\gamma|y-y_0|}}{\gamma}\right)\mathrm{e}^{-\mathrm{i}k}(x-x_0)\mathrm{d}k \\ \psi_z^x &= \frac{-\mathrm{sgn}(y-y_0)}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{-\mathrm{i}\gamma|y-y_0|}\right)\mathrm{e}^{-\mathrm{i}k(x-x_0)}\mathrm{d}k \end{aligned} \tag{5}$$

where $\nu = \sqrt{k_{\mathrm{p}}^2 - k_z^2 - k^2}$ with $(\mathrm{Imag}(\nu) \leq 0)$, $\gamma = \sqrt{k_{\mathrm{s}}^2 - k_z^2 - k^2}$ with $(\mathrm{Imag}(\gamma) \leq 0)$, and the integration is with respect to the horizontal wave number, k, along the x direction.

In order to transform the integral into a summation, consider an infinite number of such sources distributed along the x direction, at equal intervals L_x. The above compressional and rotational potentials can then be written as

$$\begin{aligned} \phi^x &= E_a\sum_{n=-\infty}^{n=+\infty}\left(\frac{k_n}{\nu_n}E_b\right)E_d \\ \psi_x^x &= 0 \\ \psi_y^x &= E_a k_z\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_c}{\gamma_n}\right)E_d \\ \psi_z^x &= -\mathrm{sgn}(y-y_0)E_a\sum_{n=-\infty}^{n=+\infty}(E_c)E_d \end{aligned} \tag{6}$$

where $E_a = 1/(2\rho\omega^2 L_x)$, $E_b = \mathrm{e}^{-\mathrm{i}\nu_n|y-y_0|}$, $E_c = \mathrm{e}^{-\mathrm{i}\gamma_n|y-y_0|}$, $E_d = \mathrm{e}^{-\mathrm{i}k_n(x-x_0)}$.
$\nu_n = \sqrt{k_{\mathrm{p}}^2 - k_z^2 - k_n^2}$ with $(\mathrm{Imag}(\nu_n) \leq 0)$, $\gamma_n = \sqrt{k_{\mathrm{s}}^2 - k_z^2 - k_n^2}$ with $(\mathrm{Imag}(\gamma_n) \leq 0)$, $k_n = \frac{2\pi}{L_x}n$ which can in turn be approximated by a finite sum of equations, N.

The Green's functions can be expressed in terms of the compressional and rotational potentials, ϕ^x, ψ_x^x, ψ_y^x and ψ_z^x, from which the following three components of displacement can be obtained,

$$\begin{aligned} G_{xx}^{\mathrm{full}} &= E_a\sum_{n=-N}^{n=+N}\left[\frac{-\mathrm{i}k_n^2}{\nu_n}E_b + \left(-\mathrm{i}\gamma_n - \frac{\mathrm{i}k_z^2}{\gamma_n}\right)E_c\right]E_d \\ G_{yx}^{\mathrm{full}} &= E_a\sum_{n=-N}^{n=+N}[-\mathrm{i}\,\mathrm{sgn}(y-y_0)k_nE_b \\ &\qquad + \mathrm{i}\,\mathrm{sgn}(y-y_0)k_nE_c]E_d \\ G_{zx}^{\mathrm{full}} &= E_a\sum_{n=-N}^{n=+N}\left[\frac{-\mathrm{i}k_zk_n}{\nu_n}E_b + \frac{\mathrm{i}k_zk_n}{\gamma_n}E_c\right]E_d \end{aligned} \tag{7}$$

The Green's functions for a half-space can be expressed as the sum of the source terms which are the same as those in the full-space and the surface terms which are necessary to satisfy the free-surface conditions (see Fig. 1b). These surface terms can be expressed in a form similar to that of the source term, namely

$$\phi_0^x = E_a \sum_{n=-\infty}^{n=+\infty} \left(\frac{k_n}{v_n} E_{b0} A_n^x \right) E_d$$
$$\psi_{x0}^x = 0$$
$$\psi_{y0}^x = E_a k_z \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{c0}}{\gamma_n} B_n^x \right) E_d \tag{8}$$
$$\psi_{z0}^x = -E_a \sum_{n=-\infty}^{n=+\infty} \left(E_{c0} C_n^x \right) E_d$$

where $E_{b0} = e^{-iv_n y}, E_{c0} = e^{-i\gamma_n y}$, A_n^x, B_n^x, and C_n^x, are as yet unknown coefficients to be determined from the appropriate boundary conditions, so that the field produced simultaneously by the source and surface terms should produce $\sigma_{yx} = 0$, $\sigma_{yz} = 0$ and $\sigma_{yy} = 0$ at $y = 0$.

The imposition of the three stated boundary conditions for each value of n leads to a system of three equations in the three unknown constants. While this procedure is straightforward, the detailed derivations are lengthy, and are therefore omitted. Only the final system of equations is presented,

$$\begin{bmatrix} -2k_n^2 & -k_z^2 & k_n^2 - \gamma_n^2 \\ -2 & 1 & 1 \\ \frac{-k_s^2}{v_n} - \frac{2v_{zn}^2}{v_n} & 0 & 2\gamma_n \end{bmatrix} \begin{bmatrix} A_n^x \\ B_n^x \\ C_n^x \end{bmatrix} = \begin{bmatrix} -2k_n^2 E_{b1} + \left(-k_s^2 + 2k_n^2\right) E_{c1} \\ -2E_{b1} + 2E_{c1} \\ \left(\frac{k_s^2}{v_n} + \frac{2v_{zn}^2}{v_n} \right) E_{b1} - 2\gamma_n E_{c1} \end{bmatrix} \tag{9}$$

where $v_{zn} = \sqrt{-k_z^2 - k_n^2}$, $E_{b1} = e^{-iv_n y_0}$, $E_{c1} = e^{-i\gamma_n y_0}$.

Having obtained the constants, we may compute the motions associated with the surface terms by means of the equations relating potentials and displacements. In essence, this requires the consideration of Eq. 8 and the application of partial derivatives to go from potentials to displacements. The Green's functions for a half-space are then given by the sum of the source terms and these surface terms. After carrying out this procedure, one obtains expressions for the half-space of the form:

$$G_{xx}^{\text{half}} = G_{xx}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[A_n^x \frac{-ik_n^2}{v_n} E_{b0} + \left(-i\gamma_n C_n^x - \frac{ik_z^2}{\gamma_n} B_n^x \right) E_{c0} \right] E_d$$
$$G_{yx}^{\text{half}} = G_{yx}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-ik_n A_n^x E_{b0} + ik_n C_n^x E_{c0} \right] E_d$$
$$G_{zx}^{\text{half}} = G_{zx}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-ik_z k_n}{v_n} A_n^x E_{b0} + \frac{ik_z k_n}{\gamma_n} B_n^x E_{c0} \right] E_d \tag{10}$$

The expressions for the Green's function for two-and-a-half dimensional full space G_{xx}^{full}, G_{yx}^{full} and G_{zx}^{full} can be defined in explicit form, as listed in the Appendix (Tadeu and Kausel, 2000). Additionally, the surface term can be separated into two parts, one of which corresponds to the image source solution. This image source part, $G_{x0x0}^{\text{full}}, G_{y0x0}^{\text{full}}$ and G_{z0x0}^{full}, can be calculated again in closed form (see Appendix).

$$G_{xx}^{\text{half}} = G_{xx}^{\text{full}} + G_{x0x0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\left(A_n^x - E_{b1} \right) \frac{-ik_n^2}{v_n} E_{b0} + \left(-i\gamma_n \left(C_n^x - E_{c1} \right) - \frac{ik_z^2}{\gamma_n} \left(B_n^x - E_{c1} \right) \right) E_{c0} \right] E_d$$
$$G_{yx}^{\text{half}} = G_{yx}^{\text{full}} + G_{y0x0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-ik_n \left(A_n^x - E_{b1} \right) E_{b0} + ik_n \left(C_n^x - E_{c1} \right) E_{c0} \right] E_d$$
$$G_{zx}^{\text{half}} = G_{zx}^{\text{full}} + G_{z0x0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-ik_z k_n}{v_n} \left(A_n^x - E_{b1} \right) E_{b0} + \frac{ik_z k_n}{\gamma_n} \left(B_n^x - E_{c1} \right) E_{c0} \right] E_d \tag{11}$$

Expressions for the strains and stresses may be obtained from G_{ij} by means of the well-known equations relating strains and displacements.

The corresponding expressions for forces applied along the y and z directions can be obtained in a similar manner. The derivation of these solutions is presented in the following sections, in a condensed form.

Load acting in the direction of the *y*-axis

The discrete form of the displacement potentials, resulting from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the y direction, are given by the expressions,

$$\phi^y = E_a \sum_{n=-N}^{n=+N} \left[\text{sgn}(y - y_0) E_b \right] E_d$$
$$\psi_x^y = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E_c}{\gamma_n} \right) E_d \tag{12}$$
$$\psi_y^y = 0$$
$$\psi_z^y = E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E_c \right) E_d$$

The Green's functions for a two-and-a-half dimensional full space are thus,

$$G_{xy}^{\text{full}} = G_{yx}^{\text{full}} = E_a \sum_{n=-N}^{n=+N} \left[-i\,\text{sgn}(y - y_0) k_n E_b + i\,\text{sgn}(y - y_0) k_n E_c \right] E_d$$

$$G_{yy}^{\text{full}} = E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}v_n E_b + \left(\frac{\mathrm{i}v_{zn}^2}{\gamma_n} \right) E_c \right] E_d$$

$$G_{zy}^{\text{full}} = E_a \sum_{n=-N}^{n=+N} [-\mathrm{i}\,\mathrm{sgn}(y-y_0) k_z E_b + \mathrm{i}\,\mathrm{sgn}(y-y_0) k_z E_c] E_d \tag{13}$$

These surface terms can be expressed in the form,

$$\begin{aligned}
\phi_0^y &= E_a \sum_{n=-N}^{n=+N} \left[E_{b0} A_n^y \right] E_d \\
\psi_{x0}^y &= E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E_{c0}}{\gamma_n} C_n^y \right) E_d \\
\psi_{y0}^y &= 0 \\
\psi_{z0}^y &= E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E_{c0} B_n^y \right) E_d
\end{aligned} \tag{14}$$

The imposition of the three stated boundary conditions ($\sigma_{yx} = 0, \sigma_{yz} = 0$ and $\sigma_{yy} = 0$ at $y = 0$) for each value of n leads to a system of three equations,

$$\begin{bmatrix} -2v_n & \frac{-k_n^2}{\gamma_n} + \gamma_n & \frac{-k_z^2}{\gamma_n} \\ -2v_n & \frac{-k_n^2}{\gamma_n} & \frac{-k_z^2}{\gamma_n} + \gamma_n \\ (-k_s^2 - 2v_{zn}^2) & -2k_n^2 & -2k_z^2 \end{bmatrix} \begin{bmatrix} A_n^y \\ B_n^y \\ C_n^y \end{bmatrix} = \begin{bmatrix} 2v_n E_{b1} - \left(\frac{v_{zn}^2}{\gamma_n} + \gamma_n \right) E_{c1} \\ 2v_n E_{b1} - \left(\frac{v_{zn}^2}{\gamma_n} + \gamma_n \right) E_{c1} \\ (-k_s^2 - 2v_{zn}^2) E_{b1} + 2v_{zn}^2 E_{c1} \end{bmatrix} \tag{15}$$

Having obtained the amplitude of each potential, the Green's functions for a half-space are then given by the sum of the source terms and these surface terms, giving the following expressions,

$$\begin{aligned}
G_{xy}^{\text{half}} &= G_{xy}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}A_n^y k_n E_{b0} + \mathrm{i}B_n^y k_n E_{c0} \right] E_d \\
G_{yy}^{\text{half}} &= G_{yy}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}v_n A_n^y E_{b0} + \left(\frac{-\mathrm{i}k_n^2}{\gamma_n} B_n^y + \frac{-\mathrm{i}k_z^2}{\gamma_n} C_n^y \right) E_{c0} \right] E_d \\
G_{zy}^{\text{half}} &= G_{zy}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}A_n^y k_z E_{b0} + \mathrm{i}C_n^y k_z E_{c0} \right] E_d
\end{aligned} \tag{16}$$

The surface term can be separated into two parts, one of which corresponds to the image source solution.

$$\begin{aligned}
G_{xy}^{\text{half}} &= G_{xy}^{\text{full}} + G_{x0y0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}\left(A_n^y - E_{b1}\right) k_n E_{b0} + \mathrm{i}\left(B_n^y - E_{c1}\right) k_n E_{c0} \right] E_d \\
G_{yy}^{\text{half}} &= G_{yy}^{\text{full}} + G_{y0y0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}v_n \left(A_n^y - E_{b1}\right) E_{b0} + \left(\frac{-\mathrm{i}k_n^2}{\gamma_n} \left(B_n^y - E_{c1}\right) + \frac{-\mathrm{i}k_z^2}{\gamma_n} \left(C_n^y - E_{c1}\right) \right) E_{c0} \right] E_d \\
G_{zy}^{\text{half}} &= G_{zy}^{\text{full}} + G_{z0y0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}\left(A_n^y - E_{b1}\right) k_z E_{b0} + \mathrm{i}\left(C_n^y - E_{c1}\right) k_z E_{c0} \right] E_d
\end{aligned} \tag{17}$$

These expressions G_{xy}^{full}, G_{yy}^{full}, G_{zy}^{full}, G_{x0y0}^{full}, G_{y0y0}^{full} and G_{z0y0}^{full}, can be used in explicit form as shown in the Appendix. Again, expressions for the strains and stresses may be derived from G_{ij} by means of the well-known equations relating strains and displacements.

Load acting in the direction of the *z*-axis

The discrete form of the displacement potentials, resulting from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the z direction, are given by the expressions,

$$\begin{aligned}
\phi^z &= E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_b}{v_n} \right) E_d \\
\psi_x^z &= E_a \sum_{n=-N}^{n=+N} (\mathrm{sgn}(y-y_0) E_c) E_d \\
\psi_y^z &= E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_c \right) E_d \\
\psi_z^z &= 0
\end{aligned} \tag{18}$$

The Green's functions for two-and-a-half dimensional full space are then,

$$\begin{aligned}
G_{xz}^{\text{full}} = G_{zx}^{\text{full}} &= E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_z k_n}{v_n} E_b + \frac{\mathrm{i}k_z k_n}{\gamma_n} E_c \right] E_d \\
G_{yz}^{\text{full}} = G_{zy}^{\text{full}} &= E_a \sum_{n=-N}^{n=+N} [-\mathrm{i}\,\mathrm{sgn}(y-y_0) k_z E_b + \mathrm{i}\,\mathrm{sgn}(y-y_0) k_z E_c] E_d \\
G_{zz}^{\text{full}} &= E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_z^2}{v_n} E_b + \left(\frac{-\mathrm{i}k_n^2}{\gamma_n} - \mathrm{i}\gamma_n \right) E_c \right] E_d
\end{aligned} \tag{19}$$

These surface terms can be expressed in the form,

$$\begin{aligned}
\phi_0^z &= E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_{b0}}{v_n} A_n^z \right) E_d \\
\psi_{x0}^z &= E_a \sum_{n=-N}^{n=+N} \left(E_{c0} B_n^z \right) E_d \\
\psi_{y0}^z &= E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_{c0} C_n^z \right) E_d \\
\psi_{z0}^z &= 0
\end{aligned} \tag{20}$$

The imposition of the boundary conditions ($\sigma_{yx}=0, \sigma_{yz}=0$ and $\sigma_{yy}=0$ at $y=0$) for each value of n leads to the following system of three equations,

$$\begin{bmatrix} -2 & 1 & 1 \\ -2k_z^2 & k_z^2-\gamma_n^2 & -k_n^2 \\ -\left(\frac{k_s^2}{v_n}+\frac{2v_{zn}^2}{v_n}\right) & 2\gamma_n & 0 \end{bmatrix} \begin{bmatrix} A_n^z \\ B_n^z \\ C_n^z \end{bmatrix} = \begin{bmatrix} 2(-E_{b1}+E_{c1}) \\ -2k_z^2E_{b1}+\left(k_z^2-\gamma_n^2-k_n^2\right)E_{c1} \\ \left(\frac{k_s^2}{v_n}+\frac{2v_{zn}^2}{v_n}\right)E_{b1}-2\gamma_nE_{c1} \end{bmatrix} \tag{21}$$

Having obtained the amplitude of each potential, the Green's functions for a half-space are then given by the sum of the source terms and these surface terms, giving the following expressions,

$$\begin{aligned} G_{xz}^{\text{half}} &= G_{xz}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_zk_n}{v_n} A_n^z E_{b0} + \frac{\mathrm{i}k_zk_n}{\gamma_n} C_n^z E_{c0} \right] E_d \\ G_{yz}^{\text{half}} &= G_{yz}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}k_z A_n^z E_{b0} + \mathrm{i}B_n^z k_z E_{c0} \right] E_d \\ G_{zz}^{\text{half}} &= G_{zz}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_z^2}{v_n} A_n^z E_{b0} \right. \\ &\qquad \left. + \left(\frac{-\mathrm{i}k_n^2}{\gamma_n} C_n^z - \mathrm{i}\gamma_n B_n^z \right) E_{c0} \right] E_d \end{aligned} \tag{22}$$

The surface term can be separated into two parts, one of which corresponds to the image source solution.

$$\begin{aligned} G_{xz}^{\text{half}} &= G_{xz}^{\text{full}} + G_{x0z0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_zk_n}{v_n} \left(A_n^z - E_{b1}\right) E_{b0} \right. \\ &\qquad \left. + \frac{\mathrm{i}k_zk_n}{\gamma_n} \left(C_n^z - E_{c1}\right) E_{c0} \right] E_d \\ G_{yz}^{\text{half}} &= G_{yz}^{\text{full}} + G_{y0z0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[-\mathrm{i}k_z\left(A_n^z - E_{b1}\right)E_{b0} \right. \\ &\qquad \left. + \mathrm{i}\left(B_n^z - E_{c1}\right) k_z E_{c0} \right] E_d \\ G_{zz}^{\text{half}} &= G_{zz}^{\text{full}} + G_{z0z0}^{\text{full}} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-\mathrm{i}k_z^2}{v_n} \left(A_n^z - E_{b1}\right) E_{b0} \right. \\ &\qquad \left. + \left(\frac{-\mathrm{i}k_n^2}{\gamma_n} \left(C_n^z - E_{c1}\right) - \mathrm{i}\gamma_n \left(B_n^z - E_{c1}\right) \right) E_{c0} \right] E_d \end{aligned} \tag{23}$$

These expressions G_{xy}^{full}, G_{yy}^{full}, G_{zy}^{full}, G_{x0y0}^{full}, G_{y0y0}^{full} and G_{z0y0}^{full}, can be used in explicit form as shown in the Appendix. Again, expressions for the strains and stresses may be derived from G_{ij} by means of the well-known equations relating strains and displacements.

Notice that, if $k_z=0$ is used, the two-dimensional Green's function for plane strain line-loads is recovered, agreeing with those provided by Kawase and Aki (1989).

Validation of the solution

The expressions described in the previous sections were implemented and validated by applying them to the calculation of the three displacement fields generated by a spatially harmonic varying line load in the z direction in a half-space. The results obtained by the proposed Green's functions are compared with those arrived at by using the BEM to discretize the free surface, together with the Green's functions for a full space. The BEM code has been validated for the case of a circular inclusion, for which analytical solutions exist.

Complex frequencies with a small imaginary part of the form $\omega_c=\omega-\mathrm{i}\eta$ (with $\eta=0.7(2\pi/T)$) are used to avoid an unlimited discretization of the free surface (Bouchon and Aki, 1977 and Phinney, 1965). Boundary elements are only required to the extent that they make a significant contribution to the response for a certain value of damping. These elements are distributed along the surface up to a spatial distance, L_{dist} from the center, given by $L_{\text{dist}}=\alpha T$. This gives a discretized surface with a length $2L_{\text{dist}}$. Many simulations were performed to study the effect of varying the size of boundary elements on the accuracy of the response. The performance was found to be better when smaller elements were placed close to where the response is required. Boundary elements of varying size were therefore used, with the shorter elements placed nearer to the center of the discretized surface.

The scheme used in this work to determine the placement and size of the boundary elements makes use of a geometrical construction, by which an auxiliary circular arc is divided into equal segments according to a previously defined ratio between the wavelength of the dilatational waves and the length of boundary elements. The boundary elements are then defined on the topographic surface by the vertical projection of these segments. The radius of the required circular arc, R is bigger than $(2L_{\text{dist}})/2$ and is placed tangent to the topographic surface at its boundary discretization end, thereby avoiding the existence of improperly small boundary elements. In this work, R is assumed to be $[(2L_{\text{dist}})/2]/\cos 10°$ (see Fig. 2).

A harmonic point source was applied to the half space medium ($\alpha=4208$ m/s, $\beta=2656$ m/s with $\rho=2140\,\text{kg/m}^3$), at the source point ($x=1.0$ m,

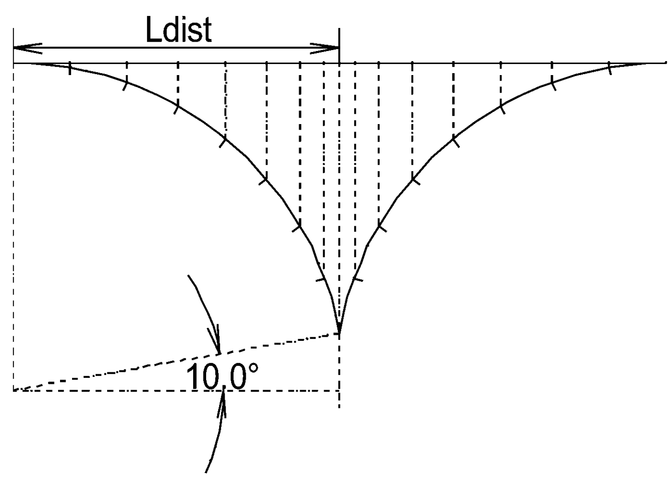

Fig. 2. Definition of the boundary elements

$y = 2.0$ m), acting along the directions x, y and z independently. Computations are performed in the frequency range [2.50, 320.0 Hz] with a frequency increment of 2.5 Hz. The scattered displacement field G_{ij}^{surf} (surface terms), the displacement in the i direction due to a load acting along j, is computed at a receiver point placed at $x = 3.0$ m and $y = 5.0$ m. The imaginary part of the frequency has been set to $\eta = 0.7(2\pi/T)$ with $T = 0.0466$ s.

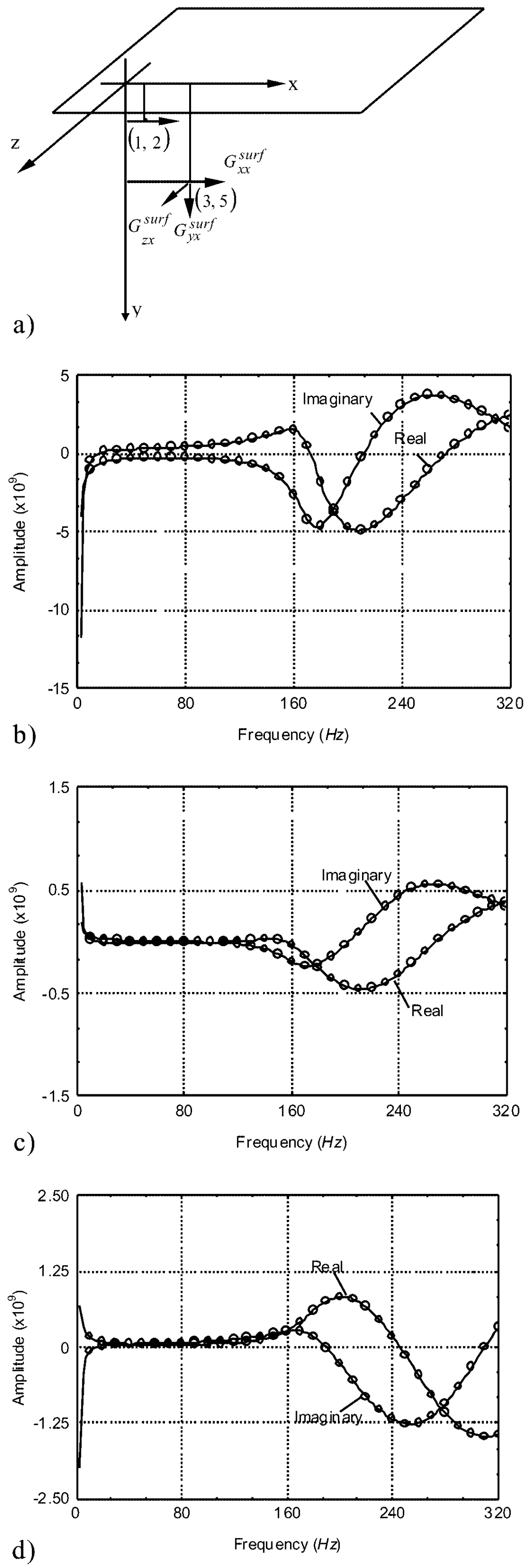

Fig. 3a–d. Spatially sinusoidal harmonic line load along the z direction in a half-space, applied in the x direction: **a** geometry of the problem; **b** G_{xx}^{surf} solutions; **c** G_{yx}^{surf} solutions; **d** G_{zx}^{surf} solutions

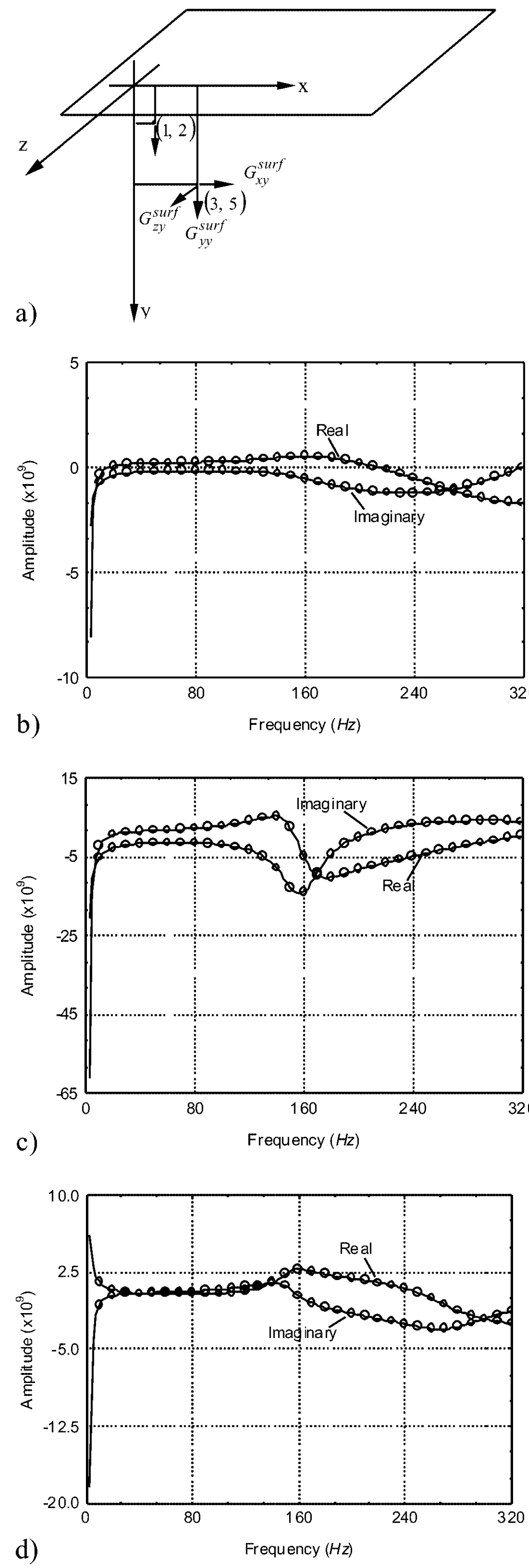

Fig. 4a–d. Spatially sinusoidal harmonic line load along the z direction in a half-space, applied in the y direction: **a** geometry of the problem; **b** G_{xy}^{surf} solutions; **c** G_{yy}^{surf} solutions; **d** G_{zy}^{surf} solutions

To illustrate the correctness of the analytical expressions, the results are computed for a single value of k_z ($k_z = 0.4$ rad/m). Figures 3–5 display the real and imaginary parts of the displacements. The solid lines represent the analytical responses, while the marked dots correspond to the boundary element method (BEM) solution. This was obtained for a very large number of boundary elements defined by the ratio between the wavelength of the incident waves and the length of the boundary elements, which was kept to a minimum of 18. These calculations are restricted to low frequencies because, for higher frequencies, the BEM solution would require the use of a very large number of boundary elements, which would make its solution impossible, owing to computational cost. It may further be mentioned that the BEM solutions provided in the present examples were obtained by limiting the discretization of the free topographical surface, through the use of complex frequencies, in such a way as to diminish the contribution of the waves generated at sources placed at the end of the discretization.

Clearly, the agreement between these two solutions is excellent. Tests with loads and receivers placed at other points produced equally accurate results.

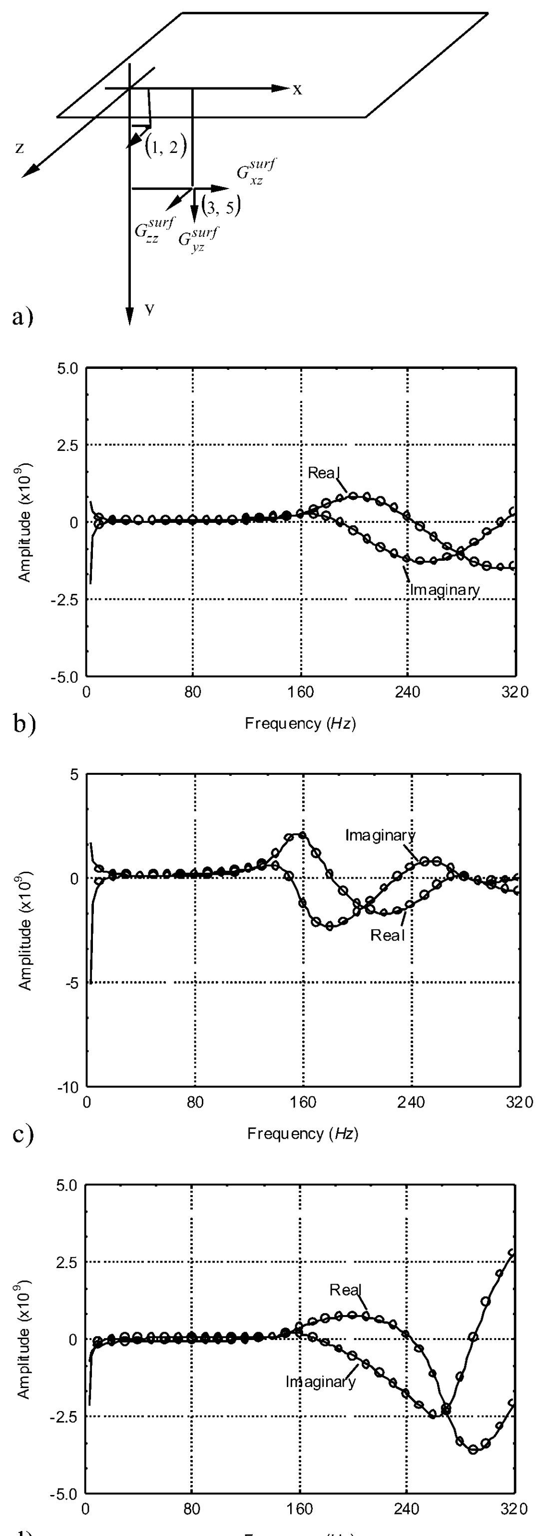

Fig. 5a–d. Spatially sinusoidal harmonic line load along the z direction in a half-space, applied in the z direction: **a** geometry of the problem; **b** G^{surf}_{xz} solutions; **c** G^{surf}_{yz} solutions; **d** G^{surf}_{zz} solution

Conclusions

A fully analytical solution for the steady state response of a spatially sinusoidal, harmonic line load in a homogeneous three-dimensional half-space has been obtained. The final expressions were validated by comparing them with numerical results computed via the BEM. An excellent agreement between the two solutions was found when the free surface was discretized with a large number of boundary elements.

The analytical solutions presented in this paper are interesting in themselves. They provide the displacement, strain or stress at a point buried in a half-space illuminated by a spatially sinusoidal, harmonic line load buried in a half-space and excited somewhere in the medium. The solutions applied in conjunction with numerical methods such as the BEM make the discretization of the free surface unnecessary, and may prove to be very useful in many engineering applications, such as the propagation of ground surface waves generated by seismic sources.

Appendix

The Green's function for a two-and-a-half dimensional full-space

$$G^{\mathrm{full}}_{xx} = \frac{\mathrm{i}}{4\rho\omega^2}\left[k_s^2 H_{0\beta} - \frac{1}{r}B_1 + \left(\frac{x-x_0}{r}\right)^2 B_2\right] \quad \text{(A1)}$$

$$G^{\mathrm{full}}_{yy} = \frac{\mathrm{i}}{4\rho\omega^2}\left[k_s^2 H_{0\beta} - \frac{1}{r}B_1 + \left(\frac{y-y_0}{r}\right)^2 B_2\right] \quad \text{(A2)}$$

$$G^{\mathrm{full}}_{zz} = \frac{\mathrm{i}}{4\rho\omega^2}\left[k_s^2 H_{0\beta} - k_z^2 B_0\right] \quad \text{(A3)}$$

$$G^{\mathrm{full}}_{xy} = G^{\mathrm{full}}_{yx} = \frac{\mathrm{i}}{4\rho\omega^2}\left(\frac{x-x_0}{r}\right)\left(\frac{y-y_0}{r}\right)B_2 \quad \text{(A4)}$$

$$G^{\mathrm{full}}_{xz} = G^{\mathrm{full}}_{zx} = \frac{-k_z}{4\rho\omega^2}\left(\frac{x-x_0}{r}\right)B_1 \quad \text{(A5)}$$

$$G^{\mathrm{full}}_{yz} = G^{\mathrm{full}}_{zy} = \frac{-k_z}{4\rho\omega^2}\left(\frac{y-y_0}{r}\right)B_1 \quad \text{(A6)}$$

with $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

$H_{n\alpha} = H_n^{(2)}(k_\alpha r)$

$H_{n\beta} = H_n^{(2)}(k_\beta r)$

$B_n = k_\beta^n H_{n\beta} - k_\alpha^n H_{n\alpha}$

For the image source, the Green's functions (G_{x0x0}^{full}, G_{y0y0}^{full}, G_{z0z0}^{full}, $G_{x0y0}^{\text{full}}, \ldots$) are obtained using the above expressions, but replacing $(y - y_0)$, with $(y + y_0)$.

References

Tadeu AJB, Kausel E (2000) Green's functions for two-and-a-half dimensional elastodynamic problems. J. Eng. Mech. ASCE 126(10): 1093–1097

Lamb H (1904) On the propagation of tremors at the surface of an elastic solid. Phil. Trans. Roy. Soc. London A203: 1–42

Bouchon M (1979) Discrete wave number representation of elastic wave fields in three-space dimensions. J. Geophys. Res. 84: 3609–3614

Kim J, Papageorgiou AS (1993) Discrete wavenumber boundary element method for 3-D scattering problems. J. Eng. Mech. ASCE 119(3): 603–624

Kawase H (1988) Time-domain response of a semicircular canyon for incident SV, P and Rayleigh waves calculated by the discrete wavenumber boundary element method. Bull. Seismol. Soc. America 78: 1415–1437

Kawase H, Aki K (1989) A study on the response of a soft basin for incident S, P and Rayleigh waves with special reference to the long duration observed in Mexico City. Bull. Seismol. Soc. America 79(5): 1361–1382

Gradshteyn, Ryzhik Tables of Integrals, Series, and Products

Pedersen HA, Sánchez-Sesma FJ, Campillo M (1994) Three-dimensional scattering by two-dimensional topographies. Bull. Seismol. Soc. America 84: 1169–1183

Papageorgiou AS, Pei D (1998) A discrete wavenumber boundary element method for study of 3-D response of 2-D scatterers. Earthquake Eng. Struct. Dyn. 27: 619–638

Bouchon M, Aki K (1977) Time-domain transient elastodynamic analysis of 3D solids by BEM. Int. J. Numer. Meth. Eng. 26: 1709–1728

Phinney RA (1965) Theoretical calculation of the spectrum of first arrivals in the layered elastic medium. J. Geophys. Res. 70: 5107–5123