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Particular solutions of Helmholtz-type operators using higher order polyhrmonic splines

A. S. Muleshkov, M. A. Golberg, C. S. Chen

Abstract We obtain explicit analytical particular solutions for Helmholtz-type operators, using higher order splines. These results generalize those in Golberg, Chen and Rashed (1998) and Chen and Rashed (1998) for thin plate splines. This enables one to substantially improve the accuracy of algorithms for solving boundary value problems for Helmholtz-type equations.

1 Introduction

Following the work of Nardini and Brebbia (1982), there has been increasing interest in using the Dual Reciprocity Method (DRM) to solve partial differential equations (PDEs) by boundary methods (Partridge et al. 1992 and Golberg and Chen 1997). Using the DRM enables one to obtain 'boundary – only' formulations for inhomogeneous, nonlinear and time-dependent problems by eliminating the domain integral which typically occurs in integral equation approaches (Partridge et al. 1992 and Golberg and Chen 1997). The key ingredient in doing this is the ability to analytically calculate particular solutions for various linear PDEs $Lu = f$. This is usually done by approximating f by a series $\sum_{j=1}^{N} a_j\varphi_j$ and then solving $L\Psi_j = \varphi_j$, $1 \leq j \leq N$, where $\{\varphi_j\}$ is an appropriate set of linearly independent basis functions. Hence, the choice of $\{\varphi_j\}$ is important, and the analysis given in Golberg et al. (1998b) shows that $\{\varphi_j\}$ need to provide an accurate approximation to f and should be of a form so that $L\Psi_j = \varphi_j$ can be solved analytically (Partridge et al. 1992, Golberg and Chen 1997 and Golberg et al. 1998b). Recent research indicates that the theory of radial basis functions (rbfs) provides a firm foundation for the first problem (Golberg et al. 1998b, Golberg and Chen 1994 and Golberg et al. 1996), while the latter depends on the nature of L. Classically, in the DRM, $\varphi(r) = 1 + r$ has been used for the basis, but better choices exist such as the thin plate splines (TPS) $\varphi(r) = r^2 \log r$ in $\mathbb{R}^2$ or multiquadrics $\varphi(r) = \sqrt{r^2 + c^2}$ (Golberg and Chen 1997, Golberg and Chen 1994 and Golberg 1995). If $L = \Delta$, the Laplacian, then $L\Psi_j = \varphi_j$, can be obtained in these cases by repeated integration (Patridge et al. 1992, Golberg and Chen 1997), but for other operators, such as Helmholtz-type operators, $L_\pm = \Delta \pm \lambda^2$, this has proven difficult (Golberg and Chen 1994, 1997). A significant result along these lines were given by Chen and Rashed (1998a) where analytic formulas were given for Ψ_j when φ_j was a TPS. These formulas have proved very useful in developing efficient mesh-free algorithms for solving the diffusion equation in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Chen et al. 1998b).

However, since the convergence rate of TPS interpolants is not large, $O(h|\log h|)$, h = minimum separation distance (Powell 1994), in the uniform norm and $O(h^2)$ in the L_2 norm (Jumarhon et al. 1997), it is of considerable interest to find other rbfs whose convergence rate is better than TPS. For Poisson's equation, Golberg, Chen and Karur (1996) have shown that considerable improvement can be obtained (up to three orders of magnitude) by using multiquadrics, but there are difficult and unresolved problems concerning the choice of the shape parameter c and analytic particular solutions are known only for $L = \Delta$. In this paper, generalizing the results in Chen and Rashed (1998a), we show how to obtain analytic particular solutions when $\varphi(r)$ are higher order Duchon splines (Duchon 1976). Since such functions can achieve L_2 convergence rates of $O(h^n)$ for an nth order spline, they can provide high order accuracy, comparable to the multiquadrics, and particular solutions for $L_\pm\Psi = \varphi$ can be obtained without numerical integration. Use of these formulas is expected to improve the efficiency of algorithms given in Chen et al. (1998b) and Golberg (1995).

We begin with a brief discussion of some problems which lead naturally to the solution of inhomogeneous Helmholtz-type equations and the need for higher order approximation of f. Following this, we discuss the results in Chen and Rashed (1998a) and indicate our approach for obtaining these based on a generalization of the annihilator method, well-known for ordinary differential equations. This is possible because higher order splines are fundamental solutions of the iterated Laplacian Δ^n (Powell 1990). A brief discussion of higher order splines is followed by our main results. Since there are four distinct cases, one for each choice of signs and dimension 2 or 3, we present our results in detail in the two dimensional case for $L_- = \Delta - \lambda^2$. In the remaining cases, the formulas will be stated and guidelines provided for their proofs. Finally,

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we present the results of a numerical experiment justifying our approach.

2 Time-dependent problems and Helmholtz-type operators

As motivation for our work we consider boundary value problems for the diffusion equation

$$\Delta u(P,t) = u_t(P,t), \quad P \in D \subseteq \mathbb{R}^d, \quad d = 2,3 \ . \tag{1}$$

Such boundary value problems can often be solved by using time-dependent fundamental solutions (Partridge et al. 1992), but this approach can be time-consuming due to the need to evaluate domain integrals. To improve the efficiency of integral equation algorithms, recent research has focused on using time-independent fundamental solutions instead (Partridge et al. 1992, Chen et al. 1998b and Zhu et al. 1994). Generally, two approaches have been taken: (i) elimination of the time dependence by using the Laplace transform (Partridge et al. 1992, Chen et al. 1998b) and (ii) using finite differences for u_t (Wrobel et al. 1986 and Chapko et al. 1997).

For (i), we define the Laplace transform of u, U by

$$U(P,s) = \int_0^\infty \mathrm{e}^{-st} u(P,t)\,\mathrm{d}t \tag{2}$$

and applying this to (1) gives

$$\Delta U(P,s) - sU(P,s) = -u_0(P) \tag{3}$$

where

$$u_0(P) = u(P,0) \tag{4}$$

is the given initial condition. If $u_0 \neq 0$, then (3) is an *inhomogeneous modified Helmholtz equation* with $\lambda = \sqrt{s}$. Transforming boundary conditions as well (Chen et al. 1998b), one arrives at a boundary value problem which can be solved using standard boundary integral equation techniques. Numerical inversion of the transform gives the solution in time (Chen et al. 1998b).

However, in these formulations, domain integrals of the form

$$\int_D G(P,Q;s) u_0(Q)\,\mathrm{d}v \tag{5}$$

appear where $G(P,Q;s)$ is the fundamental solution of $\Delta - s$. Since the domain integrals are costly to calculate, it is desirable to avoid them. In addition, if one adds a source term $f(P,t)$ to the left hand side of (1) the same problems occur and much work in this area assumes $f = u_0 = 0$ or $f = 0$ and $\Delta u_0 = 0$ (Zhu et al. 1994, Wrobel et al. 1986 and Chapko et al. 1997).

To overcome these difficulties, Chen et al. (1998b) approximated the right hand side of (3) by thin plate splines and the particular solutions were obtained using the results in Chen and Rashed (1998a). Since the accuracy of the algorithm is limited by the convergence rate of the TPS, it is desirable to have better approximating bases. As we shall see, higher order splines $r^{2n}\log r (n \geq 1)$ in $\mathbb{R}^2$ and $r^{2n-1} (n \geq 1)$ in $\mathbb{R}^3$ are a suitable choice.

As an alternative to the Laplace transform one can use finite differencing in 't'. For example, defining

$$u_n(P) = u(P, n\tau), \quad \tau > 0, \quad n = 0,1,2,\ldots \tag{6}$$

where τ is the time-step, and approximating

$$u_t(P, n\tau) \simeq \frac{u(P,n\tau) - u(P,(n-1)\tau)}{\tau} \tag{7}$$

gives v_n, the approximation to u_n, as the solution to

$$\Delta v_n(P) = \frac{v_n(P) - v_{n-1}(P)}{\tau} \ . \tag{8}$$

Rearranging (8) gives

$$\Delta v_n(P) - \frac{v_n(P)}{\tau} = \frac{-v_{n-1}(P)}{\tau} \ . \tag{9}$$

From (9), it follows that v_n satisfies a modified Helmholtz equation with $\lambda^2 = 1/\tau$ and $f = -v_{n-1}/\tau$. Hence, taking $v_0 = u_0$, boundary value problems for (1) can be reduced to solving a sequence of Helmholtz-type equations. This method was apparently due to Rothe (Chapko et al. 1997) and has recently been analyzed in some detail by Chapko et al. (1997). Again, they were limited to zero initial conditions and source terms.

As might be expected, the analysis in Chapko et al. (1997) showed that the error in using (8) is $\mathrm{O}(\tau)$ so that further accuracy requires higher order time differencing. A popular method in the BEM literature is the θ-method (Partridge et al. 1992). Here, we approximate $(t_n = n\tau)$

$$u(P,t) \simeq \theta u(P,(n+1)\tau) + (1-\theta)u(P,n\tau), \quad 0 \leq \theta \leq 1, \quad t_n \leq t \leq t_{n+1} \ , \tag{10}$$

and

$$\Delta u(P,t) \simeq \theta \Delta u_{n+1}(P) + (1-\theta)\Delta u_n(P), \quad t_n \leq t \leq t_{n+1}. \tag{11}$$

Using (10)–(11) in (1) and denoting the approximation to $u_n(P)$ by $v_n(P)$, we get

$$\theta \Delta v_{n+1}(P) + (1-\theta)\Delta v_n(P) = \frac{v_{n+1}(P) - v_n(P)}{\tau} \ , \tag{12}$$

so that

$$\Delta v_{n+1}(P) - \frac{v_{n+1}(P)}{\theta\tau} = \frac{-\theta v_n(P) - (1-\theta)\Delta v_n(P)}{\theta\tau} \ . \tag{13}$$

For $\theta = 1/2$ we get the Crank-Nicholson scheme (Partridge et al. 1992)

$$\Delta v_{n+1}(P) - \frac{2v_{n+1}(P)}{\tau} = \frac{-2v_n(P) - \Delta v_n(P)}{\tau} \ . \tag{14}$$

Again we see that this method requires the solution of a sequence of Helmholtz equations. Moreover, Lubich and Schneider (1992) considered high order discretization methods generalizing (9) and (13) of the form

$$\Delta v_{n+1} - \delta_{n+1} v_{n+1} = \sum_{k=0}^{n} \delta_{n-k} v_k$$

where $\{\delta_k\}$ are determined using A-stable multistep methods for ordinary differential equations. Such schemes are expected to require higher order spatial discretization to maintain accuracy. In fact, as will be shown in Example 2 higher order splines can be useful in accelerating the convergence of the low order time-stepping method in Eq. (9). Su and Tabarrok (1997) considered a similar time-differencing approach for a variety of other time-dependent PDEs including the wave equation, the diffusion-convection equation and nonlinear equations such as Burger's equation. Numerically, each algorithm leads to solving a sequence of Helmholtz-type equations with derivative terms on the right hand side. Hence, we expect the results in this paper to have wide applicability beyond those already discussed.

3 The annihilator method

As we indicated in the Introduction, to find approximate particular solutions u_p to

$$Lu(P) = f(P) \tag{15}$$

we approximate f by $\hat{f}$ where

$$\hat{f}(P) = \sum_{j=1}^{N} a_j \varphi_j(P), \quad P \in \mathbb{R}^d, \quad \mathrm{d} = 2, 3, \tag{16}$$

and $\{a_j\}$ are obtained by some surface fitting technique (Golberg and Chen 1997 and Golberg 1995). In the BEM this has usually been done by interpolation (Golberg and Chen 1997 and Golberg 1995) and then

$$u_p(P) = \sum_{j=1}^{N} a_j \Psi_j(P) \tag{17}$$

where

$$L\Psi_j(P) = \varphi_j(P) \ . \tag{18}$$

For numerical efficiency, it is best to solve (18) analytically (Golberg et al. 1998b). For $L = \Delta$, this can often be done by repeated integration. However, for $L_\pm$ this is usually not possible (Zhu 1993). To overcome this difficulty, we consider another approach, usually called the *annihilator method* in the context of ordinary differential equations (Derrick and Grossman 1976). Here we assume that there exists a linear partial differential operator M which satisfies

$$M\varphi_j = 0 \tag{19}$$

and commutes with L; i.e., $ML = LM$. Then,

$$ML\Psi_j = M\varphi_j = 0 = LM\Psi_j \ . \tag{20}$$

If the solution sets $V = \{v : \ Lv = 0\}$ and $W = \{w : \ Mw = 0\}$ are finite and disjoint, then

$$\Psi_j = \sum_{k=1}^{s} b_k \beta_k + \sum_{k=1}^{t} c_k \gamma_k + z \tag{21}$$

where $\{\beta_k\}$ is a basis for V, $\{\gamma_k\}$ is a basis for W and $Lz = 0$. The coefficients $\{b_k\}$ and $\{c_k\}$ are determined by requiring $L\Psi_j = \varphi_j$ and additional regularity conditions (Golberg and Chen 1997 and Golberg et al. 1999). Before proceeding with our general results, we illustrate this procedure by rederiving the results in Chen and Rashed (1998a). For simplicity we consider the case L_- in $\mathbb{R}^2$.

In Chen et al. (1998b) f was approximated by thin plate splines. Letting $r_j = \|P - Q_j\|$ be the Euclidean distance between P and Q_j where $\{Q_j\}$ is a set of interpolation points, a TPS is of the form

$$\hat{f}(P) = \sum_{j=1}^{N} a_j r_j^2 \log r_j + a_{N+1} + a_{N+2} x + a_{N+3} y \ . \tag{22}$$

As is well known, the interpolation problem

$$\hat{f}(Q_j) = f(Q_j), \quad 1 \le j \le N \tag{23}$$

with the constraints $(Q_j = (x_j, y_j))$

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} a_j x_j = \sum_{j=1}^{N} a_j y_j = 0 \tag{24}$$

has a unique solution for every set of distinct noncollinear points in $\mathbb{R}^2$ (Golberg 1995 and Duchon 1976). Hence, particular solutions to $L_- u = \hat{f}$ can be obtained by solving

$$L_- \Psi_j(P) = r_j^2 \log r_j, \quad 1 \le j \le N \tag{25}$$

and

$$L_- \Psi_{N+1} = 1, \quad L_- \Psi_{N+2} = x, \quad L_- \Psi_{N+3} = y \ . \tag{26}$$

The latter can easily be found by the method of undetermined coefficients as in Chen and Rashed (1998a)

$$\Psi_{N+1} = \frac{-1}{\lambda^2}, \quad \Psi_{N+2} = \frac{-x}{\lambda^2}, \quad \Psi_{N+3} = \frac{-y}{\lambda^2} \ . \tag{27}$$

To solve (25), we use the annihilator method. First, we observe that it suffices to solve (Golberg et al. 1999)

$$L_- \Psi(P) = r^2 \log r, \quad r = \|P\| \ , \tag{28}$$

so that (28) reduces to the ordinary differential equation

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}\Psi(r)}{\mathrm{d}r} \right) - \lambda^2 \Psi(r) = r^2 \log r \ . \tag{29}$$

Then $\Psi_j = \Psi(\|P - Q_j\|)$, $1 \le j \le N$ (Chen et al. 1998b).

Now, making use of the fact that

$$\Delta^2 r^2 \log r = 0, \quad r > 0 \ , \tag{30}$$

Ψ can be obtained by solving $\Delta^2 L_- \Psi = 0$. For radially symmetric solutions this is equivalent to solving

$$\Delta_r^2 (\Delta_r - \lambda^2) \Psi = 0 \tag{31}$$

where

$$\Delta_r u(r) = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}u(r)}{\mathrm{d}r} \right) \ . \tag{32}$$

Since Δ_r^2 and $\Delta_r - \lambda^2$ commute, and the solution spaces for $(\Delta_r - \lambda^2) v = 0$ and $\Delta_r^2 w = 0$ are finite dimensional, Ψ can be obtained by solving

$$\Delta_r^2 w = 0, \quad \left(\Delta_r - \lambda^2\right) v = 0 \ . \tag{33}$$

Since $\Delta_r - \lambda^2$ is a Bessel operator (Derrick and Grossman 1976),

$$v(r) = AI_0(\lambda r) + BK_0(\lambda r) \tag{34}$$

where I_0 and K_0 are Bessel functions of order zero. Since Δ_r^2 is a multiple of an Euler operator, we look for solutions of the form $w = r^p$ with the characteristic exponents p to be determined (Derrick and Grossman 1976).

Since $\Delta_r r^p = p^2 r^{p-2}$, $\Delta_r^2 r^p = p^2(p-2)^2 r^{p-4}$, p must satisfy the *characteristic equation* $p^2(p-2)^2 = 0$. Hence, $p = 0$ or $p = 2$ and the general solution of $\Delta_r^2 w = 0$ is (Derrick and Grossman 1976)

$$w(r) = a + b\log r + cr^2 + dr^2 \log r \ . \tag{35}$$

Thus,

$$\Psi(r) = AI_0(\lambda r) + BK_0(\lambda r) + a + b\log r + cr^2 + dr^2 \log r . \tag{36}$$

The coefficients $\{A, B, a, b, c, d\}$ are found by requiring $\left(\Delta_r - \lambda^2\right)\Psi = r^2 \log r$ and the condition that Ψ be continuous at $r = 0$. One solution is given in Chen and Rashed (1999) by

$$\Psi(r) = \begin{cases} -\frac{4}{\lambda^4} - \frac{4\log r}{\lambda^4} - \frac{r^2 \log r}{\lambda^2} - \frac{4K_0(\lambda r)}{\lambda^4}, & r \neq 0 \ , \\ -\frac{4}{\lambda^4} + \frac{4\gamma}{\lambda^4} + \frac{4}{\lambda^4}\log\left(\frac{\lambda}{2}\right), & r = 0 \ , \end{cases} \tag{37}$$

where $\gamma \simeq 0.5772156649015328$ is Euler's constant.

4 Higher order splines

To achieve higher convergence rates for $\hat{f}$, we consider using the higher order splines (Golberg 1995, Jumarhon et al. 1997 and Duchon 1976)

$$\varphi_j^{[n]}(P) = r_j^{2n} \log r_j, \quad n \geq 1, \ \text{in } \mathbb{R}^2 \tag{38}$$

and

$$\varphi_j^{[n]}(P) = r_j^{2n-1}, \quad n \geq 1, \ \text{in } \mathbb{R}^3 \tag{39}$$

Then

$$\hat{f}(P) = \sum_{j=1}^{n} a_j \varphi_j^{[n]}(P) + p_n \tag{40}$$

where p_n is a polynomial of total degree n. As for TPS, the coefficients $\{a_j\}$ and p_n can be determined by interpolating f by $\hat{f}$ on $\{Q_j\}$ where $\{Q_j\}$ is a unisolvent set of points for polynomial interpolation and the coefficients of p_n satisfy the constraints

$$\sum_{j=1}^{N} a_j b_i(P_i) = 0, \quad 1 \leq i \leq l_n \ , \tag{41}$$

where

$$l_n = \binom{n+d}{d}, \quad d = 2, 3 \ , \tag{42}$$

is the dimension of $\mathscr{P}_n$, the set of polynomials of degree $\leq n$, and $\{b_j\}$ is a basis for $\mathscr{P}_n$. Usually, $\{b_j\}$ are taken to be monomials. As was shown by Duchon (1978), this problem has a unique solution. If $\{Q_j\} \subseteq D$, a compact subset of $\mathbb{R}^d$, then (Duchon 1978)

$$\left\| f - \hat{f} \right\|_2 \leq ch^n \tag{43}$$

where

$$h = \max_{P \in \mathbb{R}^d} \min_{Q \in D} \|P - Q\| \tag{44}$$

and c depends on D but not on $\{Q_j\}$.

To find particular solutions for $L_\pm$ when φ_j is an nth order spline, we proceed as for TPS and get

$$\Psi_j(P) = \Psi\left(\|P - Q_j\|\right) + \chi(P) \tag{45}$$

where

$$L_\pm \Psi(r) = r^{2n} \log r \quad \text{in } \mathbb{R}^2 \ , \tag{46}$$

$$L_\pm \Psi(r) = r^{2n-1} \quad \text{in } \mathbb{R}^3 \ , \tag{47}$$

and

$$L_\chi = p_n, \quad \text{in } \mathbb{R}^d, d = 2, 3 \ . \tag{48}$$

Then the particular solution u_p is given by

$$u_p(P) = \sum_{j=1}^{N} a_j \Psi_j(P) + \chi(P) \ . \tag{49}$$

For brevity, we work out the details for L_- in $\mathbb{R}^2$; for the remaining cases, we merely quote the results which can be verified by the reader. We begin with (46).

First, we observe that $\Delta_r^{n+1} \varphi_j = 0$, $r > 0$, since $\varphi\left(\|P - Q_j\|\right)$ is the fundamental solution for Δ_r^{n+1} (Golberg 1995). Applying Δ_r^{n+1} to (46), solutions can be found by solving

$$\Delta_r^{n+1}\left(\Delta_r - \lambda^2\right)\Psi = 0 \ . \tag{50}$$

As for TPS,

$$\Psi = v + w \tag{51}$$

where

$$\Delta_r^{n+1} w = 0, \quad \left(\Delta_r - \lambda^2\right) v = 0 \ . \tag{52}$$

As for TPS, v is given by (34), so it suffices to solve $\Delta_r^{n+1} w = 0$. Since Δ_r^{n+1} is a multiple of an Euler operator (Golberg et al. 1999) we look for solutions of $\Delta_r^{n+1} w$ in the form $w = r^q$ where q is the characteristic exponent. Using $\Delta_r r^q = q^2 r^{q-2}$ repeatedly, gives

$$\Delta_r^{n+1} r^q = q^2(q-2)^2 \cdots (q-2n)^2 r^{q-2n} \ . \tag{53}$$

Hence, the characteristic equation is,

$$q^2(q-2)^2 \cdots (q-2n)^2 = 0 \tag{54}$$

and the characteristic exponents are $q = 0, 2, 4, \ldots, 2n$. Since the roots are double, the theory of the Euler equation shows that the solution space W is spanned by $\{r^{2k}, r^{2k}\log r\}$, $0 \leq k \leq n$ (dim $W = 2n+2$) (Derrick and Grossman 1976) so that

$$\Psi(r) = AI_0(\lambda r) + BK_0(\lambda r) + w(r) \tag{55}$$

where

$$w(r) = \sum_{k=1}^{n+1} c_k r^{2k-2} \log r + \sum_{k=1}^{n+1} d_k r^{2k-2} \quad . \tag{56}$$

We now need to choose $A, B, \{c_k\}, \{d_k\}$ so that $L_{-}\Psi = r^{2n} \log r$ with Ψ having the maximum differentiability. For this, we observe that I_0 is analytic, but K_0 is not. In fact (Abramowitz and Stegun 1965),

$$K_0(\lambda r) = \sum_{k=0}^{\infty} \mu_k r^{2k} - \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!!} r^{2k} \log r \tag{57}$$

where $(2k)!! \equiv 2 \cdot 4 \cdot 6 \cdots 2k = 2^k k!$,

$$\mu_0 = \log\left(\frac{2}{\lambda}\right) - \gamma \tag{58}$$

and

$$\mu_k = \left[\log\left(\frac{2}{\lambda}\right) - \gamma + \sum_{j=1}^{k}\left(\frac{1}{j}\right)\right] \frac{\lambda^{2k}}{[(2k)!!]^2}, \quad k \geq 1 \quad . \tag{59}$$

To obtain maximum differentiability, we choose $\{c_k\}$ in (56) to cancel the log terms in (57). Doing this gives

$$c_{k+1} = \frac{B\lambda^{2k}}{[(2k)!!]^2}, \quad 0 \leq k \leq n \quad . \tag{60}$$

To determine $\{d_k\}$, we use the fact that Ψ must satisfy $L_{-}\Psi = r^{2n} \log r$. Since $L_{-}I_0 = L_{-}K_0 = 0$, it suffices to have $L_{-}w = r^{2n} \log r$. Using the fact that $\Delta_r(r^{2k}) = k^2 r^{2k-2}$, $\Delta_r(r^{2k} \log r) = 4k^2 r^{2k-2} \log r + 4kr^{2k-2}$ and $\Delta_r \log r = 0$, gives

$$\begin{aligned}\left(\Delta_r - \lambda^2\right) w(r) = &\sum_{k=1}^{n}\left(4k^2 c_{k+1} - \lambda^2 c_k\right) r^{2k-2} \\ &- \lambda^2 c_{n+1} r^{2n} \log r \\ &+ \sum_{k=1}^{n}\left(4kc_{k+1} - \lambda^2 d_k + k^2 d_{k+1}\right) r^{2k-2} \\ &- \lambda^2 d_{n+1} r^{2n} \quad .\end{aligned} \tag{61}$$

At first glance, it seems that the available equations would not be sufficient for evaluation of the unknown coefficients. Fortunately, using (60), we observe that

$$4k^2 c_{k+1} - \lambda^2 c_k = \frac{4k^2 B\lambda^{2k}}{[(2k)!!]^2} - \frac{\lambda^2 B\lambda^{2k-2}}{[(2k-2)!!]^2} = 0 \quad . \tag{62}$$

So (61) becomes

$$\begin{aligned}&\sum_{k=1}^{n}\left(4kc_{k+1} - \lambda^2 d_k + 4k^2 d_{k+1}\right) r^{2k-2} \\ &\quad - \lambda^2 d_{n+1} r^{2n} - \lambda^2 c_{n+1} r^{2n} \log r = r^{2n} \log r \quad .\end{aligned} \tag{63}$$

Comparing coefficients gives

$$-\lambda^2 d_{n+1} = 0 \Longrightarrow d_{n+1} = 0 \quad , \tag{64}$$

$$-\lambda^2 c_{n+1} = 1 \Longrightarrow c_{n+1} = \frac{-1}{\lambda^2} \quad , \tag{65}$$

and

$$4kc_{k+1} - \lambda^2 d_k + 4k^2 d_{k+1} = 0, \quad 1 \leq k \leq n \quad . \tag{66}$$

From (60) and (65),

$$c_{n+1} = \frac{B\lambda^{2n}}{[(2n)!!]^2} = \frac{-1}{\lambda^2} \tag{67}$$

so that

$$B = -\frac{[(2n)!!]^2}{\lambda^{2n+2}} \quad . \tag{68}$$

Also from (60),

$$c_1 = B \quad . \tag{69}$$

It remains to solve (66).

From (60) and (68)

$$c_{k+1} = \frac{B\lambda^{2k}}{[(2k)!!]^2} = -\frac{[(2n)!!]^2}{[(2k)!!]^2} \lambda^{2k-2n-2} \quad . \tag{70}$$

It follows that

$$c_k = -\frac{[(2n)!!]^2}{[(2k-2)!!]^2} \lambda^{2k-2n-4} \quad . \tag{71}$$

Substituting (71) into the left hand side of (66) and multiplying by $[(2k-2)!!]^2/\lambda^{2k+2}$ it becomes

$$\frac{-[(2n)!!]^2}{\lambda^{2n+4} k} - \frac{[(2k-2)!!]^2 d_k}{\lambda^{2k}} + \frac{(2k)!! d_{k+1}}{\lambda^{2k+2}} = 0 \quad . \tag{72}$$

Let

$$\delta_k = \frac{[(2k-2)!!]^2 d_k}{\lambda^{2k}}, \quad 1 \leq k \leq n+1 \quad , \tag{73}$$

then from (64), $\delta_{n+1} = 0$ and

$$\frac{-[(2n)!!]^2}{k\lambda^{2k+4}} - \delta_k + \delta_{k+1} = 0 \quad .$$

Thus,

$$\delta_k = \delta_{n+1} + \sum_{j=k}^{n}\left(\delta_j - \delta_{j+1}\right), \quad 1 \leq k \leq n \tag{74}$$

so that

$$\delta_k = -\frac{[(2n)!!]^2}{\lambda^{2n+4}} \sum_{j=k}^{n}\left(\frac{1}{j}\right) \tag{75}$$

and

$$d_k = -\frac{[(2n)!!]^2}{[(2k-2)!!]^2} \lambda^{2k-2n-4} \sum_{j=k}^{n}\left(\frac{1}{j}\right), \quad 1 \leq k \leq n \quad . \tag{76}$$

From (71) and (76)

$$d_k = c_k \sum_{j=k}^{n}\left(\frac{1}{j}\right), \quad 1 \leq k \leq n \quad . \tag{77}$$

To complete our derivation, we need to solve $L_{-}\chi = p_n$. It can be shown that

$$\chi(x,y) = -\sum_{k=0}^{[n/2]} \frac{1}{\lambda^{2k+2}} \Delta^k p_n(x,y) \quad (78)$$

which can be verified by direct differentiation.

Summarizing, a particular solution to $L_{-}\Psi(r) = r^{2n}\log r$ is given by

$$\Psi(r) = AI_0(\lambda r) + BK_0(\lambda r) + \sum_{k=1}^{n+1} c_k r^{2k-2}\log r + \sum_{k=1}^{n} d_k r^{2k-2} \quad (79)$$

where

$$\begin{cases} B = -\frac{[(2n)!!]^2}{\lambda^{2n+2}} \\ c_k = -\frac{[(2n)!!]^2}{[(2k-2)!!]^2}\lambda^{2k-2n-4}, & 1 \le k \le n+1 \ , \\ d_k = c_k \sum_{j=k}^{n}\left(\frac{1}{j}\right), & 1 \le k \le n \ . \end{cases} \quad (80)$$

Since A is arbitrary, it can be chosen as zero as done in Chen and Rashed (1998a). Hence, u_p in (49) is given by using $\Psi_j = \Psi(\|P - Q_j\|)$ where Ψ is given by (80) and χ by (78).

The remaining cases for L_{+} in $\mathbb{R}^2$ and $L_{\pm}$ in $\mathbb{R}^3$ can be obtained using an analysis similar to that above. We quote the results which the reader can verify by differentiation.

For L_{+} in $\mathbb{R}^2$ we have

$$\Psi(r) = AJ_0(\lambda r) + BY_0(\lambda r) + \sum_{k=1}^{n+1} c_k r^{2k-2}\log r + \sum_{k=1}^{n} d_k r^{2k-2} \quad (81)$$

where

$$\begin{cases} B = \frac{\pi}{2}\frac{(-1)^{n+1}[(2n)!!]^2}{\lambda^{2n+2}}, \\ c_k = \frac{(-1)^{n+k+1}[(2n)!!]^2\lambda^{2k-2n-4}}{[(2k-2)!!]^2}, & 1 \le k \le n+1, \\ d_k = c_k \sum_{j=k}^{n}\left(\frac{1}{j}\right), & 1 \le k \le n \ , \end{cases} \quad (82)$$

where Y_0 is the Bessel function of second kind of order 0. Again, since A can be chosen arbitrarity, it is convenient to set $A = 0$.

5 Particular solutions in $\mathbb{R}^3$

To calculate the particular solutions for $L_{\pm}$ in $\mathbb{R}^3$ we have to solve $(\Delta_r \pm \lambda^2)v = 0$ and $\Delta_r^{n+1}w = 0$, since $\Delta_r^{n+1}r^{2n-1} = 0$. It is easily shown that the solution to $(\Delta_r - \lambda^2)v = 0$ is (Golberg et al. 1999)

$$v(r) = A\frac{\cosh(\lambda r)}{r} + B\frac{\sinh(\lambda r)}{r} \quad (83)$$

and that for $(\Delta_r + \lambda^2)v = 0$ is

$$v(r) = A\frac{\cos(\lambda r)}{r} + B\frac{\sin(\lambda r)}{r} \ . \quad (84)$$

As in $\mathbb{R}^2$, the remaining problem is to solve $\Delta_r^{n+1}w = 0$. Again one can show that this is a multiple of an Euler equation so we look for solutions of the form $w = r^p$. Since $\Delta_r r^p = (p+1)pr^{p-2}$ we find on repeated differentiation that the characteristic polynomial is

$$(p+1)p(p-1)\cdots(p-2n) = 0 \ . \quad (85)$$

Thus, the characteristic exponents are $p = -1, 1, 2, \ldots, 2n$ so the solution space W is spanned by $\{r^k\}$, $-1 \le k \le 2n$. Hence, Ψ, the particular solution of $(\Delta_r - \lambda^2)\Psi = r^{2n+1}$, is of the form

$$\Psi(r) = A\frac{\cosh(\lambda r)}{r} + B\frac{\sinh(\lambda r)}{r} + \sum_{k=-1}^{2n} a_k r^k \ . \quad (86)$$

To obtain solutions which are regular at $r = 0$ we use the Taylor series expansions of $\cosh(\lambda r)$ and $\sinh(\lambda r)$ at $r = 0$ and comparing coefficients gives

$$B = 0, \quad A = \frac{(2n)!}{\lambda^{2n+2}}, \quad a_{2k} = 0,$$
$$a_{2k-1} = \frac{-(2n)!}{(2k)!\lambda^{2n+2k+2}}, \quad 0 \le k \le n \ . \quad (87)$$

Thus,

$$\Psi(r) = \frac{(2n)!\cosh(\lambda r)}{r\lambda^{2n+3}} - \sum_{k=0}^{n}\frac{(2n)!}{(2k)!}\frac{r^{2k-1}}{\lambda^{2n-2k+2}} \ . \quad (88)$$

Hence, u_p in (49) is given by using $\Psi_j = \Psi(\|P - Q_j\|)$ where Ψ is given by (88) and χ by (78) with (x,y) replaced by (x,y,z).

A similar argument for L_{+} gives

$$\Psi(r) = \frac{(-1)^{n+1}(2n)!}{r\lambda^{2n+2}}\cos(\lambda r) + \sum_{k=0}^{n}\frac{(2n)!}{(2k)!}\frac{(-1)^{n+k}r^{2k-1}}{\lambda^{2n-2k+2}} \quad (89)$$

and u_p as in (49) where χ is given by (78) with (x,y) replaced by (x,y,z).

6 Computational aspects

Even though general formulas for particular solutions are given in the previous sections, for convenience we give a list of Ψ explicitly for Helmholtz-type equations for the two dimensional case as shown in Tables I and II. In Table III, particular solutions for polynomial basis functions are also given.

From the theoretical point of view, we prefer to use as high order splines as possible due to their higher convergence rate. However, from a computational point of view, numerical round-off errors may pose some limitation for implementing the higher order splines. As a result, within machine precision, we try to push the order of the splines as high as possible.

We have observed that the singularities in Eq. (77), (81), (88) and (89) have been nicely canceled out. To be more specific, (77) can be expanded and rewritten as

Table I. Ψ corresponding to given φ for a modified Helmholtz operator L_- in $\mathbb{R}^2$

φ	Ψ
$r^2 \log r$	$\begin{cases} -\frac{4}{\lambda^4}(K_0(\lambda r)+\log r) - \frac{r^2\log r}{\lambda^2} - \frac{4}{\lambda^4}, & r>0 \\ \frac{4}{\lambda^4}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{4}{\lambda^4}, & r=0 \end{cases}$
$r^4 \log r$	$\begin{cases} -\frac{64}{\lambda^6}(K_0(\lambda r)+\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{16}{\lambda^2}+r^2\right) - \frac{8r^2}{\lambda^4} - \frac{96}{\lambda^6}, & r>0 \\ \frac{64}{\lambda^6}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{96}{\lambda^6}, & r=0 \end{cases}$
$r^6 \log r$	$\begin{cases} -\frac{2304}{\lambda^8}(K_0(\lambda r)+\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{576}{\lambda^4}+\frac{36r^2}{\lambda^2}+r^4\right) & \\ \quad -\frac{12r^2}{\lambda^4}\left(\frac{40}{\lambda^2}+r^2\right) - \frac{4224}{\lambda^8}, & r>0 \\ \frac{2304}{\lambda^8}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{4224}{\lambda^8}, & r=0 \end{cases}$
$r^8 \log r$	$\begin{cases} -\frac{147456}{\lambda^{10}}(K_0(\lambda r)+\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{36864}{\lambda^6}+\frac{2304r^2}{\lambda^4}+\frac{64r^4}{\lambda^2}+r^6\right) & \\ \quad -\frac{r^2}{\lambda^4}\left(\frac{39936}{\lambda^4}+\frac{1344r^2}{\lambda^2}+16r^4\right) - \frac{307200}{\lambda^{10}}, & r>0 \\ \frac{147456}{\lambda^{10}}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{307200}{\lambda^{10}}, & r=0 \end{cases}$
$r^{10} \log r$	$\begin{cases} -\frac{14745600}{\lambda^{12}}(K_0(\lambda r)+\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{3686400}{\lambda^8}+\frac{230400r^2}{\lambda^6}+\frac{6400r^4}{\lambda^4}+\frac{100r^6}{\lambda^2}+r^8\right) & \\ \quad -\frac{r^2}{\lambda^4}\left(\frac{4730880}{\lambda^6}+\frac{180480r^2}{\lambda^4}+\frac{2880r^4}{\lambda^2}+20r^6\right) - \frac{33669120}{\lambda^{12}}, & r>0 \\ \frac{14745600}{\lambda^{12}}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{33669120}{\lambda^{12}}, & r=0 \end{cases}$

$$\Psi(r) = B\sum_{k=0}^{\infty} a_k r^{2k} - B\sum_{k=n+1}^{\infty} \frac{\lambda^{2k}}{[(2k)!!]^2} r^{2k}\log r + \sum_{k=1}^{n} d_k r^{2k-2} \tag{90}$$

From (90), we notice that $\Psi \in C^{2n+1}$. Furthermore, for computational purposes, we prefer to use (90) for small r, since only a few terms in the summation are required. For larger r, we switch to Eq. (77) by using a library subroutine to compute K_0.

Similarly, for L_+ in the two dimensional case, we have the expansion

$$\Psi(r) = B\sum_{k=0}^{\infty} a_k r^{2k} + \frac{2B}{\pi}\sum_{k=n+1}^{\infty} \frac{(-1)^k\lambda^{2k}}{[(2k)!!]^2} r^{2k}\log r + \sum_{k=1}^{n} d_k r^{2k-2} \ . \tag{91}$$

For L_-, in the three dimensional case,

$$\Psi(r) = \sum_{k=n+1}^{\infty} \frac{(2n+1)!}{(2k+1)!}\frac{r^{2k}}{\lambda^{2n-2k+2}} \ . \tag{92}$$

For L_+, in the three dimensional case,

$$\Psi(r) = \sum_{k=n+1}^{\infty} \frac{(2n+1)!}{(2k+1)!}\frac{(-1)^{n+k+1}r^{2k}}{\lambda^{2n-2k+2}} \ . \tag{93}$$

7 Numerical examples

To demonstrate the effectiveness of the higher order splines, we give an example of the modified Helmholtz equation with Dirichlet boundary conditions and another one for the diffusion equation.

Example 1. Consider the following interior Dirichlet problem for the modified Helmholtz equation

Table II. Ψ corresponding to a given φ for the Helmholtz operator L_+ in 2D

φ	Ψ
$r^2 \log r$	$\begin{cases} \frac{2}{\lambda^4}(\pi Y_0(\lambda r)-2\log r) - \frac{r^2\log r}{\lambda^2} - \frac{4}{\lambda^4}, & r>0 \\ -\frac{4}{\lambda^4}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{4}{\lambda^4}, & r=0 \end{cases}$
$r^4 \log r$	$\begin{cases} -\frac{32}{\lambda^6}(\pi Y_0(\lambda r)-2\log r) + \frac{r^2\log r}{\lambda^2}\left(-\frac{16}{\lambda^2}+r^2\right) - \frac{8r^2}{\lambda^4} + \frac{96}{\lambda^6}, & r>0 \\ -\frac{64}{\lambda^6}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) + \frac{96}{\lambda^6}, & r=0 \end{cases}$
$r^6 \log r$	$\begin{cases} \frac{1152}{\lambda^8}(\pi Y_0(\lambda r)-2\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{576}{\lambda^4}-\frac{36r^2}{\lambda^2}+r^4\right) & \\ \quad +\frac{12r^2}{\lambda^4}\left(\frac{40}{\lambda^2}-r^2\right) - \frac{4224}{\lambda^8}, & r>0 \\ -\frac{2304}{\lambda^8}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{4224}{\lambda^8}, & r=0 \end{cases}$
$r^8 \log r$	$\begin{cases} -\frac{73728}{\lambda^{10}}(\pi Y_0(\lambda r)-2\log r) - \frac{r^2\log r}{\lambda^2}\left(-\frac{36864}{\lambda^6}+\frac{2304r^2}{\lambda^4}-\frac{64r^4}{\lambda^2}+r^6\right) & \\ \quad +\frac{r^2}{\lambda^4}\left(-\frac{39936}{\lambda^4}+\frac{1344r^2}{\lambda^2}-16r^4\right) + \frac{307200}{\lambda^{10}}, & r>0 \\ -\frac{147456}{\lambda^{10}}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) + \frac{307200}{\lambda^{10}}, & r=0 \end{cases}$
$r^{10} \log r$	$\begin{cases} \frac{7372800}{\lambda^{12}}(\pi Y_0(\lambda r)-2\log r) - \frac{r^2\log r}{\lambda^2}\left(\frac{3686400}{\lambda^8}-\frac{230400r^2}{\lambda^6}+\frac{6400r^4}{\lambda^4}-\frac{100r^6}{\lambda^2}+r^8\right) & \\ \quad +\frac{r^2}{\lambda^4}\left(\frac{4730880}{\lambda^6}-\frac{180480r^2}{\lambda^4}+\frac{2880r^4}{\lambda^2}-\frac{20r^6}{\lambda^4}\right) - \frac{33669120}{\lambda^{12}}, & r>0 \\ -\frac{7372800}{\lambda^{12}}\left(\gamma+\log\left(\frac{\lambda}{2}\right)\right) - \frac{33669120}{\lambda^{12}}, & r=0 \end{cases}$

Table III. Particular solution for polynomial terms

φ	Ψ
1	$-\frac{1}{\lambda^2}$
x	$-\frac{x}{\lambda^2}$
y	$-\frac{y}{\lambda^2}$
x^2	$-\frac{x^2}{\lambda^2}-\frac{2}{\lambda^4}$
xy	$-\frac{xy}{\lambda^2}$
y^2	$-\frac{y^2}{\lambda^2}-\frac{2}{\lambda^4}$
x^3	$-\frac{x^3}{\lambda^2}-\frac{6x}{\lambda^4}$
x^2y	$-\frac{x^2y}{\lambda^2}-\frac{2y}{\lambda^4}$
xy^2	$-\frac{xy^2}{\lambda^2}-\frac{2x}{\lambda^4}$
y^3	$-\frac{y^3}{\lambda^2}-\frac{6y}{\lambda^4}$
x^4	$-\frac{x^4}{\lambda^2}-\frac{12x^2}{\lambda^4}-\frac{24}{\lambda^6}$
x^3y	$-\frac{x^3y}{\lambda^2}-\frac{6xy}{\lambda^4}$
x^2y^2	$-\frac{x^2y^2}{\lambda^2}-\frac{2x^2}{\lambda^4}-\frac{2y^2}{\lambda^4}-\frac{8}{\lambda^6}$
xy^3	$-\frac{xy^3}{\lambda^2}-\frac{6xy}{\lambda^4}$
y^4	$-\frac{y^4}{\lambda^2}-\frac{12y^2}{\lambda^4}-\frac{24}{\lambda^6}$
x^5	$-\frac{x^5}{\lambda^2}-\frac{20x^3}{\lambda^4}-\frac{120x}{\lambda^6}$
x^4y	$-\frac{x^4y}{\lambda^2}-\frac{12x^2y}{\lambda^4}-\frac{24y}{\lambda^6}$
x^3y^2	$-\frac{x^3y^2}{\lambda^2}-\frac{6xy^2}{\lambda^4}-\frac{2x^3}{\lambda^4}-\frac{24x}{\lambda^6}$
x^2y^3	$-\frac{x^2y^3}{\lambda^2}-\frac{6x^2y}{\lambda^4}-\frac{2y^3}{\lambda^4}-\frac{24y}{\lambda^6}$
xy^4	$-\frac{xy^4}{\lambda^2}-\frac{12xy^2}{\lambda^4}-\frac{24x}{\lambda^6}$
y^5	$-\frac{y^5}{\lambda^2}-\frac{20y^3}{\lambda^4}-\frac{120y}{\lambda^6}$

$$(\Delta-\lambda^2)u(x,y)=(e^x+e^y)(1-\lambda^2),\ \ (x,y)\in D, \tag{94}$$

$$u(x,y)=e^x+e^y, \quad (x,y)\in\partial D, \tag{95}$$

where $D\cup\partial D=\{(x,y):0\le x,y\le 1\}$.

The solution to (94) and (95) is given by

$$u(x,y)=e^x+e^y, \quad (x,y)\in D\cup\partial D\ . \tag{96}$$

First, we find an approximate particular solution $\hat{u}_p$ of (94) by using higher order splines. We choose a 36 point uniform grid on the unit square to interpolate the forcing term in (94). In this example we also choose $\lambda^2=25$. After finding $\hat{u}_p$, we let

$$\hat{u}=v+\hat{u}_p \tag{97}$$

and Eqs. (94) and (95) reduce to the following homogeneous equation

$$(\Delta-\lambda^2)v(x,y)=0, \quad (x,y)\in D\ , \tag{98}$$

$$v(x,y)=e^x+e^y-\hat{u}_p, \quad (x,y)\in\partial D\ . \tag{99}$$

Equations (98) and (99) can be solved by various boundary methods (Golberg and Chen 1997). Here we choose the method of fundamental solutions (MFS) due to its exponential convergence rate. In the MFS we choose 16 uniformly distributed collocation points on ∂D and the same number of source points on the fictitious boundary which is a circle with center at (0, 0) and radius 8. For the MFS, we refer readers to references (Golberg et al. 1998b and Golberg et al. 1996) for details. Once v is computed, $\hat{u}$ can be found from (97).

We observe the absolute errors of approximate solutions $\hat{u}$ along the line on $y=0.4$; i.e., $\{(x,0.4):0\le x\le 1\}$. In Fig. 1, we show the results of the absolute errors of $\hat{u}$ in a logrithamic scale using splines with order 1 through 5. Here we denote S* as splines of order $*$. The numerical results in Fig. 1 show the accuracy of the higher order splines which improve on TPS up to three orders of magnitude. In Fig. 2, we show the profile of the overall relative errors using S4. As one can see, the maximum relative error is within 7×10^{-7}. Comparing with TPS, the results are remarkable with very little additional computational cost.

Example 2. Consider the following boundary value problems for the heat equation

$$u_t(x,y,t)=\frac{1}{k}\Delta u(x,y,t), \quad (x,y)\in D\subseteq\mathbb{R}^2, \quad t>0$$

$$u(x,y,t)=0, \quad (x,y)\in\partial D,$$

$$u(x,y,0)=1, \quad (x,y)\in D,$$

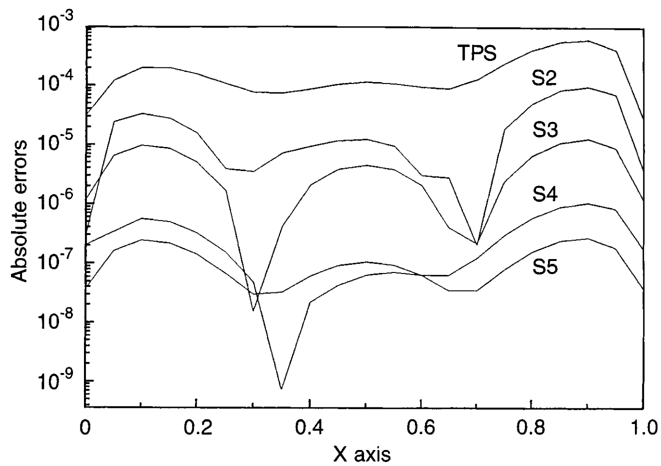

Fig. 1. Absolute errors of u along the line $y=0.4$ using higher order splines

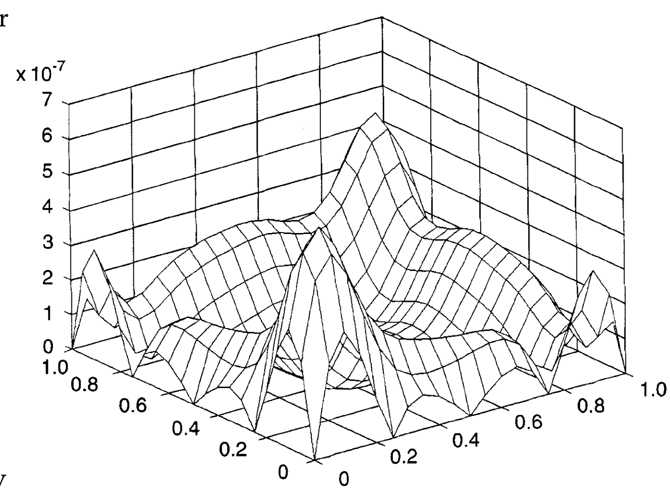

Fig. 2. Profile of relative errors for u using splines of order 4

Table IV. Relative errors (%) at seven interior points

x	y	TPS	S2	S3	S4
0.00	0.00	3.93	1.70	1.35	1.01
0.10	0.00	8.00	1.71	0.83	0.40
0.10	0.10	11.50	2.07	0.34	0.39
0.05	0.05	6.04	1.69	1.09	0.68
0.05	0.15	13.00	2.44	0.11	0.58
0.15	0.15	13.47	0.53	1.44	0.49

where $D = (-0.2, 0.2) \times (-0.2, 0.2)$. The analytical solution was given by Carslaw and Jaeger (1959).

To illustrate the effectiveness of higher order polyharmonic splines, we utilize the first order time difference scheme as shown in (9). To interpolate the source term in (9), we chose 25 and 16 uniformly distributed interior points and boundary points respectively. The DRM is then used to find the particular solutions. To evaluate the homogeneous solution for each time step, we apply the MFS in which 16 uniformly distributed collocation points on the boundary and the same number of source points on a circle with center at the origin and radius $r = 3.0$ have been selected. The numerical solutions were observed on 25 interior points at the final time step $T = 0.9$ with $k = 5.8 \times 10^{-3}$, $\tau = 0.025$, and 36 time steps. In Table IV, we show the relative errors (%) using various orders of polyharmonic splines. Due to the symmetry, we only show the numerical results of relative errors at seven different interior points in $[0, 0.2] \times [0, 0.2]$. In Table IV, S2, S3 and S4 denote the polyharmonic splines of orders 2, 3 and 4 respectively.

The numerical results in Table IV show that we can improve the accuracy by simply using higher polyharmonic splines instead of increasing the number of interpolation points.

8
Conclusions

We have generalized previous work using TPS for finding particular solutions to Helmholtz-type operators by using higher order splines. As the mathematical theory suggests, substantially increased accuracy can be obtained using higher splines with little additional work. These results are ready to implement for solving nonlinear, diffusion, diffusion-reaction and other types of partial differential equations.

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