Computational Mechanics 22 (1998) 42–49 © Springer-Verlag 1998

A simple, accurate scheme for the numerical evaluation of integrals with complex singularity poles

N. A. Dumont, M. A. M. Noronha

Abstract The procedure proposed herein is to the authors' best knowledge the only mathematically consistent technique for dealing with general quasi-singularities that occur in the boundary integral equation formulations. Its implementation results in a robust code and implies in no additional computational effort.

1 Introduction

The integrals obtained in the frame of a boundary integral equation formulation generally involve some kind of singularity. This fact has been perceived from the very begin of the developments on the subject and, as a consequence, a series of research works has been undertaken with the aim of handling the problem adequately. The first papers dealt with singular and hypersingular integrals, but since recently more attention is being given to quasi-singular integrals, which occur when the singularity pole (source node) is close to (but not on) the boundary segment over which the integral is being evaluated. A quasi-singularity is a more subtle issue than an actual singularity, particularly because its effects are more difficult to be assessed. Moreover, while an actual singularity may be dealt with indirectly, either by means of some spectral properties of the matrices involved in the formulation (use of rigid body motions or of constant stress states, for instance) or as a consequence of some reformulation (non-singular boundary integral representations), there are no alternatives in case of quasi-singularities other than applying a suitable integration scheme. An inadequate numerical treatment of quasi-singular integrals may increase substantially the global computational effort, in practical applications, and still not result in a satisfactory improvement of the accuracy. The quasi-singular effect depends upon many factors, like geometric shape, discretization, boundary element formulation (resulting in singular or hypersingular formulations, for instance) and fundamental solution used. The usual discretization of thin-shaped structures almost unavoidably leads to integrals with quasi-singularity problems, since some nodes may be placed too close to some boundary segments.

Some research works done at PUC-Rio (Dumont and de Souza, 1992; Dumont, 1994; Noronha, 1994; Noronha and Dumont, 1995) suggest that, independently of the kind of singularity or quasi-singularity and of the dimensionality of an integral that appears in a boundary element formulation, all integration cases may be advantageously dealt with semi-analytically, as a sum of a simple, adequately chosen singular or quasi-singular integral, which can only be evaluated analytically, and a general, but regular, integral, which may be evaluated numerically. Differently from previous quadrature schemes, this numerical evaluation is to be accomplished exclusively along abscissas given as roots of Lagrangean polynomials, independently from the kind and the degree of the singularity or quasi-singularity.

The present paper may be seen as a continuation of (Dumont, 1994), in the sense that it re-investigates the general integration scheme proposed for integrals with a complex pole of quasi-singularity and represents it in a closed, ready to apply form based on a novel generalized series expansion that is so easy to obtain as a conventional Taylor series (which, by the way, plays no rule herein). After this presentation, it becomes evident that all integration schemes proposed by other authors are to be viewed as unnecessary, rough approximations of the proposed technique, since they are also less effective in terms of both code robustness and computational effort.

One shall be restricted to two-dimensional problems. Generalization to three-dimensional problems is not straightforward, since it requires a transformation to a polar coordinates system. Besides that, however, it involves the same considerations done herein (Dumont, 1995, 1996).

2 General singularity types

The effectiveness of a boundary integral equation formulation relies on some singularity function that is present in the required fundamental solution. Depending on the type of differential equation that is being dealt with in a problem, this singularity function may assume different shapes, such as $\ln(\rho), 1/\rho$ or $1/\rho^2$, in which ρ is the distance between the source point and a field point over a boundary segment.

Consider an integration interval, properly normalized (see (Dumont, 1995, 1996) for the case of finite-part inte-

Communicated by T. A. Cruse, 6 February 1998

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This research work was supported by the Brazilian Ministry of Sciences and Technology, and by the agencies CAPES and CNPq.

grals), expressed in terms of an integration variable ξ (in a local coordinate system also used to describe the element), which spans from 0 to 1 (Fig. 1). According to the relative position between the source pole and the boundary segment, following singularity cases may occur:

1. If the source point is located on the integration interval, it gives rise to an actual singularity (point A in Fig. 1).
   1.1 If the singularity is related to $1/\rho^m$ (in which the exponent m is a positive number greater than or equal to 1), the corresponding integral must be dealt with as a finite-part integral, in general (possibly as part of a Cauchy principal value, if $m = 1$).
   1.2 If the singularity is related to either $\ln(\rho)$ or $1/\rho^m$, in which m is a positive number smaller then 1, one is dealing with an improper integral.
2. If the source point is given by $\xi = \xi_0$ located outside the integration interval, but on the ideal curve obtained as an extension of the integration interval, there is a real quasi-singularity pole (point B in Fig. 1).
3. Otherwise, the source point can only be expressed as $\xi_0 = a \pm bi$. It is the case of a complex quasi-singularity pole (point C in Fig. 1), the subject of this paper.

The denomination "complex quasi-singularity pole" was introduced by Dumont (1994). It is of paramount importance to recognize the existence of a complex quasi-singularity pole, since usual procedures for dealing with real singularities or quasi-singularities cannot be applied herein. The effect of a complex quasi-singularity may be stronger than that of a real quasi-singularity (Dumont, 1994). However, since an adequate numerical treatment of a complex quasi-singularity demands less integration points to achieve a given accuracy, it is always more effective than the treatment of a real quasi-singularity.

In practical applications, all singularity cases outlined above may appear combined in double or even multiple poles of singularity and quasi-singularity (Noronha, 1998; Dumont and Noronha, 1996). This occurs explicitly in the evaluation of the flexibility matrix [F] of the hybrid boundary element method (Dumont, 1989), but may also occur implicitly and inadvertently in the conventional boundary element formulation, in case of curved elements. Moreover, complex quasi-singularities may arise artificially, as a consequence of the introduction of some coordinates transformation, in case of strongly distorted elements, and even in the conventional finite element formulation, owing to a Jacobian in the denominator.

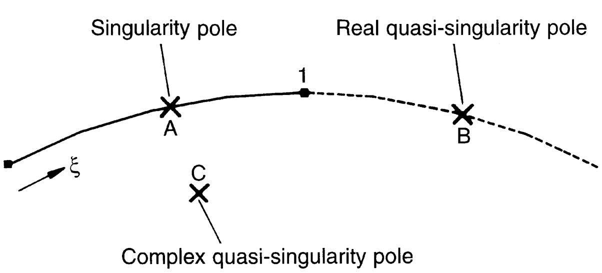

Fig. 1. General singularity poles

3 Problem proposition

For the sake of brevity, one shall be restricted, in this paper, to the case of a single complex quasi-singularity pole, for integrals of the kind

$$\int_0^1 \frac{1}{\rho^{2m}} f(\xi)\,\mathrm{d}\xi = \int_0^1 \frac{1}{(w\overline{\rho}^2)^m} f(\xi)\,\mathrm{d}\xi \equiv \int_0^1 \frac{1}{w^m} g(\xi)\,\mathrm{d}\xi \tag{1}$$

In this equation, the second-degree polynomial in the denominator of the last integral

$$w = (\xi - a + bi)(\xi - a - bi) \equiv \xi^2 - 2a\xi + a^2 + b^2 \tag{2}$$

appears in the definition of the distance ρ from the pole of quasi-singularity $\xi_0 = a \pm bi$ to a point ξ of the integration interval on the boundary:

$$\rho^2 = w\overline{\rho}^2 \tag{3}$$

in which $\overline{\rho}$ is also a polynomial, the (generally complex) roots of which, in terms of ξ, are very far from the integration interval ($\overline{\rho}$ is a constant in case of a straight boundary segment). The expressions outlined in equation (1), in terms of the square of the distance ρ, always occur in case of complex quasi-singularities, for some positive integer number m, both in the evaluation of the elements of the matrix **H** and in the determination of displacements and stresses at internal points (in which case m may be equal to 3 in some integrals (Dumont, 1994)). In eq. (1), $g(\xi) \equiv f(\xi)/\overline{\rho}^{2m}$ is an analytical function (a density function), in the sense that it may be well represented by a low-degree polynomial (in other words, $g(\xi)$ itself is not a source of quasi-singularity), as it generally occurs in the boundary element methods (Dumont, 1994; Noronha, 1994). For integrals such as the one that occurs in the matrix **G**, the singularity would be given by $\ln(w)$, instead of $1/w^m$. However, the knowledge one gains in dealing with equation (1) may be easily generalized for singularities given by non-algebraic kernels (Dumont and de Souza, 1992; Dumont, 1994; Noronha, 1994; Noronha and Dumont, 1995). The article by Noronha and Dumont (1995) is specially worth being consulted, since it illustrates that singularities related to Bessel functions and general curved boundaries may be dealt with elegantly and efficiently.

4 On the accurate quadrature scheme for complex quasi-singularities

Since $g(\xi)$ is analytical both in the integration interval and in its vicinity (it does not involve quasi-singularities), it may be well approximated by a polynomial $p(\xi)$ of degree $2n - 1 + 2m$, the values of which coincide with the values of $g(\xi)$ not only at each of the n integration points (since one shall end up with a quadrature scheme) but also at ξ_0. Moreover, the $m - 1$ first derivatives of $p(\xi)$ equal the corresponding derivatives of $g(\xi)$ at the singularity point. Then, one may write

$$\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi = \int_0^1 \frac{g(\xi) - \mathrm{gser}(w, \xi, m)}{w^m}\,\mathrm{d}\xi + \int_0^1 \frac{\mathrm{gser}(w, \xi, m)}{w^m}\,\mathrm{d}\xi \tag{4}$$

or

$$\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi \approx \overset{n}{\mathrm{GL}}\int_0^1 \frac{g(\xi)-\mathrm{gser}(w,\xi,m)}{w^m}\,\mathrm{d}\xi + \int_0^1 \frac{\mathrm{gser}(w,\xi,m)}{w^m}\,\mathrm{d}\xi \tag{5}$$

in which $\overset{n}{\mathrm{GL}}\int$ means a Gauss–Legendre quadrature with n points and $\mathrm{gser}(w,\xi,m)$ may be interpreted as either the remainder polynomial of the synthetic division (as defined by Hamming (1962) for $m=1$) between the approximating polynomial $p(\xi)$ and w^m or a series expansion of the function $g(\xi)$ about the complex conjugate couple $\xi_0 = a \pm bi$ with m terms. The general expression of gser is

$$\mathrm{gser}(w,\xi,m) = \sum_{k=1}^{m}(R_{2k-1}\xi + R_{2k})w^{k-1} \tag{6}$$

The coefficients R_{2k-1} and R_{2k} may be determined recursively (Dumont, 1994) by

$$R_{2k-1} = b_{k-1}/b \tag{7}$$

and

$$R_{2k} = a_{k-1} - aR_{2k-1} \tag{8}$$

in which a and b define the singularity pole $\xi_0 = a \pm bi$ whereas a_{k-1} and b_{k-1} are respectively the real and imaginary parts of the regular integrand in eq. (5) for $m \equiv k-1$, evaluated at $\xi_0 = a+bi$, that is,

$$a_{k-1} + b_{k-1}i \equiv \left.\frac{g(\xi)-\mathrm{gser}(w,\xi,k-1)}{w^{k-1}}\right|_{\xi=a+bi} \tag{9}$$

For $k>1$, the evaluation of eq. (9) has to be carried out by applying L'Hospital's rule, since both numerator and denominator vanish at $\xi_0 = a \pm bi$.
The series $\mathrm{gser}(w,\xi,m)$ differs entirely from a Laurent series, but it converges to a Taylor series as b tends to zero. The success of the quadrature scheme (5) depends on the possibility of obtaining the exact, analytical value of the last integral at the right-hand side. The accuracy of the quadrature scheme in equation (5) increases with the value of m and the strongness of the singularity (that is, as b tends to zero).
Equation (5) may be further transformed into

$$\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi \approx \overset{n}{\mathrm{GL}}\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi + \left(\int_0^1 \frac{\mathrm{gser}(w,\xi,m)}{w^m}\,\mathrm{d}\xi - \overset{n}{\mathrm{GL}}\int_0^1 \frac{\mathrm{gser}(w,\xi,m)}{w^m}\,\mathrm{d}\xi\right) \tag{10}$$

or

$$\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi \approx \overset{n}{\mathrm{GL}}\int_0^1 \frac{g(\xi)}{w^m}\,\mathrm{d}\xi + \sum_{i=1}^{2m} C_i R_i \tag{11}$$

According to equation above one may evaluate the quasi-singular integral at the left-hand side as it were regular and then add $2m$ correction terms, all of them evaluated at the quasi-singularity pole. In this equation, the constants R_i depend on $g(\xi)$ and its derivatives at ξ_0, as indicated in eqs. (7), (8) and (9). The constants C_i, however, depend only on the geometry of the problem (Dumont, 1994):

$$C_{2k-1} = \int_0^1 w^{k-1}\xi\,\mathrm{d}\xi - \sum_{i=1}^{n}\xi_i w^{k-1}(\xi_i)h_i \tag{12}$$

$$C_{2k} = \int_0^1 w^{k-1}\,\mathrm{d}\xi - \sum_{i=1}^{n} w^{k-1}(\xi_i)h_i \tag{13}$$

in which ξ_i and h_i are respectively abscissas and weights of the Gauss–Legendre quadrature with n points and referred to the interval (0, 1) – there is of course no restriction for using the interval $(-1,1)$. The evaluation of the constants C_i, according to the equations above, is straightforward (some cases involving non–algebraic singularities are dealt with by Dumont (1994), Noronha (1994, 1998), Noronha and Dumont (1995)).

In equation (5), the analytical function $g(\xi)$ was substituted, for the sake of numerical integration, by a polynomial of degree $2n-1+2m$. This degree is an accuracy measure of the proposed integration scheme for the regularized part of the quasi-singular integral, according to equation (11). There is no approximation involved in the correction terms, which account analytically for the singularity. In this sense, the proposed scheme cannot be matched by any other procedure.

5
A Qualitative comparison of the proposed accurate technique with some other approaches

5.1
Preliminary considerations

A general singularity or quasi-singularity owes its hazardous effect to the fact that it cannot be adequately approximated by a polynomial. As a consequence, accuracy of a Gauss–Legendre quadrature does not necessarily improve with an increasing number of integration points (Dumont, 1994). The only feasible way to overcome the difficulty presented by general singularities is first to diagnose it and second to treat it with mathematical adequacy. Unfortunately, the history of the still young boundary integral equation formulations is prodigal in bad examples of the mathematical treatment of singularities. The first example was possibly the research work by Kutt (1975), whose merit in "rediscovering" Hadamard's finite-part integrals (Hadamard, 1923) is unquestionable, but who introduced a digression in terms of an adequate quadrature scheme, as pointed out by Dumont and de Souza (1992), since even in such a case the traditional Gauss–Legendre scheme cannot be matched, if adequately applied. As a logical continuation of the research work reported by Dumont and de Souza (1992), the first author introduced in reference (Dumont, 1994) the concept of "complex singularity", which is of paramount importance in which concerns dealing with the most inconspicuous, though most frequent source of inaccuracies in any implementation of a boundary element method. The idea of subtracting and adding a term in order to regularize an integrand is at least as old as Hadamard's achievements and is being used since the beginning by almost all boundary element researchers. The merit of reference (Dumont, 1994) is to demonstrate that complex singularities have to be dealt with in a more general frame than real singularities. Notwithstanding, it also demonstrates that complex singularities may be dealt with exactly and at the expense of negligible computational effort, no matter the type of the singularity kernel, the geometry of the boundary and the distance from the singularity pole to the boundary segment over which the integral is to be evaluated.

Besides the fact that the proposed technique involves no approximation in dealing with the singularities and is generally applicable, it is highly efficient, since the Gauss–Legendre quadrature with n points evaluates exactly an integrand approximated by a polynomial of degree $2n - 1 + 2m$. As a consequence, general elasticity and potential problem formulations only require 1 integration point for the exact evaluation of all kinds of integrals over straight boundaries (Dumont, 1994; Noronha, 1994). On the other hand, if the normalized integral (1) refers to a curved boundary segment, the function $g(\xi)$ can only be approximated by a polynomial of a not so low degree, because it now involves a non-quasi-singular part $\overline{\rho}$ of the radius ρ. However, this just means that one should increase the number of integration points for an adequate evaluation of the regularized part of equation (1), according to equation (11) (if one is a perfectionist, one might consider dealing with multiple complex singularity poles – the poles of both w and $\overline{\rho}^2$ – which results in $g(\xi)$ being represented by a very low-degree polynomial at the expense of some more analytical manipulation (Dumont, 1994; Noronha, 1998; Dumont and Noronha, 1996).

In the following one shall briefly discuss the disadvantages of the most important alternatives of dealing with complex singularities available in the technical literature (only a few key articles are referred to), as compared with the accurate technique proposed [See also Zhang(1992)].

5.2
Element subdivision (Lachat and Watson, 1976)

In which respects both real and complex quasi-singularities, this is the only possible panacea, since it always work and may be generally implemented, unregarded the nature of the quasi-singularity kernel. The reason for that is quite simple: through element subdivision one may choose an integration interval as small as necessary for the distance ρ to the quasi-singularity pole to become relatively large. Then, this is not a technique of dealing with quasi-singularities, but rather a technique of avoiding the singularity. Its main disadvantage is that it may become too time consuming, especially if the imaginary part of $\xi_0 = a \pm bi$ tends to zero.

5.3
Coordinate transformation (Telles, 1987)

This technique consists in finding a way of directly canceling the quasi-singularity. However, it is not guaranteed that the resulting integrand becomes a regular function, in the sense that it may be well represented by a polynomial of reasonably low degree. This technique also may become too time consuming, since convergence with increasing number of integration points (the abscissas of which are no longer the roots of a Legendre polynomial) may be very bad, particularly if the imaginary part of $\xi_0 = a \pm bi$ tends to zero.

5.4
Use of spectral properties of the matrices involved in the formulation

It consists in avoiding some numerical evaluations involving quasi-singularities by applying the matrix formulation to some simple displacement states of the elastic body, such as rigid body motions and constant stress states. It was called a "superposition" procedure by Crotty Sisson (1990), who refers to Ushijima's thesis (1993), and has been applied in combination with the element subdivision technique. This is an approach with limited advantages, as one may deprehend by trying to apply it to calculate either stresses or displacements at the internal point **p** of Fig. 2. Since **p** is a complex singularity pole close to point **B**, the integral that involves a shape function based on point **B** (with unitary value thereon and zero value at any other nodal point) could be obtained by means of some spectral property of the matrices involved, provided that the integrals involving the shape functions based on all other nodal points are known with sufficient accuracy. But this is exactly what does not happen in general, as in case of Fig. 2, in which **p** is also a remarkable source of quasi-singularity for integrals involving the shape functions related to the nodes **A** and **C**. A still worse case is represented in Fig. 2 by the internal point **q**, which is simultaneously close to several nodal points and boundary segments.

5.5
Regularization by means of "tangent planes" and Taylor-series expansions

The first work on this subject was probably done by Hayami (1988). Among other researchers, Cruse and Aithal (1993) also present some contribution on this kind of techniques. This and other related schemes of subtracting and adding a term are just approximations of the proposed accurate technique presented herein: otherwise they would coincide with the achievements of reference (Dumont, 1994).

As a matter of fact, the use of a Taylor-series expansion about $\xi = a$ in equation (4), instead of gser, does not guarantee a low-degree polynomial approximation for the integrand of the first integral at the right-hand side, unless $b = 0$. This is illustrated in Fig. 3, for $g(\xi) = x^3$,

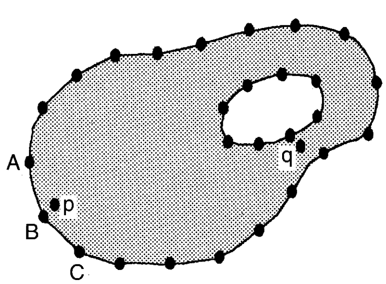

Fig. 2. An illustration of internal points that are close to several nodal points and boundary segments at same time

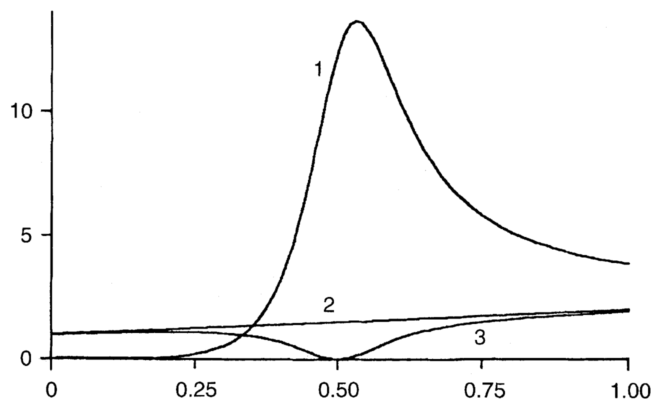

Fig. 3. Illustration of the poor performance of a regularization scheme with a Taylor-series expansion (curve 3). Curve 1 is the integrand with the original quasi-singularity. Curve 2 is the correctly regularized integrand

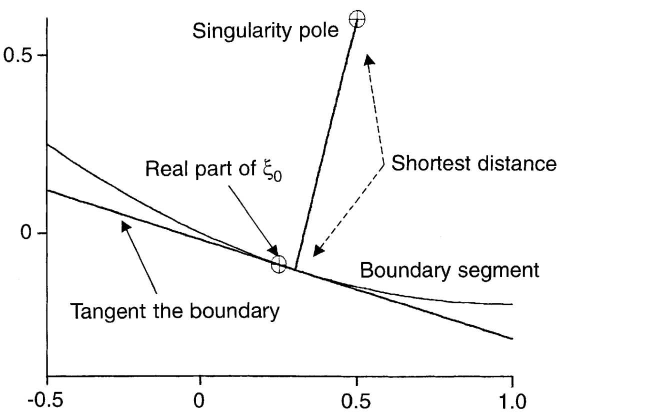

Fig. 4. Illustration that the point $\xi = a_t$ of a boundary segment closest to the singularity pole is not necessarily the real part of $\xi_0 = a \pm bi$

$a = 0.5, b = 0.1$ and $m = 2$. The curves displayed are the integrands 1) $g(\xi)/w^m$; 2) $g(\xi) - \text{gser}(w, \xi, m))/w^m$; and 3) $(g(\xi) - \text{taylor}(g, a, m))/w^m$. The first curve illustrates the occurrence of a quasi-singularity. One may clearly note that the third curve is as weird as the first. The second curve is a low-degree polynomial.

Another important consideration is related to curved boundaries. In the technical literature, the Taylor-series expansion is performed about a point $\xi = a_t$ of the boundary that is closest to the quasi-singularity pole (x_0, y_0). Given the radius $\rho = \sqrt{(x(\xi) - x_0)^2 + (y(\xi) - y_0)^2}$, one solves for ξ such that

$$\frac{\partial \rho(\xi)}{\partial \xi} = 0 \tag{14}$$

However, the correct procedure is to look for the smallest complex roots $\xi_0 = a \pm bi$ that satisfy the equation

$$\rho(\xi) = 0 \tag{15}$$

It is true and trivial that $a_t = a$ in case of straight boundary segments, but not for curved ones. Figure 4 illustrates this difference, for a boundary segment given by $(x = \xi,\ y = 0.2\xi^2 - 0.4\xi)$ and singularity pole $(x_0 = 0.5, y_0 = 0.6)$. The roots of $\rho(\xi)$ given according to equation (15) are $\xi_0 = a \pm bi = 0.253875 \pm 0.824678$, whereas equation (14) yields $\xi = a_t = 0.304317$, a value that is close, but different from the real part of the first solution.

6
Numerical Examples

Many examples of application of the proposed technique are displayed in references [2, 3, 4] (besides that, Dumont and de Souza (1992) present some numerical examples for singularities and real quasi-singularities). Following academic examples also illustrate the excellency of the results one may achieve.

6.1
An assessment of the accuracy of the matrix [H]

This first example illustrates the accuracy one may achieve in the evaluation of the elements of the matrix **[H]**, as given in the boundary element formulation (Noronha, 1994; Noronha and Dumont, 1995). Several matrices **[H]** corresponding to two hollow plates were generated using a code developed in language C. Eight straight linear elements were used in the first model, with a total of eight nodal points, as outlined in Fig. 5, whereas the example of Fig. 6 corresponds to eight curved quadratic elements with

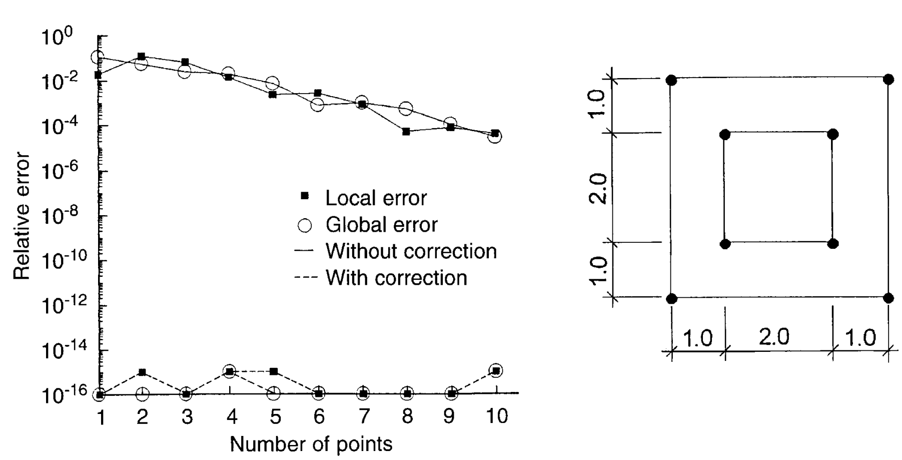

Fig. 5. Quasi-singularity errors of the **[H]** matrix for a model with straight elements

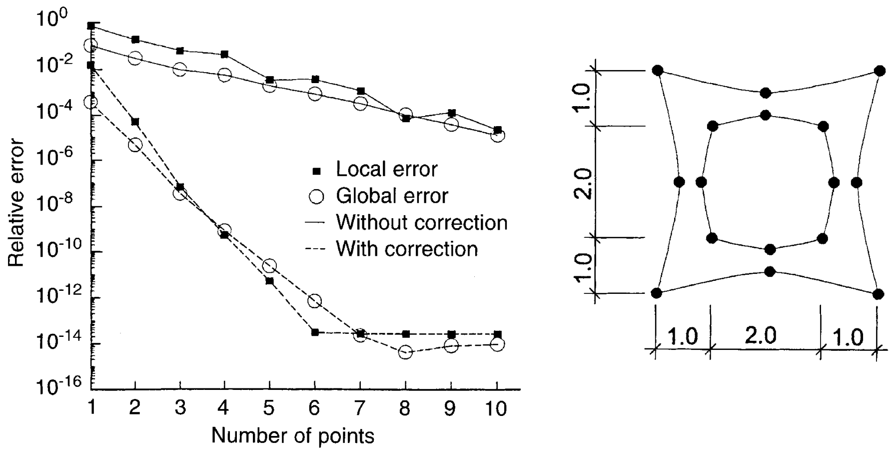

Fig. 6. Quasi-singularity errors of the [**H**] matrix for a model with curved elements

a total of 16 nodes. Strong quasi-singularities are present in both models.

The integrals required for the determination of matrix [**H**] were first evaluated by merely using a Gauss–Legendre quadrature rule, as if they were well behaved, in which regards quasi-singularities (integrals in terms of Cauchy principal value were always properly evaluated). A series of analyses considering 1 through 10 integration points was undertaken with the aim of studying the convergence of the results. Two error norms were considered in the analysis: a local error norm, represented by circles in Figs. 5 and 6, related to the maximum relative error detected in the evaluation of each element of the matrix [**H**]: and a global error norm related to the average error of all elements of [**H**] (squares in Figs. 5 and 6). These results, represented by solid lines in Figs. 5 and 6, are compared with the results evaluated in the frame of the proposed technique (dashed lines), obtained from the previous one by just adding the adequate correction terms, according to eq. (11). Target results were the best-calculated results considering corrections, as given by a one-point rule in Fig. 5 (exact) and a 10-point rule in Fig. 6.

One may observe in these examples that no satisfactory convergence is obtainable in the frame of a usual Gauss–Legendre quadrature. However, only one integration point is required for arriving at the analytical results, in case of straight boundary segments, using the proposed technique (Fig. 5). The evaluation along curved boundary elements (Fig. 6) demands more computational effort, due to the presence of the (non-constant) polynomial term $\bar{\rho}$ in the denominator of the integrand, according to eq. (3) . Notwithstanding, an accuracy of about 9 figures could be achieved with as few as 3 or 4 integration points.

6.2
Analysis of a thick-wall cylinder under internal pressure

This illustration consists in a plane-strain analysis of a thick-wall cylinder (Fig. 7a), the dimensions of which are a = 25.4 mm and b = 50.8 mm. It has a Young modulus E = 34473.8 N/mm^2, Poisson's ratio $v = 0.3$ and is submitted to an internal pressure $p_i = 3.447$ N/mm^2. Taking advantage of symmetry, one quarter of the cylinder is discretized using 26 nodal points and 13 quadratic elements (Fig. 7b): 5 and 4 curved elements along the exterior

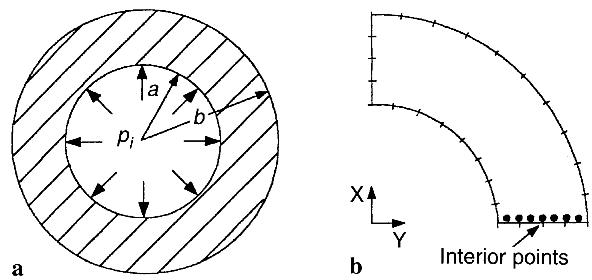

Fig. 7. (a) Thick-wall cylinder with internal pressure (b) discretization

and interior surfaces, respectively, and 2 straight elements along each side. This example was analyzed by Shiue (1991) exactly as outlined above. The analysis with the proposed technique was carried out using an object-oriented code in language C++ (BEMTECH) developed in the frame of a cooperation program between the universities PUC-Rio and Stuttgart (Wirnitzer, 1996; Noronha et al., 1966).

Figure 8 shows the radial displacement and stress results evaluated at 11 interior points, all with the same ordinate $y = 0.0254$ mm and equally spaced between abscissas $x = 25.654$ mm and $x = 50.546$ mm. Some of these points are outlined in Fig. 7b. The complex quasi-singularity has a strong effect, in this case, since the distance from the internal points to the boundary, as related to the length of each horizontal element, is equal to 1/500.

The program BEMTECH used one Gaussian point along each straight element and three Gaussian points along each curved element, as it is required, in this case, according to the results of the previous example, for a reasonably accurate evaluation of all integrals – not only the integrals required in establishing the matrix equations, but also the ones required for the evaluation of both displacement and stress results at the internal points. As a matter of comparison of the computational effort, Shiue (1991) reported the use of 128 Gaussian points along each element, for the evaluations at internal points.

Results of radial displacements at the internal points, as obtained by Shiue, by using the proposed technique and analytically, are displayed in Fig. 8a. It is worth noticing that the analytical results cannot be considered as the

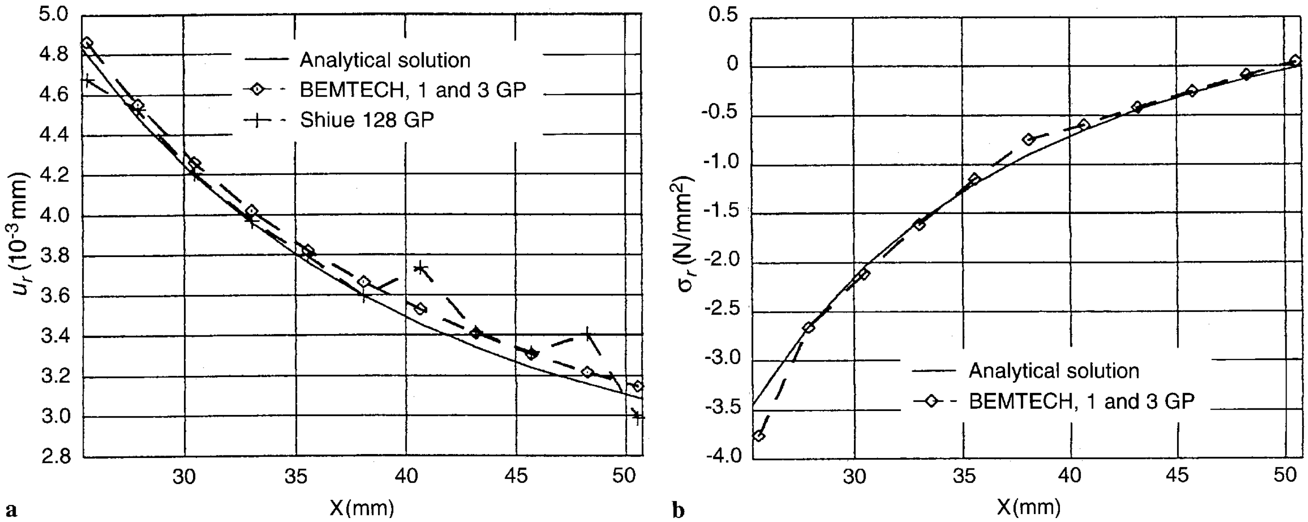

Fig. 8. Results at the interior points: (a) radial displacements and (b) radial stresses

correct ones, for the sake of comparing the exactness of the numerical evaluations, since they are referred to a circular cylinder, whereas the numerical analyses have been performed for a cylinder with an only approximately circular shape.

Figure 8b displays the results of radial stresses at the same internal points obtained with BEMTECH, using analytical results as the target ones (these results are not given by Shiue). The jumps in these results are due to the approximations involved in the boundary element formulation, not to integration errors.

Conclusion

The proposed technique is a two-step procedure, as it is unavoidably usual in the literature. In the first step, which is the core of the technique, a mathematically exact transformation is performed. This yields a well-behaved integral to be evaluated, in a second step, by means of a Gauss–Legendre quadrature, for the sake of both practicality and generality. As a consequence of the mathematical exactness of the first step, the higher the effect of a quasi-singularity, the fewer are the number of quadrature points required in the second step for achieving a given accuracy. It is a very difficult task to assess objectively the computational costs involved in the present technique, as compared to other techniques or to no technique at all. In the academic examples illustrated by Figs. 5 and 6, for instances, the additional computational effort for considering the correction terms is negligible in all cases (less than one percent, in terms of time). However, such a comparison is not objective, since one is confronting high accurate results with very bad ones. As a rule of thumb, one may say that, in general, for an expected accuracy, the relative additional computational effort decreases with a) the increasing strongness of the (quasi-) singularity, b) the increasing number of quadrature points (required in the second step), and c) the increasing number of boundary elements (integration intervals). A more efficient integration technique would be only conceivable in terms of trying to improve the numerical quadrature of the well-behaved part of the quasi-singular integral. In some simplecases, as for undistorted boundary elements, an analytical integration may be performed instead of a Gauss–Legendre quadrature. However, a Gauss–Legendre quadrature is always easier to implement and is as accurate as the analytical integration, for an adequate number of points. In case of very distorted boundary elements, the so-called well-behaved part of the quasi-singular integral may contain itself some non–negligible quasi-singularity. This could be dealt with efficiently either by extending the present technique for multiple poles of singularity (Dumont, 1994; Dumont and Noronha, 1996; Noronha, 1998) or by adequately refining the boundary element mesh.

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