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An assessment of the spectral properties of the matrix G used in the boundary element methods

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Abstract This paper outlines the correct boundary element formulation concerning the fact that arbitrary rigid body displacements, which are inherent to any fundamental solution, should actually have no influence on the accuracy of the final results. A related, and most important, aspect is that, in such an improved formulation, the forces described along the boundary are always in balance, independently of approximation. Implementation of the proposed modifications in the traditional equations, for the sake of achieving spectral consistency of the matrices involved, is always simple and possibly inexpensive, depending on the application. The formulation is generally valid for any static elasticity or potential problem in two or three dimensions, for either finite or infinite domains and considering body forces. It is demonstrated that this formulation yields a constrained equation system, which is mathematically equivalent to the problem proposed and solved by Bott and Duffin (1953) for electrical networks, in the frame of the theory of generalized inverses. The present paper proposes adequately and solves exactly a problem that has been hanging for decades. The author suggests that its main achievements be incorporated into the fundamentals of the boundary element methods.

1 Introduction

The results obtained in a two-dimensional (traditional) boundary element formulation vary with the scale chosen to describe a problem. The researchers relate this fact to the presence of the logarithm term in the fundamental solution. This is only the more conspicuous aspect of the fact, which is also verified in a three-dimensional formulation, that adding a constant to a fundamental solution does affect the final results and could even contribute to some ill-conditioning. It is also well known that, differently from the finite element method and independently of computational precision the (traditional) boundary element formulation yields non-equilibrated solutions for both two-dimensional and three-dimensional problems, unless the results coincide with the analytical ones. A number of research works have been done on these subjects in the last years, with no concludent results (the most complete investigation to date is possibly the one by Telles and De Paula, 1991; see also Phan-Thien and Fan, 1995).

The author introduced in the year 1987 a hybrid stress boundary element formulation based on the Hellinger-Reissner potential (Dumont, 1987, 1989). The key to the developments started by the author was the assumption that the flexibility matrix **F** one arrives at in the formulation is singular. This occurs as a consequence of some physical interpretations. But this assumption, or, better expressed, this physical substantiation, is still not well accepted or understood by many researchers.

In the present paper, which is a continuation of a short communication (Dumont, 1996), the author does not attempt to convince his colleagues that the singularity of the flexibility matrix **F** is not such an undesirable feature. On the contrary, a matrix singularity (or, expressing it in a more suitable way, a well established and understood matrix spectral property – that may arise in a formulation independently of the fact that the underlying fundamental solution or the consequent boundary integral equation may involve some singularity) is a welcome property to be taken advantage of, as it notoriously occurs in case of the matrix **H** of the conventional boundary element method. Properly obtained, the matrix **G** of a consistently formulated boundary element method is or should be also singular. This is a conceptually welcome feature, as it will be demonstrated presently.

2 Some basic considerations on the fundamental solutions

Consider the fundamental solution of a generic two- or three-dimensional elasticity problem (particularization to potential problems is straightforward), expressed in terms of displacements u_i^* measured at a given point and at a given coordinate direction "i" of the domain, caused by some arbitrary, (not necessarily) singular force p_m^* acting according to a given degree of freedom "m" (the index "m" characterizes both a point and a direction in the domain):

$$u_i^* = u_{im}^* p_m^* + u_{is}^{\mathrm{r}} r_s \equiv (u_{im}^* + u_{is}^{\mathrm{r}} C_{sm}) p_m^* \tag{1}$$

This fundamental solution is usually given in the literature by the function u_m^* alone, implicitly related to unitary forces p_m^*. The complete representation of eq. (1) is both mathematically and physically more adequate, since it is stated for an arbitrary (not unitary) singular force p_m^* (in which the symbol "$*$" means "fundamental solution") and a term is added to take into account the arbitrary rigid

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body displacements, as characterized by the superscript "r". In the rigid body displacement functions u^{r}_{is},"s" refers to the rigid body displacement being interpolated (there may be 1 "rigid body displacement" for potential problems and 3 or 6 for 2D or 3D problems, respectively). The quantities r_s are arbitrary constants, which may be correlated to the arbitrary singular forces p^*_m through some arbitrary matrix C_{sm} of constants. In this paper, subscripts "m" and "n" refer to degrees of freedom of discretized quantities; subscripts "s" and "t" refer to rigid body displacements; and subscripts "i" and "j" are related to the coordinate directions.

The stresses at a given point of the domain are obtained from eq. (1) as

$$\sigma^*_{ij} = \sigma^*_{ijm} p^*_m \tag{2}$$

One may verify that, as a property of a fundamental solution,

$$\sigma^*_{ij,j} = \sigma^*_{ijm,j} p^*_m = 0 \tag{3}$$

everywhere in the domain, except in a vicinity Ω_0 of the point of application of the singular force p^*_m, where

$$\int_{\Omega_0} \sigma^*_{ijm,j} \, \mathrm{d}\Omega = \begin{cases} 1 & \text{if "}i\text{" and "}m\text{" refer to} \\ & \text{the same degree of freedom} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

From the stresses in eq. (2) one may derive the traction forces along a given boundary Γ of the domain as

$$t^*_i = p^*_{im} p^*_m \tag{5}$$

3
The traditional boundary element equation

The matrix equations of the traditional boundary element method may be stated, starting from minimum residual considerations and making use of eqs. (1) and (5), as

$$\begin{aligned} & p^*_m \left(\int_\Gamma p^*_{im} u_{in} \, \mathrm{d}\Gamma - \int_\Omega \sigma^*_{ijm,j} u_{in} \, \mathrm{d}\Omega \right) d_n \\ & = p^*_m \left(\int_\Gamma u^*_{im} t_{in} \, \mathrm{d}\Gamma \right) t_n + p^*_m \left(\int_\Gamma C_{sm} u^{\mathrm{r}}_{is} t_{in} \, \mathrm{d}\Gamma \right) t_n \\ & \quad + p^*_m \left(\int_\Omega u^*_{im} b_i \, \mathrm{d}\Omega \right) + p^*_m \left(\int_\Omega C_{sm} u^{\mathrm{r}}_{is} b_i \, \mathrm{d}\Omega \right) \end{aligned} \tag{6}$$

in which u_{in} and t_{in} are interpolation functions for displacements u_i, in terms of some nodal parameters d_n, and traction forces t_i, in terms of some nodal parameters t_n, respectively (usually $u_{in} \equiv t_{in}$):

$$\left. \begin{aligned} u_i &= u_{in} d_n \\ t_i &= t_{in} t_n \end{aligned} \right\} \quad \text{along } \Gamma \tag{7}$$

The domain integrals at the right-hand side in eq. (6) take body forces b_i into account.

Considering that p^*_m is arbitrary, eq. (6) leads to the known matrix equation

$$\mathbf{Hd} = \mathbf{Gt} + \mathbf{b} \tag{8}$$

In equation above,

$$\mathbf{H} \equiv H_{mn} = \int_\Gamma p^*_{im} u_{in} \, \mathrm{d}\Gamma - \int_\Omega \sigma^*_{ijm,j} u_{in} \, \mathrm{d}\Omega \tag{9}$$

is given by the expression in the first brackets in eq. (6), supposing that the singularities and quasi-singularities of the boundary integral have been properly dealt with and observing eqs. (3) and (4) for the correct interpretation of the domain integral.

The matrix

$$\mathbf{G} \equiv G_{mn} = \int_\Gamma u^*_{im} t_{in} \, \mathrm{d}\Gamma \tag{10}$$

is given by the boundary integral in the second brackets of eq. (6), an improper integral that may also present some quasi-singularities (Dumont, 1994).

The terms $\mathbf{d} \equiv d_n$ and $\mathbf{t} \equiv t_n$ in eq. (8) are vectors corresponding to boundary displacement and traction parameters, respectively. Their elements may be either prescribed or taken as unknowns, according to the boundary conditions.

Finally, the vector of nodal displacements equivalent to the body forces $\mathbf{b}$ in eq. (8) are expressed as

$$\mathbf{b} \equiv b_m = \int_\Omega u^*_{im} b_i \, \mathrm{d}\Omega \tag{11}$$

Although this is not the subject of the present article, the author presents in Appendix I a simple way of expressing vector $\mathbf{b}$ in terms of boundary integrals alone, by means of a virtual work statement, as developed in the frame of the hybrid boundary element formulation (Dumont, 1994; Carvalho, 1990).

However, eq. (6) can only lead to eq. (8) if the terms related to the rigid body displacements u^{r}_{is} vanish, for arbitrary p^*_m and C_{sm}, that is, if

$$\int_\Gamma u^{\mathrm{r}}_{is} t_{in} \, \mathrm{d}\Gamma t_n + \int_\Omega u^{\mathrm{r}}_{is} b_i \, \mathrm{d}\Omega \equiv 0 \tag{12}$$

This equation means that the assumed traction forces along the boundary should be in equilibrium with the body forces as a premise (the total work done by the traction forces and the body forces on the virtual rigid body displacements u^{r}_{is} along the boundary and in the domain, respectively, is equal to zero). It seems that this fact has not been adequately dealt with in the literature, since a premise can only be taken for granted if it is actually satisfied.

4
Constructing a spectrally admissible matrix G

4.1
Some preliminary considerations

Equation (12) may be represented in matrix notation as

$$\mathbf{R}^{\mathrm{T}} \mathbf{t} + \mathbf{b}^{\mathrm{r}} = 0 \tag{13}$$

in which

$$\mathbf{R} \equiv R_{ns} = \int_\Gamma u^{\mathrm{r}}_{is} t_{in} \, \mathrm{d}\Gamma \tag{14}$$

$$\mathbf{b}^{\mathrm{r}} \equiv b_s = \int_\Omega u^{\mathrm{r}}_{is} b_i \, \mathrm{d}\Omega \tag{15}$$

are a rectangular matrix with as many columns as the number of rigid body displacements u_{is}^{r} and a vector of equivalent nodal displacements obtained in terms of the (mixed) virtual work done by the body forces on u_{is}^{r}, respectively.

One may define a rectangular matrix $\mathbf{Z}$, the columns of which are an orthogonal basis of the columns of $\mathbf{R}$, that is, such that

$$\mathbf{Z}^{\mathrm{T}}\mathbf{Z} = \mathbf{I} \tag{16}$$

where $\mathbf{I}$ is an identity matrix, and $\mathbf{ZZ}^{\mathrm{T}}$ is idempotent, that is,

$$(\mathbf{ZZ}^{\mathrm{T}})(\mathbf{ZZ}^{\mathrm{T}}) = \mathbf{ZZ}^{\mathrm{T}} \tag{17}$$

The idempotent matrix $\mathbf{ZZ}^{\mathrm{T}}$ is the *orthogonal projector* on the space of the inadmissible, unbalanced traction force parameters $\mathbf{t}$ (see Ben-Israel and Greville (1980) for the definition of *orthogonal projector*, specially the footnote on page 51). For elasticity problems, the rigid body displacement functions u_{is}^{r} may be defined in infinite ways. Moreover, the orthogonal basis $\mathbf{Z}$ may be obtained from $\mathbf{R}$ in more than one way. However, the resulting idempotent matrix $\mathbf{ZZ}^{\mathrm{T}}$ is unique, as demonstrated in Appendix II. It follows from the definition of $\mathbf{Z}$ that

$$\mathbf{R} = \mathbf{Z}\boldsymbol{\lambda} \tag{18}$$

in which $\boldsymbol{\lambda}$ is a non-singular square matrix readily obtained from eqs. (16) and (18) as

$$\boldsymbol{\lambda} = \mathbf{Z}^{\mathrm{T}}\mathbf{R} \tag{19}$$

If the traction force parameters $\mathbf{t}$ satisfy eq. (13), a condition for eq. (8) to be valid, it follows from eqs. (18) and (19) that

$$\mathbf{Z}^{\mathrm{T}}\mathbf{t} + \boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{\mathrm{r}} = \mathbf{0} \tag{20}$$

Pre-multiplying equation above by $\mathbf{Z}$ and subtracting $\mathbf{t}$ from both sides yields the condition that $\mathbf{t}$ must satisfy to ensure the validity of eq. (8):

$$\mathbf{t} = (\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}})\mathbf{t} - \mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{\mathrm{r}} \tag{21}$$

If this relationship is valid, then eq. (8) should be rewritten as

$$\mathbf{Hd} = \mathbf{G}(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}})\mathbf{t} + (\mathbf{b} - \mathbf{GZ}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{\mathrm{r}}) \quad \text{or} \quad \mathbf{Hd} = \mathbf{G}_{\mathrm{a}}\mathbf{t} + \mathbf{b}_{\mathrm{a}} \tag{22}$$

in which

$$\mathbf{G}_{\mathrm{a}} \equiv \mathbf{G}(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}}) \tag{23}$$

is the *admissible* part of the matrix $\mathbf{G}$, obtained through the orthogonal projection given by $(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}})$ and $\mathbf{b}_{\mathrm{a}}$ is what might be called the vector of *admissible* nodal displacements related to the body forces.

The admissible matrix $\mathbf{G}_{\mathrm{a}}$, as defined in eq. (23), is obviously singular. It is worth establishing that

$$\mathrm{Rank}(\mathbf{G}_{\mathrm{a}}) = \mathrm{rank}(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}}) \tag{24}$$

a feature that cannot be demonstrated mathematically. The matrix $\mathbf{G}$ is a flexibility-type transformation matrix, which must always yield some non-trivial equivalent nodal displacement vector to any set of traction force parameters $\mathbf{t}$, if one is dealing with an elastic body. Owing to this physical property, $\mathbf{G}$ is always non-singular. However, depending on the set of rigid body displacement functions u_{is}^{r} that appear in the definition of the fundamental solution, as given in eq. (1), $\mathbf{G}$ may become singular or ill conditioned (this may happen even if one explicitly sets $u_{is}^{r} = 0$ in eq. (1), unless u_{im}^{*} in this equation is also set as orthogonal to u_{is}^{r}, for each degree of freedom). Regardless the condition of matrix $\mathbf{G}$, the rank of matrix $\mathbf{G}_{\mathrm{a}}$ is always well defined according to eq. (24), since $\mathbf{G}_{\mathrm{a}}$ is by construction independent of the rigid body displacement functions u_{is}^{r}. The conventional boundary element formulation relies on the fact that the matrix $\mathbf{G}$ is non-singular and hopefully not ill conditioned. All considerations of the present paper are based on the effectively reliable premise expressed by eq. (24).

4.2
An alternative way of arriving at equation (22)

Starting from eq. (6), one could think in obtaining the matrix $\mathbf{C} \equiv C_{sm}$ in such a way that, in absence of body forces, the nodal displacements equivalent to any set of traction force parameters defined by the basis $\mathbf{Z}$ be equal to zero:

$$(\mathbf{G} + \mathbf{CR}^{\mathrm{T}})\mathbf{Z} = \mathbf{0} \tag{25}$$

Making use of eq. (18), this equation may be transformed into

$$(\mathbf{G} + \mathbf{C}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}})\mathbf{Z} = \mathbf{0} \tag{26}$$

from which follows the expression for the constants $\mathbf{C}$:

$$\mathbf{C} = -\mathbf{GZ}\boldsymbol{\lambda}^{-\mathrm{T}} \tag{27}$$

Substitution of $\mathbf{C}$ in eq. (6), according to its expression in eq. (27), yields the same eq. (22).

4.3
Solving equation (22)

Equation (22) seems to be useless for the sake of arriving at an equation system of the shape

$$\mathbf{Ax} = \mathbf{y} \tag{28}$$

with a vector $\mathbf{y}$ that gathers the known displacement and traction force parameters, besides the body force parameters, since the boundary conditions should be expressed in terms of the admissible parameters $(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}})\mathbf{t}$, which are completely unknown unless all the elements of $\mathbf{t}$ are known (Section 7 of this paper brings some more considerations on this specific subject).

However, one might attempt to solve eq. (22) for the admissible traction parameters $\mathbf{t}$:

$$\mathbf{t} = \mathbf{G}_{\mathrm{a}}^{(-1)}(\mathbf{Hd} - \mathbf{b}_{\mathrm{a}}) \tag{29}$$

An apparent difficulty in obtaining eq. (29) lies in the fact that $\mathbf{G}_{\mathrm{a}}$, as introduced in eq. (22), is singular. Fortunately, the equation system (22) corresponds mathematically to the problem that Bott and Duffin (1953) have proposed and solved (Ben-Israel and Greville, 1980), as outlined in Appendix III. According to that, and based on the experience gained in the development of the hybrid boundary element method (Dumont, 1987, 1989), the author proposes following restricted inverse for $\mathbf{G}_a$:

$$\mathbf{G}_a^{(-1)} = (\mathbf{I} - \mathbf{Z}\mathbf{Z}^T)(\mathbf{G}_a + \mathbf{Z}\gamma\mathbf{Z}^T)^{-1} \tag{30}$$

which is more general than the Bott-Duffin inverse, since it involves an in principle arbitrary symmetric matrix γ, that may be chosen in order to ensure that the elements of $\mathbf{Z}\gamma\mathbf{Z}^T$ and $\mathbf{G}_a$ are of the same magnitude, thus avoiding round-off errors during the numerical computations. Since $\mathbf{G}_a$ and $\mathbf{Z}\gamma\mathbf{Z}^T$ are complementary matrices ($\mathbf{G}_a\mathbf{Z}\boldsymbol{\lambda}\mathbf{Z}^T \equiv \mathbf{0}$), $\mathbf{G}_a + \mathbf{Z}\gamma\mathbf{Z}^T$ is always well conditioned (see eq. (24) and subsequent considerations).

Instead of the equation above, one might express

$$\mathbf{G}_a^{(-1)} = (\mathbf{I} - \mathbf{Z}\mathbf{Z}^T)\mathbf{G}^{-1} \tag{31}$$

but the matrix $\mathbf{G}$, as defined in eq. (10), may become ill conditioned, as already discussed, whereas $\mathbf{G}_a + \mathbf{Z}\gamma\mathbf{Z}^T$ is always well conditioned, for adequate γ.

It is evident from the definition of $\mathbf{G}_a^{(-1)}$ in eq. (30) that

$$\begin{aligned} \mathrm{Rank}(\mathbf{G}_a^{(-1)}) &= \mathrm{rank}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^T) \\ &\geq \mathrm{rank}(\mathbf{G}_a^{(-1)}\mathbf{H}) \leq \mathrm{rank}(\mathbf{H}) \end{aligned} \tag{32}$$

However, in order to demonstrate that eq. (29) is both physically and mathematically consistent one has to succeed in demonstrating that, in fact,

$$\begin{aligned} \mathrm{Rank}(\mathbf{G}_a^{(-1)}) &= \mathrm{rank}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^T) \\ &= \mathrm{rank}(\mathbf{G}_a^{(-1)}\mathbf{H}) = \mathrm{rank}(\mathbf{H}) \end{aligned} \tag{33}$$

This demonstration is not straightforward and shall be accomplished as a consequence of the considerations made at the end of the next section.

5
A spectrally consistent stiffness-type matrix

One may define a vector $\mathbf{p}$ of nodal forces that are equivalent in terms of virtual work to the traction force parameters $\mathbf{t}$ on the boundary

$$\mathbf{p} = \mathbf{L}\mathbf{t} \tag{34}$$

in which

$$\mathbf{L} \equiv L_{mn} = \int_\Gamma u_{im} t_{in} \, \mathrm{d}\Gamma \tag{35}$$

Then it follows from eqs. (29) and (34)

$$\mathbf{p} = \mathbf{L}\mathbf{G}_a^{(-1)}\mathbf{H}\mathbf{d} - \mathbf{L}\mathbf{G}_a^{(-1)}\mathbf{b}_a \tag{36}$$

in which

$$\mathbf{L}\mathbf{G}_a^{(-1)}\mathbf{H} \equiv \mathbf{K} \tag{37}$$

is a stiffness-type matrix. There is no reason to believe that this matrix should be symmetric, or at least less non-symmetric, in general, than the stiffness-type matrix $\mathbf{L}\mathbf{G}^{-1}\mathbf{H}$. The criticisms expressed by Dumont (1986, 1987) are still valid in case of an admissible matrix $\mathbf{G}_a$. However, the matrix $\mathbf{K}$, as given in eq. (37), has improved spectral properties that make sure that, in the equilibrium eq. (36), the equivalent nodal forces $\mathbf{p}$ are always in balance. This shall be demonstrated in the following.

Let the columns of a rectangular matrix $\mathbf{W} \equiv W_{ns}$ be a basis of the nodal displacements $\mathbf{d}$ related to rigid body displacements. For the moment, one can only say that $\mathbf{W}$ and $\mathbf{Z}$ have the same dimension. For a finite domain, it follows from eq. (8) that, necessarily,

$$\mathbf{H}\mathbf{W} = \mathbf{0} \tag{38}$$

which is a feature related to the physical nature of the fundamental solution. On the other hand, the rigid body displacement functions u_{is}^r may be described along the boundary Γ as a linear combination of the displacement interpolation functions u_{in}, introduced in eq. (7), and W_{ns}:

$$u_{is}^r = u_{im} W_{mt} \omega_{ts} \tag{39}$$

in which $\omega \equiv \omega_{ts}$ is a non-singular square matrix that transforms W_{mt} into the nodal displacements related to u_{is}^r. Then, it follows from eqs. (14), (35) and (39) that

$$\mathbf{R} = \mathbf{L}^T\mathbf{W}\omega \tag{40}$$

and, according to eq. (18),

$$\mathbf{L}^T\mathbf{W} = \mathbf{Z}\boldsymbol{\lambda}\omega^{-1} \tag{41}$$

that is, the columns of $\mathbf{L}^T\mathbf{W}$ lie in the space spanned by the rows of $\mathbf{Z}$ and, as a consequence,

$$\mathbf{W}^T\mathbf{L}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^T) = \omega^{-T}\boldsymbol{\lambda}^T\mathbf{Z}^T(\mathbf{I} - \mathbf{Z}\mathbf{Z}^T) \equiv \mathbf{0} \tag{42}$$

It may be noticed that, since $\mathbf{L}$, as given in eq. (35), is by construction a nonsingular matrix, it follows from eq. (41) that

$$\mathrm{Rank}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) = \mathrm{Rank}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^T) \tag{43}$$

This is in conformity with the assertion, made before, that $\mathbf{W}$ and $\mathbf{Z}$ have the same dimension. Then, given the definitions of $\mathbf{G}_a^{(-1)}$ in eq. (30) and $\mathbf{K}$ in eq. (37), one gets from the orthogonality conditions expressed in eqs. (38) and (42) that

$$\mathbf{W}^T\mathbf{K} = \mathbf{K}\mathbf{W}^T = \mathbf{0} \tag{44}$$

As a consequence, the equivalent nodal forces $\mathbf{p}$ of eq. (36) are always self-equilibrated.

The verification of the properties of the stiffness-type matrix $\mathbf{K}$, as given in eq. (37), is not complete, since its rank is still not determined. From eqs. (42), (30) and (38) one may express the matrix $\mathbf{K}$ in eq. (37) as

$$\begin{aligned} \mathbf{K} &\equiv (\mathbf{I} - \mathbf{W}\mathbf{W}^T)\mathbf{L}\mathbf{G}_a^{(-1)}\mathbf{H}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) \\ &= (\mathbf{I} - \mathbf{W}\mathbf{W}^T)\mathbf{L}(\mathbf{G}_a^{(-1)} + \mathbf{Z}\mathbf{Z}^T)\mathbf{H}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) \\ &\equiv (\mathbf{I} - \mathbf{W}\mathbf{W}^T)\mathbf{A}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) \end{aligned} \tag{45}$$

in which $\mathbf{A} \equiv \mathbf{L}(\mathbf{G}_a^{(-1)} + \mathbf{Z}\mathbf{Z}^T)\mathbf{H}$ is a matrix such that $\mathbf{A}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) + \mathbf{W}\mathbf{W}^T$ is nonsingular. Then, according to item "c" of the theorem demonstrated in Appendix III,

$$\mathrm{Rank}\big((\mathbf{I} - \mathbf{W}\mathbf{W}^T)\mathbf{A}(\mathbf{I} - \mathbf{W}\mathbf{W}^T)\big) = \mathrm{rank}\big((\mathbf{I} - \mathbf{W}\mathbf{W}^T)\big) \tag{46}$$

and, as a consequence,

$$\mathrm{Rank}(\mathbf{K}) = \mathrm{rank}(\mathbf{I} - \mathbf{W}\mathbf{W}^T) \tag{47}$$

Considering the definition of $\mathbf{K}$ in eq. (37), eq. (47) also demonstrates the equalities stated in eq. (33), which had remained unproved.

6
Spectral properties for an infinite domain

For a cavity in an infinite domain, it may be demonstrated (Brebbia, Telles and Wrobel, 1984) that the boundary element matrix equation (8) could be expressed as

$$(\overline{\mathbf{H}} + \mathbf{H}_\infty)\mathbf{d} = (\overline{\mathbf{G}} + \mathbf{G}_\infty)\mathbf{t} + (\overline{\mathbf{b}} + \mathbf{b}_\infty) \qquad (48)$$

in which the matrices with upper bars are the result of integration along the inner boundary (or multiple boundaries) of the infinite domain and are related to the matrices given in the main body of the paper, for the complementary interior problem, in the following way:

$$\overline{\mathbf{H}} = \mathbf{I} - \mathbf{H} \qquad (49)$$

$$\overline{\mathbf{G}} = -\mathbf{G} \qquad (50)$$

One is assuming that the domain integrals required in the evaluation of both body forces vectors $\overline{\mathbf{b}}$ of eq. (48), above, and $\bar{\mathbf{b}}$, of eq. (60), below, are expressible in terms of boundary integrals. The matrices with an "∞" as a subscript are the result of integration carried out along the boundary placed at infinite and should, owing to physical reasons, (almost! – see considerations below) always satisfy the identity

$$\mathbf{H}_\infty \mathbf{d} \equiv \mathbf{G}_\infty \mathbf{t} + \mathbf{b}_\infty \qquad (51)$$

for any applied body forces and any set of nodal displacements and traction forces, as (not so) well explained in the technical literature (Brebbia, Telles and Wrobel, 1984).

Then, for a cavity in an infinite domain, the boundary element matrix equation may be expressed simply as

$$\overline{\mathbf{H}}\mathbf{d} = \overline{\mathbf{G}}\mathbf{t} + \overline{\mathbf{b}} \qquad (52)$$

The matrix, $\overline{\mathbf{H}}$, as given in eq. (49), is no longer singular, since, for a basis $\mathbf{W}$ of the rigid body displacements, one gets from eq. (38)

$$\overline{\mathbf{H}}\mathbf{W} = \mathbf{W} \qquad (53)$$

As a consequence, there seems to be a paradox in eq. (52), since rigid body displacements in the cavity are transformable into traction forces along its boundary (the matrix $\overline{\mathbf{G}}$ is also nonsingular), causing a state of stresses around the cavity. Conversely, application of unbalanced forces in a cavity provokes general (not only rigid body) displacements in the domain, according to eq. (52). The most surprising of all that is the fact that these uncontrollable results vary with the rigid body displacement function u^{r}_{in} of the fundamental solution, eq. (1).

There is only one way out of this apparent paradox. It consists in stating that eq. (52) is valid only for *admissible* displacements,

$$\mathbf{d}_\mathrm{a} = (\mathbf{I} - \mathbf{W}\mathbf{W}^\mathrm{T})\mathbf{d} \qquad (54)$$

which are orthogonal to rigid body motions and are therefore the only ones capable or yielding a stress field, and *admissible*, or balanced traction forces,

$$\mathbf{t}_\mathrm{a} = (\mathbf{I} - \mathbf{G}\mathbf{G}^\mathrm{T})\mathbf{t} \qquad (55)$$

the only ones capable of yielding deformation.

This assertion is based on the fact that eq. (51) – a premise for expressing eq. (52) – does not hold for rigid body displacements, since, no matter how $\mathbf{H}_\infty$ is obtained, it is the result of an integral along a closed boundary that envelopes all source points and, as a consequence,

$$\mathbf{H}_\infty \mathbf{W} = -\mathbf{W} \qquad (56)$$

According to that, following cases are to be considered.

a) Prescribed mixed (Cauchy) boundary conditions. This is the general case. The same concepts that gave rise to eq. (22) apply to eq. (52):

$$\overline{\mathbf{H}}_\mathrm{a}\mathbf{d} = \overline{\mathbf{G}}_\mathrm{a}\mathbf{t} + \overline{\mathbf{b}}_\mathrm{a} \qquad (57)$$

where

$$\overline{\mathbf{H}}_\mathrm{a} = \overline{\mathbf{H}}(\mathbf{I} - \mathbf{W}\mathbf{W}^\mathrm{T}) \qquad (58)$$

$$\overline{\mathbf{G}}_\mathrm{a} = \overline{\mathbf{G}}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^\mathrm{T}) \qquad (59)$$

$$\overline{\mathbf{b}}_\mathrm{a} = \overline{\mathbf{b}} - \mathbf{G}\mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\overline{\mathbf{b}}_r \qquad (60)$$

in which $\overline{\mathbf{b}}$ and $\overline{\mathbf{b}}_\mathrm{a}$ may be obtained as explained in Appendix I. Rigid body displacements and unbalanced traction forces, as they are canceled out in eq. (57), are meaningless quantities in this static formulation. The results at internal points are obtained as usual, by means of Somigliana's displacement and stress identities.

b) Prescribed displacements along the whole boundary (Dirichlet boundary conditions). Equation (57) still holds. The vector of the balanced traction forces is obtained from this equation by making use of the Bott-Duffin inverse $\mathbf{G}^{(-1)}_\mathrm{a}$, according to eq. (30). A rigid body displacement field corresponding to the inadmissible vector $\mathbf{W}\mathbf{W}^\mathrm{T}\mathbf{d}$, which has been canceled out in eq. (57), may be added to the displacement results obtained at internal points.

c) Prescribed traction forces along the entire boundary (Neumann boundary conditions). Equation (57) is also applicable in this case, as part of the complete solution. The rigid body displacement field remains unknown (only relative displacements are evaluated, as usual). Now, the equation system (57) is solved by means of the Bott-Duffin inverse of $\overline{\mathbf{H}}_\mathrm{a}$. The unbalanced part $\mathbf{Z}\mathbf{Z}^\mathrm{T}\mathbf{t}$ of the traction forces cannot be transformed into any displacements. However, the stress field it generates may be evaluated directly using Somigliana's stress identity. Summarizing, in this case an admissible displacement vector $\mathbf{d}_\mathrm{a}$ is obtained by solving the constrained system (57) for the given traction forces vector $\mathbf{t}$ and afterwards the results at internal points are evaluated considering the known parameters $\mathbf{d}_\mathrm{a}$ and $\mathbf{t}$ (and eventual body forces).

7
An alternative development

7.1
Preliminary considerations

In this paper, the subject in question seems to have been theoretically covered to exhaustion. However, one or two more manipulations of the matrices $\mathbf{G}$ and $\mathbf{G}^{(-1)}_\mathrm{a}$ may be of academic interest.

Both eqs. (30) and (31) may be generic expressed as

$$\mathbf{G}_{\mathrm{a}}^{(-1)} = (\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}})\mathbf{G}_{g}^{(-1)} \tag{61}$$

in which $\mathbf{G}_g$ stands for either the nonsingular, well-conditioned matrix $(\mathbf{G}_{\mathrm{a}} + \mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}})$ or the nonsingular, but possibly not so well conditioned matrix $\mathbf{G}$. Then, one may obtain from eq. (61)

$$\begin{aligned}\mathbf{G}_{\mathrm{a}}^{(-1)} &= \mathbf{G}_{g}^{(-1)} - \mathbf{G}_{g}^{(-1)}\mathbf{G}_{g}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}_{g}^{(-1)} \\ &= \mathbf{G}_{g}^{(-1)}(\mathbf{I} - \mathbf{G}_{g}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}_{g}^{(-1)})\end{aligned} \tag{62}$$

in which $(\mathbf{I} - \mathbf{G}_{g}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}_{g}^{(-1)})$ is an idempotent matrix, although not an orthogonal projector (it is generally nonsymmetrical). As a consequence, eq. (29) may be rewritten as

$$\mathbf{t} = \mathbf{G}_{g}^{(-1)}(\mathbf{I} - \mathbf{G}_{g}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}_{g}^{(-1)})(\mathbf{Hd} - \mathbf{b}_{\mathrm{a}}) \tag{63}$$

which yields, since $\mathbf{G}_{g}^{(-1)}$ is nonsingular,

$$\mathbf{G}_{g}\mathbf{t} = (\mathbf{I} - \mathbf{G}_{g}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}_{g}^{(-1)})(\mathbf{Hd} - \mathbf{b}_{\mathrm{a}}) \tag{64}$$

Since the matrix $\mathbf{G}_g$ is non-singular, equation above is valid only if $\mathbf{t}$ is a vector of balanced forces. As a consequence, it is possible to rearrange eq. (64), for mixed boundary conditions, as in eq. (28), and still get balanced traction forces (note that, since $\mathbf{H}$ is singular, one cannot obtain from eq. (28) a vector $\mathbf{d}$ which is orthogonal to rigid body displacements, except in some particular cases). In this sense, eq. (64) might be considered as an improvement of eq. (22), for $\mathbf{G}_g \equiv (\mathbf{G}_{\mathrm{a}} + \mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}})$.

Unfortunately, however, establishing eq. (64) requires the complete computation of the matrix $\mathbf{G}_g$ and at least the solution of a system of the type $\mathbf{G}_g^{\mathrm{T}}\mathbf{X} = \mathbf{Z}$. As a consequence, eq. (64) does not seem more promising than eq. (29), in terms of both computational time and storage allocation. If one is aiming at expressing a stiffness-type matrix, such as in eq. (37), then $\mathbf{G}_{\mathrm{a}}^{(-1)}$, as given in eq. (30) is definitely better than eq. (62).

7.2 Another way of building a constrained system

Although the expression (62) of the Bott-Duffin inverse is not practical, as compared with the standard one, it may be academically interesting to obtain it in a different way than according to the steps presented in items 4.1 and 4.2.

Disregarding the body forces, for the sake of brevity, one may express the boundary element eq. (6) as

$$\mathbf{Gt} = \mathbf{Hd} + \text{residuum} \tag{65}$$

in which the residuum is to be expressed in terms of the unbalanced forces $\mathbf{Z}^{\mathrm{T}}\mathbf{t}$. Considering eqs. (25–27), it is evident that eq. (65) might look like

$$\mathbf{G}\mathbf{t}_1 = \mathbf{Hd} - \mathbf{G}\mathbf{Z}\boldsymbol{\chi}\mathbf{Z}^{\mathrm{T}}\mathbf{t}_0 \tag{66}$$

in which $\mathbf{t}_1$ is an improved solution, $\boldsymbol{\chi}$ is a matrix of unknowns to be determined in some way and $\mathbf{t}_0$ is a first approximation, obtained from eq. (65) by disregarding the residuum:

$$\mathbf{t}_0 = \mathbf{G}^{-1}\mathbf{Hd} \tag{67}$$

According to that, the improved solution is given by

$$\mathbf{G}\mathbf{t}_1 = \mathbf{Hd} - \mathbf{G}\mathbf{Z}\boldsymbol{\chi}\mathbf{Z}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{Hd} = (\mathbf{I} - \mathbf{G}\mathbf{Z}\boldsymbol{\chi}\mathbf{Z}^{\mathrm{T}}\mathbf{G}^{-1})\mathbf{Hd} \tag{68}$$

from which follows

$$\mathbf{t}_1 = (\mathbf{I} - \mathbf{Z}\boldsymbol{\chi}\mathbf{Z}^{\mathrm{T}})\mathbf{G}^{-1}\mathbf{Hd} \tag{69}$$

If one imposes that $\boldsymbol{\chi} = \mathbf{I}$ in eq. (69), the traction forces $\mathbf{t}_1$ turn out to be already in balance. As a consequence, one may express eq. (65) according to eq. (68) for $\boldsymbol{\chi} = \mathbf{I}$:

$$\mathbf{Gt} = (\mathbf{I} - \mathbf{G}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}^{-1})\mathbf{Hd} \tag{70}$$

which coincides with what was expressed in eq. (64) for $\mathbf{G}_g \equiv \mathbf{G}$.

7.3 An approximation of equation (70)

Telles and De Paula (1991) used a similar procedure for dealing with the subject of this paper (actually, the procedure just outlined above was motivated by this reference). However, they considered, for some unexplained reason, the basis $\mathbf{W}$ of the rigid body displacements in the expression of the residuum, obtaining as starting point.

$$\mathbf{G}\mathbf{t}_1 = \mathbf{Hd} - \mathbf{W}\boldsymbol{\chi}\mathbf{R}^{\mathrm{T}}\mathbf{t}_0 \tag{71}$$

instead of eq. (66). Then, following steps similar to the ones outlined in eqs. (67–70), they arrived at

$$\mathbf{Gt} = (\mathbf{I} - \mathbf{W}(\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{W})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1})\mathbf{Hd} \tag{72}$$

in which $(\mathbf{I} - \mathbf{W}(\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{W})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1})$ is also an idempotent matrix (not an orthogonal projector). The equation above is an approximation of eq. (70) and, as a consequence, the stiffness-type matrix obtained by Telles and De Paula

$$\mathbf{K}^{u} = \mathbf{L}\mathbf{G}^{-1}(\mathbf{I} - \mathbf{W}(\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{W})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1})\mathbf{H} \tag{73}$$

is also an approximation of the matrix given in eq. (37). Fortunately, one may demonstrate, according to the achievements of the present paper, that

$$\mathrm{Rank}(\mathbf{K}^{u}) = \mathrm{rank}(\mathbf{I} - \mathbf{W}\mathbf{W}^{\mathrm{T}}) \tag{74}$$

that is, $\mathbf{K}^{u}$ is also spectrally consistent.

8 Final considerations

8.1 Preliminaries

It was demonstrated that the rigid body displacement functions, which are part of the fundamental solution of a boundary element method, cannot be left out of the formulation, if one is aiming at establishing a consistent system of equations involving balanced forces.

All achievements of the present paper started from sound well established physical principles – which are the basis of the traditional boundary element formulation – and were developed according to demonstrated mathematical statements.

As a consequence, eqs. (22), (30) and (37) are to be considered as definitive achievements in terms of a consistent boundary element formulation – a formulation that adequately takes into account the spectral properties of the matrices involved as well as the equilibrium of the resulting forces.

It was demonstrated through three different ways (items 4.1, 4.2, and 7.2) that an adequate treatment of the matrix **G** in the boundary element method leads unavoidably to one and the same constrained system, the mathematical expression of which had been already established and dealt with by Bott and Duffin (1953). Notwithstanding, all Bott and Duffin's mathematical accomplishments that are relevant to this paper have been re-investigated and confirmed.

8.2
On the feasibility of an equation of the type Ax = y

Establishing an equation of the type

$$\mathbf{Ax} = \mathbf{y} \tag{28}$$

in which the vectors **y** and **x** gather the known and unknown quantities, respectively, is not a straightforward step starting from the system

$$\mathbf{Hd} = \mathbf{G}(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}})\mathbf{t} + (\mathbf{b} - \mathbf{GZ}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{\mathrm{r}}) \quad \text{or} \quad \mathbf{Hd} = \mathbf{G}_{\mathrm{a}}\mathbf{t} + \mathbf{b}_{\mathrm{a}} \tag{22}$$

if one is looking for balanced forces, but can be accomplished from the alternative formulation

$$\mathbf{G}_g\mathbf{t} = (\mathbf{I} - \mathbf{G}_g\mathbf{ZZ}^{\mathrm{T}}\mathbf{G}_g^{(-1)})(\mathbf{Hd} - \mathbf{b}_{\mathrm{a}}) \tag{64}$$

at the expense of some additional computational effort.

Using either equation

$$\mathbf{Hd} = \mathbf{Gt} + \mathbf{b} \tag{8}$$

or eq. (64) for arriving at eq. (28) is a matter of choice and depends on what the analyst is aiming at. One may think of examples in which the blind application of eq. (8) leads to very bad results, as compared to the consistent formulation. However, it is not methodologically correct to choose an impractical example to illustrate how relatively good a formulation might be. In terms of arriving at an equation of the type of eq. (28), for a reasonably formulated and discretized problem, the present formulation does not mean a breakthrough: it suffices that the analyst knows that eq. (8) generally leads to unbalanced forces and that one has to be careful in which concerns ill-conditioning of the system.

Nonetheless, a simple illustration is given in the following item (Dumont, 1996).

8.3
A simple numerical example

Consider the solution of the Laplace equation on a rectangular domain, according to Fig. 1. The boundary is discretized with a total of 8 constant elements for both potential **u** and gradient **t**. The applied boundary conditions are $u = 0$ along the edges $x = 0$ and $y = 0$, $u_x = 0$ along the edge $x = .5$ and $u = x(1 - x)$ along $y = 1$. The results obtained according to eqs. (8) and (22) are represented in Fig. 2 (crosses and circles, respectively), as compared with the analytical solution $u(x, y) = .02234116360\ sinh(\pi y)\ sin(\pi x) + .1542330835\,10^{-5}\ sinh(3\pi y)\,sin(3\pi x)\ + .6221263291\,10^{-9}\,sinh(5\pi y)sin(5\pi x) + .1543617478\,10^{-19}\ sinh(2\pi y)sin(2\pi x) + \ldots$. The first part of the graphic represents gradients across the edge $y = 0$; then follow potentials along $x = .5$ and gradients across $y = 1$ and $x = 0$. This example was repeated for different scales. The results obtained with eq. (22) presented always the same degree of approximation and were always self-equilibrated. The same kind of results was observed for different boundary elements and discretization meshes. The results of Fig. 2 are reproduced in Table 1. Two extra columns are added with the results obtained for the dimensions of the problem outlined in Fig. 1 multiplied by 1000. An accuracy of eight figures was used in all calculations.

8.4
On a consistent stiffness-type matrix

A stiffness-type matrix

$$\mathbf{LG}_{\mathrm{a}}^{(-1)}\mathbf{H} \equiv \mathbf{K} \tag{37}$$

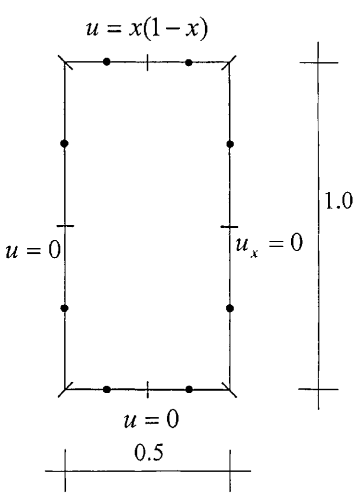

Fig. 1. Rectangular domain and discretization with eight constant elements for the solution of the Laplace equation.

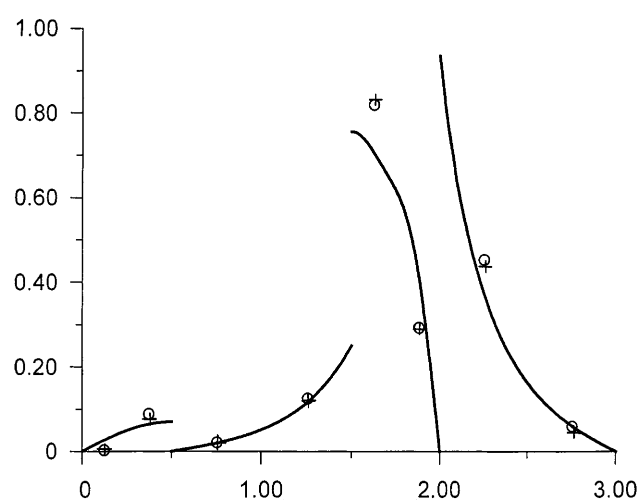

Fig. 2. Results obtained along the boundary for the solution of the Laplace equaiton: $-u_y$ at $y = 0$, u at $x = .5$, u_y at $y = 1$ and $-u_x$ at $x = 0$.

Table 1. Values of potentials and gradients calculated at the nodal points for example of Fig. 1

	Analytical values	Values acoording to eq. (8)		Values according to eq. (22)	
		Sides × 1	Sides × 1000	Sides × 1	Sides × 1000
Gradients	−.268727E-01	−.632023E-02	−.161494E-04	−.383530E-02	−.383503E-05
along $y = 0$	−.648386E-01	−.772419E-01	−.943112E-04	−.880434E-01	−.880431E-04
Potentials	.193991E-01	.182500E-01	.198856E-01	.212514E-01	.212514E-01
along $x = .5$	.115934E 00	.121802E 00	.123438E 00	.124804E 00	.124804E 00
Gradients	.704797E 00	.833699E 00	.816630E-03	.822898E 00	.822898E-03
along $y = 1$	.424514E 00	.291004E 00	.281175E-03	.293489E 00	.293489E-03
Gradients	−.376110E 00	−.436932E 00	−.444930E-03	−.451610E 00	−.451611E-03
along $x = 0$	−.610455E-01	−.459658E-01	−.539645E-04	−.606441E-01	−.606441E-04

is obtained at the expense of almost negligible additional computational effort, as compared with the traditional, inconsistent formulation. Telles and De Paula (1991) suggested a matrix

$$\mathbf{K}^u = \mathbf{LG}^{-1}(\mathbf{I} - \mathbf{W}(\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{W})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{G}^{-1})\mathbf{H} \tag{73}$$

which is to be considered as an approximation of $\mathbf{K}$ in eq. (37), although it is more complicated and less reliable (owing to the questionable conditioning of matrix $\mathbf{G}$). Use of eq. (37) is strongly recommended, since $\mathbf{K}$ has been obtained according to sound physical and mathematical bases.

A numerical illustration of the advantages of using eq. (37) is not methodologically adequate, since, for a well formulated and discretized problem, a remarkable gain in accuracy cannot be demonstrated, although convenient examples may always be thought of. As a matter of fact, one cannot agree that it is adequate to compare results obtained through "symmetrized" consistent and inconsistent $\mathbf{K}$-matrices (Telles and De Paula, 1991), since "symmetrization", as already demonstrated by Dumont (1986, 1987), is a conceptual mistake (see also Li, Han, Mang and Torzicky, 1986). As a consequence, an example that shows that a big mistake yields bigger errors than a lesser one is a tautology.

9 Conclusions

This paper outlines the correct boundary element formulation in respect to the role played by the rigid body displacements that are present in any fundamental solution. It is not expected a remarkable improvement of the numerical results obtained in the frame of the present achievements, as compared with the results of either the traditional, inconsistent formulation or any other formulation, for any well formulated and discretized example. Since the present paper proposes adequately and solves exactly a basic pending problem, the author suggests that the main achievements outlined herein – concerning eqs. (22), (30) and (37) – be incorporated in the textbooks on boundary elements.

Appendix I – a simple way of expressing the contribution of domain forces in terms of boundary integrals

The last two terms involving domain integrals in eq. (6) represent the work done by the body forces b_i on the fundamental displacement solution:

$$\int_\Omega u_i^* b_i \, \mathrm{d}\Omega \tag{I.1}$$

One may start by finding a particular solution of the equilibrium differential equation

$$\sigma^{\mathrm{p}}_{ji,j} + b_i = 0 \quad \text{in} \quad \Omega \tag{I.2}$$

in which σ^{p}_{ji} is an arbitrary (as simple and convenient as possible) particular solution, whenever it exists analytically. Then, one may substitute b_i by $-\sigma^{\mathrm{p}}_{ji,j}$ in eq. (I.1) and, after application of integration by parts, arrive at

$$\begin{aligned}\int_\Omega u_i^* b_i \, \mathrm{d}\Omega &= -\int_\Omega u_i^* \sigma^{\mathrm{p}}_{ji,j} \, \mathrm{d}\Omega \\ &= -\int_\Omega (u_i^* \sigma^{\mathrm{p}}_{ji})_{,j} \, \mathrm{d}\Omega + \int_\Omega u^*_{i,j} \sigma^{\mathrm{p}}_{ji} \, \mathrm{d}\Omega\end{aligned} \tag{I.3}$$

For an elastic medium, the stresses σ^{p}_{ji} are related to a displacement field u^{p}_i such that $\sigma^{\mathrm{p}}_{ji} = C_{ijkl} u^{\mathrm{p}}_{k,l}$. Then, the integrand of the last integral in eq. (I.3) may undergo the transformations

$$u^*_{i,j}\sigma^{\mathrm{p}}_{ji} = u^*_{i,j} C_{ijkl} u^{\mathrm{p}}_{k,l} = \sigma^*_{lk} u^{\mathrm{p}}_{k,l} \equiv \sigma^*_{ji} u^{\mathrm{p}}_{i,j} \tag{I.4}$$

which involve no approximations, since both particular and fundamental solutions are analytical in Ω. Considering that, one may carry out further transformations in eq. (I.3), by applications of integration by parts:

$$\begin{aligned}\int_\Omega u_i^* b_i \, \mathrm{d}\Omega &= -\int_\Omega (u_i^* \sigma^{\mathrm{p}}_{ji})_{,j} \, \mathrm{d}\Omega + \int_\Omega \sigma^*_{ji} u^{\mathrm{p}}_{i,j} \, \mathrm{d}\Omega \\ &= -\int_\Omega (u_i^* \sigma^{\mathrm{p}}_{ji})_{,j} \, \mathrm{d}\Omega + \int_\Omega (\sigma^*_{ji} u^{\mathrm{p}}_i)_{,j} \, \mathrm{d}\Omega \\ &\quad - \int_\Omega \sigma^*_{ji,j} u^{\mathrm{p}}_i) \, \mathrm{d}\Omega\end{aligned} \tag{I.5}$$

The next step consists in applying Gauss' theorem to the first two integrals at the right-hand side of equation above, thus yielding

$$\begin{aligned}\int_\Omega u_i^* b_i \, \mathrm{d}\Omega &= -\int_\Gamma \sigma^{\mathrm{p}}_{ji} \eta_j u_i^* \, \mathrm{d}\Gamma + \int_\Gamma \sigma^*_{ji} \eta_j u^{\mathrm{p}}_i \, \mathrm{d}\Gamma \\ &\quad - \int_\Omega \sigma^*_{ji,j} u^{\mathrm{p}}_i \, \mathrm{d}\Omega\end{aligned} \tag{I.6}$$

in which η_j are the cosine directors of the outward normal to the elementary surface $\mathrm{d}\Gamma$.

Finally, one makes use of eqs. (1) and (2) to relate eq. (I.6) to the arbitrary singular forces p_m^* of the fundamental solution:

$$\int_\Omega u_i^* b_i \, d\Omega$$
$$= -p_m^* \left(\int_\Gamma \sigma_{ji}^{p} \eta_j u_{im}^* \, d\Gamma \right) - p_m^* \left(\int_\Gamma \sigma_{ji}^{p} \eta_j u_{is}^{r} C_{sm} \, d\Gamma \right)$$
$$+ p_m^* \left(\int_\Gamma p_{im}^* u_i^{p} \, d\Gamma - \int_\Omega \sigma_{ijm,j}^* u_i^{p} \, d\Omega \right) \qquad (I.7)$$

The arbitrary integration constants that arise in the passage from $u_{i,j}^{p}$ to u_i^{p} in eq. (I.5) do not affect the terms in the last brackets of eq. (I.7), since, as a property of a fundamental solution,

$$\int_\Gamma p_{im}^* \, d\Gamma - \int_\Omega \sigma_{ijm,j}^* \, d\Omega \equiv 0 \qquad (I.8)$$

As a consequence of the development above, the last two terms of eq. (6) may be substituted by the terms of eq. (I.7), which involve only boundary integrals, considering eq. (4). According to that, eqs. (11) and (15) may be re-written as

$$\mathbf{b} \equiv b_m = -\int_\Gamma \sigma_{ji}^{p} \eta_j u_{im}^* \, d\Gamma + \int_\Gamma p_{im}^* u_i^{p} \, d\Gamma$$
$$- \int_\Omega \sigma_{ijm,j}^* j u_i^{p} \, d\Omega \qquad (I.9)$$

$$\mathbf{b}^{r} \equiv b_s = -\int_\Gamma \sigma_{ji}^{p} \eta_j u_{is}^{r} \, d\Gamma \qquad (I.10)$$

For a cavity in an infinite domain (item 6), one may obtain vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{b}}^{r}$ in the same way as outlined above, just considering that the outward normal vector to Γ is now reversed.

The formulation just outlined in this Appendix is strictly speaking not original. It serves the only purpose of demonstrating that rigid body terms, which may be present in the expression of the displacement functions u_{im}^* and u_i^p, do not influence the expression of the equivalent nodal displacements $\mathbf{b}$ and $\mathbf{b}^{r}$, as given by eqs. (I.9) and (I.10). A thorough treatment of body forces in the boundary element methods may be found in (Partridge et al, 1992), for instance.

Appendix II – a simple theorem on orthogonal projectors

Consider two arbitrary rectangular matrices $\mathbf{Z}$ and $\mathbf{X}$, the columns of which represent two different orthogonal bases of a given space, that is, such that

$$\mathbf{Z}^{T}\mathbf{Z} = \mathbf{X}^{T}\mathbf{X} = \mathbf{I} \qquad (II.1)$$

Theorem: $\mathbf{X}\mathbf{X}^{T} \equiv \mathbf{Z}\mathbf{Z}^{T}$.
Proof: The matrix $\mathbf{X}$ may be obtained from $\mathbf{Z}$ through a linear transformation represented by a nonsingular square matrix $\boldsymbol{\lambda}$:

$$\mathbf{X} = \mathbf{Z}\boldsymbol{\lambda} \qquad (II.2)$$

Since

$$\mathbf{X}^{T}\mathbf{X} = \mathbf{I} \qquad (II.3)$$

it follows from eqs. (II.2) and (II.1) that also

$$\boldsymbol{\lambda}^{T}\boldsymbol{\lambda} = \mathbf{I} \qquad (II.4)$$

Then, using eq. (II.2), one may conclude that

$$\mathbf{X}\mathbf{X}^{T} = \mathbf{Z}\boldsymbol{\lambda}\boldsymbol{\lambda}^{T}\mathbf{Z}^{T} = \mathbf{Z}\mathbf{Z}^{T} \qquad (II.5)$$

since the square matrix $\boldsymbol{\lambda}$ is also orthogonal, according to eq. (II.4), and, as a consequence, $\boldsymbol{\lambda}\boldsymbol{\lambda}^{T} = \mathbf{I}$.

Appendix III – some features of the bott-duffin inverse

Consider a square matrix $\mathbf{G}$, an arbitrary vector $\mathbf{y}$ with the same dimension of $\mathbf{G}$, and a subspace $\mathbf{S}_z$ represented by an orthogonal projector

$$\mathbf{P}_z = \mathbf{I} - \mathbf{Z}\mathbf{Z}^{T} \qquad (III.1)$$

The boundary element equation (22) may be expressed as

$$\mathbf{G}\mathbf{t} - \mathbf{Z}\mathbf{Z}^{T}\mathbf{t} = \mathbf{y} \qquad (III.2)$$

or, exactly as Bott and Duffin formulated for the first time [10], as the constrained system

$$\mathbf{G}\mathbf{t} + \mathbf{t}_\perp = \mathbf{y} \quad \text{with} \quad \mathbf{t} \in \mathbf{S}_z \quad \text{and} \quad \mathbf{t}_\perp \in \mathbf{S}_{z^\perp} \qquad (III.3)$$

which may be further transformed into the system

$$(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})\mathbf{x} = \mathbf{y} \qquad (III.4)$$

thus yielding

$$\mathbf{t} = \mathbf{P}_z\mathbf{x} \quad \text{and} \quad \mathbf{t}_\perp = \mathbf{P}_{z^\perp}\mathbf{x} = \mathbf{y} - \mathbf{G}\mathbf{P}_z\mathbf{x} \qquad (III.5)$$

If the matrix $(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})$ is nonsingular, the solution

$$\mathbf{t} = \mathbf{P}_z(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})^{-1}\mathbf{y}, \quad \mathbf{t}_\perp = \mathbf{y} - \mathbf{G}\mathbf{t} \qquad (III.6)$$

is unique. The matrix

$$\mathbf{G}_{a}^{(-1)} = \mathbf{P}_z(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})^{-1} \qquad (III.7)$$

is the Bott-Duffin inverse of $\mathbf{G}$. The superscript "a" of $\mathbf{G}_{a}^{(-1)}$ stands in this paper for *admissible*, meaning the constrained, admissible part of an otherwise arbitrary inverse matrix of the transformation indicated in eq. (22), for a singular matrix $\mathbf{G}_a$. The solution $\mathbf{t}$, according to eq. (III.6), is the *admissible* part of the solution one is looking for in eq. (III.3). In the present paper, it corresponds to the part of the traction forces, which are in equilibrium along the boundary.

Theorem (Bott-Duffin [9], apud Ben-Israel [8]): Let $(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})$ be nonsingular. Then
a) The Equation

$$\mathbf{G}\mathbf{t} + \mathbf{t}_\perp = \mathbf{y} \quad \text{with} \quad \mathbf{t} \in \mathbf{S}_z \quad \text{and} \quad \mathbf{t}_\perp \in \mathbf{S}_{z^\perp} \qquad (III.3)$$

has for every $\mathbf{t}$ the unique solution

$$\mathbf{t} = \mathbf{G}_{a}^{(-1)}\mathbf{y} \quad \text{and} \quad \mathbf{t}_\perp = (\mathbf{I} - \mathbf{G}\mathbf{G}_{a}^{(-1)})\mathbf{y} \qquad (III.8)$$

in which $\mathbf{G}_{a}^{(-1)}$ is given by eq. (III.7).
b) $\mathbf{G}$, $\mathbf{P}_z$ and $\mathbf{G}_{a}^{(-1)}$ satisfy

$$\mathbf{P}_z = \mathbf{G}_{a}^{(-1)}\mathbf{G}\mathbf{P}_z = \mathbf{P}_z\mathbf{G}\mathbf{G}_{a}^{(-1)} \qquad (III.9)$$

$$\mathbf{G}_{a}^{(-1)} = \mathbf{P}_z\mathbf{G}_{a}^{(-1)} = \mathbf{G}_{a}^{(-1)}\mathbf{P}_z \qquad (III.10)$$

c) Rank($\mathbf{P}_z\mathbf{G}\mathbf{P}_z$) = rank ($\mathbf{P}_z$).

Proof

a) This follows from the equivalence of eq. (III.3) and eq. (III.4) together with eq. (III.5), which is unique for a nonsingular matrix $(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp})$.

b) From eq. (III.7), $\mathbf{P}_z\mathbf{G}_\mathrm{a}^{(-1)} = \mathbf{G}_\mathrm{a}^{(-1)}$. Postmultiplying $\mathbf{G}_\mathrm{a}^{(-1)}(\mathbf{G}\mathbf{P}_z + \mathbf{P}_{z^\perp}) = \mathbf{P}_z$ by $\mathbf{P}_z$ gives $\mathbf{G}_\mathrm{a}^{(-1)}\mathbf{G}\mathbf{P}_z = \mathbf{P}_z$. Therefore $\mathbf{G}_\mathrm{a}^{(-1)}\mathbf{P}_{z^\perp} = \mathbf{0}$ and $\mathbf{G}_\mathrm{a}^{(-1)}\mathbf{P}_z = \mathbf{G}_\mathrm{a}^{(-1)}$. Multiplying the expression of $\mathbf{t}_\perp$ in eq. (III.8) by $\mathbf{P}_z$ gives $(\mathbf{P}_z - \mathbf{P}_z\mathbf{G}\mathbf{G}_\mathrm{a}^{(-1)})\mathbf{y} = 0$ for all $\mathbf{y}$, thus $\mathbf{P}_z = \mathbf{P}_z\mathbf{G}\mathbf{G}_\mathrm{a}^{(-1)}$.
From these results it follows that the Bott-Duffin inverse $\mathbf{G}_\mathrm{a}^{(-1)}$ is a $\{1,2\}$-inverse of $(\mathbf{P}_z\mathbf{G}\mathbf{P}_z)$:

$$\text{1-inverse: } (\mathbf{P}_z\mathbf{G}\mathbf{P}_z)\mathbf{G}_\mathrm{a}^{(-1)}(\mathbf{P}_z\mathbf{G}\mathbf{P}_z) = (\mathbf{P}_z\mathbf{G}\mathbf{P}_z) \qquad \text{(III.11)}$$

$$\text{2-inverse: } \mathbf{G}_\mathrm{a}^{(-1)}(\mathbf{P}_z\mathbf{G}\mathbf{P}_z)\mathbf{G}_\mathrm{a}^{(-1)} = \mathbf{G}_\mathrm{a}^{(-1)} \qquad \text{(III.12)}$$

c) The matrices $\mathbf{G}_\mathrm{a}^{(-1)}$ and $(\mathbf{P}_z\mathbf{G}\mathbf{P}_z)$ have the same rank, as it may be verified by inspection of eqs. (III.11) and (III.12). Then, since $\mathrm{rank}(\mathbf{G}_\mathrm{a}^{(-1)}) = \mathrm{rank}\,(\mathbf{P}_z)$, according to eq. (III.7), it follows that $\mathrm{rank}(\mathbf{P}_z\mathbf{G}\mathbf{P}_z) = \mathrm{rank}(\mathbf{P}_z)$.

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