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Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties

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Abstract This paper presents the local boundary integral formulation for an elastic body with nonhomogeneous material properties. All nodal points are surrounded by a simple surface centered at the collocation point. Only one nodal point is included in each the sub-domain. On the surface of the sub-domain, both displacements and traction vectors are unknown generally. If a modified fundamental solution, for governing equation, which vanishes on the local boundary is chosen, the traction value is eliminated from the local boundary integral equations for all interior points. For every sub-domain, the material constants correspond to those at the collocation point at the center of sub-domain. Meshless and polynomial element approximations of displacements on the local boundaries are considered in the numerical analysis.

1 Introduction

The boundary element method (BEM) has become an efficient and popular alternative to the finite element method (FEM). Nevertheless, it is still believed that the FEM is more versatile and appropriate, mainly when geometrical and material nonlinearities and nonhomogeneous material properties are analysed. This can be explained by the fact that the fundamental solutions for the governing equations of such problems are not available, in general. If Kelvin's fundamental solution for a homogeneous material problem is used for the global domain, the global boundary integral equations have to be supplemented by the integral representations for displacement gradients in the unique boundary-domain intergral formulation (Sladek et al. 1993). Displacement gradients at interior points are expressed in terms of the displacements and tractions taken at the boundary nodes. Special regularization techniques are required for the evaluation of nearly singular integrals in points lying close to the boundary. If the gradients of material properties, that occur in the domain integral, are significant, the domain integral is dominant in the BIE. To eliminate the dominance of such domain integral in the global integral equation formulation, an alternative local boundary integral equation (LBIE) formulation is adopted in this paper for an elastic body with nonhomogeneous material properties. The LBIE has been recently introduced by Atluri et al. (1998) for potential problems. Numerical results obtained for a domain with homogeneous properties were not sensitive to the selection of the sub-domain size. Then, the size of the sub-domain can be selected to be sufficiently small. Assuming that the present properties in each sub-domain are homogeneous, and are equal to those at the center point of the sub-domain, a Kelvin type fundamental solution is assumed for each small sub-domain. In such an approach the dominance of the domain integral is substantially eliminated, and the displacement gradients in the local domain can be expressed in a differential form. Then, the LBIE is sufficient for a unique integral formulation of a nonhomogeneous problem. On the surface of sub-domain both displacements and traction vectors are unknown if the standard Kelvin fundamental solution (Balas et al. 1989) is used. If a 'companion solution' is introduced to the Kelvin fundamental solution, so as to give a zero value on the other hand, to the final fundamental solution for the displacement on the local boundary, the traction quantity is eliminated in the LBIE, for all interior points. The displacements on the local boundary and in the interior of the sub-domain are approximated either by the moving least-square (MLS) in a meshless implementation, or by using the standard polynomial interpolation within the domain elements. The essential idea of MLS interpolants is that it is only necessary to construct an array of nodes in the domain under consideration (Belytschko et al. 1994). Thus, the method is completely element-free. The standard polynomial elements are constructed in such a way that the nodes that are immediately adjacent to the collocation point create the nodal points of the element over which the displacements are approximated. Then, the whole sub-domain and its boundary are included into the element. One attractive property of the MLS, and specially created polynomial elements, is their continuity of displacements as well as strains. Numerical implementation is illustrated, using problem of road, whose quadrilateral cross-section has nonhomogeneous material properties, and which is subjected to a uniform tension.

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Boundary-domain integral equations for a body with nonhomogeneous material properties

Consider an isotropic and linear elastic continuum, whose Young's modulus depends on the Cartesian co-ordinates, and whose Poisson's ratio is a constant. Moreover, we shall assume that the Young's modulus is given by a differentiable function $E(x)$. Under these assumptions, we can write the tensor of material coefficient as

$$c_{ijkl}(x) = \mu(x)c^0_{ijkl}, \quad \mu(x) = \frac{E(x)}{2(1+\nu)} \tag{1}$$

where

$$c^0_{ijkl} = 2\nu/(1-2\nu)\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

Recall, that ν should be replaced everywhere (except Eq. (1)) by $\nu/(1+\nu)$ in the case of plane stress problems. The tensor c^0_{ijkl} corresponds to a homogeneous, isotropic and linear elastic continuum with shear modulus $\mu_0 = 1$ and Poisson ratio ν.

The stress tensor can be expressed in terms of displacement gradients by

$$\sigma_{ij}(x) = c_{ijkl}(x)u_{k,l}(x) \tag{2}$$

and the equilibrium equations become

$$\left(c_{ijkl}u_{k,l}\right)_{,j} = -X_i \tag{3}$$

Hence and from (1), we may write

$$c^0_{ijkl}u_{k,lj} = -\frac{1}{\mu(x)}X_i - \frac{\mu_{,j}}{\mu}(x)c^0_{ijkl}u_{k,l}(x) \tag{4}$$

Eventually, substituting for the tensor of material constants, c^0_{ijkl}, one obtains the expression corresponding to a nonhomogeneous isotropic medium:

$$\mu u_{i,kk} + \mu\frac{1}{1-2\nu}u_{k,ki} = -X_i - \mu_{,i}\frac{2\nu}{1-2\nu}u_{k,k} - \mu_{,j}(u_{i,j} + u_{j,i}) \tag{5}$$

Apparently, it is impossible to find the closed form fundamental solution for the operator

$$c^0_{ijkl}\left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_j} + \frac{\mu_{,j}(x)}{\mu(x)}\frac{\partial}{\partial x_l}\right)$$

in general.

On the other hand, the fundamental displacements $U_{km}(r)$, for an elastic homogeneous continuum (the Kelvin solution for $\mu = 1$), satisfy the equation:

$$c^0_{ijkl}\partial_j\partial_l U_{km}(y-x) = -\delta_{im}\delta(x-y) \tag{6}$$

and the corresponding fundamental tractions are given by

$$T_{im}(\eta, y) = c^0_{ijkl}n_j(\eta)U_{km,l}(\eta - y)$$

Following the derivation of the boundary-domain formulation (Sladek et al. 1993), the integral representation of displacements in a nonhomogeneous elastic medium can be written as

$$u_k(y) = \int_\Gamma \lfloor t^*_1(\eta)U_{ik}(\eta - y) - u_i(\eta)T_{ik}(\eta, y)\rfloor \mathrm{d}\Gamma_\eta + \int_\Omega g_i(x)U_{ik}(x-y)\mathrm{d}\Omega_x + W_k(y) \tag{7}$$

where the modified traction vector is defined by

$$t_i(\eta) = \mu(\eta)t^*_i(\eta) \quad \text{or} \quad t^*_i(\eta) = n_j(\eta)c^0_{ijkl}u_{k,l}(\eta) \tag{8}$$

and

$$W_k(y) = \int_\Omega \frac{1}{\mu(x)}X_i(x)U_{ik}(x-y)\mathrm{d}\Omega_x$$

$$g_i(x) = \frac{1}{\mu(x)}\left\{\frac{2\nu}{1-2\nu}\mu_{,i}(x)u_{j,j}(x) + \mu_{,j}(x)[u_{i,j}(x) + u_{j,i}(x)]\right\} \tag{9}$$

Due to the singular behaviour of the kernel T_{ik}, the accuracy of the numerical computation of displacements deteriorates near the boundary. This singularity can be removed by using the integral identity (Balas et al. 1989)

$$\int_\Gamma T_{ik}(\eta, y)\mathrm{d}\Gamma_\eta = -\delta_{ik} \tag{10}$$

In view of (10), we can perform the limit $y \to \zeta \in \Gamma$ in Eq. (7), and derive the nonsingular integral equation (Sladek et al. 1993)

$$\int_\Gamma [u_i(\eta) - u_i(\zeta)]T_{ik}(\eta, \zeta)\mathrm{d}\Gamma_\eta - \int_\Gamma t^*_i(\eta)U_{ik}(\eta - \zeta)\mathrm{d}\Gamma_\eta = \int_\Omega g_i(x)U_{ik}(x-\zeta)\mathrm{d}\Omega_x + W_k(\zeta) \tag{11}$$

The boundary integral equation (11) has to be supplemented by the integral representation of displacement gradients at interior points, in order to derive a unique set of equations, which describe a boundary value problem in a finite body with nonhomogeneous material properties. Although the problem of singularities has been resolved successfully in such a formulation, the discretization of both the boundary and interior domain is required (Sladek et al. 1993). Consequently, two sets of coupled algebraic equations, for boundary and interior unknowns, have to be solved.

Another approach is to use the local boundary integral equations, valid on the boundaries of simple circular domains around each of the (randomly) distributed nodal points within the analysed domain. The resulting set of algebraic equations is sparse.

We have presented above, a boundary integral equation for a finite nonhomogeneous elastic body Ω bounded by the boundary Γ. If, instead of the entire domain Ω of the given problem, we consider a sub-domain Ω_s, which is located entirely inside Ω and contains the point y, Eq. (7) becomes

$$u_k(y) = \int_{\partial\Omega_s} \lfloor t^*_i(\eta)U_{ik}(\eta - y) - u_i(\eta)T_{ik}(\eta, y)\rfloor \mathrm{d}\Gamma_\eta + \int_{\Omega_s} g_i(x)U_{ik}(x-y)\mathrm{d}\Omega_x + W_k(y) \tag{12}$$

where $\partial\Omega_s$ is the boundary of the sub-domain Ω_s.

On the artificial boundary $\partial\Omega_s$, both displacement and traction vectors are unknown. In order to get rid of the traction vector in the integral over $\partial\Omega_s$, the concept of a 'companion solution' can be utilized successfully (Atluri et al. 1999). The companion solution is associated with the fundamental solution U_{ik} and is defined as the solution to the following equations:

$$\begin{aligned} \tilde{\sigma}_{ij,j} &= 0 \quad \text{on } \Omega'_s \\ \tilde{U}_{ik} &= U_{ik} \quad \text{on } \partial\Omega'_s \end{aligned} \tag{13}$$

where Ω'_s and $\partial\Omega'_s$ are the same as those defined in Fig. 1. As usual, Ω'_s is taken as a circle in the present implementation.

The modified test function $U^*_{ik} = U_{ik} - \tilde{U}_{ik}$ has to satisfy the governing equation (6). Then, the integral representation (12) is valid also for modified fundamental solution U^*_{ik}. On a circle $\partial\Omega_s$, this fundamental solution is zero due to the second condition (13). Hence, we can write

$$\begin{aligned} u_k(y) = &-\int_{\partial\Omega_s} u_i(\eta) T^*_{ik}(\eta, y)\mathrm{d}\Gamma_\eta \\ &+ \int_{\Omega_s} g_i(x) U^*_{ik}(x-y)\mathrm{d}\Omega_x + W_k(y) \end{aligned} \tag{14}$$

for the source point y located inside Ω and

$$\begin{aligned} u_k(\zeta) &+ \int_{L_s} u_i(\eta) T^*_{ik}(\eta,\zeta)\mathrm{d}\Gamma_\eta + \lim_{y\to\zeta}\int_{\Gamma_s} u_i(\eta) T^*_{ik}(\eta, y)\mathrm{d}\Gamma_\eta \\ &- \int_{\Gamma_s} t^*_i(\eta) U^*_{ik}(\eta,\zeta)\mathrm{d}\Gamma_\eta = \int_{\Omega_s} g_i(x) U^*_{ik}(x-\zeta)\mathrm{d}\Omega_x + W_k(\zeta) \end{aligned} \tag{15}$$

for the source point located on the global boundary $\zeta \in \Gamma_s \subset \Gamma$. Note that $\partial\Omega_s = L_s \cup \Gamma_s$ with $\Gamma_s = \partial\Omega_s \cap \Gamma$. The BIE (15) is written in the limit form in contrast to Eq. (11). Such an expression of the BIE is appropriate (Sladek et al. 1999), if the unknown displacements are known only digitally as it appears in the case of the MLS approximation. The explicit expression of the modified test function can be found in (Atluri et al. 1999).

The introduction of the companion solution in this case mainly aims at simplifying the formulation and reducing the computational cost. The unknown traction vector $t^*_i(\eta)$ on Γ_s (see Fig. 1) can be considered as independent variable, and a simple approximation scheme can be used. Thus both displacements, as well as tractions at the global boundary may appear in the final algebraic equations as independent unknown variables. If the direct differentiation of displacement approximation is used in Eq. (8) for the traction vector, only one unknown (displacement) will appear in the final algebraic equations.

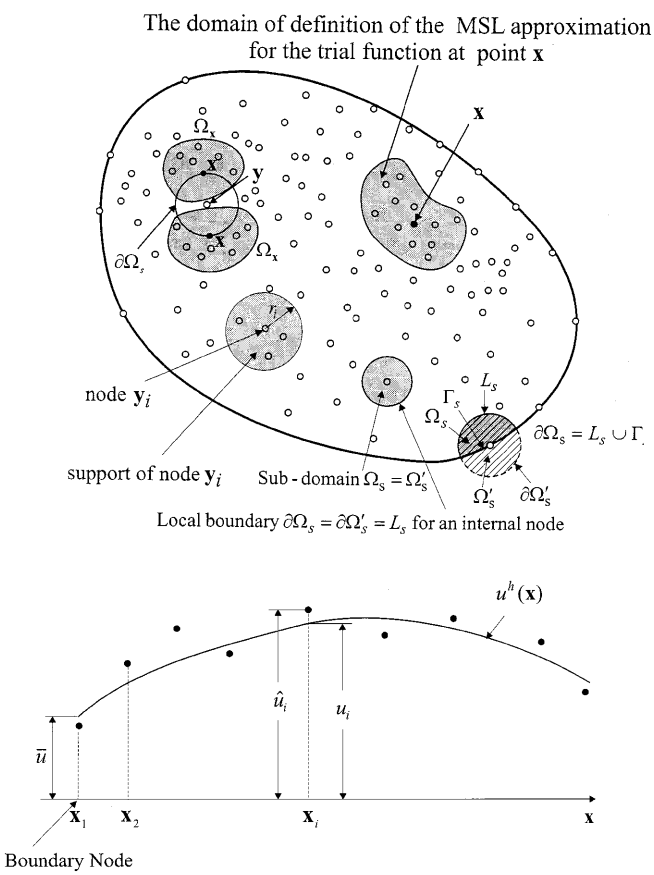

Fig. 1. Local boundaries, the support of nodes and the domain of definition of the MLS approximation

3 The MLS approximation scheme

In general, a meshless method uses a local interpolation to represent the trial function with the values (or the fictitious values) of the unknown variable at some randomly located nodes. The moving least squares (MLS) approximation may be considered as one of such schemes, and is used in the current work. Consider a sub-domain Ω_x, the neighbourhood of a point x and denoted as the domain of definition of the MLS approximation for the trial function at x, which is located in the problem domain Ω. To approximate the distribution of function u in Ω_x, over a number of randomly located nodes $\{x_i\}, i = 1, 2, \ldots, n$, the MLS approximant $\mathbf{u}^h(x)$ of $u, \forall\, x \in \Omega_x$, can be defined by

$$\mathbf{u}^h(x) = \mathbf{p}^{\mathrm{T}}(x)\mathbf{a}(x) \quad \forall\, x \in \Omega_x \tag{16}$$

where $\mathbf{p}^{\mathrm{T}}(x) = [p_1(x), p_2(x), \ldots, p_m(x)]$ is a complete monomial basis of order m; and $\mathbf{a}(x)$ is a vector containing coefficients $a_j(x), j = 1, 2, \ldots, m$ which are functions of the space coordinates $\mathbf{x} = [x^1, x^2, x^3]^{\mathrm{T}}$. For example, for a 2-d problem

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = \lfloor 1, x^1, x^2 \rfloor, \quad \text{linear basis } m = 3 \tag{17a}$$

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = \lfloor 1, x^1, x^2, (x^1)^2, x^1x^2, (x^2)^2 \rfloor, \quad \text{quadratic basis } m = 6 \tag{17b}$$

The coefficient vector $\mathbf{a}(x)$ is determined by minimizing a weighted discrete L_2 norm, defined as

$$J(x) = \sum_{i=1}^{n} w_i(x)[\mathbf{p}^{\mathrm{T}}(x_i)\mathbf{a}(\mathbf{x}) - \hat{u}_i] \tag{18}$$

where $w_i(x)$ is the weight function associated with the node i, with $w_i(x) > 0$. Recall that n is the number of nodes in Ω_x for which the weight functions $w_i(x) > 0$ and $\hat{u}_i$ are the fictitious nodal values, and not the nodal values of the unknown trial function $\mathbf{u}^h(x)$ in general (Atluri et al. 1999). The stationarity of J in Eq. (18) with respect to $\mathbf{a}(x)$ leads to the following linear relation between $\mathbf{a}(x)$ and $\hat{\mathbf{u}}$

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\hat{\mathbf{u}} \tag{19}$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)$$

$$\mathbf{B}(\mathbf{x}) = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \ldots, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)]$$

Solving for $\mathbf{a}(x)$ from Eq. (19) and substituting it into Eq. (16) give a relation

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{\Phi}^{\mathrm{T}}(\mathbf{x}) \cdot \hat{\mathbf{u}} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\hat{u}_i; \quad u^h(x_i) \equiv u_i \neq \hat{u}_i \ , \tag{20}$$

where

$$\mathbf{\Phi}^{\mathrm{T}}(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})$$

Both Gaussian and spline weight functions with compact supports can be considered in a numerical analysis (Zhu et al. 1998; Atluri et al. 1999). The Gaussian weight function can be written as

$$w_i(\mathbf{x}) = \begin{cases} \exp[-(d_i/c_i)^2] - \exp[-(r_i/c_i)^2] & 0 \le d_i \le r_i \\ 0 & d_i \ge r_i \end{cases} \tag{21}$$

where $d_i = |\mathbf{x} - \mathbf{x}_i|$; c_i is a constant controlling the shape of the weight function w_i and r_i is the size of support. The size of support r_i should be large enough to have a sufficient number of nodes covered in the domain of definition to ensure the regularity of matrix $\mathbf{A}$.

Substituting the MLS approximation (20) for displacements into the modified traction vector t_i^* defined by Eq. (8), we get

$$\mathbf{t}^*(\eta) = \mathbf{N}(\eta)\mathbf{D}\sum_{j=1}^{n} \mathbf{B}_j(\eta)\hat{u}_j \tag{22}$$

where matrix $\mathbf{N}$ correspond to normal vector

$$\mathbf{N}(\eta) = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}$$

the stress-strain matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \frac{2}{1-2v}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & (1-2v)/2 \end{bmatrix}$$

for plane strain problem

$$\mathbf{D} = \frac{2}{1-v}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix}$$

for plane stress problem

and

$$\mathbf{B}_j(\eta) = \begin{bmatrix} \phi_{j,1} & 0 \\ 0 & \phi_{j,2} \\ \phi_{j,2} & \phi_{j,1} \end{bmatrix}$$

Now, the local boundary integral equations (14) and (15) considered at nodal points ζ_i result in the linear system of algebraic equations for unknown fictitious nodal values

$$\begin{aligned} &\sum_{j=1}^{n} \phi_j(\zeta_i)\hat{u}_j + \sum_{j=1}^{n} \int_{L_s} \mathbf{T}^*(\eta, \zeta_i)\phi_j(\eta)\mathrm{d}\Gamma_\eta \hat{u}_j \\ &- \sum_{j=1}^{n} \lim_{y\to\zeta_i} \int_{\Gamma_{st}} \mathbf{T}^*(\eta, y)\phi_j(\eta)\mathrm{d}\Gamma_\eta \hat{u}_j \\ &+ \sum_{j=1}^{n} \int_{\Gamma_{su}} \mathbf{U}^*(\eta, \zeta_i)\mathbf{N}(\eta)\mathbf{D}\mathbf{B}_j(\eta)\mathrm{d}\Gamma_\eta \hat{u}_j \\ &- \sum_{j=1}^{n} \int_{\Omega_s} \mathbf{U}^*(x, \zeta_i)\frac{1}{\mu(x)}\mathbf{G}_j(x)\mathrm{d}\Gamma_x \hat{u}_j \\ &= \int_{\Gamma_{su}} \mathbf{T}^*(\eta, \zeta_i)\bar{\mathbf{u}}(\eta)\mathrm{d}\Gamma_\eta - \int_{\Gamma_{st}} \mathbf{U}^*(\eta, \zeta_i)\bar{\mathbf{t}}^*(\eta)\mathrm{d}\Gamma_\eta \end{aligned} \tag{23}$$

where Γ_{st} and Γ_{su} are the traction and displacement boundary sections of Γ_s with $\Gamma_s = \Gamma_{st} \cup \Gamma_{su}$, the prescribed quantities are denoted by bar and L_s is a part of the local boundary $\partial\Omega_s$, which is not located on the global boundary Γ. For those interior nodes located inside the domain $\Omega, L_s \equiv \partial\Omega_s$ and the boundary integrals over Γ_{su} and Γ_{st} vanish in Eq. (23).

The explicit expression of the matrix $\mathbf{G}_j$ is given by

$$\mathbf{G}_j(x) = \begin{bmatrix} 2\frac{1-v}{1-2v}\mu_{,1}\phi_{j,1} + \mu_{,2}\phi_{j,2} & \frac{2v}{1-2v}\mu_{,1}\phi_{j,2} + \mu_{,2}\phi_{j,1} \\ \frac{2v}{1-2v}\mu_{,2}\phi_{j,1} + \mu_{,1}\phi_{j,2} & 2\frac{1-v}{1-2v}\mu_{,2}\phi_{j,2} + \mu_{,1}\phi_{j,1} \end{bmatrix}$$

If displacement components are prescribed at a boundary point $\zeta_i \in \Gamma$, the discretized LBIE (23) can be replaced by the approximation formula

$$\mathbf{u}(\zeta_i) = \sum_{j=1}^{n} \phi_j(\zeta_i)\hat{u}_j \tag{24}$$

Thus, ζ_i in Eq. (23) is either an interior node or nodal point on Γ_{st}. That is why the integral of the kernel $\mathbf{T}^*$ over Γ_{su} is not singular. The limit of the singular integral over Γ_{st} can be evaluated numerically, by using a regular quadrature accurately, provided that an optimal transformation of the integration variable is employed before the integration (Sladek and Sladek 1998; Sladek et al. 1999).

4
Polynomial approximation elements

The local boundary integral equations (14) and (15) can also be transformed into a system of algebraic equations, by using a standard polynomial approximation of displacements over certain domain. In contrast to the MLS approach the nodal displacements are computed directly in this case. To analyse a boundary value problem for a nonhomogeneous elastic body, the nodal points are to be spread over the whole domain including the boundary. Around each nodal point ζ a domain element S_ζ is constructed for approximation purposes. Neighbouring nodes are utilized in such a construction as shown in Fig. 2.

Over the domain element, the displacement can be expressed approximately in terms of their values at nodal points and interpolation polynomials as follows (Balas et al. 1989)

$$u_i(\eta) = \sum_{a=0}^{n} u_i(\eta^a) N^a(\xi_1, \xi_2) \tag{25}$$

where n is a number of element nodes. Note that such an approximation is employed in the LBIE approach only over the sub-domain Ω_s (Fig. 2). Tractions and displacement gradients are obtained from the approximation (25) by differentiation. Thus, the only unknowns are the nodal values of displacements.

Now, the singular LBIE collocated at $\zeta \in \Gamma$ can be rewritten in a regularized form, since the polynomials N^a are known in a closed form, in contrast to $\phi_i(x)$ in the MLS approach. Similar to Eq. (11) we can write

$$\int_{\partial\Omega_s} [u_i(\eta) - u_i(\zeta)] T^*_{ik}(\eta, \zeta) d\Gamma_\eta - \int_{\Gamma_s} t^*_i(\eta) U^*_{ik}(\eta - \zeta) d\Gamma_\eta = \int_{\Omega_s} g_i(x) U^*_{ik}(x - \zeta) d\Omega_x + W_k(\zeta) \tag{26}$$

where the integral identity (10) has been utilized with integration over $\partial\Omega_s$. The LBIE collocated at interior nodes ζ can be obtained from (14) by simply changing $y = \zeta$, and in view of the above mentioned identity, this LBIE is formally equivalent with Eq. (26). If the collocation point ζ_i is lying on the boundary Γ, the isoparametric coordinates (ξ_1, ξ_2) are transformed into polar coordinates (ρ, φ) with its origin at ζ_i. In performing such a transformation new polynomials P^a are defined on S_ζ as (Balas et al. 1989). Then, we can write along $\partial\Omega_s$:

$$[u_i(\eta) - u_i(\zeta)] = \begin{cases} \rho \sum_{a=1}^{n} u_i(\eta^a) P^a(\rho, \varphi) & \text{for } \zeta \in \Gamma \\ \sum_{a=1}^{n} u_i(\eta^a) P^a(\xi_i, \xi_2) - u_i(\zeta) & \text{for } \zeta \notin \Gamma \end{cases} \tag{27}$$

Finally, the nonsingular LBIE given by Eq. (26) takes the discretized form

$$-\mathbf{u}(\zeta) \int_{L_s} \mathbf{T}^*(\eta, \zeta_i) d\Gamma_\eta + \sum_{a=1}^{n} \mathbf{u}(\eta^a) \Bigg[\int_{L_s} \mathbf{T}^*(\eta, \zeta_i) P^a d\Gamma_\eta + \int_{\Gamma_s} \rho P^a \mathbf{T}^*(\eta, \zeta_i) d\Gamma_\eta - \int_{\Gamma_s} \mathbf{U}^*(\eta, \zeta_i) \mathbf{N}(\eta) \mathbf{D} \mathbf{B}^a(\eta) d\Gamma_\eta - \int_{\Omega_s} \mathbf{U}^*(x, \zeta_i) \frac{1}{\mu(x)} \mathbf{G}^a(x) d\Gamma_x \Bigg] = 0 \tag{28}$$

Recall that the integrals over Γ_s disappear if $\zeta_i \notin \Gamma$. Note that the definitions of the matrices $\mathbf{B}^a$ and $\mathbf{G}^a$ are the same as in the MLS approximation case $(\mathbf{B}_j, \mathbf{G}_j)$. Only the digital shape functions ϕ_j are replaced by the polynomial shape functions P^a.

5
Numerical example

Consider a quadrilateral cross-section $(2a \times 2a)$ subjected to a uniform tension $\sigma_{22} = p$ in the x_2-direction, assuming a non-homogeneous material behaviour:

$$\mu = \mu_0(1 + \alpha|x_2|)^2, \quad \mu_0 = \frac{E_0}{2(1+v)}$$

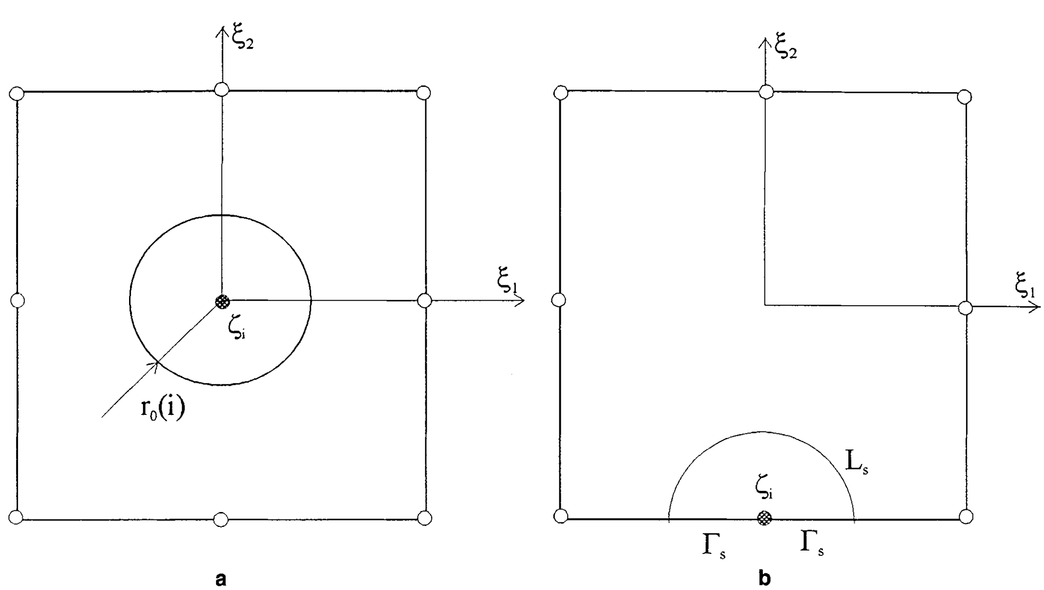

Fig. 2a,b. Quadrilateral element for approximation of displacements on the local boundary $\partial\Omega_s$. **a** Interior node, **b** boundary node with $E_0 = 1.23 \times 10^5$ MPa, $\nu = 0.35$ and $\alpha = 0.5$. Owing to the symmetry, it is sufficient to analyse only a quarter of the cross section. Plane strain conditions are assumed in this example. Several equidistant node distributions, with a total number of nodes $N = 121, 49, 25$ are considered in numerical analysis.

A comparison of both the numerical methods (LBIE with MLS approximations and polynomial element approximations, respectively) with the FEM results (Sladek et al. 1993), for vertical and horizontal displacements is presented in Fig. 3, with $N = 121$. Quite a good agreement of results is observed for the FEM and the LBIE/polynomial element results. The LBIE/MLS results are very sensitive to the selection of free parameters (c_i, r_i) in the weight function (22). The dependence of the traction norm error, defined as

$$r = \left[\int_0^a \sigma_{22}(x_1, 0)\mathrm{d}x_1 - \mathrm{pa}\right] \frac{1}{\mathrm{pa}} \times 100\,[\%] \;,$$

on the weight function parameter c_i and the size of support r_i is illustrated by Fig. 4.

Next, $c_i = 0.074$ and $r_i = 0.8$ are considered. The stress σ_{22} distribution along x_1-axis is presented in Fig. 5.

The stress norm error is 1.7% for the (LBIE/polynomial element) results, and -1.73% for the (LBIE/MLS) results. The influence of the mesh size on the accuracy of the LBIE element analysis can be observed on Fig. 6. The displacement error is related to the FEM results at the point $(0., a)$. Similarly, the stress σ_{22} is compared with FEM stress at $(0., 0.)$ in the relative traction error. The highest accuracy is received for the finest mesh as could be expected.

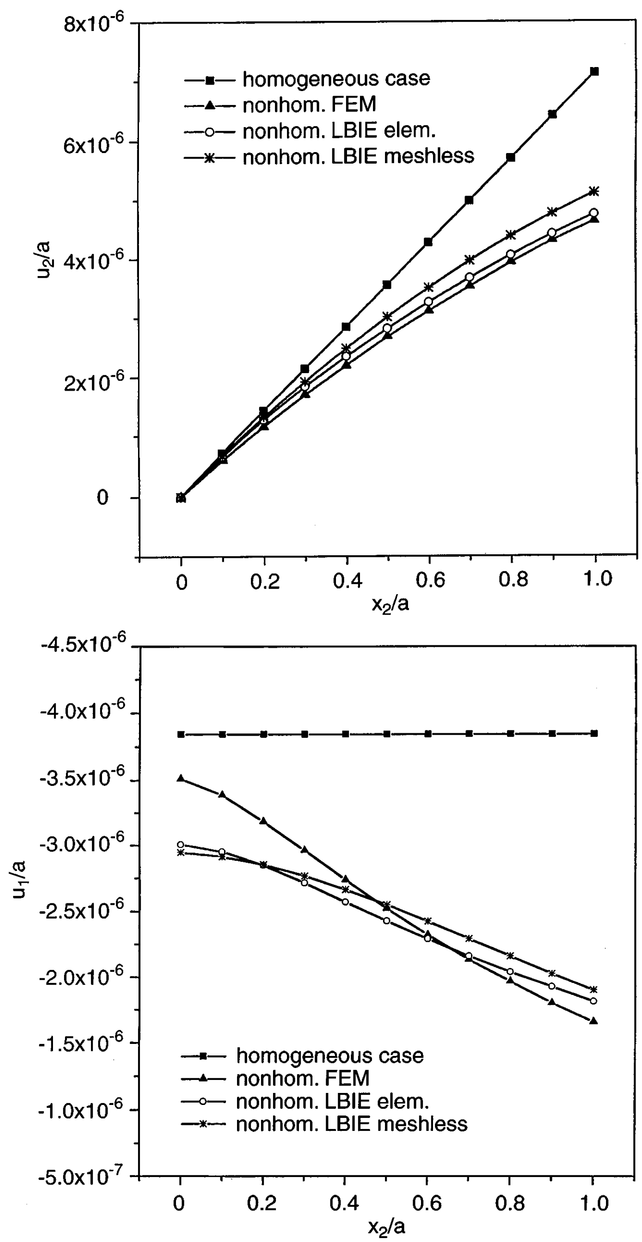

Fig. 3. Vertical and horizontal displacements along x_2-axis and lateral side of the square ($x_1 = a$), respectively.

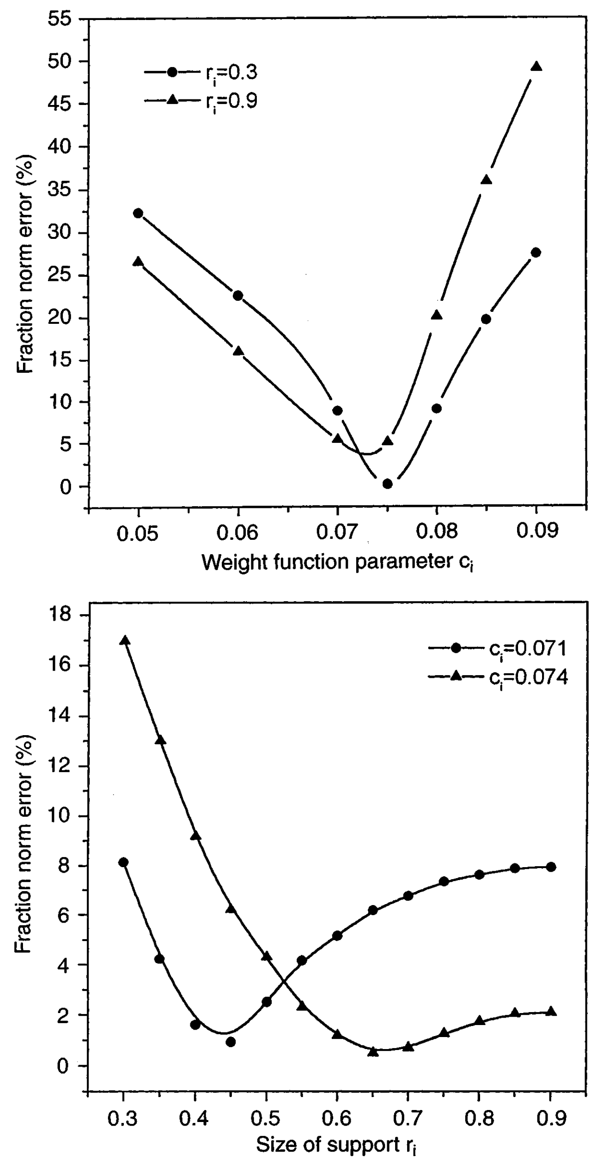

Fig. 4. Dependence of the traction norm error on the weight function parameter c_i and size of support r_i, respectively

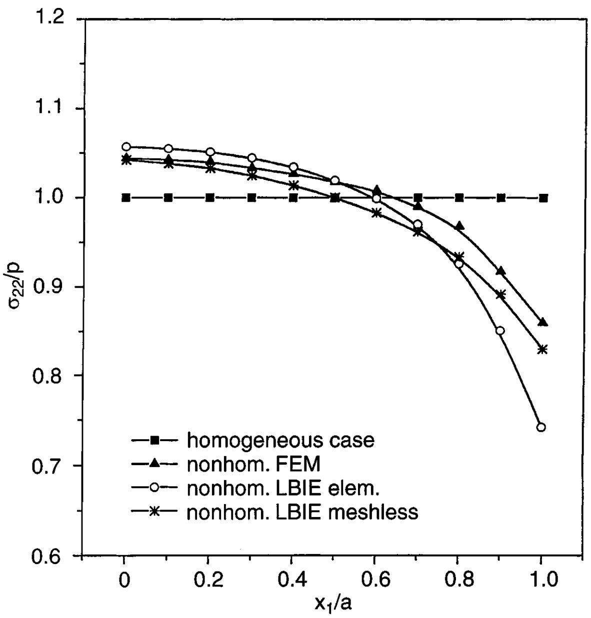

Fig. 5. Stress σ_{22} distribution along x_1-axis

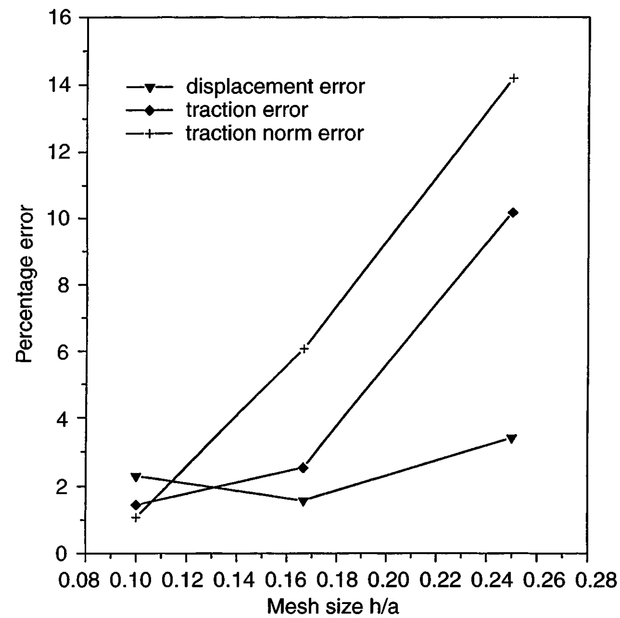

Fig. 6. Influence of mesh size on the accuracy of the LBIE element analysis

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