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Fuzzy structural analysis using α-level optimization

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Abstract In this paper new concepts and developments are presented for structural analysis involving uncertain parameters. Based on a classification of the uncertainties in structural analysis the uncertainty "fuzziness" is identified and its quantification is demonstrated. On the basis of fuzzy set theory a general method for fuzzy structural analysis is developed and formulated in terms of the α-level optimization with the application of a modified evolution strategy. Every known analysis algorithm for the realistic simulation of load-bearing behavior may be applied in the fuzzy structural analysis in the sense of a deterministic fundamental solution. By way of example, geometrically and physically nonlinear algorithms are adopted in the presented study as a deterministic fundamental solution for the analysis of steel and reinforced concrete structures. The paper also describes coupling between α-level optimization and the deterministic fundamental solution.

1 Uncertainties in structural analysis

The realistic analysis of structures requires reliable (input) data as well as suitably-matched computational models; as a rule the data and the model contain uncertainties. In contrast to deterministic structural analysis, fuzzy structural analysis takes account of these data and model uncertainties.

The geometrical, material and loading data required for structural analysis are more or less characterized by uncertainties. It is necessary to appropriately take these uncertainties into consideration.

When carrying out structural analysis the basic concept of the so-called "toolbox philosophy" (see Hung T. Nguyen, 1997) is followed. The principal strategy adopted for solving a problem is defined by the problem itself. The aim of this approach is to supplement established probabilistic methods in such a way that uncertainties in their natural form (characteristics) may be more appropriately accounted for. In this respect the different developments do not directly contradict each other but rather constitute an entirety (see Hung T. Nguyen, 1997; Elishakoff, 1995). In Elishakoff (1995) the relationship between the probabilistic, fuzzy and convex modeling approaches is formulated as "uncertainty triangle".

According to Bothe (1993) uncertainty is the gradual assessment of the truth content of a postulation, which may be referred to the occurrence of a defined event. Depending on the particular cause of their occurrence the various types of uncertainty may be distinguished from each other and described using suitably-matched "tools".

If an event (regarding its occurrence), as a random result of a test, may be observed on an almost unlimited number of occasions under constant boundary conditions, this concerns a stochastic uncertainty which may be described and investigated using the methods of probability analysis. If the boundary conditions are (apparently) subject to arbitrary fluctuations, a comprehensive system overview is lacking, or the number of observations are only available to a limited extent, an information deficit exists. This type of uncertainty is referred to as informal uncertainty. In contrast to the latter, lexical uncertainty is characterized by linguistic variables representing quantified verbal postulations (see Stransky, 1999).

The cause of an uncertainty determines its characteristics. The uncertain characteristic randomness is assigned to stochastic uncertainty which may be mathematically described with the aid of random variables. Informal and lexical uncertainties are described by the uncertainty characteristic fuzziness. The uncertainty characteristic fuzzy randomness arises when the statistical description of a random variable is informally or lexically uncertain.

Randomness, fuzziness and fuzzy randomness may occur in the form of data uncertainty as well as model uncertainty. If the uncertainty is directly included in the input parameters, this implies the existence of data uncertainty. If a consideration of uncertainties leads to uncertain models, this implies the existence of model uncertainty.

The characteristics of the uncertainty determines the particular form of a structural analysis as well as the assessment of the structural safety of a construction, as often carried out subsequently on the basis of different concepts. The method of probabilistics, fuzzy set theory, convex modeling and – as a synthesis of the probabilistic method and fuzzy set theory – the theory of fuzzy random variables are available as "tools" for this purpose.

Randomness is dealt with using probabilistic methods, fuzzy randomness is handled using the theory of fuzzy

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random variables, and fuzziness requires the application of fuzzy set theory. The non-assessed uncertainty accounted for in convex modeling may be treated as a special case of fuzziness. Theoretical treatments of convex modeling are given e.g. in Ben-Haim (1996) and Elishakoff et al. (1994) present an application for solving optimization problems with uncertain input parameters.

The established probabilistic methods are dealt with e.g. in Spaethe (1987, 1992) or Thoft-Christensen and Baker (1982); a comprehensive state-of-the-art report is given in Schüeller (1997); time-dependent processes are reported in Rackwitz (1999). Although the algorithms are mainly applicable to the safety assessment of structures, the authors are also aware of applications for structural optimization and structural analysis.

Besides the deterministic and semi-probabilistic safety concepts widely incorporated into standards, which are nevertheless highly restrictive and inflexible with regard to assessment, there is an increasing tendency to apply probabilistic approximation solutions according to first order reliability method (FORM) and second order reliability method (SORM). These yield the reliability index β as a measure of structural reliability. The mathematically more complex probabilistic "exact" solution obtained from an analytical or numerical integration or a simulation yields the failure probability P_f. Time-dependent loading processes may be directly included in the reliability analysis. In order to approximately account for uncertainties in models of structural systems the method of stochastic finite elements was developed.

A basis for the application of probabilistic-oriented methods is the validity of statistical laws for stochastic input parameters. As the input data necessary for this purpose are only available to a limited extent, it is not possible to reliably define the required distribution functions and statistical parameters. Regarding the uncertainties (presumed to be stochastic) in the statistical information a probability distribution function can at best be assumed; with the aid of Bayes theorem it is possible to update estimates (see e.g. Stange, 1977 or Benjamin and Cornell, 1970). This criticism of the probabilistic approach provides a starting point for investigating alternative methods of accounting for uncertainties – in the sense of the toolbox philosophy – more accurately.

Human mistakes and errors in the manufacture, use and maintenance of constructions as well as investigations under changing boundary conditions (e.g. investigation of material samples) represent subjective influences on structural parameters. This gives rise to data as well as models possessing uncertainties which exhibit no or only partly stochastic characteristics. The assessment of structural parameters is both objective and subjective in character. Depending on the type and extent of the available information it is possible to apply fuzzy set theory or the theory of fuzzy random variables. If statistical properties are entirely lacking, fuzzy set theory is then applied. Uncertainties are taken into account with the aid of "assessed intervals".

The theoretical basics of fuzzy set theory are explained e.g. in Zadeh (1965); Bandemer and Gottwald (1989) or Zimmermann (1992). Mapping of the fuzzy input values onto the result space is based on the extension principle in combination with the Cartesian product.

In problem-solving applications in structural statics α-discretization is advantageous for the numerical processing of the fuzzy analysis. Wood et al. (1992) adopt this method for solving special problems in structural design; Bonarini and Bontempi (1994) describe the solution of differential equations containing fuzzy parameters. Methods relating to more general fuzzy analysis strategies are presented e.g. in Möller and Beer (1997a), Möller et al. (1999).

Various methods are available for applying fuzzy set theory to assess structural safety. On the basis of possibility theory (see e.g. Dubois and Prade, 1986) the failure possibility Π_f is used to describe the safety level; examples of this are given by Möller (1997), Möller and Beer (1997b), Möller et al. 1999.

Brown (1979), Brown et al. (1984) recommend the consideration of influences resulting from human actions, which are not accounted for in the assessment of operative structural safety. A consideration of uncertainties in the mechanical model, similar to the method adopted in probabilistics (e.g. using stochastic finite elements), is widely discussed in the literature. For example, Bardossy and Bogardi (1989), Kam and Brown (1984) consider models for estimating service life. A method for estimating the damping and vibration behavior of systems is presented in Soize (1995). Fuzzy limit state functions, which may result from uncertainties in limit states, are considered e.g. in Ching-Hsue Cheng and Don-Lin Mon (1993). Applications of Bayes algorithm for processing additional information in combination with fuzzy parameters are described in Geyskens et al. (1998), Cheung-Bin Lee and Ju-Won Park (1997) or Chou and Jie Yuan (1993).

As an alternative to a purely probabilistic and possibilistic approach, probability theory in combination with uncertain input information may be applied.

The probability of a fuzzy event is dealt with in Bothe (1993), Bandemer and Gottwald (1989) or Zadeh (1968); the computation is carried out by evaluating a Lebesgue integral. The description of special problems such as e.g. yield surfaces in plasticity theory (see Klisinski, 1988) with the aid of fuzzy parameters leads to uncertain limit states, which may be interpreted and evaluated as fuzzy events. In Bardossy and Bogardi (1989) an application is presented for fatigue problems in service life analysis; crisp values for P_f as well as fuzzy failure probabilities may be computed.

The theory of fuzzy random variables provides a mathematical basis for taking account of uncertainties due to randomness and fuzziness simultaneously (see e.g. Kwakernaak, 1978, 1979; Puri and Ralescu, 1986 or Wang Guangyuan and Zhang Yue, 1992). In Liu Yubin et al. (1997) the application of fuzzy random variables in reliability theory is described. The individual random variables possess (besides their randomness) an additional uncertainty, which may be described by means of fuzzy set theory. This approach leads to uncertain probability density and probability distribution functions, uncertain limit state functions and, as a result of reliability analysis, to fuzzy values for the failure probability and the reliability index. In Qiang Song et al. (1997) time-dependent processes are also considered.

All procedures for taking account of uncertainties in structural analysis are linked to deterministic fundamental solutions obtained from a mechanical analysis of the system. The computed results only yield reliable prognoses regarding the behavior of the structure provided the applied deterministic basic model is capable of describing the load-bearing behavior in a sufficiently realistic manner.

The numerical simulation of the system behavior of structures under consideration of geometrical and physical nonlinearities is the subject of intensive research. A realistic description of nonlinearities including loading and damage processes is of utmost importance, particularly when simulating the behavior of reinforced concrete structures subject to general loading processes. The efficient, deterministic basic model adopted here for the numerical simulation of the load-bearing behavior of plane (prestressed) reinforced concrete bar structures is described in Müller et al. (1995, 1996, 1998), this model satisfies the expected requirements. An application example is presented in Möller et al. (1997). The coupling of this model with algorithms for stochastic analysis is described in Möller et al. (1999).

In the following, only informal and lexical uncertainties possessing the characteristic fuzziness are dealt with. The differences between data uncertainties and model uncertainties are considered. Fuzzy set theory forms the mathematical basis for the latter. The overall approach to take account of, describe, process and evaluate fuzziness may be subdivided into the following:

- Fuzzification
- Fuzzy structural analysis
- Safety assessment or defuzzification

Fuzzification and fuzzy structural analysis are described in greater detail in the following.

2 Modeling of fuzzy structural parameters

2.1 Classification of data uncertainty and model uncertainty

As a criterion for distinguishing between data uncertainty and model uncertainty, assignment of the uncertainty to the input values or the model is formulated. A definition of the model concept is used to concretize the distinction between data uncertainty and model uncertainty. In the mathematical sense every model represents an abstract algebra. It may be interpreted as a mapping operator which maps the input values onto the result values. A model is thus a self-contained entity which processes information.

Model uncertainty implies uncertainty in the mapping operator. This is induced by uncertain structural parameters which are exclusively effective within the model and are thus referred to as fuzzy model parameters. Fuzzy model parameters are not explicitly mapped onto result values but only influence the mapping itself. Model uncertainty is created in the abstraction process, the result of which is the model. A model possessing model uncertainty is referred to as an uncertain model. It is characterized by the fact that crisp input values lead to uncertain model responses.

All uncertain structural parameters which explicitly enter the model as input values are referred to as fuzzy input values. These include all input values external to the model which have no influence on the model itself but may be mapped onto the result values by means of the model. Data uncertainty is not included in the abstraction process model creation.

The major problem is the demarcation of the model in any given case. A model comprised of several sub-models may itself be a sub-model of a higher-ranking model system. The models of a model system are ordered according to hierarchy, whereby interactive relationships may exist. The decision as to which sub-models may be grouped to form a model system cannot be made globally. The specification of model limits is a subjective decision based on objective information. Accordingly, the distinction between data uncertainty and model uncertainty is uncertain in terms of a fuzzy criterion. In structural statics different mathematical descriptions of real objects and relationships may be considered as models which are interlinked.

2.2 Fuzzification of uncertain input and model parameters

Fuzzification is understood to be the specification of the membership function $\mu(x)$ of an uncertain set $\tilde{A}$. The result of fuzzification is the fuzzy value $\tilde{x}$. During fuzzification informal and lexical uncertainties possessing the characteristic fuzziness are assessed. These uncertainties may be uncertainties in the physical structural parameters as well as uncertainties in the planning, construction, utilization and damage processes as well as the maintenance and strengthening processes. Human mistakes and errors as well as uncertain boundary conditions are subjective influences on these processes. By means of fuzzification it is possible to describe mathematically the effects of these uncertainties on physical structural parameters (see e.g. Stransky, 1999). In fuzzy structural analysis all uncertainties are accounted for in the form of fuzzy values.

In order to specify the membership function of an uncertain set it is not possible to state a general algorithm. The only requirement is compliance with the conditions for membership functions in accordance with the appendix. Additional suggestions and descriptions for the purpose of fuzzification serve as orientation aids, which may be implemented by the user according to the problem concerned.

The conceptual starting point for fuzzification is the definition of the concept uncertain set. This presupposes a fundamental set $\mathbf{X}$ with elements $x \in \mathbf{X}$ and a subset $A \subseteq \mathbf{X}$. The subset A may be described by a postulation or may represent an event. In order to specify a membership function $\mu_A(x)$ the membership of the elements x of the subset A is assessed gradually.

The fundamental set $\mathbf{X}$ may be constructed using physical structural parameters or linguistic variables. For the purpose of structural analysis, however, numerical

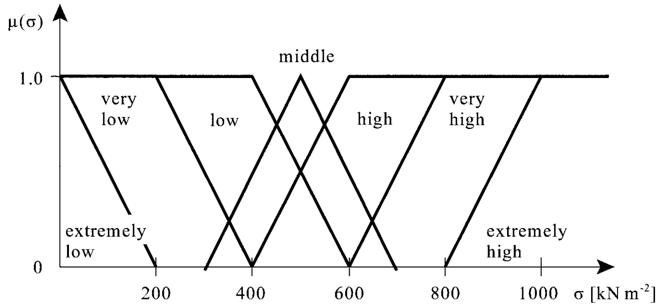

Fig. 1. Assessment of the linguistic variable load-bearing capacity of the foundation soil

values are required. It is therefore necessary to transform linguistic variables into numerical values (see Stransky, 1999). When assessing the foundation soil it is possible to adopt e.g. the linguistic variable load-bearing capacity. An assessment of the load-bearing capacity is carried out by assigning degrees (linguistic values) high, medium and low in combination with a selection of modifiers such as extreme and very. These degrees must then be referred to a numerical scale, which e.g. represents a sustainable soil pressure. This approach is illustrated in Fig. 1.

Membership functions for physical structural parameters may be specified on the basis of samples. Besides the measured values it is thereby necessary to take account of the extent of sampling, possible errors in measurement and other inaccuracies, estimates by experts or expert groups, experience gained from comparable problems and additional information where necessary. The membership functions obtained represent a subjective assessment reflecting objective conditions. It is appropriate to select simple functional forms for $\mu_A(x)$, e.g. linear or polygonal functions.

When specifying the membership functions a distinction may be drawn between two principal approaches. If the fundamental set **X** is a discrete finite set with known elements, each element may then be directly assessed using a membership value. Connecting the points defined in this way by interpolation functions or a point-to-point polygon yields the membership function.

In the case of a continuous or infinite fundamental set **X** a quasi-infinite number of points must be assessed. In this case a crisp set representing a kernel set of the uncertain set may first be defined (Fig. 2). The boundary regions of this crisp kernel set are subsequently "smeared" by assigning membership values $\mu_A(x) < 1$ to the near-boundary elements and allowing the branches of the $\mu_A(x)$ to approach $\mu_A(x) = 0$ monotonically beyond the boundaries of the crisp kernel set. By this means, elements which are not members of the crisp kernel set, but nevertheless lie "in proximity", are assigned membership values $\mu_A(x) > 0$. This approach may be extended if several crisp kernel sets for different membership levels (see α-level sets) are selected and the $\mu_A(x)$ are defined in level increments. The branches of the membership functions may thereby be described using different functional forms. Existing data comprised of samples may be utilized in the form of histograms. Smooth histograms are valuable as a "first draft" for the sought $\mu_A(x)$.

The fuzzification method is demonstrated by the example of the compression strength β_D of sandstone. Figure 2 shows the recorded and normalized histogram for 16 investigated sample blocks and two fuzzification alternatives.

3
Fuzzy structural analysis

Fuzzy structural analysis implies the analysis of a structure with the aid of a crisp (or uncertain) algorithm applied to fuzzy values for input and model parameters. In fuzzy structural analysis the deterministic algorithms for statical and dynamic computations are adopted as a deterministic fundamental solution.

3.1
Deterministic fundamental solution

In the following, a geometrically and physically nonlinear analysis algorithm for plane (also prestressed) bar structures after Müller et al. (1995, 1996, 1998) is applied by way of example as the deterministic fundamental solution for fuzzy structural analysis.

By means of this realistic simulation algorithm complex loading processes are treated in an incremental-iterative manner under consideration of all essential nonlinearities. In addition to the evaluation of equilibrium conditions for the displaced system, the geometrical nonlinearities are taken into account by including the quadratic terms in the strain-displacement relationships. This permits the investigation of system states characterized by large displacements and moderate rotations.

Physical nonlinearities, especially reinforced concrete nonlinearities, are accounted for by applying uniaxial, explicitly formulated material laws for the determination of stress at cross-sectional points. Account is thereby taken of progressive material damage, cyclic loading and

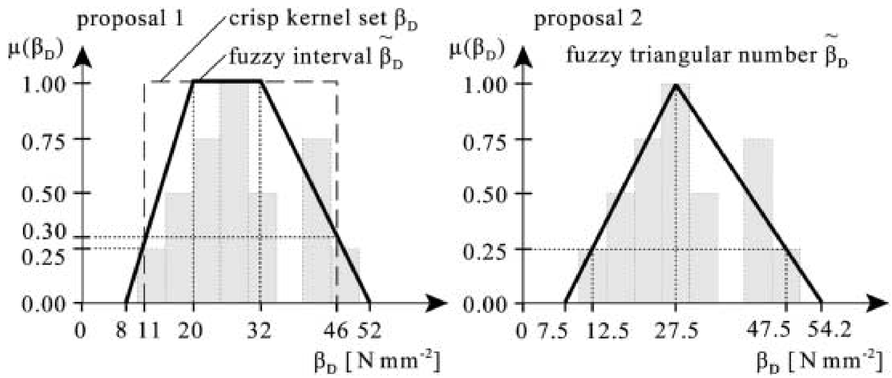

Fig. 2. Proposals for the membership function $\mu(\beta_D)$ of the compression strength of sandstone

unloading (including the Bauschinger effect), monotonic and cyclic hardening of steel, contact forces associated with crack closure in concrete, and tension stiffening.

Internal force interaction is alternatively included in the analysis with the aid of an M-N interaction model or an M-N-Q interaction model. The cross-sectional stiffnesses are determined numerically; for this purpose the cross-section is subdivided into layers, whereby reinforcement steel forms separate layers. In order to construct the stiffness matrices and compute the responses in bars, integration increments are specified along each bar and the system of differential equations for the bars is numerically integrated. It is presupposed that the bar cross-sections remain plane.

The incremental-iterative treatment of the loading process is carried out by means of a modified Newton-Raphson method or a pathfollowing algorithm with generalized displacement control (normal plane iteration). In the case of dynamic investigations a modified Newmark operator after Argyris and Mleynek (1988) is applied for the numerical time-step integration. The dynamic equilibrium condition is evaluated iteratively for every bar integration point at each time step.

3.2 Aim of fuzzy structural analysis

The aim of fuzzy structural analysis is to map fuzzy input values and fuzzy model values onto result values with the aid of an analysis algorithm (deterministic fundamental solution).

The results $\tilde{z}_j$ of the fuzzy structural analysis are also fuzzy values. They may be computed from the n fuzzy input values $\tilde{x}_i$ and the p fuzzy model values $\tilde{m}_r$ by means of the extension principle in combination with the Cartesian product between uncertain sets (see e.g. Zadeh, 1965; Bothe, 1993 or Möller et al., 1999). A distinction is thereby made between two operators:

1. The mapping operator $\underline{z} = (z_1, \ldots, z_j, \ldots, z_m) = f(x_1, \ldots, x_i, \ldots, x_n)$, which here represents the analysis algorithm, transforms all points $\underline{x}$ in the space of the fuzzy input values $\tilde{x}_i$ (x-space) into the space of the fuzzy result values $\tilde{z}_j$ (z-space).
2. The max-min operator determines the membership values $\mu(\underline{z})$ for the result points.

The mapping operator $f(\underline{x})$ represents the computational model M for the fuzzy structural analysis. The fuzzy model values $\tilde{m}_r$ are introduced into the model M, which, owing to

$$\tilde{f} = \tilde{\mathrm{M}}(\tilde{m}_1, \ldots, \tilde{m}_r, \ldots, \tilde{m}_p) \tag{1}$$

becomes the uncertain mapping operator $\tilde{f}$. The fuzzy model values $\tilde{m}_r$ are also treated according to the rules of fuzzy set theory. The uncertain mapping operator $\tilde{f}$ is determined from Eq. (1) as a mapping of the fuzzy model values $\tilde{m}_r$ onto $\tilde{f}$. This mapping is treated mathematically in the same way as the mapping of the fuzzy input values $\tilde{x}_i$ onto the fuzzy result values $\tilde{z}_j$. For this reason only the processing of the fuzzy input values $\tilde{x}_i$ using the crisp mapping operator f is described in the following.

Applying the extension principle, the fuzzy input values $\tilde{x}_i$ in the x-space form the uncertain input set $\tilde{\underline{X}}$ and are mapped onto the uncertain result set $\tilde{\underline{Z}}$ in the z-space. The fuzzy result values $\tilde{z}_j$ are contained in $\tilde{\underline{Z}}$.

The extension principle is hardly practicable in the case of complex mapping operators, as its application requires discretization of the support of the uncertain input set $\tilde{\underline{X}}$ – e.g. using a point mesh. This leads to numerical problems (see Möller et al., 1999).

In order to develop a suitable method for processing fuzzy input values and fuzzy model parameters the concept of α-discretization is adopted. Procedures which exploit the special properties of the mapping operator or additional information concerning the mapping are suggested e.g. in Bonarini and Bontempi (1994) or Wood et al. (1992). In the following a method is derived which permits the use of mapping operators without special properties.

3.3 α-Discretization of fuzzy values

Fuzzy values may be discretized with the aid of α-level sets (see Fig. 3). The α-level sets $A_i, \alpha_k, \ i = 1, \ldots, n$ of the fuzzy input values $\tilde{A}_1, \ldots, \tilde{A}_i, \ldots, \tilde{A}_n$ form the n-dimensional crisp subspace $\underline{X}_{\alpha_k}$ of the x-space. For $\alpha_k = 0$ the crisp support subspace is obtained. The crisp subspace $\underline{X}_{\alpha_k}$ for the two fuzzy input values $\tilde{x}_1 = \tilde{A}_1$ and $\tilde{x}_2 = \tilde{A}_2$ and the α-level α_k is presented in Fig. 4. If the fuzzy input values are convex uncertain sets, between which no interaction exists, an n-dimensional hypercuboid is obtained. Non-convex fuzzy input values lead to a disjoint subspace $\underline{X}_{\alpha_k}$. If interaction exists, the shape of the subspace $\underline{X}_{\alpha_k}$ generally departs from the shape of the n-dimensional hypercuboid; the formation of "voids" in $\underline{X}_{\alpha_k}$ is possible.

Between the two subspaces $\underline{X}_{\alpha_i}$ and $\underline{X}_{\alpha_k}$ for α_i and α_k the following relationship holds

$$\underline{X}_{\alpha_i} \subseteq \underline{X}_{\alpha_k} \quad \forall\ \alpha_i, \alpha_k | \alpha_i, \alpha_k \in [0; 1], \ \alpha_i \geq \alpha_k \tag{2}$$

It follows that all subspaces $\underline{X}_{\alpha_k}$ are contained in the support subspace $\underline{X}_{\alpha_k=0}$. If α_k may take on all real values in the interval [0; 1], the entirety of $\underline{X}_{\alpha_k}$ then forms the uncertain input set $\tilde{\underline{X}}$; α_k is equal to the membership values of the subspaces $\underline{X}_{\alpha_k}$. For selected values $\alpha_k \in [0;\ 1]$, on the other hand, the uncertain input set $\tilde{\underline{X}}$ is discretized (Fig. 4).

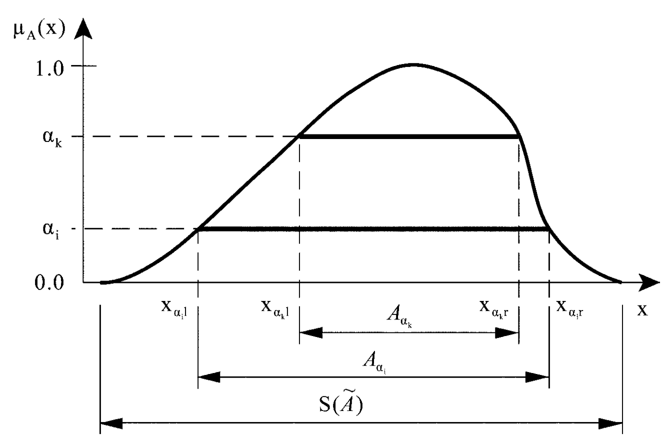

Fig. 3. α-level sets

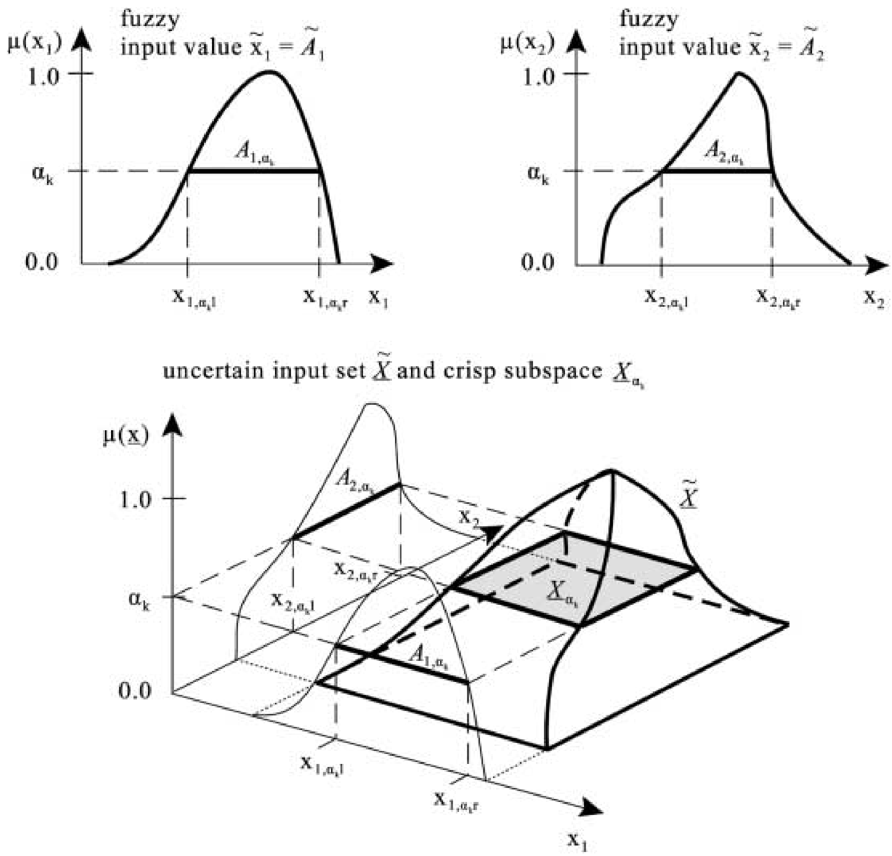

Fig. 4. Uncertain input set $\tilde{\underline{X}}$ and crisp subspace $\underline{X}_{\alpha_k}$ for the two fuzzy input values

3.4
α-Level optimization

All fuzzy input values are discretized using the same number of α-levels α_k, $k = 1, \ldots, r$. For each fuzzy input value $\tilde{x}_i = \tilde{A}_i$ on the level α_k the α-level set A_{j,α_k} is then assigned to $\tilde{x}_i$, and all A_{i,α_k} form the crisp subspace $\underline{X}_{\alpha_k}$. With the aid of the mapping operator $\underline{z} = f(x_1, \ldots, x_n)$ it is possible to compute elements of the α-level sets B_{j,α_k} of the fuzzy result values $\tilde{z}_j = \tilde{B}_j$, $j = 1, \ldots, m$ on the α-level α_k. The mapping of all elements of $\underline{X}_{\alpha_k}$ yields the crisp subspace $\underline{Z}_{\alpha_k}$ of the z-space.

Once the largest element $z_{j,\alpha_k r}$ and the smallest element $z_{j,\alpha_k l}$ of the α-level set B_{j,α_k} have been found, two points of the membership function $\mu(z_j) = \mu_{B_j}(z_j)$ are known (Fig. 5). In the case of convex fuzzy result values the $\mu(z_j)$ are thus completely described. The determination of $z_{j,\alpha_k r}$ and $z_{j,\alpha_k l}$ replaces the max-min operator of the extension principle. The search for the smallest and largest elements may be formulated as an optimization problem. The objective functions

$$z_j = f_j(x_1, \ldots, x_n) \Rightarrow \mathrm{Max}|(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k} \quad (3)$$

$$z_j = f_j(x_1, \ldots, x_n) \Rightarrow \mathrm{Min}|(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k} \quad (4)$$

must be satisfied. The requirements $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$ represent the restrictions of the optimization problem.

Equations (3) and (4) are satisfied by the optimum points $\underline{x}_{\text{opt}}$. For each fuzzy result value precisely two optimum points in the crisp subspace $\underline{X}_{\alpha_k}$ belong to each α-level α_k. The optimization task according to Eqs. (3) and (4) for all α-levels α_k and all fuzzy result values $\tilde{z}_j$ is referred to as α-level optimization. In order to solve the α-level optimization problem special properties of the mapping operator $\underline{z} = f(x_1, \ldots, x_n)$ may be used; these include uniqueness, biuniqueness, continuity, monotonicity and dimensionality of the x-space and z-space.

If the mapping operator has no special properties, the optimum points $\underline{x}_{\text{opt}}$ are located arbitrarily in $\underline{X}_{\alpha_k}$; otherwise the search for the $\underline{x}_{\text{opt}}$ may be limited to parts of $\underline{X}_{\alpha_k}$ – e.g. on the "boundary".

If

(1) every crisp subspace $\underline{X}_{\alpha_k}$ is coherent and
(2) the mapping operator is continuous and unique,

the fuzzy result values $\tilde{z}_j$ are then convex uncertain sets. If no interaction exists between the fuzzy input values $\tilde{x}_i$, condition (1) is satisfied when all $\tilde{A}_i = \tilde{x}_i$ are convex uncertain sets. If condition (2) is not complied with, the α-level optimization yields envelope curves of the actual membership functions of the fuzzy result values.

3.5
Fuzzy structural analysis procedure – optimization strategy

The optimization problem according to Eqs. (3) and (4) is characterized by the following:

(1) The optimization variables are continuous.

(2) The objective function is generally in the form of an algorithm, i.e. only implicit. It is only possible to formulate the objective function explicitly in simple cases.

(3) The value range of the $x_1, \ldots, x_n$ is defined on every α-level by the crisp subspace $\underline{X}_{\alpha_k}$. If no interaction exists between the fuzzy input values $\tilde{x}_i$, the subspace $\underline{X}_{\alpha_k}$ is then bounded by hyperplanes, which are perpendicular to one another and each perpendicular to one coordinate axis x_i.

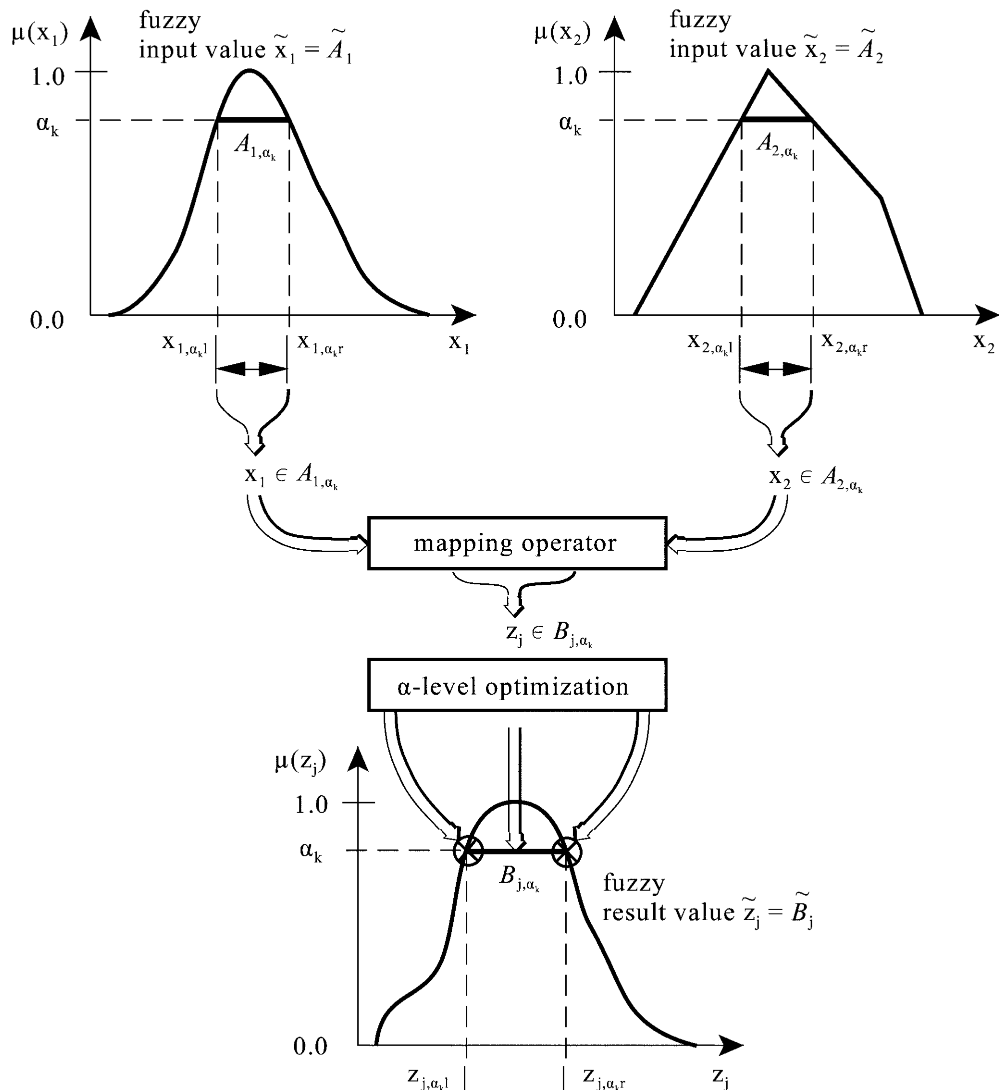

Fig. 5. Mapping of the fuzzy input values $\tilde{x}_1$ and $\tilde{x}_2$ onto the fuzzy result value $\tilde{z}_j$ by mapping of all α-level sets A_{i,α_k} onto the α-level sets B_{j,α_k}

(4) Between the subspaces $\underline{X}_{\alpha_k} \mid \alpha_k \in [0;\ 1]$ the relationship $\underline{X}_{\alpha_i} \subseteq \underline{X}_{\alpha_k} \mid \alpha_i \geq \alpha_k$ exists.

All result values

$$\underline{z}_{\alpha_i} = f(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in \underline{X}_{\alpha_i} \tag{5}$$

of the subspace $\underline{Z}_{\alpha_i}$ are therefore also result values $\underline{z}_{\alpha_k}$ in the subspace $\underline{Z}_{\alpha_k}$ with $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$. For every result point $\underline{z}_t$ belonging to the subspace $\underline{Z}_{\alpha_i}$, $\mu(z_t) \geq \alpha_i$ holds. Owing to the fact that $\alpha_i \geq \alpha_k$, the condition $\mu(\underline{z}_t) \geq \alpha_k$ is also satisfied, i.e. all points $\underline{z}_t \in \underline{Z}_{\alpha_i}$ also belong to the α-level α_k; $\underline{z}_t \in \underline{Z}_{\alpha_k}$ holds (Fig. 6).

(5) The optimization problem for each of the m fuzzy result values $\tilde{z}_j$ on each of the r α-levels α_k must be solved twice, i.e. $(2 \cdot \mathrm{m} \cdot \mathrm{r})$ times in total.

Features (1)–(4) characterize an optimization problem without special properties which might be exploited numerically. The extent of the numerical computation is mainly determined by the objective function representing the mechanical model adopted (e.g. linear, nonlinear). The optimum points may lie within the subspace $\underline{X}_{\alpha_k}$ as well as on the "boundary" or in the "corners".

As hardly any information is available a priori the method of solution should be independent of assumptions concerning the position of the optimum points. Owing to feature (5) multiple solution of the optimization problem is necessary. The α-level optimization thus demands a robust optimization technique which is independent of the type and behavior of the objective function or restrictions and is capable of reliably finding global optima. Standard optimization methods are only partly suitable for this purpose. For this reason a compromise solution is developed by combining evolution strategy, the gradient method and the Monte-Carlo method.

The combination of directed and non-directed search techniques is advantageous compared with a purely directed search technique when seeking global optima; a mixed technique is less sensitive in relation to less "well-behaved" objective functions. By taking advantage of existing information concerning the behavior of the objective function the number of "unnecessary" computations of objective function values (leading to poorer results) is reduced. If the available information is insufficient, random-oriented methods are applied. The developed method of solution is described in the following and illustrated graphically by way of example in Fig. 7.

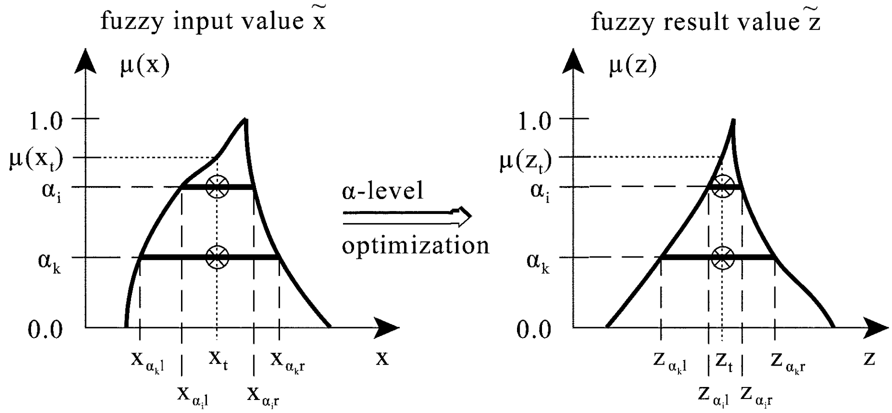

Fig. 6. Mapping of the fuzzy input value $\tilde{x}$ onto the fuzzy result value $\tilde{z}$ by α-level optimization with $z_t = f(x_t)$ as an element of the α-level sets for α_i and α_k

The optimization problem is described by the input values $x_i \in \tilde{x}_i$, $i = 1, \ldots, n$, the result value $z_j \in \tilde{z}_j$ and the algorithm for computing the objective function value (e.g. statical or dynamic analysis). The restrictions $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$ are obtained from the interaction relationships between the fuzzy input values $\tilde{x}_i$; these restrict the definition region of the objective function $z_j = f_j(x_1, \ldots, x_n)$. If no interaction exists between the convex fuzzy input values $\tilde{x}_i$, the permissible domain is obtained as an n-dimensional hypercuboid (subspace) in the x-space. In Fig. 7 the permissible domain $\underline{X}_{\alpha_k}$ for the α-level $\alpha_k = \alpha_1 = 0$ (support subspace) is shown as a rectangle.

The basis for the compromise solution is a (1+1) evolution strategy, which is matched to the optimization problem by modification. The starting point for the optimization is specified with aid of uniformly distributed random numbers U $\in$ [0; 1) (or their realizations u_i) in the subspace $\underline{X}_{\alpha_k}$ (permissible domain for α_k); this serves as the first parent point $\underline{x}^{[0]}$ (represented as ○ in Fig. 7).

$$x_i^{[0]} = x_{i,\,\alpha_k l} + u_i \cdot \left(x_{i,\,\alpha_k r} - x_{i,\,\alpha_k l}\right); \quad i = 1, \ldots, n \qquad (6)$$

A mutation of the properties of the parent point is then simulated by generating an offspring point in its proximity according to the random principle. This represents a departure from the classical evolution strategy procedure. For the offspring points $\underline{x}^{[q+1]}$ a maximum and minimum distance

$$\max\_d_i = \max\left|x_i^{[q+1]} - x_i^{[q]}\right|; \quad i = 1, \ldots, n \qquad (7)$$

$$\min\_d_i = \min\left|x_i^{[q+1]} - x_i^{[q]}\right|; \quad i = 1, \ldots, n \qquad (8)$$

from the parent point $\underline{x}^{[q]}$ is specified directionally (in terms of coordinates). The subspace defined by $\max\_d_i$ and $\min\_d_i$ assigned to the parent point $\underline{x}^{[q]}$ (local search domain) is an n-dimensional hypercuboid. The specification of $\max\_d_i$ and $\min\_d_i$ by compressing the support subspace is advantageous.

$$\max\_d_i = c_1 \cdot \left(x_{i,\,\alpha_1=0r} - x_{i,\,\alpha_1=0l}\right) \,|\, 0 < c_1 < 1; \quad i = 1, \ldots, n \qquad (9)$$

$$\min\_d_i = c_2 \cdot \max\_d_i \,|\, 0 < c_2 < 1; \quad i = 1, \ldots, n \qquad (10)$$

If no interaction exists between the fuzzy input values $\tilde{x}_i$, a similarity relationship exists between the local search domain and the support subspace $\underline{X}_{\alpha_1=0}$ (permissible domain for $\alpha_1 = 0$). If interaction exists, the local search domain may be oriented to the form of $\underline{X}_{\alpha_1=0}$ using the restrictions for the $\tilde{x}_i$. The distance bounds $\max\_d_i$ and $\min\_d_i$ are independent of the α-level α_k. Figure 7 shows the bounds of the local search domains assigned to the parent points $\underline{x}^{[q]}$ (indicated by ⊕) as dotted rectangles (for $\min\_d_i$) and solid rectangles (for $\max\_d_i$).

The first offspring point $\underline{x}^{[q+1]}$ of $\underline{x}^{[q]}$ is generated within this local search domain (between the dotted and solid lines in Fig. 7) by means of an uniform distribution of U $\in$ [0; 1).

$$x_i^{[q+1]} = x_i^{[q]} + 2 \cdot (u_i - 0.5) \cdot \max\_d_i; \quad i = 1, \ldots, n \qquad (11)$$

For at least one i the condition

$$\min\_d_i \le d_i = \left|x_i^{[q+1]} - x_i^{[q]}\right|; \quad 1 \le i \le n \qquad (12)$$

must be fulfilled. Based on the parent point $\underline{x}^{[q]}$ an offspring point $\underline{x}^{[q+1]}$ is first determined. A test is carried out to check whether its objective function value $z_j^{[q+1]} = f_j(x_1^{[q+1]}, \ldots, x_n^{[q+1]})$ is an improvement compared with $z_j^{[q]} = f_j(x_1^{[q]}, \ldots, x_n^{[q]})$ of the parent point. If an improvement is obtained, advancement is made along the randomly selected direction until no further improvement in the objective function value is possible.

$$\underline{x}^{[q+r+1]} = 2 \cdot \underline{x}^{[q+r]} - \underline{x}^{[q+r-1]} \left| \begin{array}{ll} z_j^{[q+r]} < z_j^{[q+r-1]} & \text{for minimum search} \\ z_j^{[q+r]} > z_j^{[q+r-1]} & \text{for maximum search} \\ r = 1, 2, 3, \ldots & \end{array} \right. ; \qquad (13)$$

The step increment thereby remains constant. The last improved point $\underline{x}^{[q+r+1]}$ then becomes the current parent point. In Fig. 7 these step sequences are indicated by the points ⊕.

If the offspring point $\underline{x}^{[q+1]}$ determined randomly from Eqs. (11) and (12) does not lead to an improvement in the

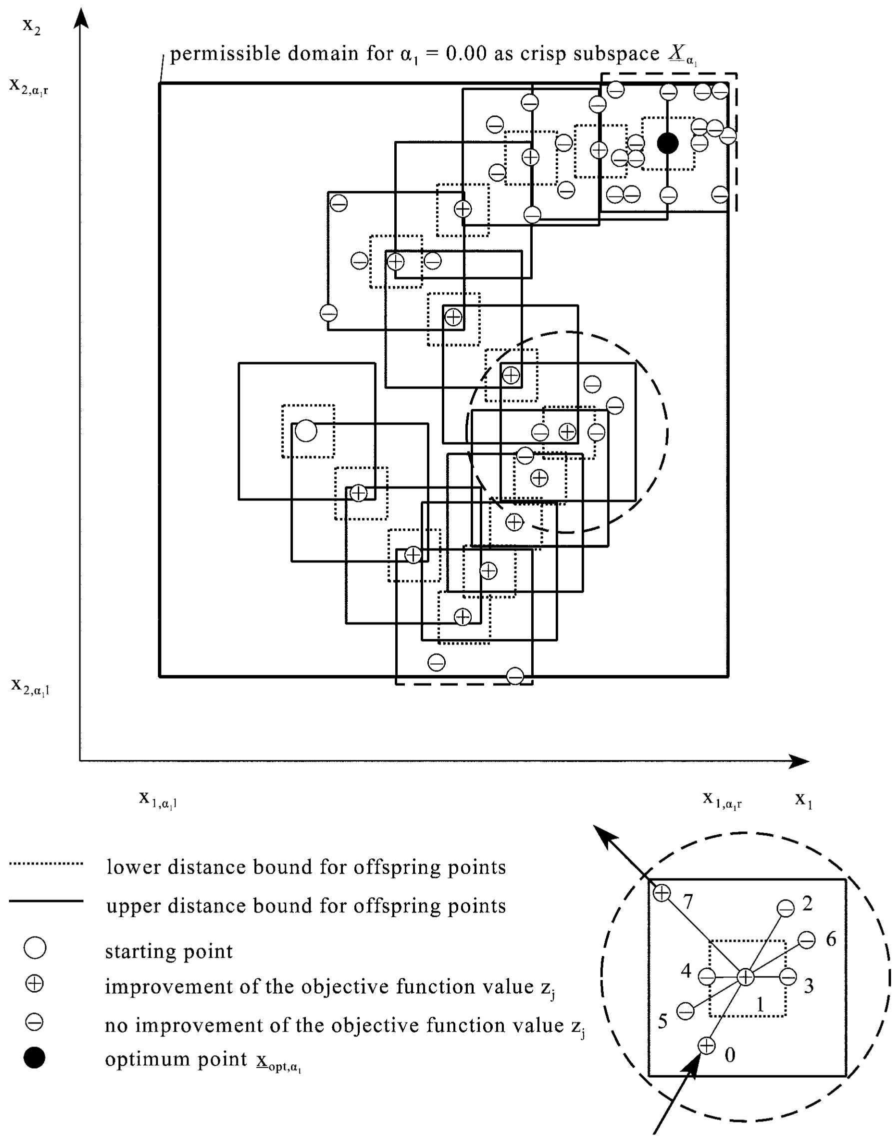

Fig. 7. Modified evolution strategy, general run for $\alpha_k = \alpha_1 = 0.00$

objective function value $\underline{z}_j^{[q+1]}$ compared with $\underline{z}_j^{[q]}$ (denoted by $\ominus$ in Fig. 7), the next offspring point $\underline{x}^{[q+2]}$ is positioned at the same distance as $\underline{x}^{[q+1]}$ from the parent point $\underline{x}^{[q]}$ in the opposite direction.

$$\underline{x}^{[q+2]} = 2 \cdot \underline{x}^{[q]} - \underline{x}^{[q+1]} \left| \begin{array}{ll} z_j^{[q+1]} \geq z_j^{[q]} & \text{for minimum search} \\ z_j^{[q+1]} \leq z_j^{[q]} & \text{for maximum search} \end{array} \right. \qquad (14)$$

If the offspring point $\underline{x}^{[q+2]}$ shows no further improvement in the value of $z_j^{[q+2]}$, the point $\underline{x}^{[q+3]}$ is determined randomly on the basis of $\underline{x}^{[q]}$ (as in the computation of $\underline{x}^{[q+1]}$) using Eqs. (11) and (12). The point $\underline{x}^{[q+3]}$ is evaluated in the same way as for $\underline{x}^{[q+1]}$. This procedure is presented on a large scale in Fig. 7, which also shows the sequence of the placed points. The points 0, 1 and 2 are a result of advancements along the initial direction with improvement of the functional values of the objective function; these are obtained from Eq. (13). Points 3, 5 and 7 were defined randomly, whereas 4 and 6 were determined in a targeted direction from points 3 and 5 using Eq. (14).

If no improvement in the objective function value $z_j^{[q]}$ is achieved after a given number n_P of point pairs $(\underline{x}^{[q+2r+1]}, \underline{x}^{[q+2r+2]})$ with $r = 0, 1, 2 \ldots$, the distance bounds min $\_d_i$ and max $\_d_i$ are reduced. After further unsuccessful steps the search is considered to be at an end and the last parent point is then interpreted as being the optimum point $\underline{x}_{\text{opt},\alpha_k}$ (plotted as ● in Fig. 7).

On the boundary of the permissible domain (subspace $\underline{X}_{\alpha_k}$) the search algorithm distinguishes between randomly placed points computed from Eqs. (11) and (12) and directed specified points obtained from Eq. (13) or (14).

If the "randomly placed point" $\underline{x}^{[q+1]}$ does not satisfy all of the restrictions $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$, the respective coordinates $x_i^{[q+1]}$ (of those which do not adhere to the restrictions) are corrected. In this case the coordinates of the boundary of $\underline{X}_{\alpha_k}$ are chosen for $x_i^{[q+1]}$. If no interaction exists between the $\tilde{x}_i$, the following holds

$$x_{i(\text{new})}^{[q+1]} = x_{i,\,\alpha_k l} \Big| x_i^{[q+1]} < x_{i,\,\alpha_k l}; \quad i = 1, \ldots, n \qquad (15)$$

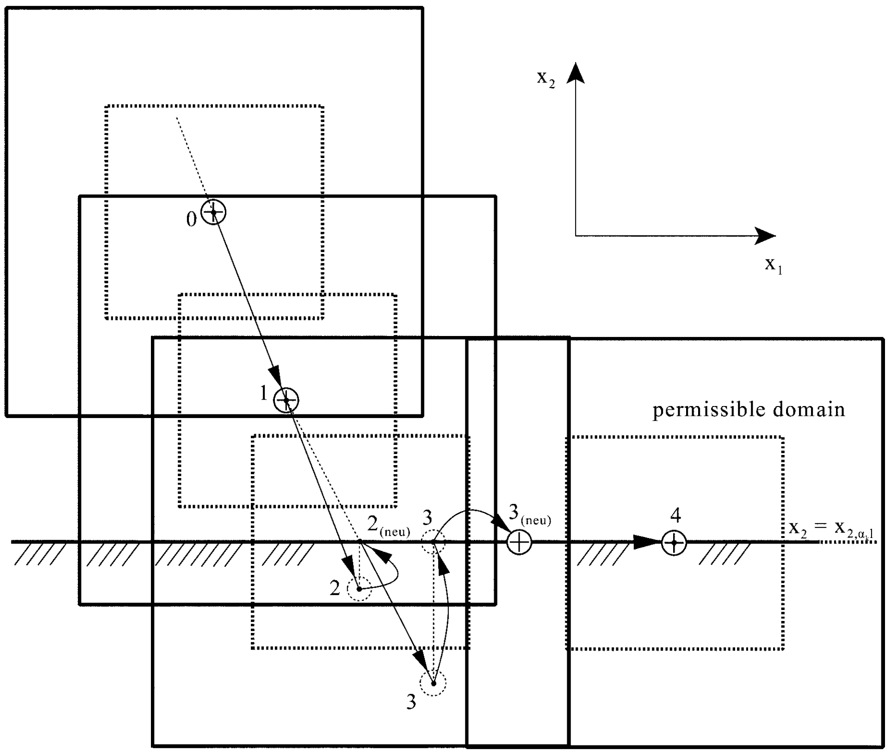

Fig. 8. Modified evolution strategy, behavior of the search algorithm along the boundary of the permissible domain x_{α_k}

$$x_{i(\text{new})}^{[q+1]} = x_{i,\,\alpha_k r} \Big| x_i^{[q+1]} > x_{i,\,\alpha_k r}; \quad i = 1, \ldots, n \tag{16}$$

If the point $\underline{x}_{(\text{new})}^{[q+1]}$ found in this way does not comply with the distance bound min $\_d_i$ according to Eq. (12), this point is rejected. The coordinate search is continued with the random determination of $\underline{x}^{[q+1]}$ (Eqs. (11) and (12)).

If the restrictions $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$ for the "directed specified point" $\underline{x}^{[q+r]}$ are not completely fulfilled, the coordinates $x_i^{[q+r]}$ concerned are placed (as for randomly placed points) on the boundary of $\underline{X}_{\alpha_k}$ (Fig. 8). If no interaction exists between the $\tilde{x}_i$, Eqs. (15) and (16) may be applied. The distance bound min $\_d_i$ is then checked using Eq. (12). For this purpose the coordinate $x_i^{[q+r]}$ is determined, which falls short of the minimum distance min $\_d_i$ by the least relative distance

$$d_{\text{rel,min}} = \max_{i=1,\ldots,n} \left(\frac{d_i}{\min\_d_i} \right) \tag{17}$$

If $d_{\text{rel,min}} \geq 1.0$, no coordinate $x_i^{[q+r]}$ falls short of the distance bound min $\_d_i$; the point $\underline{x}_{(\text{new})}^{[q+r]}$ has been found. If the requirement of minimum distance is not complied with (i.e. $d_{\text{rel,min}} < 1.0$), all d_i values (see Eq. (12)) are proportionally increased by the factor (Fig. 8)

$$c_3 = \frac{1}{2} \cdot \left(\frac{1}{d_{\text{rel,min}}} + \frac{1}{d_{\text{rel,max}}} \right) \tag{18}$$

in which

$$d_{\text{rel,max}} = \max_{i=1,\ldots,n} \left(\frac{d_i}{\max\_d_i} \right) \tag{19}$$

is the minimum relative deficit below the maximum distance max $\_d_i$. In accordance with Eq. (10) $d_{\text{rel,max}}$ corresponds to the same coordinate $x_i^{[q+r]}$ as $d_{\text{rel,min}}$. The point $\underline{x}_{(\text{new})}^{[q+r]}$ is determined from

$$\underline{x}_{(\text{new})}^{[q+r]} = \underline{x}^{[q+r-1]} + c_3 \cdot \left(\underline{x}^{[q+r]} - \underline{x}^{[q+r-1]} \right) \tag{20}$$

If no interaction exists between the $\tilde{x}_i$, the coordinates $x_i^{[q+r]}$, which are already boundary values of $\underline{X}_{\alpha_k}$ (i.e. for which $x_i^{[q+r]} = x_{i,\alpha_k l}$ or $x_i^{[q+r]} = x_{i,\alpha_k r}$), are retained and no longer considered in Eq. (20), see Fig. 8. On account of Eq. (18) the coordinate $x_{i(\text{new})}^{[q+r]}$ belonging to $d_{\text{rel,min}}$ is thus placed exactly between min $\_d_i$ and max $\_d_i$. In the case of "no interaction between the $\tilde{x}_i$" the distance bound max $\_d_i$ is retained for all $i = 1, \ldots, n$.

For the point $\underline{x}_{(\text{new})}^{[q+r]}$ compliance with the restrictions $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$ is checked; individual coordinates of $\underline{x}_{(\text{new})}^{[q+r]}$ are corrected as required, e.g. using Eqs. (15) and (16). If the point $\underline{x}_{(\text{new})}^{[q+r]}$ thereby lies in a corner of $\underline{X}_{\alpha_k}$ (i.e. of the permissible domain), a check is no longer necessary for the distance bounds and the objective function value for $\underline{x}_{(\text{new})}^{[q+r]}$ is computed. Otherwise, the distance bounds min $\_d_i$ and max $\_d_i$ are rechecked. If all conditions are satisfied, $\underline{x}_{(\text{new})}^{[q+r]}$ is evaluated.

If max $\_d_i$ is exceeded, the value of the corresponding coordinate $x_{i(\text{new})}^{[q+r]}$ must be reduced in such a way that the equals sign holds in

$$\max\_d_i \geq d_i = \left| x_i^{[q+r]} - x_i^{[q+r-1]} \right| \tag{21}$$

Compliance with the restrictions is rechecked; coordinates are corrected where necessary and the procedure is continued using Eq. (17). This procedure is repeated as required until all conditions for $\underline{x}_{(\text{new})}^{[q+r]}$ are satisfied or a corner position in $\underline{X}_{\alpha_k}$ is attained.

If the condition $\min\_d_i$ is not complied with, only the steps according to Eqs. (17)–(20) are included in the sequence to be repeated.

When searching for optimum points $\underline{x}_{\mathrm{opt}}$, use may be made of the inclusion properties of the subspaces $\underline{X}_{\alpha_k}$ and $\underline{X}_{\alpha_i}$ for $\alpha_k \leq \alpha_i$ according to Eq. (2). If it is necessary to check the point $\underline{x}^{[q]}$ on the α-level $\alpha_k \leq \alpha_i$ for optimality (which has already been evaluated for $\alpha = \alpha_i$), the existing functional value $z_j^{[q]}$ for $\alpha = \alpha_i$ is then used. A recomputation of $z_j^{[q]}$ is no longer necessary; see feature (4) of the optimization problem.

The use of existing points $\underline{x}^{[q]}$ with a known objective function value $z_j^{[q]}$ leads to an improved efficiency of the method. An endeavor is thus made to include existing points in the "proximity" of a newly-located point in the search for an extreme value. The proximity is defined with the aid of a direction-dependent distance Δd_i referred to the distance bound $\max\_d_i$.

$$\Delta d_i = c_4 \cdot \max\_d_i \mid 0 < c_4 < 1; \quad i = 1, \ldots, n \tag{22}$$

For every newly-placed point $\underline{x}^{[q]}$ a test is carried out to check whether this point lies in the proximity of the known point $\underline{x}^{[p]}$ with the known objective function value $z_j^{[p]}$. If

$$\Delta d_i \geq \left| x_i^{[q]} - x_i^{[p]} \right|; \quad 1, \ldots, n \tag{23}$$

then

$$\underline{x}^{[q]} = \underline{x}^{[p]} \tag{24}$$

is specified and the optimum search is continued.

All points which have already been evaluated in one extreme value search – i.e. optimization for one α-level, one fuzzy result value and one optimization objective (minimum or maximum) – are designated by $\underline{x}^{[\bar{p}]}$. These points $\underline{x}^{[\bar{p}]}$ are unable to produce improved results within the same extreme value search. It is further assumed that an improvement in the results is no longer possible using points in the proximity of the $\underline{x}^{[\bar{p}]}$ points. For these randomly or directed specified points $\underline{x}^{[q]}$ determined in this extreme value search a test is carried out using Eq. (23) to check whether a point $\underline{x}^{[\bar{p}]}$ is located in their proximity. If this is the case, $\underline{x}^{[q]}$ is not evaluated; the optimization search is continued with a new point $\underline{x}^{[q]}$.

The described procedure combines elements of different optimization methods (evolution strategy, gradient method, Monte-Carlo method). The random specification of offspring points within the distance bounds $\min\_d_i$ and $\max\_d_i$ corresponds to the simple Monte-Carlo method. In the event that no interaction exists between the $\tilde{x}_i$ and that the local search domains bounded by $\min\_d_i$ and $\max\_d_i$ form n-dimensional hypercuboids, the most probable, randomly selected search direction runs parallel to one of the spatial diagonals of the permissible domain. In the case of uniform distribution of the randomly selected points $\underline{x}^{[q+r]}$ within the local search domain assigned to the point $\underline{x}^{[q]}$, the functional value of the probability density function is

$$\begin{aligned} p_0 &= f(\underline{x}^{[q+r]}) \\ &= \frac{1}{\prod_{i=1}^{n}(2 \cdot \max\_d_i) - \prod_{i=1}^{n}(2 \cdot \min\_d_i)} \\ &= \frac{1}{2^n \cdot (1 - c_2^n) \cdot \prod_{i=1}^{n}(\max\_d_i)} \end{aligned} \tag{25}$$

If the angles between the coordinate axes x_i and the vector $(\underline{x}^{[q+r]} - \underline{x}^{[q]})$ are designated by φ_i, the functional value of the probability density function for the randomly selected search direction as a function of φ_i may be stated as $f(\underline{\varphi}^{[q+r]})$. The line determined by the direction of the vector $(\underline{x}^{[q+r]} - \underline{x}^{[q]})$ originating in $\underline{x}^{[q]}$ intersects the planes $x_i = x_i^{[q]} \pm \min\_d_i$ and $x_i = x_i^{[q]} \pm \max\_d_i$. The line segment Δl between the two intersection points for $\min\_d_i$ and $\max\_d_i$ (for the same i), located at the shortest distance from point $\underline{x}^{[q]}$, runs within the local search domain (Fig. 9).

The product of the length Δl and the constant p_0 yields the functional value of the probability density function

$$\begin{aligned} f\underline{\varphi}^{[q+r]}) &= \frac{p_0 \cdot (\max\_d_i - \min\_d_i)}{|\cos\,\varphi_i|} \\ &= \frac{p_0 \cdot (1 - c_2) \cdot \max\_d_i}{|\cos\,\varphi_i|} \end{aligned} \tag{26}$$

The values of $\min\_d_i$, $\max\_d_i$ and $\cos\varphi_i$ must be referred to the coordinates x_i for which $x_i = x_i^{[q]} \pm \min\_d_i$ and

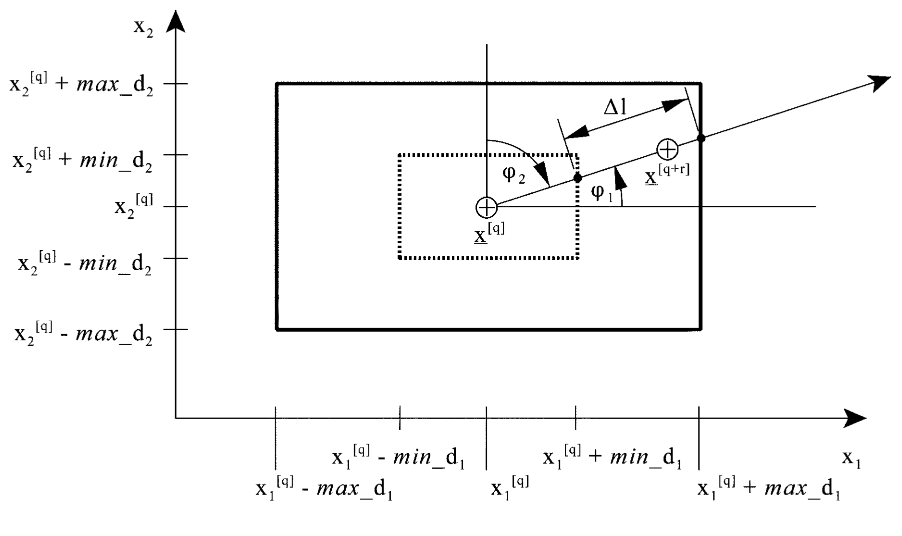

Fig. 9. Random determination of the search direction

$x_i = x_i^{[q]} \pm \max\_d_i$ form the limit points for Δl. The denominator in Eq. (26) is the absolute value of the direction cosine between x_i and $(\underline{x}^{[q+r]} - \underline{x}^{[q]})$. For $\cos\varphi_i = 1$, i.e. in the direction $\pm x_i$, the function $f(\underline{\varphi}^{[q+r]})$ possesses local minima; $f(\underline{\varphi}^{[q+r]})$ increases with decreasing $\cos\varphi_i$. For angles $\underline{\varphi}^{[q+r]}$, for which the reference coordinate x_i changes (edges or corners of the local search domain), "kinks" occur in the function $f(\underline{\varphi}^{[q+r]})$; local maxima of the functional value occur here. The global maximum of $f(\underline{\varphi}^{[q+r]})$ lies in the direction of a corner of the local search domain.

The search for optimum points is thus strongly guided towards the corners of the permissible domain (i.e. towards the corners of the subspace $\underline{X}_{\alpha_k}$), where the $\underline{x}_{\text{opt}}$ are located in the case of monotonic mappings. The direction of the largest departure from $\underline{X}_{\alpha_k}$ is preferred compared with other directions. The larger the factor c_2 then the less pronounced are these characteristics.

The progressive search in a fixed direction for improvement of the solution corresponds to an improved gradient method. The random determination of this direction avoids the numerical computations necessary to determine the gradient.

The flexibility of the algorithm is realized by means of variable control parameters. These must be specified for the particular problem concerned. The following control parameters were introduced:

(1) Maximum number n_P of randomly specified offspring points (starting from the same parent point) without improvement of the result.
(2) Number of refinement stages n_f for the distance bounds $\min\_d_i$ and $\max\_d_i$.
(3) Maximum directional step increment $\max\_d_i$ relating to the support of the fuzzy input value $\tilde{x}_i$; factor c_1 in Eq. (9).
(4) Minimum directional step increment $\min\_d_i$ relating to $\max\_d_i$ according to (3); factor c_2 in Eq. (10).
(5) Maximum directional distance Δd_i for the reuse of existing points instead of the newly-placed ones, referred to the maximum step increment according to (3); factor c_4 in Eq. (22).
(6) Reduction of the maximum step increment for each refinement stage of the distance bounds, referred to the respective current maximum step increment according to (3); factor c_5 in

$$\max\_d_{i(\text{new})} = c_5 \cdot \max\_d_i \tag{27}$$

(7) Termination limits for the relative improvement of the result in the last n_z steps, referred to the maximum difference in the functional values of the objective function within an extreme value search; factor c_6 in

$$\min\_\Delta z_j = c_6 \cdot \max\left|z_j^{[q]} - z_j^{[p]}\right|; \quad q \neq p \tag{28}$$

Termination occurs when

$$\min\_\Delta z_j > \max_{r=1,\ldots,n_z}\left|z_j^{[q]} - z_j^{[q+r]}\right| \left| \begin{array}{ll} z_j^{[q+r]} > z_j^{[q]} & \text{for maximum search} \\ z_j^{[q+r]} < z_j^{[q]} & \text{for minimum search} \end{array} \right. . \tag{29}$$

If the parameter according to (1) is active, a jump occurs into the next refinement step for the distance bounds. If the number of refinement steps according to (2) is also exhausted, the optimization is terminated and the optimum point is taken to be localized. Alternatively, termination occurs when the criterion according to (7) becomes active.

Due to the direct or indirect orientation of all distance dimensions at the support of the respective fuzzy input value $\tilde{x}_i$ (independent of the α-level) a decrease in the defined distances with increasing α is avoided. The computational effort and the accuracy of the results are thus the same for all α_k, i.e. all points defining the membership function of a fuzzy result value are determined to the same accuracy. For large α_k the permissible domain $\underline{X}_{\alpha_k}$ may be very small compared with the support subspace $\underline{X}_{\alpha_l=0}$ and hence also very small compared with the local search domain. For an extreme value search with coarse accuracy requirements this may mean that (starting from the first parent point $\underline{x}^{[0]}$) all additional points in the permissible domain $\underline{X}_{\alpha_k}$ do not satisfy the requirement for the minimum distance $\min\_d_i$ according to Eq. (12). The optimization is then (corresponding to the selected control parameters) continued with the next refinement step or terminated. An orientation of the distance dimensions to the respective permissible domain $\underline{X}_{\alpha_k}$ would then lead to an increase in the search accuracy and thus to an increase in the computational effort with increasing α_k.

The identification of global optima is not always guaranteed even using well-matched parameters. A post-computation is thus carried out in order to raise the success probability. After the completion of all optimizations for the selected α-levels all of the stored results z_j of the points $\underline{x}$ considered (observing the respective restrictions $(x_1, \ldots, x_n) \in \underline{X}_{\alpha_k}$) are rechecked for optimality. When improving the results the algorithm is restarted at the "best" points found. At the end of these additional computations the post-computation is repeated. When no (significant) improvement is obtained, the α-level optimization is considered to be completed.

The principle of the post-computation is illustrated in Fig. 10 for the α-level $\alpha_k = 0.40$. All points $\underline{x}$ are plotted for which the objective function values z_j are known from previous optimizations; the point ● has not yet been evaluated. The optimum points $\underline{x}_{\text{opt}}$ identified in the optimizations on the five α-levels considered (here: only for one optimization objective in each case) are denoted by ▲. All points $\underline{x}$ to be checked for optimality for $\alpha_k = 0.40$ are represented by ○; these lie in the marked (solid lines) permissible domain (subspace $\underline{X}_{\alpha_k}$). A comparison of the results z_j obtained at points ○ with the current optimum obtained so far (at ▲) yields an improved result at point ⊕. Repeated optimization for $\alpha_k = 0.40$ with the identified point ⊕ results in the new optimum point $\underline{x}_{\text{opt}}$ indicated by ●. The final post-computation leads to no improvement in the result.

The combination of the modified evolution strategy with a post-computation for each fuzzy result value on all selected α-levels leads to a qualitatively improved solution of the α-level optimization. The probability of finding global optima increases considerably. Based on the inclu-

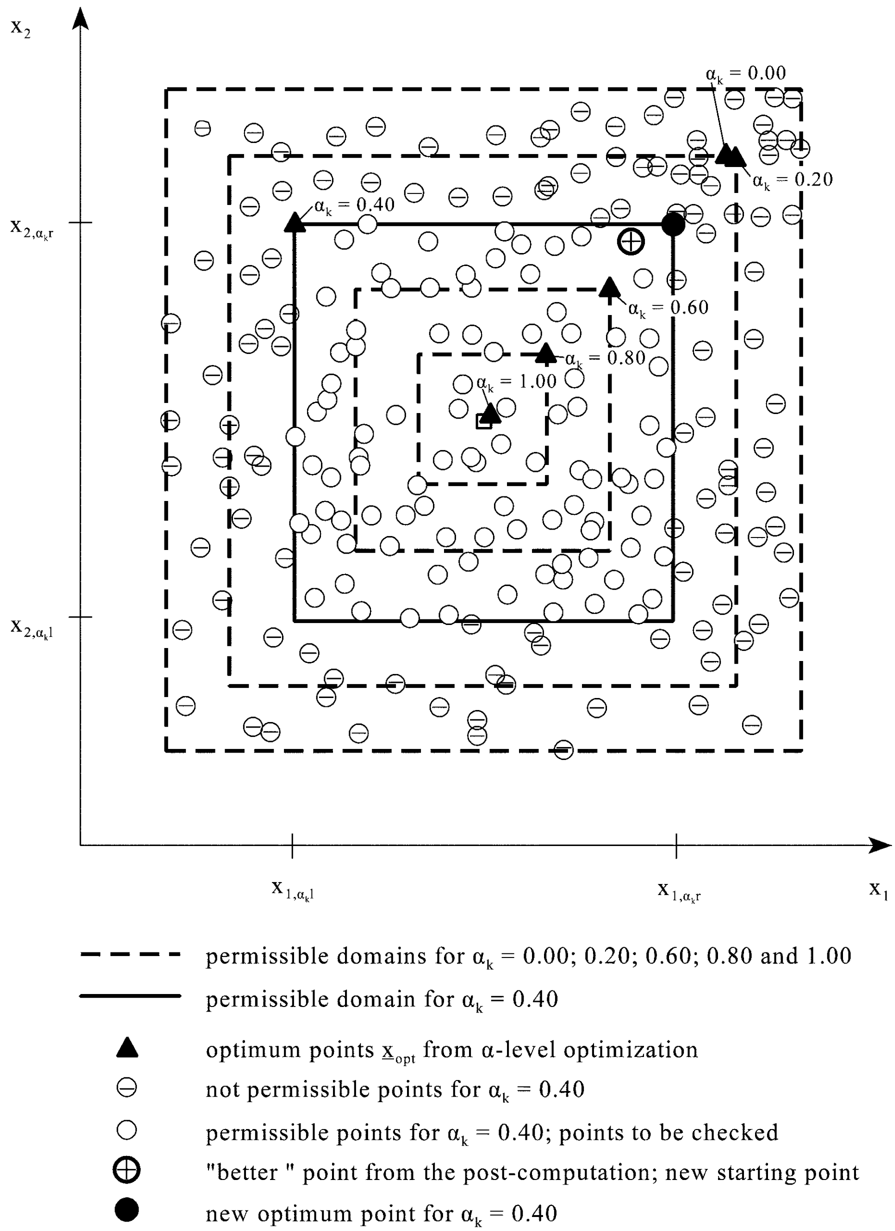

Fig. 10. Post-computation for $\alpha_k = 0.40$

sion relationship between the subspaces $\underline{X}_{\alpha_i}$ and $\underline{X}_{\alpha_k}$ for different α_i and α_k according to Eq. (2) it may be concluded that the magnitudes of the computed optima $z_{j,\alpha_k l}$ and $z_{j,\alpha_k r}$ behave monotonically in relation to the associated α-values α_k. In view of the latter the convexity of the fuzzy result values $\tilde{z}_j$ is guaranteed.

4 Examples

4.1 Statical fuzzy structural analysis

The effects of different fuzzy values are investigated for the plane reinforced concrete frame shown in Fig. 11. The system is modeled using three bars. Fifty integration increments are chosen for each bar and each cross-section is subdivided into 60 layers. The geometrically and physically nonlinear analysis is carried out using the material laws for reinforcement steel and concrete after Oetes (see Müller et al., 1995). Tension stiffening and the effects of stirrup reinforcement are accounted for in the concrete material law.

The loading process is comprised of dead weight, horizontal load P_H, vertical nodal loads $\nu \cdot P_{V0}$ and the line load $\nu \cdot p_0$. After applying dead weight the horizontal load P_H is introduced; P_{V0} and p_0 are finally increased incrementally using the load factor ν.

Investigation I – model uncertainty, computation of a fuzzy system reaction and the fuzzy failure load

When modeling the structure it is necessary to define the arrangement of the reinforcement steel. The tolerances in the laying of reinforcement steel are accounted for in the investigation as fuzzy values. These represent an example of model uncertainty.

The distances h_1, h_2 and h_3 (Fig. 12) between the cross-sectional boundaries and the position of the rein-

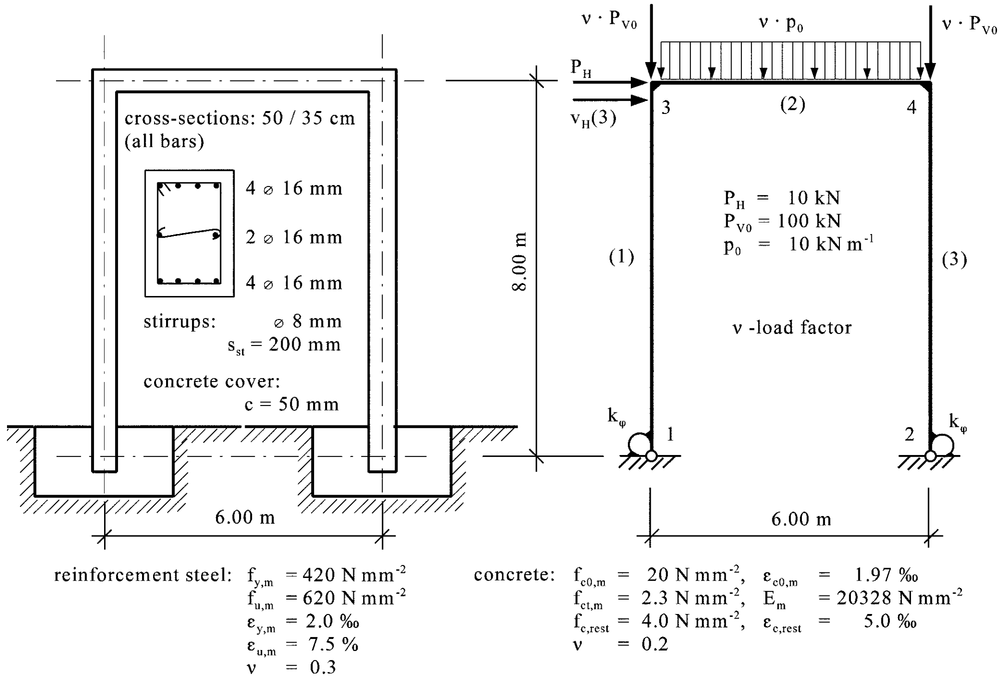

Fig. 11. Reinforced concrete frame (plane); structure, cross-sections, materials, statical system with loading

forcement steel are prescribed at each end of the bars. In the cross bar the reinforcement arrangement is also specified in the middle of the bar. The distances h_1, h_2 and h_3 are considered to be independent of each other; the reinforcement arrangements at the various prescribed locations are also independent of each other. Assuming a tolerance of ±5 mm in the laying of the reinforcement steel, h_1, h_2 and h_3 become fuzzy values. For the case in question a simple fuzzification of the values h_i is applied. The fuzzy model values $\tilde{h}_i$ are described by the fuzzy triangular numbers

$$\tilde{h}_i = \langle h_{i,\mathrm{soll}} - 5,\ h_{i,\mathrm{soll}},\ h_{i,\mathrm{soll}} + 5 \rangle\ [\mathrm{mm}] \qquad (30)$$

A consideration of the fuzzy model values $\tilde{h}_i$ when modeling the bars leads to the uncertain model reinforcement arrangement. The system shown in Fig. 11 contains 21 fuzzy model values.

The remaining model and input values are assumed to be deterministic. The rotational spring stiffness k_φ relating to the fixing of the columns in the foundation soil is specified by

$$k_\varphi = 5.0\,\mathrm{MNm\ rad^{-1}} \qquad (31)$$

The described loading process is simulated up to global system failure. The fuzzy result values are the horizontal displacement of node 3 $\tilde{v}_H(3)$ for all increments of the loading process and the load factor $\tilde{\nu}_{g+p}$ (g + p = structural analysis under consideration of geometrical and physical nonlinearities) for the fuzzy failure load. The three α-levels $\alpha_1 = 0.00$, $\alpha_2 = 0.50$ and $\alpha_3 = 1.00$ are chosen for the computation. The fuzzy load-displacement dependency for $\tilde{v}_H(3)$ is presented in Fig. 13; the load factor $\tilde{\nu}_{g+p}$ for the fuzzy failure load is approximately described by the fuzzy triangular number

$$\tilde{\nu}_{g+p} = \langle 6.27,\ 6.43,\ 6.59 \rangle \qquad (32)$$

Investigation II – data uncertainty, computation of a fuzzy system reaction

The investigation of the frame shown in Fig. 11 is repeated with altered structural parameters. The reinforcement arrangement is deterministically assumed to be in the

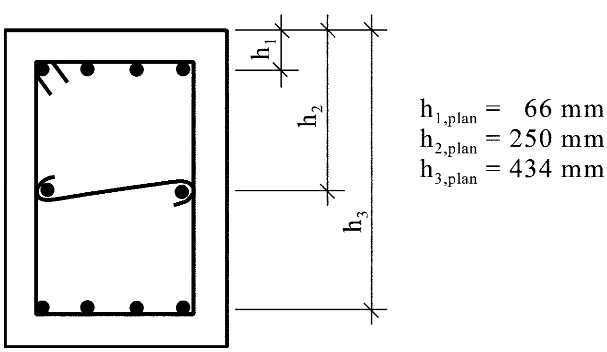

Fig. 12. Planned arrangement of the reinforcement

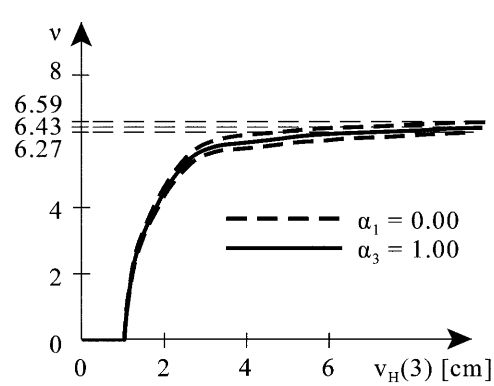

Fig. 13. Fuzzy load-displacement dependency for $\tilde{v}_H(3)$ and $\alpha_1 = 0.00$ and $\alpha_3 = 1.00$

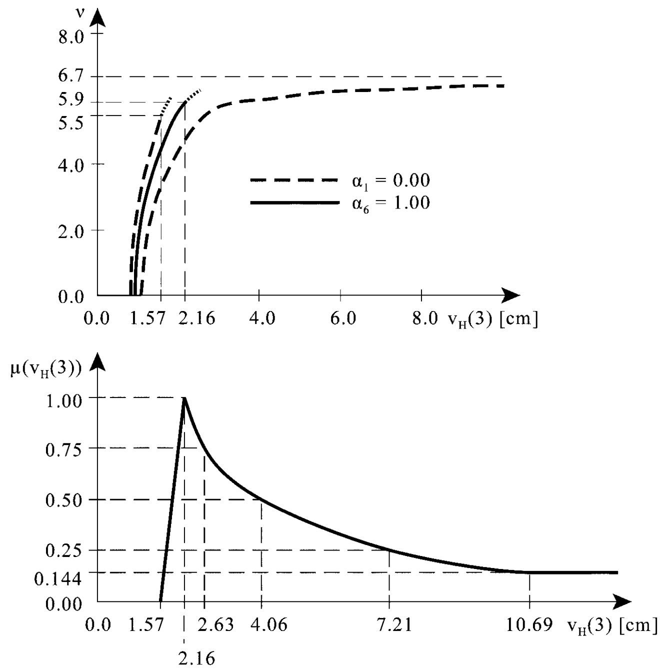

Fig. 14. Fuzzy load-displacement dependency for $\tilde{v}_H(3)$ and $\alpha_1 = 0.00$ and $\alpha_6 = 1.00$; fuzzy result $\tilde{v}_H(3)$

planned position. The fuzzy input values are now the load factor $\tilde{\nu}$ and the rotational spring stiffness $\tilde{k}_\varphi$. These are modeled as fuzzy triangular numbers

$$\tilde{\nu} = \langle 5.5,\ 5.9,\ 6.7 \rangle \tag{33}$$

and

$$\tilde{k}_\varphi = \langle 5.0,\ 9.0,\ 13.0 \rangle\ [\text{MNm rad}^{-1}] \tag{34}$$

the values of $\tilde{\nu}$ and $\tilde{k}_\varphi$ describe data uncertainty.

The fuzzy result value is the horizontal displacement of node 3 $\tilde{v}_H(3)$. The investigation is carried out for the α-levels $\alpha_1 = 0.00$, $\alpha_2 = 0.144$, $\alpha_3 = 0.25$, $\alpha_4 = 0.50$, $\alpha_5 = 0.75$ and $\alpha_6 = 1.00$. Figure 14 shows the fuzzy load-displacement dependency for $\alpha_1 = 0.00$ and $\alpha_6 = 1.00$ and the fuzzy result $\tilde{v}_H(3)$. For $\alpha_1 = 0.00$ global system failure occurs before the attainment of $\nu_{\alpha_l r} = 6.7$; the search for maximum $v_H(3)$ yields the result $v_H(3)_{\alpha_l r} \to \infty$ on this α-level.

Investigation III – data uncertainty, computation of the fuzzy failure load

The load factor $\tilde{\nu}_{g+p}$ of the fuzzy failure load for global system failure (for geometrically and physically nonlinear behavior) is computed as the fuzzy result value. In contrast to investigation II, only the rotational spring stiffness according to Eq. (34) enters the analysis as a fuzzy input value. The loads P_{V0} and p_0 are incrementally increased with ν until global system failure is attained. For the purpose of comparison the (quasi-crisp) failure load represented by ν_p, under exclusive consideration of physical nonlinearities, and the fuzzy failure load represented by $\tilde{\nu}_g$, under exclusive consideration of geometrical nonlinearities, are computed. The results for $\tilde{\nu}_{g+p}$ and ν_p are compared in Fig. 15. The load factor $\tilde{\nu}_g$ is approximately described by the fuzzy triangular number

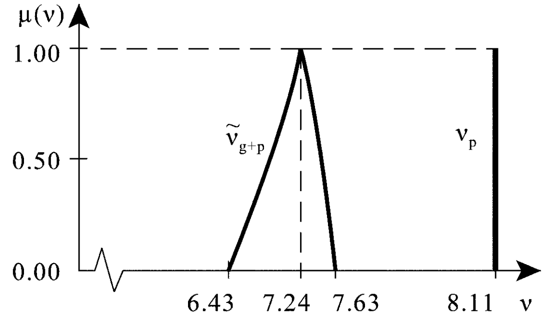

Fig. 15. Load factors $\tilde{\nu}_{g+p}$ and ν_p of fuzzy failure loads for global system failure; considered nonlinearities: p = physical, g+p = geometrical and physical

$$\tilde{\nu}_g = \langle 27.15,\ 32.70,\ 37.60 \rangle \tag{35}$$

This example demonstrates the decisive influence of the deterministic fundamental solution on the quality of the results.

4.2 Dynamic fuzzy structural analysis

The plane reinforced concrete frame with a prestressed cross bar according to Fig. 16 is investigated under dynamic transient loading. The system is modeled using three bars. 49 integration increments are chosen for the cross bar and 40 integration increments for the columns. Each of the cross-sections are subdivided into 60 layers. The geometrically and physically nonlinear analysis is carried out using the material laws after Ma, Bertero and Meskouris, Krätzig (see Meskouris et al., 1988) for reinforcement steel and concrete. Contactforces associated with fracture closure and the effects of stirrup reinforcement are accounted for in the concrete material law; tension stiffening is neglected in this case.

The structure is constructed from prefabricated parts and the frame corners are rigidly connected on site. The fixture of the columns in the foundation soil is modeled by means of linear-elastic rotational springs (Fig. 17).

The simulated loading process, including system modification, is comprised of the following components:

1. Simultaneous prestressing of all tendons in the cross bar according to the specified prestressing force without the effects of dead weight, grouting of the conduits.
2. Application of the dead weight of the columns, hinged connection of the columns and the cross bar at the frame corners and application of the dead weight of the cross bar.
3. Transformation of the hinged joints at the frame corners into rigid connections.
4. Application of additional translational mass at the frame corners and along the cross bar (statical loads P_V and $p_{R,V}$ in Fig. 17).
5. Introduction of dynamic loading ($P_H(t)$, $p_{R,H}(t)$ and $p_{S,H}(t)$ in Fig. 17) due to the horizontal acceleration a according to the normalized load-time function.

The stiffness of the rotational springs (the same at each column base) and the horizontal acceleration are modeled by the fuzzy triangular numbers

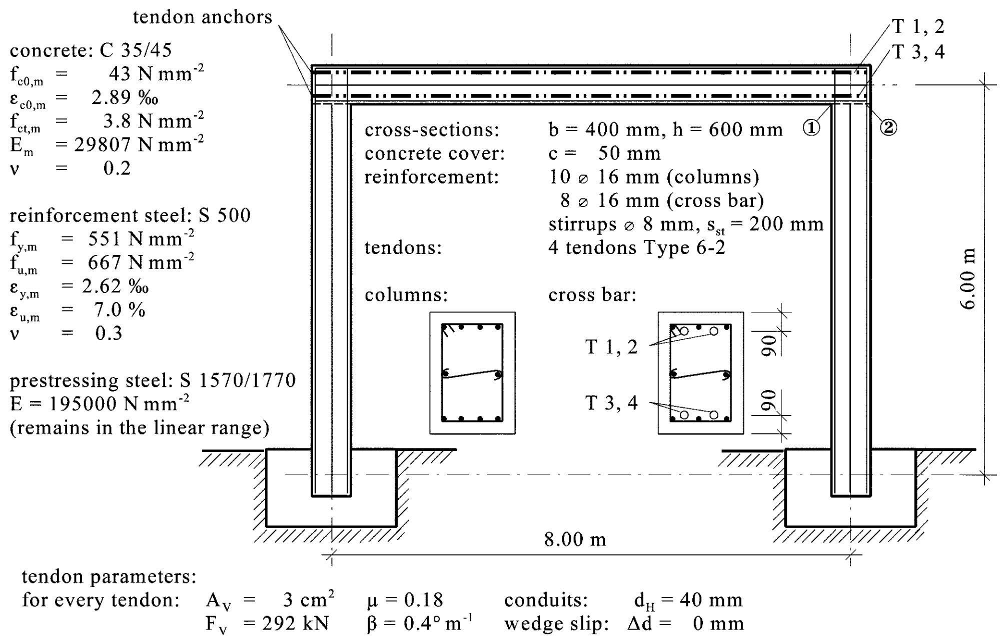

Fig. 16. Reinforced concrete frame (plane); structure, cross-sections, materials

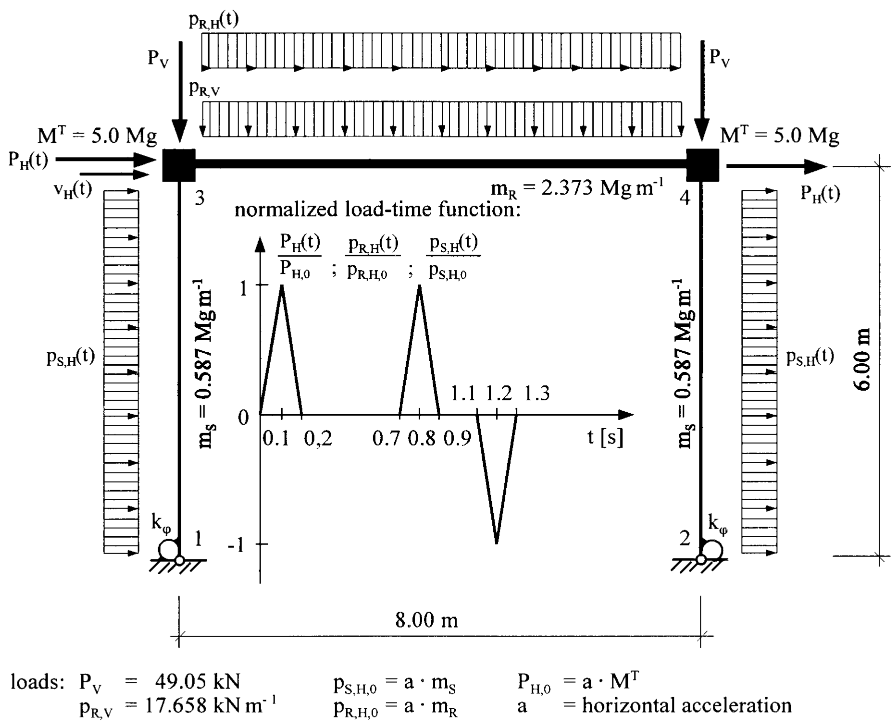

Fig. 17. Statical system (final state) with statical and dynamic loading

$$\tilde{k}_\varphi = \langle 7.0,\ 9.0,\ 11.0\rangle\ [\text{MNm rad}^{-1}] \tag{36}$$

and

$$\tilde{a} = \langle 0.35,\ 0.40,\ 0.45\rangle\ [g] \tag{37}$$

For the purpose of determining a suitable time step Δt for the numerical time-step integration the first three natural angular frequencies for the linear case and $k_\varphi = 9.0$ MNm rad^{-1} were computed to be $\omega_1 = 14.48\ \text{s}^{-1}$, $\omega_2 = 68.43\ \text{s}^{-1}$ and $\omega_3 = 108.64\ \text{s}^{-1}$. It is assumed that the prescribed loading excites the system exclusively at the first natural frequency. The time step for the numerical integration is $\Delta t = 0.025$ s. The time dependency of the horizontal fuzzy displacement $\tilde{v}_H(t)$ of the left-hand frame corner up to $t = 2.5$ s is plotted in Fig. 18. For the purpose of comparison the results of a linear deterministic investigation for $\mu(v_H(t)) = 1.0$ are also plotted.

The fuzzy result for the magnitude of the largest bending moment $\tilde{M}_S$ (absolute value) at the right-hand column base is plotted in Fig. 19 and compared with the results of the linear analysis for $\mu(M_S) = 1.0$. The stress-strain dependencies for the inner concrete layer (① in Fig. 16) and the outer reinforcement layer (② in Fig. 16) for the right-hand column head are also plotted in

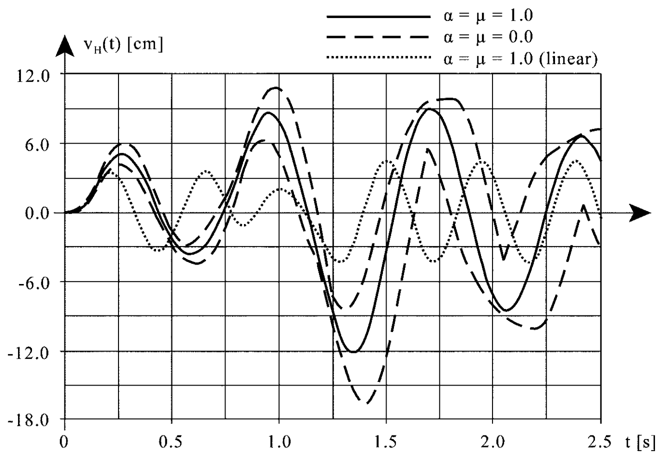

Fig. 18. Time dependency of the fuzzy displacement $\tilde{v}_H(t)$

Fig. 19 for $k_\varphi = 7000$ kNm rad^{-1} and a = 0.45 g ($\mu = 0$) up to $t = 2.5$ s. The material laws for hysteresis material behavior include the effects of material damping (Fig. 18).

5 Conclusions

The present paper indicates the way in which uncertain input values and model values may be described and applied in fuzzy structural analysis. Existing uncertainty is quantified on the basis of fuzzy set theory, with the inclusion of expert knowledge. These (partly subjective) assessed uncertainties in the fuzzy input values and fuzzy model values are mapped onto the fuzzy result values by means of special fuzzy analysis algorithms; the assessment of uncertainties is thereby retained. For the deterministic fundamental solution coupled with fuzzy analysis all known algorithms for structural analysis may be implemented.

The uncertain results of fuzzy structural analysis permit an improved assessment of load-bearing behavior under consideration of uncertainties. These may also provide a starting point for safety assessment on the basis of new concepts. Besides the application of possibility theory for evaluating the uncertainty fuzziness, a combination of fuzziness and randomness (based on the theory of fuzzy random variables) may also be successfully implemented on the basis of fuzzy first order reliability method (FFORM); see Möller et al. (1999).

Appendix

Several mathematical definitions

In classical set theory the membership of elements in relation to a set is assessed in binary terms on the basis of a crisp condition. An element either belongs or does not belong to the set under consideration. As a further development of classical set theory, fuzzy set theory permits the gradual assessment of the membership of elements in relation to a set; this is described with the aid of a membership function. An uncertain set is defined as follows:

Definition 1 – uncertain set
If $\mathbf{X}$ represents a fundamental set and x are the elements of this fundamental set, to be assessed according to an uncertain postulation and assigned to a subset A of $\mathbf{X}$, the set

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in \mathbf{X}\} \qquad \text{(A1)}$$

is referred to as the uncertain set of $\mathbf{X}$. $\mu_A(x)$ is the membership function (characteristic function) of the uncertain set $\tilde{A}$. The uncertain set $\tilde{A}$ is also referred to as a fuzzy value $\tilde{x}$.

The following holds for the functional values of the membership function $\mu_A(x)$

$$\mu_A(x) \geq 0 \quad \forall\, x \in \mathbf{X} \qquad \text{(A2)}$$

If

$$\sup_{x \in \mathbf{X}}[\mu_A(x)] = 1 \qquad \text{(A3)}$$

holds, the membership function is referred to as normalized.

Definition 2 – support of an uncertain set
The support $S(\tilde{A})$ of an uncertain set $\tilde{A}$ is a crisp set. This contains the elements

$$S(\tilde{A}) = \{x \in \mathbf{X} \mid \mu_A(x) > 0\} \qquad \text{(A4)}$$

Definition 3 – convex uncertain set
An uncertain set $\tilde{A}$ is referred to as convex if its membership function $\mu_A(x)$ monotonically decreases on each side of the maximum value, i.e. when the following applies

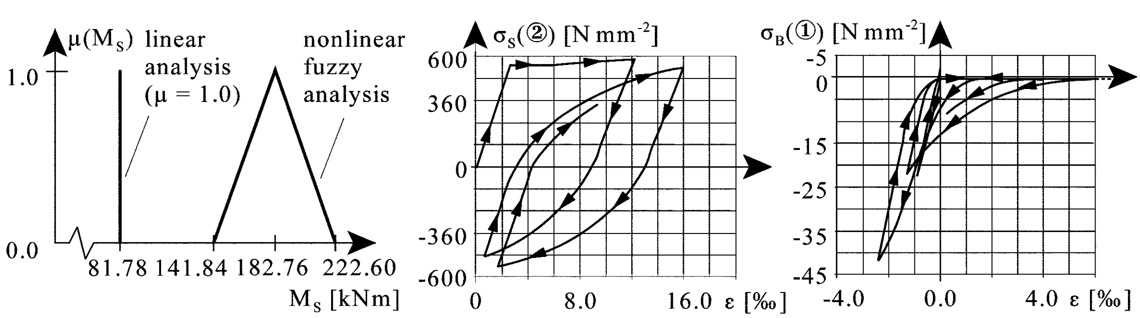

Fig. 19. Largest fuzzy bending moment $\tilde{M}_S$ (absolute value) at the right-hand column base; stress-strain dependencies for concrete (at ①) and steel (at ②)

$$\mu_A(x_2) \geq \min[\mu_A(x_1), \mu_A(x_3)]$$
$$\forall\, x_1, x_2, x_3 \in \mathbf{X} \text{ mit } x_1 \leq x_2 \leq x_3 \quad \text{(A5)}$$

Definition 4 – fuzzy number
A fuzzy number $\tilde{a}$ is a convex, normalized uncertain set $\tilde{A} \subseteq \mathbb{R}$, whose membership function is at least segmentally continuous and has the functional value $\mu_A(x) = 1$ at precisely one of the x values. This point x is referred to as the mean value of the fuzzy number. Fuzzy numbers with a linear membership function are referred to as fuzzy triangular numbers. These may be represented with the aid of the number triplet $\tilde{a} = \langle x_1, x_2, x_3 \rangle$, whereby x_1 and x_3 are the interval bounds of the support and x_2 is the mean value with the functional value $\mu_A(x_2) = 1$.

Definition 5 – α-level set
From the uncertain set $\tilde{A}$ the crisp sets

$$A_{\alpha_k} = \{x \in \mathbf{X} \,|\, \mu_A(x) \geq \alpha_k\} \quad \text{(A6)}$$

may be extracted for real numbers $\alpha_k \in [0, 1]$. These crisp sets are called α-level sets. All α-level sets A_{α_k} are crisp subsets of the support $S(\tilde{A})$. For several α-level sets of the same uncertain set $\tilde{A}$ the following holds:

$$A_{\alpha_k} \subseteq A_{\alpha_i} \quad \forall\, \alpha_i, \alpha_k \in [0, 1] \text{ with } \alpha_i \leq \alpha_k \quad \text{(A7)}$$

If the uncertain set $\tilde{A}$ is convex, each α-level set A_{α_k} is an interval $[x_{\alpha_k l}; x_{\alpha_k r}]$ in which

$$x_{\alpha_k l} = \min[x \in \mathbf{X} \,|\, \mu_A(x) \geq \alpha_k] \quad \text{(A8)}$$

$$x_{\alpha_k r} = \max[x \in \mathbf{X} \,|\, \mu_A(x) \geq \alpha_k] \quad \text{(A9)}$$

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