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Energy approach for static and linearized dynamic studies of elastic structures containing incompressible liquids with capillarity: a theoretical formulation

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Abstract In the context of the study of structures coupled with internal liquids, we present in this article a theoretical work to treat the coupling between the structure elasticity and the surface tension phenomenon, which has not been the object of specific studies before (according to the authors knowledge). Considering an incompressible and inviscid liquid in an elastic container, an energy approach is used to obtain a variational formulation of the small amplitude vibrations of the coupled problem around the nonlinear static equilibrium position. The incompressibility of the liquid supposed inviscid and the contact condition at the fluid-structure interface are introduced by Lagrange multipliers. Gravitational forces and surface tensions are both taken into account considering their associated potential energies.

Keywords Variational formulation · Least action principle · Fluid structure interaction · Hydroelasticity · Wall contact condition · Incompressibility · Sloshing · Surface tension

1 Introduction

The motion of liquids contained in satellites, probes or space stations can influence the vibrational behavior of the main structure, and disturb the stabilization procedures or the trajectory controller. This liquid sloshing phenomenon has been studied for many years [1,10,14,16], and the estimation of the inner liquid eigenmodes and eigenfrequencies, right from the design phase, is still an important topic.

In previous works, the effect of the gravity on the vibrations of a coupled fluid-structure system [18] and the effect of capillary forces on the liquid sloshing in a rigid container [9] have been studied independently. The aim of this work is to establish a unified formulation taking into account the structure elasticity, the gravity effects and the capillary forces considering an inviscid and incompressible liquid.

Since the coupled local equations of the problem are not known a priori, we propose to use an energy method. This approach also allows to take into account the fluid incompressibility or the contact condition at the fluid-structure interface with Lagrange multipliers [4]. A similar idea of starting with an augmented Lagrangian approach to FSI, and then formally (i.e., using variational arguments) eliminating the Lagrange multipliers to obtain an appropriately coupled FSI problem only in terms of the primal variables was proposed in [2,3]. In [2] and [12] the resultant formulation was applied to the simulation of wind turbine FSI using FEM for aerodynamics and isogeometric analysis [13] for structural mechanics. It should here be emphasized that the primal variable which is retained for the fluid will be only the displacement field of the free surface. For the liquid domain, we will use a potential displacement field.

Using the Hamilton's principle of least action, the variational formulation of the non linear static problem is first presented (Sect. 3). Then, the dynamic equations of the system and its linearization using the fluid and structure displacements are presented (Sect. 4). In Sect. 5, the fluid potential displacement is introduced to eliminate the Lagrange multipliers (Sects. 6, 7). Finally, a discretization by the

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finite element method gives the matrix equation of the system (Sect. 8).

2 Notations

We first use the fluid and structure displacements to describe the system, three different states can be distinguished (see Fig. 1):

The natural state which is a virtual configuration: no forces are applied on the fluid-structure system. The free surface is horizontal. Elements related to this configuration are indexed by the sign ^{0}.
The equilibrium state is the reference state which is determined by the static study (Sect. 3) and the elements related to this state are indexed by the sign *. By applying forces associated to a potential energy, the system reaches its equilibrium shape. The fluid and structure displacements corresponding to this transformation are respectively written $\boldsymbol{U}^{\boldsymbol{F}}_*$ and $\boldsymbol{U}^{\boldsymbol{S}}_*$. Due to surface tensions, the free surface at the equilibrium state is curved (it forms a meniscus).
The dynamical state is the state at all time t: a shaking force $\boldsymbol{f}$ is applied on the structure (surface Σ_f).[1] For sake of brevity, we suppose here that the applied force is in equilibrium. The fluid and structure displacements are written $\boldsymbol{U}^{\boldsymbol{F}}$ and $\boldsymbol{U}^{\boldsymbol{S}}$.

The volume of fluid Ω_F is delimited by the free surface Γ and the fluid-structure interface Σ. Let γ be the intersection between Γ and Σ, γ is also called the *triple line*. The fluid position is given by the vector field $\boldsymbol{X}$ with $\boldsymbol{X} = \boldsymbol{X}^0 + \boldsymbol{U}^{\boldsymbol{F}}$ and $\boldsymbol{X}^0$ is the position of the fluid at its natural state. The solid domain is written Ω_S and the solid–gas interface is Σ_G. $\boldsymbol{n}_\Sigma$ and $\boldsymbol{n}_\Gamma$ are the outward unit normals to the fluid on Σ and Γ. On the triple line, we will use the basis illustrated by Fig. 2: $\boldsymbol{t}_\gamma$ is tangent to γ, $\boldsymbol{v}_\Sigma$ and $\boldsymbol{v}_\Gamma$ are the unit vectors tangent to Σ and Γ such that $\boldsymbol{t}_\gamma = \boldsymbol{v}_\Sigma \wedge \boldsymbol{n}_\Sigma$ and $\boldsymbol{t}_\gamma = \boldsymbol{n}_\Gamma \wedge \boldsymbol{v}_\Gamma$.

3 Static study

Let $\mathcal{L}$ be the Lagrangian of the system defined by: $\mathcal{L} = \mathcal{T} - \mathcal{V} + W_{ext}$, where $\mathcal{T}$ is the kinetic energy, $\mathcal{V}$ the potential energy and W_{ext} is the work done by the external load $\boldsymbol{f}$ on Σ_f. We first consider the static problem in which $\mathcal{T}$ and W_{ext} are null. The potential energy is composed of the gravitational potential energy of the fluid E_g, the elastic potential energy of the structure E_{elas} and the surface energies written $E_{\sigma_{tot}}$. Let $\boldsymbol{Q} = (\boldsymbol{U}^{\boldsymbol{F}}, \boldsymbol{U}^{\boldsymbol{S}})$ be the displacement unknown of the problem, $\boldsymbol{\delta q} = (\boldsymbol{\delta u}^{\boldsymbol{F}}, \boldsymbol{\delta u}^{\boldsymbol{S}})$ a small variation of $\boldsymbol{Q}$ (kinematically admissible). From the principle of minimum total potential energy, a mechanical system shall find an equilibrium position which minimizes its total potential energy. The physically admissible solution $\boldsymbol{Q}^*$ is such that the variation of $\mathcal{L}$ is null for any $\boldsymbol{\delta q}$:

$$\left.\frac{\partial \mathcal{L}}{\partial \boldsymbol{Q}}\right|_{\boldsymbol{Q}=\boldsymbol{Q}^*} (\boldsymbol{\delta q}) = 0, \quad \forall \boldsymbol{\delta q} \tag{1}$$

In the rest, we will use the notation $\delta_{\boldsymbol{Q}}[\,\mathcal{L}\,]^*(\boldsymbol{\delta q})$ to represent the derivative of $\mathcal{L}$ with respect to $\boldsymbol{Q}$ in the direction $\boldsymbol{\delta q}$ around the equilibrium state $\boldsymbol{Q}^*$. Moreover, the variation in (1) will be done under constraints using Lagrange multipliers added to the Lagrangian. The first constraint is the incompressibility of the liquid introduced by the term $\mathcal{I}$. Similarly, the coupling between the structure and the fluid will be taken into account by the term $\mathcal{C}$ such that:

$$\mathcal{L} = -\left(E_{elas} + E_g + E_{\sigma_{tot}}\right) + \mathcal{C} + \mathcal{I} \tag{2}$$

In sects. 3.1–3.5, we determine the first order variation of those energies in order to express the variational formulation of the static problem in Sect. 3.6.

3.1 Gravitational potential energy of the fluid

By definition, the gravitational potential energy is given by:

$$E_{pes}(\boldsymbol{U}^{\boldsymbol{F}}) = \rho^F g \int_{\Omega_F} Z \, \mathrm{d}\Omega_F = \rho^F g \int_{\Omega_F^0} Z \, \det(\boldsymbol{F}) \, \mathrm{d}\Omega_F^0 \tag{3}$$

where $Z = \boldsymbol{X} \cdot \boldsymbol{i}_z$ and $\boldsymbol{i}_z$ is the vertical unit vector oriented upwards. ρ^F is the mass density and g the gravity constant.[2] As the incompressibility condition is enforced in the Lagrangian via a constraint, the mass conservation will be taken into account considering that ρ^F is constant. $\boldsymbol{F}$ is the deformation gradient and the Jacobian of this transformation is given by $J = \det(\boldsymbol{F})$. The small variation of a volume $\mathrm{d}\Omega_F$ is given by (see [7, Chap. 9]):

$$\delta\,[\mathrm{d}\Omega_F](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \mathrm{div}(\boldsymbol{\delta u}^{\boldsymbol{F}})\mathrm{d}\Omega_F$$

Moreover, noting that $\delta\,[\,Z\,](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \boldsymbol{\delta u}^{\boldsymbol{F}} \cdot \boldsymbol{i}_z$, the first variation of the gravitational potential energy in direction $\boldsymbol{\delta u}^{\boldsymbol{F}}$ is given by:

$$\delta_{\boldsymbol{U}^{\boldsymbol{F}}}\left[E_{pes}\right](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \rho^F g \int_{\Omega_F} \mathrm{div}(Z\,\boldsymbol{\delta u}^{\boldsymbol{F}}) \, \mathrm{d}\Omega_F \tag{4}$$

[1] $\boldsymbol{f}$ is supposed to be a dead load.

[2] g can be assimilated to the gravity on earth or to the acceleration of the system's frame of reference.

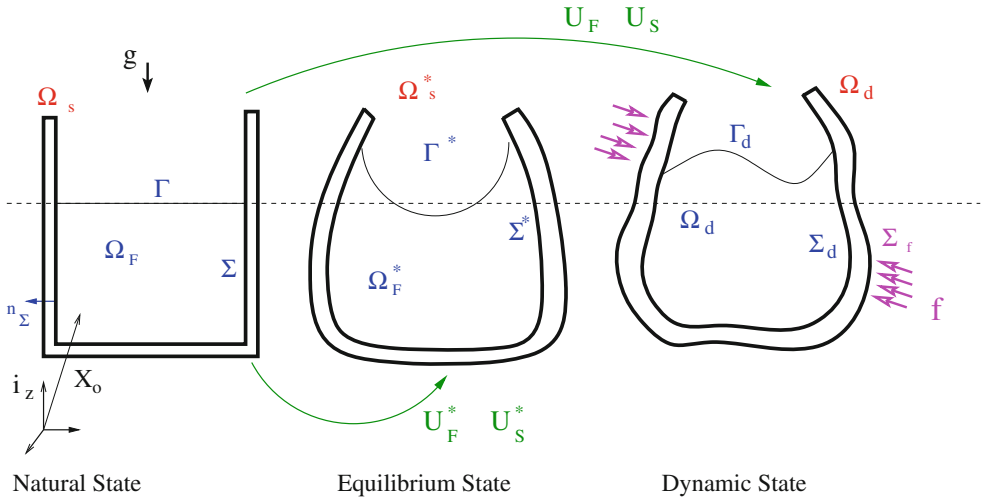

Fig. 1 Representation of the different states

Using the Green formulae:

$$\boxed{\delta\left[E_{pes}\right](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \rho^F g \int_{\Gamma\cup\Sigma} Z\boldsymbol{\delta u}^{\boldsymbol{F}} \cdot \boldsymbol{n}\, \mathrm{d}S} \tag{5}$$

where $\boldsymbol{n}$ is the outward unit normal to the volume Ω_F.

3.2 Elastic potential energy of the structure

The first variation of the elastic potential energy of the structure E_{elas} in direction $\boldsymbol{\delta u}^{\boldsymbol{S}}$ is classically given by:

$$\boxed{\delta_{\boldsymbol{U}^S}\left[E_{elas}\right](\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Omega_S} \mathrm{Tr}\left[\boldsymbol{\sigma}\left(\boldsymbol{U}^{\boldsymbol{S}}\right)\boldsymbol{\mathcal{E}}\left(\boldsymbol{\delta u}^{\boldsymbol{S}}\right)\right]\mathrm{d}\Omega_S} \tag{6}$$

where ε is the linearized strain tensor, and $\boldsymbol{\sigma}$ the Cauchy's strain tensor.

3.3 Surface energies

Let S be the interface between two media (one of them is condensed), the surface energy associated to S is a potential energy proportional to the area of S written $\mathcal{A}_S$ [20]:

$$E_S = \sigma_S \mathcal{A}_S \tag{7}$$

where σ_S is the capillary coefficient of S (supposed to be constant in this article). For a displacement field $\boldsymbol{U}$ defined on S, the first variation of the surface energy in direction $\boldsymbol{\delta u}$ can be written [7]:

$$\delta_{\boldsymbol{U}}\left[E_S\right](\boldsymbol{\delta u}) = \sigma_S \int_S \mathrm{div}_S(\boldsymbol{\delta u})\mathrm{d}S \tag{8}$$

Writing $\boldsymbol{\delta u}^{/\!/}$ the part of $\boldsymbol{\delta u}$ tangent to S and $\boldsymbol{\delta u}^{\perp}$ the normal component of $\boldsymbol{\delta u}$, we have $\boldsymbol{\delta u} = \boldsymbol{\delta u}^{/\!/} + \boldsymbol{\delta u}^{\perp}$ and:

$$\mathrm{div}_S(\boldsymbol{\delta u}) = \mathrm{div}_S(\boldsymbol{\delta u}^{/\!/}) + (\boldsymbol{\delta u}\cdot\boldsymbol{n})\mathrm{div}_S(\boldsymbol{n}) + \boldsymbol{n}\cdot\boldsymbol{\nabla}_S(\boldsymbol{\delta u}\cdot\boldsymbol{n})$$

$\boldsymbol{n}$ is the unit surface normal of S and $\boldsymbol{\nabla}_S$ the surface gradient along S.[3] $\boldsymbol{\nabla}_S$ being tangent to the surface S, the last term is null. By definition, we will write $\mathrm{div}_S(\boldsymbol{n}) = 2H$ where H is the local mean curvature of S. Finally, applying the Green formula on (8):

$$\delta\left[E_S\right](\boldsymbol{\delta u}) = \sigma_S \oint_{\partial S} \boldsymbol{\delta u}\cdot\boldsymbol{\nu}\mathrm{d}S + \sigma_S \int_S 2H\,\boldsymbol{\delta u}\cdot\boldsymbol{n}\mathrm{d}S \tag{9}$$

where ∂S is the boundary of S. Let $\boldsymbol{t}$ be the tangent vector of ∂S, $\boldsymbol{\nu}$ is such that the basis $(\boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{n})$ is the Darboux frame (see Fig. 2). Let E_Σ, E_{Σ_G} and E_Γ be the energies respectively associated to the interfaces Σ, Σ_G and Γ with the constant surface energy densities σ_L, σ_G and σ. The sum of these energies is equal to the energy $E_{\sigma_{tot}}$ in expression (2). Considering the energies E_Γ, E_Σ, E_{Σ_G} and (9), we chose to associate the energy E_Γ and the terms on the boundary of the free surface γ to the fluid displacement $\boldsymbol{U}^{\boldsymbol{F}}$. The terms corresponding to the normal displacement of the energies E_Σ and E_{Σ_G} are associated to the structure displacement $\boldsymbol{U}^{\boldsymbol{S}}$. We have:

$$\delta_{\boldsymbol{U}^F}\left[E_{\sigma_{tot}}\right](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \delta\left[\sigma\mathcal{A}_\Gamma + \sigma_L\mathcal{A}_\Sigma + \sigma_G\mathcal{A}_{\Sigma_G}\right](\boldsymbol{\delta u}^{\boldsymbol{F}})$$
$$\delta_{\boldsymbol{U}^S}\left[E_{\sigma_{tot}}\right](\boldsymbol{\delta u}^{\boldsymbol{S}}) = \delta\left[\sigma_G\mathcal{A}_{\Sigma_G} + \sigma_L\mathcal{A}_\Sigma\right](\boldsymbol{\delta u}^{\boldsymbol{S}})$$

[3] For a scalar field k, the surface gradient $\boldsymbol{\nabla}_S$ can be defined by $\boldsymbol{\nabla}_S k = \boldsymbol{\nabla}k - (\boldsymbol{\nabla}k\cdot\boldsymbol{n})\boldsymbol{n}$. The surface divergence div_S can be expressed, for a vector field $\boldsymbol{v}$ defined on S, by $\mathrm{div}_S = \boldsymbol{\nabla}\cdot\boldsymbol{v} - \boldsymbol{n}^{\mathrm{T}}[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n}$, where $\boldsymbol{D}(\boldsymbol{v})$ is the tensor of partial derivatives of vector $\boldsymbol{v}$.

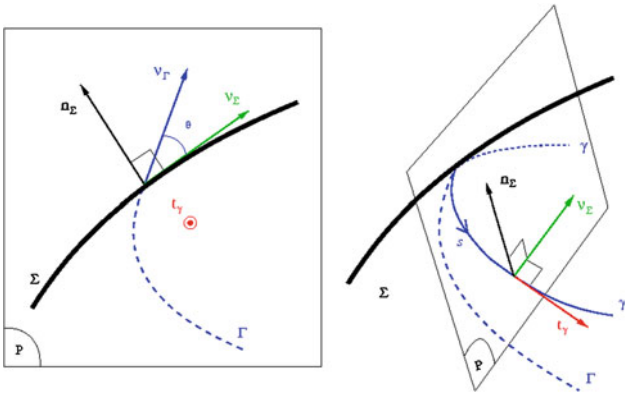

Fig. 2 Local basis (also called the Darboux frame) illustrated with the triple line γ

with:

$$\boxed{\begin{aligned}\delta\left[E_\Gamma\right](\boldsymbol{\delta u^F}) &= \sigma \oint_\gamma \boldsymbol{\delta u^F}\cdot\boldsymbol{v}_\Gamma \,\mathrm{d}\gamma + \cdots \\ &\cdots \sigma\int_\Gamma 2H_\Gamma\,\boldsymbol{\delta u^F}\cdot\boldsymbol{n}_\Gamma\,\mathrm{d}\Gamma \\ \delta\left[E_\Sigma\right](\boldsymbol{\delta u^F}) &= \sigma_L \oint_\gamma \boldsymbol{\delta u^F}\cdot\boldsymbol{v}_\Sigma \,\mathrm{d}\gamma \\ \delta\left[E_{\Sigma_G}\right](\boldsymbol{\delta u^F}) &= -\sigma_G \oint_\gamma \boldsymbol{\delta u^F}\cdot\boldsymbol{v}_\Sigma \,\mathrm{d}\gamma\end{aligned}} \tag{10}$$

$$\boxed{\begin{aligned}\delta\left[E_{\Sigma_G}\right](\boldsymbol{\delta u^S}) &= \sigma_G\int_{\Sigma_G} 2H_{\Sigma_G}\,\boldsymbol{\delta u^S}\cdot\boldsymbol{n}_{\Sigma_G}\,\mathrm{d}\Sigma_G \\ \delta\left[E_\Sigma\right](\boldsymbol{\delta u^S}) &= \sigma_L\int_\Sigma 2H_\Sigma\,\boldsymbol{\delta u^S}\cdot\boldsymbol{n}_\Sigma\,\mathrm{d}\Sigma\end{aligned}} \tag{11}$$

3.4 Incompressibilty

For a displacement field $\boldsymbol{U^F}$ defined on the fluid domain Ω_0^F, the incompressibility can be expressed by the relation $J(\boldsymbol{U^F}) = 1$. We deduce from this relation the incompressibility term $\mathcal{I}$:

$$\mathcal{I}(\Lambda, \boldsymbol{U^F}) = \int_{\Omega_0^F} \Lambda\,(J-1)\;\mathrm{d}\Omega_F^0 \tag{12}$$

where Λ is the Lagrange multiplier associated to the incompressibility and we denote by $\delta\lambda$ its virtual fluctuation. Using the expression of the variation of J given by (3.1) we have:

$$\delta_{\boldsymbol{U^F}}\left[\mathcal{I}\right](\boldsymbol{\delta u^F}) = \int_{\Omega^F} \Lambda\,\mathrm{div}(\boldsymbol{\delta u^F})\,\mathrm{d}\Omega_F \tag{13}$$

Integrating by part this expression we have:

$$\boxed{\delta\left[\mathcal{I}\right](\boldsymbol{\delta u^F}) = \int_{\Gamma\cup\Sigma} \Lambda\,\boldsymbol{\delta u^F}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{\Omega^F}\nabla\Lambda\cdot\boldsymbol{\delta u^F}\,\mathrm{d}\Omega_F} \tag{14}$$

The variation of $\mathcal{I}$ with respect to Λ in direction $\delta\lambda$ is given by:

$$\boxed{\delta_\Lambda\left[\mathcal{I}\right](\delta\lambda) = \int_{\Omega_0^F}\delta\lambda\,(J-1)\;\mathrm{d}\Omega_F^0} \tag{15}$$

3.5 Wall contact condition

3.5.1 At the interface Σ

For virtual displacements $\boldsymbol{\delta u^F}$ and $\boldsymbol{\delta u^S}$, we will consider that the coupling term on the interface Σ will be such that:

$$\boxed{\delta_{\boldsymbol{U^S},\boldsymbol{U^F}}\left[\mathcal{C}_\Sigma\right](\boldsymbol{\delta u^S},\boldsymbol{\delta u^F}) = \int_\Sigma M\,(\boldsymbol{\delta u^S}-\boldsymbol{\delta u^F})\cdot\boldsymbol{n}_\Sigma\mathrm{d}\Sigma} \tag{16}$$

where M is a Lagrange multiplier, homogeneous to a pressure. The part involved in the fluid equation (taking $\boldsymbol{\delta u^S} = 0$) is the opposite of the one involved in the structure equation (reciprocal action). One of the interests here is to uncouple the fluid and structure displacements by relaxing the contact condition through the operator $\mathcal{C}_\Sigma$. $\boldsymbol{U^S}$ and $\boldsymbol{U^F}$ are independents on Σ which simplifies the derivations.

3.5.2 On the triple line γ

The first variation of the surface energy $E_{\sigma_{tot}}$ in Sect. 3.3 illustrates the different works done by the surface forces on the interfaces Γ, Σ and Σ_G and by the line forces on the triple line γ (see (10) and (11)). Taking $\boldsymbol{v}_\Gamma = \cos(\theta)\boldsymbol{v}_\Sigma + \sin(\theta)\boldsymbol{n}_\Sigma$ (cf. Fig. 2), the projection of those line forces on $\boldsymbol{v}_\Sigma$ gives the Young's relation $\sigma_G - \sigma_L = \sigma\,\cos(\theta)$ where θ is the contact angle. Then, by projecting on $\boldsymbol{n}_\Sigma$, we notice that the liquid applies on the solid a force per unit length $\boldsymbol{f}^N_{F/S}$, whose magnitude is $\sigma\,\sin(\theta)$, pulling on γ in direction $\boldsymbol{n}_\Sigma$:

$$\boldsymbol{f}^N_{F/S} = -\sigma\,\sin(\theta)\,\boldsymbol{n}_\Sigma \tag{17}$$

In [6], the authors show, by using a microscopic scale model showing the molecular attraction between the liquid and the solid on the triple line "layer", that the normal force exerted by the solid on the liquid can be represented at a macroscopic scale by a force per unit length $\boldsymbol{f}^N_{S/F} = \sigma\,\sin(\theta)\,\boldsymbol{n}_\Sigma$. This

force compensates the effect of $\boldsymbol{f}^N_{F/S}$. We notice the singularity coming from the classic theory of elastic solids in three dimensions: a line force would be accompanied by infinite deformations. The explicit expression for the local microscopic deformation of the substrate is given in [21]. In the resolution of the static problem, the elastic deformation due to $\boldsymbol{f}^N_{F/S}$ would be negligible due to the high Young's modulus of the container. To take into account the contact condition (16) at the triple line we introduce another coupling term $\mathcal{C}_\gamma$ such that $\mathcal{C} = \mathcal{C}_\Sigma + \mathcal{C}_\gamma$:

$$\boxed{\delta\left[\mathcal{C}_\gamma\right](\boldsymbol{\delta u^S}, \boldsymbol{\delta u^F}) = \oint_\gamma M_\gamma(\boldsymbol{\delta u^S} - \boldsymbol{\delta u^F}) \cdot \boldsymbol{n}_\Sigma \, \mathrm{d}\gamma} \tag{18}$$

Here, the Lagrange multiplier M_γ is homogeneous to a force per unit length.

3.6 Variational formulation

$\mathcal{L}$ is a function of the variables $(\boldsymbol{U^F}, \boldsymbol{U^S}, \Lambda, M, M_\gamma)$. Let $\mathcal{C}^0_{\boldsymbol{U^F}} \times \mathcal{C}^0_{\boldsymbol{U^S}}$ and $\mathcal{C}_{\boldsymbol{U^F}} \times \mathcal{C}_{\boldsymbol{U^S}}$ be the spaces of admissible functions $(\boldsymbol{U^F}, \boldsymbol{U^S})$ respectively defined on $\Omega^0_F \cup \Omega^0_S$ and $\Omega_F \cup \Omega_S$. The Lagrangian is given by (2) and the static problem writes:

$$\boxed{\begin{array}{c} \exists(\boldsymbol{U^F_*}, \boldsymbol{U^S_*}, \Lambda^*, M^*, M^*_\gamma) \in \mathcal{C}^0_{\boldsymbol{U^F}} \times \mathcal{C}^0_{\boldsymbol{U^S}} \times \mathbb{R}^3, \\ \forall(\boldsymbol{\delta u^F}, \boldsymbol{\delta u^S}, \delta\lambda, \delta\mu, \delta\mu_\gamma) \in \mathcal{C}^0_{\boldsymbol{U^F}} \times \mathcal{C}^0_{\boldsymbol{U^S}} \times \mathbb{R}^3 \\ \delta_{\boldsymbol{U^F}}[\,\mathcal{L}\,]^*(\boldsymbol{\delta u^F}) = 0 \\ \delta_{\boldsymbol{U^S}}[\,\mathcal{L}\,]^*(\boldsymbol{\delta u^S}) = 0 \\ \delta_\Lambda[\,\mathcal{L}\,]^*(\delta\lambda) = 0 \\ \delta_M[\,\mathcal{L}\,]^*(\delta\mu) = 0 \\ \delta_{M_\gamma}[\,\mathcal{L}\,]^*(\delta\mu_\gamma) = 0 \end{array}} \tag{19}$$

The first two equations correspond to the fluid and structure equation, the last three corresponding to the constraints taken into account. The variational formulation (20) of the static problem is non linear ($\boldsymbol{U^F_*}$ and $\boldsymbol{U^S_*}$ can take large values). It is possible to solve the complete problem, however, we will neglect in the rest of the article the deformation of the structure due to gravity effects and the pre-stress stiffness will be defined on Σ^0 considering that Σ^* and Σ^0 are superimposed ($\boldsymbol{U^S_*} = 0$). Thus, the static study meet the one presented in [8] which is solved using the iterative method of Newton–Raphson to find the free surface shape Γ^*. We do not present this method here and we consider that the free surface at the equilibrium state and $\boldsymbol{U^F_*}$ are known.

From the fluid equation it is possible to express the Lagrange multipliers Λ^*, M^*, and M^*_γ. The first variation of the energies we consider in direction $\boldsymbol{\delta u^F}$ around the state Ω_F is given by (5), (10), (11), (14), (16) and (18). Taking $\boldsymbol{U^F} = \boldsymbol{U^F_*}$ in these expressions, we deduce the variation of $\mathcal{L}$ around the static equilibrium Ω^*_F and the first relation of (19) gives:

$$\begin{aligned} &-\int_{\Omega^F_*} \nabla\Lambda^* \cdot \boldsymbol{\delta u^F} \, \mathrm{d}\Omega^F \\ &+\int_{\Gamma^*} \left(\Lambda^* - \rho^F g\, Z^* - \sigma 2H^*_\Gamma\right) \boldsymbol{n}^*_\Gamma \cdot \boldsymbol{\delta u^F} \, \mathrm{d}\Gamma \\ &+\int_{\Sigma^*} \left(\Lambda^* - \rho^F g\, Z^* - M^*\right) \boldsymbol{n}^*_\Sigma \cdot \boldsymbol{\delta u^F} \, \mathrm{d}\Sigma \\ &+\oint_{\gamma^*} \left(-\sigma\sin(\theta) - M^*_\gamma\right) \boldsymbol{n}^*_\Sigma \cdot \boldsymbol{\delta u^F} \, \mathrm{d}\gamma = 0 \end{aligned} \tag{20}$$

With test functions null on the boundary of the fluid domain $\partial\Omega^F_*$ we obtain:

$$\nabla\Lambda^* = 0 \;\Rightarrow\; \Lambda^* \text{constant on } \Omega^F_* \tag{21}$$

Then, using (21) in (20) and taking respectively test functions $\boldsymbol{\delta u^F}$ null on Σ^* and γ^*, Γ^* and γ^* we have:

$$\Lambda^* = \rho^F g\, Z^* + \sigma 2H^*_\Gamma \quad \text{on } \Gamma^* \tag{22a}$$

$$M^* = \Lambda^* - \rho^F g\, Z^* \quad \text{on } \Sigma^* \tag{22b}$$

$$M^*_\gamma = -\sigma\sin(\theta) \quad \text{on } \gamma^* \tag{22c}$$

Considering the relation on the free surface Γ^*, the expression $\rho^F g\, Z^* + \sigma 2H^*_\Gamma$ is constant, which corresponds to the Laplace–Young equation that is generally written:

$$P - P_G = \sigma 2H^*_\Gamma \quad \text{on } \Gamma^* \tag{23}$$

where P is the pressure in the fluid at the free surface and P_G the external gas pressure. Then we have $\Lambda^* - \rho^F g\, Z^* = P - P_G$ on Γ^* and we deduce the following expressions:

$$\Lambda^* = P_{z=0} - P_G \quad \text{on } \Gamma^* \tag{24a}$$

$$M^* = P - P_G \quad \text{on } \Sigma^* \tag{24b}$$

The Lagrange multiplier M^* can be interpreted as the hydrostatic pressure on Σ^*.

4 Dynamic analysis

4.1 Nonlinear formulation

Let $\mathcal{S}$ be the action of the system:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L} dt \tag{25}$$

where t_1 and t_2 are fixed instants. The generalization of (20) in dynamics is given by the Hamilton's principle of least action: a mechanical system shall follow the trajectory which minimizes its total action. The real path $\boldsymbol{Q}$ taken by the system between times t_1 and t_2 is determined by a variational approach. $\mathcal{S}$ is a function of $\boldsymbol{Q}$ and the velocity unknown

$\dot{\boldsymbol{Q}}$. The physically admissible path of motion is such that the variation of $\mathcal{S}$ is null for any perturbation:

$$\delta[\mathcal{S}(\boldsymbol{Q},\dot{\boldsymbol{Q}})](\boldsymbol{\delta q},\dot{\boldsymbol{\delta q}})=0 \quad \forall\,(\boldsymbol{\delta q},\dot{\boldsymbol{\delta q}}) \tag{26}$$

with $\boldsymbol{\delta q}(t_1)=\boldsymbol{\delta q}(t_2)=0$. Similarly to the static study, we include the constitutive law of the fluid and the contact condition at the fluid structure interface in $\mathcal{L}$. We consider in this part the kinetic energies of the fluid E_C^F and the structure E_C^S and we have: $\mathcal{L}=\left(E_C^F+E_C^S\right)-\left(E_{pes}+E_{\sigma_{tot}}\right)+\mathcal{I}+\mathcal{C}+W_{ext}$. Thus, the principle of least action yields:

$$\boxed{\begin{aligned}
\delta[\mathcal{S}](\boldsymbol{\delta u^F},\dot{\boldsymbol{\delta u^F}})&=0 \quad \text{(a)}\\
\delta[\mathcal{S}](\boldsymbol{\delta u^S},\dot{\boldsymbol{\delta u^S}})&=0 \quad \text{(b)}\\
\delta[\mathcal{S}](\delta\lambda)&=0 \quad \text{(c)}\\
\delta[\mathcal{S}](\delta\mu)&=0 \quad \text{(d)}\\
\delta[\mathcal{S}](\delta\mu_\gamma)&=0 \quad \text{(e)}
\end{aligned}} \tag{27}$$

4.2 Linearization

As we only consider small vibrations around the equilibrium state, we linearize system (27) by writing variables $(\boldsymbol{U^F},\boldsymbol{U^S},\Lambda,M,M_\gamma)$ as:

$$\boxed{\begin{aligned}(\boldsymbol{U^F},\boldsymbol{U^S},\Lambda,M,M_\gamma)=\ &(\boldsymbol{U^F_*},\boldsymbol{U^S_*},\Lambda^*,M^*,M^*_\gamma)+\cdots\\ &\cdots(\boldsymbol{u^F},\boldsymbol{u^S},\lambda,\mu,\mu_\gamma)\end{aligned}} \tag{28}$$

where $(\boldsymbol{u^F},\boldsymbol{u^S},\lambda,\mu,\mu_\gamma)$ represent small perturbations. The linearization of relation (26) is given by its expansion around the static state:

$$\begin{aligned}\delta[\mathcal{S}](\boldsymbol{\delta q},\dot{\boldsymbol{\delta q}})=&\underbrace{\delta[\mathcal{S}]^*(\boldsymbol{\delta q},\dot{\boldsymbol{\delta q}})}_{A}+\cdots\\ &\cdots\frac{\partial\left[\delta[\mathcal{S}](\dot{\boldsymbol{\delta q}})\right]^*}{\partial\dot{\boldsymbol{Q}}}(\dot{\boldsymbol{q}})+\frac{\partial\left[\delta[\mathcal{S}](\boldsymbol{\delta q})\right]^*}{\partial\boldsymbol{Q}}(\boldsymbol{q})\end{aligned} \tag{29}$$

Considering the problem (19), the Eq. (29) can be simplified since A is null. In Sect. 4.2 we determine the linearization of the kinetic energies and of the energies presented in Sect. 3 in order to express the dynamical Eq. (27).

We introduce in the rest of this article the normal variation vector $\boldsymbol{\tau}$ defined for any displacement $\boldsymbol{u}$. Considering a surface element $\mathrm{d}S$ and its normal vector $\boldsymbol{n}$, we have[4]:

$$\boldsymbol{\tau}(\boldsymbol{u})\,\mathrm{d}S=\delta[\boldsymbol{n}\,\mathrm{d}S](\boldsymbol{u}) \tag{30}$$

[4] By definition we have: $\boldsymbol{\tau}(\boldsymbol{u})=(\nabla\cdot\boldsymbol{u})\boldsymbol{n}-{}^t\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}$.

4.2.1 Kinetic energy

By definition, the kinetic energy of the fluid E_C^F is given by:

$$E_C^F=\frac{1}{2}\int_{\Omega_F}\rho^F(\dot{\boldsymbol{U^F}})^2\,\mathrm{d}\Omega_F \tag{31}$$

Only the variation of this energy with respect to the velocity $\dot{\boldsymbol{U^F}}$ appears in (29). This gives the classical expression of the kinetic energy linearized around a fixed position Ω_F^*:

$$\frac{\partial\left[\delta\left[E_C^F\right](\dot{\boldsymbol{\delta u^F}})\right]^*}{\partial\dot{\boldsymbol{U^F}}}(\dot{\boldsymbol{u^F}})=\int_{\Omega_F^*}\rho^F\dot{\boldsymbol{u^F}}\cdot\dot{\boldsymbol{\delta u^F}}\,\mathrm{d}\Omega_F \tag{32}$$

The expression of the kinetic energy of the structure is identical, with ρ^S the solid density we have:

$$\frac{\partial\left[\delta\left[E_C^S\right](\dot{\boldsymbol{\delta u^S}})\right]^*}{\partial\dot{\boldsymbol{U^S}}}(\dot{\boldsymbol{u^S}})=\int_{\Omega_S^*}\rho^S\dot{\boldsymbol{u^S}}\cdot\dot{\boldsymbol{\delta u^S}}\,\mathrm{d}\Omega_S \tag{33}$$

4.2.2 Gravity potential energy

The non linear expression (5) depends on $\boldsymbol{U^F}$ and its linearization is given by:

$$\begin{aligned}\frac{\partial\left[\delta\left[E_{pes}\right](\boldsymbol{\delta u^F})\right]^*}{\partial\boldsymbol{U^F}}(\boldsymbol{u^F})&=\delta\left[\rho^F g\int_{\Gamma\cup\Sigma}Z\boldsymbol{\delta u^F}\cdot\boldsymbol{n}\mathrm{d}S\right]^*(\boldsymbol{u^F})\\ &=\boldsymbol{K}^g_\Gamma(\boldsymbol{u^F},\boldsymbol{\delta u^F})+\boldsymbol{K}^g_\Sigma(\boldsymbol{u^F},\boldsymbol{\delta u^F})\end{aligned} \tag{34}$$

where $\boldsymbol{K}^g_\Gamma(\boldsymbol{u^F},\boldsymbol{\delta u^F})$ and $\boldsymbol{K}^g_\Sigma(\boldsymbol{u^F},\boldsymbol{\delta u^F})$ are respectively defined on Γ^* and Σ^* such that:

$$\begin{aligned}\boldsymbol{K}^g_\Gamma(\boldsymbol{u^F},\boldsymbol{\delta u^F})=&\int_{\Gamma^*}\rho^F g(\boldsymbol{u^F}\cdot\boldsymbol{i}_z)\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Gamma\,\mathrm{d}\Gamma\\ &+\int_{\Gamma^*}\rho^F gZ^*\boldsymbol{\delta u^F}\cdot\boldsymbol{\tau}^*_\Gamma(\boldsymbol{u^F})\mathrm{d}\Gamma\end{aligned} \tag{35a}$$

$$\begin{aligned}\boldsymbol{K}^g_\Sigma(\boldsymbol{u^F},\boldsymbol{\delta u^F})=&\int_{\Sigma^*}\rho^F g(\boldsymbol{u^F}\cdot\boldsymbol{i}_z)\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Sigma\,\mathrm{d}\Sigma\\ &+\int_{\Sigma^*}\rho^F gZ^*\boldsymbol{\delta u^F}\cdot\boldsymbol{\tau}^*_\Sigma(\boldsymbol{u^F})\mathrm{d}\Sigma\end{aligned} \tag{35b}$$

4.2.3 Elastic potential energy

The linearization of the first variation of the elastic potential energy introduced in Sect. 3.2 is given by:

$$\frac{\partial\left[\delta\left[E_{elas}\right](\boldsymbol{\delta u^S})\right]^*}{\partial\boldsymbol{U^S}}(\boldsymbol{u^S})=(\boldsymbol{K}_E+\boldsymbol{K}_G)(\boldsymbol{u^S},\boldsymbol{\delta u^S}) \tag{36}$$

where $\boldsymbol{K}_E$ is the classically used elastic stiffness of the structure and $\boldsymbol{K}_G$ is the geometric stiffness linked to the prestress

applied on the structure by the fluid. These stiffnesses are given by:

$$\boldsymbol{K}_E(\boldsymbol{u}^S, \boldsymbol{\delta u}^S) = \int_{\Omega_S^*} \mathrm{Tr}\left[C\, \mathcal{E}(\boldsymbol{u}^S)\, \mathcal{E}(\boldsymbol{\delta u}^S)\right] \mathrm{d}\Omega_S \tag{37}$$

$$\boldsymbol{K}_G(\boldsymbol{u}^S, \boldsymbol{\delta u}^S) = \int_{\Omega_S^*} \mathrm{Tr}\left[\boldsymbol{D}(\boldsymbol{u}^S)\, \sigma_0(\boldsymbol{D}(\boldsymbol{\delta u}^S))^T\right] \mathrm{d}\Omega_S \tag{38}$$

where σ_0 is the stress tensor at initial state (the domain Ω_S^* is superposed to Ω_S^0), C is the fourth order tensor that characterizes the elasticity of the structure.[5]

4.2.4 Surface energies

The energy E_Γ only depends on $\boldsymbol{U}^F$, then from (10) we have:

$$\begin{aligned}\frac{\partial\left[\delta\left[E_\Gamma\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) &= \delta\left[\sigma \int_\Gamma \mathrm{div}_S(\boldsymbol{\delta u}^F)\, \mathrm{d}\gamma\right]^*(\boldsymbol{u}^F)\\ &= \sigma \oint_{\gamma^*} \boldsymbol{\delta u}^F \cdot \delta\left[\boldsymbol{v}_\Gamma\, \mathrm{d}\gamma\right]^*(\boldsymbol{u}^F) + \cdots\\ &\cdots\ \sigma \int_{\Gamma^*} \boldsymbol{\delta u}^F \cdot \delta\left[2H_\Gamma \boldsymbol{n}_\Gamma\, \mathrm{d}\Gamma\right]^*(\boldsymbol{u}^F) = \boldsymbol{K}_\Gamma^\sigma(\boldsymbol{u}^F, \boldsymbol{\delta u}^F)\end{aligned} \tag{39}$$

The energies E_{Σ_G} and E_Σ depend on $\boldsymbol{U}^F$ and $\boldsymbol{U}^S$ (see Sect. 3.3). We will write:

$$\frac{\partial\left[\delta\left[E_{\Sigma_G}\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) = -\sigma_G \oint_{\gamma^*} \boldsymbol{\delta u}^F \cdot \delta\left[\boldsymbol{v}_\Sigma\, \mathrm{d}\gamma\right]^*(\boldsymbol{u}^F)$$

$$\frac{\partial\left[\delta\left[E_\Sigma\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) = \sigma_L \oint_{\gamma^*} \boldsymbol{\delta u}^F \cdot \delta\left[\boldsymbol{v}_\Sigma\, \mathrm{d}\gamma\right]^*(\boldsymbol{u}^F)$$

$$\begin{aligned}&\frac{\partial\left[\delta\left[E_{\Sigma_G}\right](\boldsymbol{\delta u}^S)\right]^*}{\partial \boldsymbol{U}^S}(\boldsymbol{u}^S)\\ &= \sigma_G \int_{\Sigma_G^*} \boldsymbol{\delta u}^S \cdot \delta\left[2H_{\Sigma_G}\, \boldsymbol{n}_{\Sigma_G}\, \mathrm{d}\Sigma_G\right]^*(\boldsymbol{u}^S)\end{aligned} \tag{40a}$$

$$\begin{aligned}&\frac{\partial\left[\delta\left[E_\Sigma\right](\boldsymbol{\delta u}^S)\right]^*}{\partial \boldsymbol{U}^S}(\boldsymbol{u}^S)\\ &= \sigma_L \int_{\Sigma^*} \boldsymbol{\delta u}^S \cdot \delta\left[2H_\Sigma\, \boldsymbol{n}_\Sigma\, \mathrm{d}\Sigma\right]^*(\boldsymbol{u}^S)\end{aligned} \tag{40b}$$

[5] Let us remark that in case of a non linear initial deformation, an additional stiffness term must be inserted [19].

4.2.5 Incompressibility

The incompressibility $\mathcal{I}$ is a function of Λ and $\boldsymbol{U}^F$. The first variation according to each variable is given in (14) and (15). Using the relation $\nabla\Lambda^* = 0$ obtained in the static study and the fact that $\delta[\Lambda](\lambda) = \lambda$ we have:

$$\begin{aligned}&\frac{\partial\left[\delta\left[\mathcal{I}\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) + \frac{\partial\left[\delta\left[\mathcal{I}\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \Lambda}(\lambda)\\ &= \int_{\Gamma^*\cup\Sigma^*} \Lambda^*\, \boldsymbol{\delta u}^F \cdot \boldsymbol{\tau}_S^*(\boldsymbol{u}^F)\mathrm{d}S + \cdots\\ &\cdots \int_{\Gamma^*\cup\Sigma^*} \lambda\, \boldsymbol{\delta u}^F \cdot \boldsymbol{n}_S^* \mathrm{d}S - \int_{\Omega_F^*} \nabla\lambda \cdot \boldsymbol{\delta u}^F \mathrm{d}\Omega_F\end{aligned} \tag{41}$$

We introduce $\boldsymbol{K}_\Gamma^\Lambda$ and $\boldsymbol{K}_\Sigma^\Lambda$ such that:

$$\boldsymbol{K}_\Gamma^\Lambda(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = \int_{\Gamma^*} \Lambda^*\, \boldsymbol{\delta u}^F \cdot \boldsymbol{\tau}_\Gamma^*(\boldsymbol{u}^F)\mathrm{d}\Gamma \tag{42}$$

$$\boldsymbol{K}_\Sigma^\Lambda(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = \int_{\Sigma^*} \Lambda^*\, \boldsymbol{\delta u}^F \cdot \boldsymbol{\tau}_\Sigma^*(\boldsymbol{u}^F)\mathrm{d}\Sigma \tag{43}$$

Moreover, the linearization of $\delta\left[\mathcal{I}\right](\delta\lambda)$ writes:

$$\frac{\partial\left[\delta\left[\mathcal{I}\right](\delta\lambda)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) = \int_{\Omega_F^*} \delta\lambda\, \mathrm{div}(\boldsymbol{u}^F)\mathrm{d}\Omega \tag{44}$$

As only $\mathcal{I}$ depends on Λ, (44) corresponds to the linearized relation (c) of (27).

4.2.6 Wall contact condition

At the interface Σ

We consider at first the fluid equation ($\boldsymbol{\delta u}^S = 0$), the energy introducing the coupling writes:

$$\delta\left[\mathcal{C}_\Sigma\right](\boldsymbol{\delta u}^F) = -\int_\Sigma M\, \boldsymbol{\delta u}^F \cdot \boldsymbol{n}_\Sigma \mathrm{d}\Sigma \tag{45}$$

The linearization of this energy is done considering small displacement of the fluid around the interface Σ^* (the interface follows fluid particles, see Fig. 3):

$$\frac{\partial\left[\delta\left[\mathcal{C}_\Sigma\right](\boldsymbol{\delta u}^F)\right]^*}{\partial \boldsymbol{U}^F}(\boldsymbol{u}^F) = -\int_{\Sigma^*} M^*\, \boldsymbol{\delta u}^F \cdot \boldsymbol{\tau}_\Sigma^*(\boldsymbol{u}^F)\mathrm{d}\Sigma \tag{46}$$

$\mathcal{C}_\Sigma$ also depends on M, writing μ_F the small pressure fluctuations corresponding to a fluid displacement we have:

$$\frac{\partial\left[\delta\left[\mathcal{C}_\Sigma\right](\boldsymbol{\delta u}^F)\right]^*}{\partial M}(\mu_F) = -\int_{\Sigma^*} \mu_F\, \boldsymbol{\delta u}^F \cdot \boldsymbol{n}_\Sigma^* \mathrm{d}\Sigma \tag{47}$$

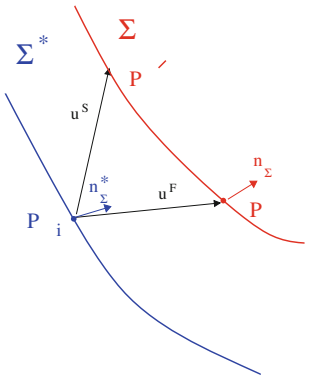

Fig. 3 Interfaces Σ^* and Σ, point P_i and its representation after a fluid displacement (P) and a structure displacement (P')

In the structure equation we write μ_S the pressure fluctuation corresponding to small displacement of the structure around Σ^*. The linearization of the coupling term is then given by:

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\Sigma\,\right](\boldsymbol{\delta u^S})\right]^*}{\partial \boldsymbol{U^S}}(\boldsymbol{u^S}) + \frac{\partial\left[\delta\left[\,\mathcal{C}_\Sigma\,\right](\boldsymbol{\delta u^S})\right]^*}{\partial M}(\mu_S) = \int_{\Sigma^*} M^*\,\boldsymbol{\delta u^S}\cdot\boldsymbol{\tau}^*_\Sigma(\boldsymbol{u^S})\mathrm{d}\Sigma + \int_{\Sigma^*}\mu_S\,\boldsymbol{\delta u^S}\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\Sigma \quad (48)$$

In Fig. 3 the surfaces Σ and Σ^* are represented. Let P be a point that belongs to Σ and displacements $\boldsymbol{u^F}$ and $\boldsymbol{u^S}$ are such that $\boldsymbol{P_iP} = \boldsymbol{u^F}$ and $\boldsymbol{P_iP'} = \boldsymbol{u^S}$. The Lagrange multiplier M^* represents the static pressure, μ_S the pressure difference between P_i and P', and μ_F the pressure difference between P_i and P. The relation between those two pressures is given by the expansion of the pressure around P [17]:

$$\mu_S = \mu_F + \nabla M^*\cdot(\boldsymbol{u^S}-\boldsymbol{u^F}) \quad (49)$$

μ_F and μ_S can be interpreted as the Lagrangian pressure fluctuation following the liquid or the solid. Finally, the linearization of $\delta\left[\mathcal{C}_\Sigma\right](\delta\mu)$ is obtained considering relations (47) and (48) written for displacements $\boldsymbol{u^F}$, $\boldsymbol{u^S}$, and $\delta\mu$:

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\Sigma\,\right](\delta\mu)\right]^*}{\partial \boldsymbol{U^F}}(\boldsymbol{u^F}) + \frac{\partial\left[\delta\left[\,\mathcal{C}_\Sigma\,\right](\delta\mu)\right]^*}{\partial \boldsymbol{U^S}}(\boldsymbol{u^S}) = \int_{\Sigma^*}\delta\mu\,\left(\boldsymbol{u^S}-\boldsymbol{u^F}\right)\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\Sigma \quad (50)$$

Since only $\mathcal{C}_\Sigma$ depends on M, then (50) corresponds to the linearized relation (d) of (27).

On the triple line γ

The same approach is used to linearize terms associated to $\mathcal{C}_\gamma$:

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\boldsymbol{\delta u^F})\right]^*}{\partial \boldsymbol{U^F}}(\boldsymbol{u^F}) = -\oint_{\gamma^*} M^*_\gamma\,\boldsymbol{\delta u^F}\cdot\delta\left[\boldsymbol{n}_\Sigma\mathrm{d}\gamma\right]^*(\boldsymbol{u^F})$$

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\boldsymbol{\delta u^F})\right]^*}{\partial M_\gamma}(\mu^F_\gamma) = -\oint_{\gamma^*}\mu^F_\gamma\,\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\gamma \quad (51)$$

For the structure equation:

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\boldsymbol{\delta u^S})\right]^*}{\partial \boldsymbol{U^S}}(\boldsymbol{u^S}) = \oint_{\gamma^*} M^*_\gamma\,\boldsymbol{\delta u^S}\cdot\delta\left[\boldsymbol{n}_\Sigma\mathrm{d}\gamma\right]^*(\boldsymbol{u^S})$$

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\boldsymbol{\delta u^S})\right]^*}{\partial M_\gamma}(\mu^S_\gamma) = \oint_{\gamma^*}\mu^S_\gamma\,\boldsymbol{\delta u^S}\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\gamma \quad (52)$$

with

$$\mu^S_\gamma = \mu^F_\gamma + \nabla M^*_\gamma\cdot(\boldsymbol{u^S}-\boldsymbol{u^F}) \quad (53)$$

As M^*_γ is constant we have $\mu^S_\gamma = \mu^F_\gamma$. The linearization of $\delta\left[\mathcal{C}_\gamma\right](\delta\mu_\gamma)$, which corresponds to the linearized relation (d) of (27), is obtained considering (51) and (52) written for displacements $\boldsymbol{u^F}$, $\boldsymbol{u^S}$, and $\delta\mu_\gamma$:

$$\frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\delta\mu_\gamma)\right]^*}{\partial \boldsymbol{U^F}}(\boldsymbol{u^F}) + \frac{\partial\left[\delta\left[\,\mathcal{C}_\gamma\,\right](\delta\mu_\gamma)\right]^*}{\partial \boldsymbol{U^S}}(\boldsymbol{u^S}) = \int_{\gamma^*}\delta\mu_\gamma\,\left(\boldsymbol{u^S}-\boldsymbol{u^F}\right)\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\gamma \quad (54)$$

We now write the fluid and structure equations (relations (a) and (b) of (27)). The new variables of this problem are $(\boldsymbol{u^F}, \boldsymbol{u^S}, \lambda, \mu, \mu_\gamma)$. We consider the fluid equation to express the Lagrange multipliers $(\lambda, \mu, \mu_\gamma)$ as a function of the fluid and structure displacements. Using the relations (24) and the linearized energies, we directly write:

$$\int_{t_1}^{t_2}\Bigg[\int_{\Omega^*_F}\left(-\rho^F\ddot{\boldsymbol{u}}^{\boldsymbol{F}}-\nabla\lambda\right)\cdot\boldsymbol{\delta u^F}\mathrm{d}\Omega_F + \int_{\Gamma^*}\left(-\rho^F g(\boldsymbol{u^F}\cdot\boldsymbol{i}_z)-\sigma\delta\left[2H_\Gamma\right]^*(\boldsymbol{u^F})+\lambda\right)\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Gamma\mathrm{d}\Gamma + \int_{\Sigma^*}\left(-\rho^F g\,(\boldsymbol{u^F}\cdot\boldsymbol{i}_z)+\lambda-\mu_F\right)\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\Sigma + \oint_{\gamma^*}\mu^F_\gamma\boldsymbol{\delta u^F}\cdot\boldsymbol{n}^*_\Sigma\mathrm{d}\gamma\Bigg]\mathrm{d}t = 0 \quad (55)$$

where the kinetic energy has been modified using an integration by part:

$$\int_{\Omega_F^*} \rho^F \dot{\boldsymbol{u}}^{\boldsymbol{F}} \cdot \boldsymbol{\delta}\dot{\boldsymbol{u}}^{\boldsymbol{F}} \mathrm{d}\Omega_F = -\int_{\Omega_F^*} \rho^F \ddot{\boldsymbol{u}}^{\boldsymbol{F}} \cdot \boldsymbol{\delta u}^{\boldsymbol{F}} \mathrm{d}\Omega_F + \underbrace{\left[\int_{\Omega_F^*} \rho^F \dot{\boldsymbol{u}}^{\boldsymbol{F}} \cdot \boldsymbol{\delta u}^{\boldsymbol{F}}\right]_{t_1}^{t_2}}_{=0}$$

As $\boldsymbol{\delta u}^{\boldsymbol{F}}(t_1) = \boldsymbol{\delta u}^{\boldsymbol{F}}(t_2) = 0$, the last term is null. Considering test functions null on the boundary of the fluid domain $\partial\Omega_*^F$ we have:

$$\nabla\lambda = -\rho^F \ddot{\boldsymbol{u}}^{\boldsymbol{F}} \quad \text{on } \Omega_F^* \tag{56}$$

Taking the curl of this expression, we notice that the linearized liquid displacements $\boldsymbol{u}^{\boldsymbol{F}}$ are irrotational since $\mathrm{curl}(\boldsymbol{u}^{\boldsymbol{F}}) = 0$.[6] Using (56) in (55) and considering test functions null on Γ^* and γ^* we have:

$$\mu_F = \lambda - \rho^F g\,(\boldsymbol{u}^{\boldsymbol{F}} \cdot \boldsymbol{i}_z) \quad \text{on } \Sigma^* \tag{57}$$

Similarly, the free surface equation is obtained considering test function null on Σ^* and γ^*:

$$\lambda = \rho^F g\,(\boldsymbol{u}^{\boldsymbol{F}} \cdot \boldsymbol{i}_z) + \sigma\,\delta\,[2H_\Gamma]^*\,(\boldsymbol{u}^{\boldsymbol{F}}) \quad \text{on } \Gamma^* \tag{58}$$

Finally, from (58), (57) and (56) we deduce that $\mu_\gamma^F = 0$. Indeed, we supposed that the contact angle is constant, otherwise, the lineic force magnitude would depend on the fluid displacement and $\mu_\gamma^F = \delta\,[\sigma \sin\theta]\,(\boldsymbol{u}^{\boldsymbol{F}})$.

5 Elimination of the Lagrange multipliers

The irrotational displacement of the fluid will be described by a potential displacement φ such that $\boldsymbol{u}^{\boldsymbol{F}} = \nabla\varphi$ on Ω_F^*. φ is defined up to an additional constant. Introducing this new variable instead of $\boldsymbol{u}^{\boldsymbol{F}}$, it is now possible to eliminate the Lagrange multipliers. However, as we will see later, it is convenient to keep the unknown $\boldsymbol{u}^{\boldsymbol{F}}$ on the fluid domain boundary (Σ, Γ, γ). From (56) we then deduce an expression of λ:

$$\lambda = -\rho^F \ddot{\varphi} + \pi(\boldsymbol{u}^{\boldsymbol{S}}) \tag{59}$$

where $\pi(\boldsymbol{u}^{\boldsymbol{S}})$ is a scalar and a linear function of $\boldsymbol{u}^{\boldsymbol{S}}$. Considering expression (59), this Lagrange multiplier can be assimilated to the eulerian pressure fluctuation [17]. The unicity of π is ensured by an unicity condition written $l(\varphi) = 0$ where l is a linear form such that $l(1) \neq 0$.

[6] From (56), $\mathrm{curl}(\ddot{\boldsymbol{u}}^{\boldsymbol{F}}) = 0$ and with appropriate zero initial condition we will consider that $\mathrm{curl}(\boldsymbol{u}^{\boldsymbol{F}}) = 0$.

λ and μ_F are functions of $\boldsymbol{u}^{\boldsymbol{F}}$ and $\boldsymbol{u}^{\boldsymbol{S}}$ and we now want to eliminate those Lagrange multipliers writing a reduced Lagrangian $\mathcal{L}$ that only depends on $(\boldsymbol{U}^{\boldsymbol{F}}, \boldsymbol{U}^{\boldsymbol{S}})$. This change of variable yields a formulation in terms of $(\boldsymbol{u}^{\boldsymbol{F}}, \boldsymbol{u}^{\boldsymbol{S}}, \varphi, \pi)$. Suppressing the Lagrange multipliers, we suppress equations (c), (d) and (e) of (27). This new problem is equivalent if we consider the incompressibility condition $\mathrm{div}(\boldsymbol{u}^{\boldsymbol{F}}) = 0$ in Ω_F^* and the boundary condition $\boldsymbol{u}^{\boldsymbol{F}} \cdot \boldsymbol{n}_\Sigma^* = \boldsymbol{u}^{\boldsymbol{S}} \cdot \boldsymbol{n}_\Sigma^*$ on Σ^*. Then the new linearized problem writes:

$$\begin{aligned}
&\frac{\partial\left[\delta\,[\,\mathcal{S}\,](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{F}}}(\boldsymbol{u}^{\boldsymbol{F}}) = 0 \quad \forall\, \boldsymbol{\delta u}^{\boldsymbol{F}} \in \mathcal{C}_{U^F} && \text{(a)}\\
&\frac{\partial\left[\delta\,[\,\mathcal{S}\,](\boldsymbol{\delta u}^{\boldsymbol{S}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{S}}}(\boldsymbol{u}^{\boldsymbol{S}}) = 0 \quad \forall\, \boldsymbol{\delta u}^{\boldsymbol{S}} \in \mathcal{C}_{U^S} && \text{(b)}\\
&\mathrm{div}(\boldsymbol{u}^{\boldsymbol{F}}) = 0 \quad \text{on } \Omega_F^* && \text{(c)}\\
&\boldsymbol{u}^{\boldsymbol{F}} \cdot \boldsymbol{n}_\Sigma^* = \boldsymbol{u}^{\boldsymbol{S}} \cdot \boldsymbol{n}_\Sigma^* = 0 \quad \text{on } \Sigma^* && \text{(d)}
\end{aligned} \tag{60}$$

Now we need to reformulate the incompressibility and coupling terms $\mathcal{I}$ and $\mathcal{C}$ that depend on the Lagrange multipliers. From (14), the first variation of $\mathcal{I}$ is given by:

$$\delta\,[\,\mathcal{I}\,](\boldsymbol{\delta u}^{\boldsymbol{F}}) = \int_{\Omega_F} \Lambda\,\mathrm{div}(\boldsymbol{\delta u}^{\boldsymbol{F}})\mathrm{d}\Omega_F + \int_{\Omega_F^0} \delta\,[\Lambda]\,(\boldsymbol{\delta u}^{\boldsymbol{F}})\,(J-1)\;\mathrm{d}\Omega_F^0 \tag{61}$$

Then its linearization writes:

$$\begin{aligned}
&\frac{\partial\left[\delta\,[\,\mathcal{I}\,](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{F}}}(\boldsymbol{u}^{\boldsymbol{F}}) + \frac{\partial\left[\delta\,[\,\mathcal{I}\,](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{S}}}(\boldsymbol{u}^{\boldsymbol{S}})\\
&= \int_{\Omega_F^*} \left(\delta\,[\,\Lambda\,]^*\,(\boldsymbol{u}^{\boldsymbol{F}}, \boldsymbol{u}^{\boldsymbol{S}})\right)\mathrm{div}(\boldsymbol{\delta u}^{\boldsymbol{F}})\mathrm{d}\Omega_F\\
&\quad + \int_{\Gamma^*\cup\Sigma^*} \Lambda^*\,\boldsymbol{\delta u}^{\boldsymbol{F}} \cdot \boldsymbol{\tau}_S^*(\boldsymbol{u}^{\boldsymbol{F}})\mathrm{d}S\\
&\quad + \int_{\Omega_F^*} \delta\,[\,\Lambda\,]^*\,(\boldsymbol{\delta u}^{\boldsymbol{F}})\,\mathrm{div}(\boldsymbol{u}^{\boldsymbol{F}})\mathrm{d}\Omega_F
\end{aligned}$$

$\delta\,[\,\Lambda\,]^*\,(\boldsymbol{u}^{\boldsymbol{F}}, \boldsymbol{u}^{\boldsymbol{S}})$ corresponds to the linearization of Λ around the reference state and is equal to $\lambda(\boldsymbol{u}^{\boldsymbol{F}}, \boldsymbol{u}^{\boldsymbol{S}})$. In comparison with (41), this new expression has an additional term that contains the constitutive law of the fluid $\mathrm{div}(\boldsymbol{u}^{\boldsymbol{F}})$ and corresponds to its weak form. This additional term yields the equation of the fluid in $\delta\varphi$. Using (59) and $\delta\,[\,\Lambda\,]^*\,(\boldsymbol{\delta u}^{\boldsymbol{F}}) = -\rho^F \delta\ddot{\varphi}$ we have:

$$\frac{\partial\left[\delta\left[\mathcal{I}\right](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{F}}}(\boldsymbol{u}^{\boldsymbol{F}}) + \frac{\partial\left[\delta\left[\mathcal{I}\right](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{S}}}(\boldsymbol{u}^{\boldsymbol{S}})$$
$$= 2\int_{\Omega_F^*} \rho^F \nabla\varphi\cdot\nabla\delta\ddot{\varphi}\,\mathrm{d}\Omega_F + \int_{\Gamma^*\cup\Sigma^*} \Lambda^*\,\boldsymbol{\delta u}^{\boldsymbol{F}}\cdot\boldsymbol{\tau}_S^*(\boldsymbol{u}^{\boldsymbol{F}})\mathrm{d}S$$
$$+ \int_{\Gamma^*\cup\Sigma^*} (-\rho^F\ddot{\varphi}+\pi)\boldsymbol{\delta u}^{\boldsymbol{F}}\cdot\boldsymbol{n}^*\mathrm{d}S$$
$$-\int_{\Gamma^*}\rho^F\delta\ddot{\varphi}\,\boldsymbol{u}^{\boldsymbol{F}}\cdot\boldsymbol{n}_\Gamma^*\mathrm{d}\Gamma - \int_{\Sigma^*}\rho^F\delta\ddot{\varphi}\,\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^*\mathrm{d}\Sigma \quad (62)$$

The coupling term $\mathcal{C}$ corresponding to the boundary condition is obtained for the fluid equation simply replacing relations (57) and (24) in (46) and (47). For the structure equation, the coupling term appears in the form given by (48). Considering the relation (49), M^* is given by (24) and noting that $\nabla M^* = -\rho^F g\,\boldsymbol{i}_z$ we have:

$$\mu_S = -\rho^F\ddot{\varphi} + \pi - \rho^F g\,(\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{i}_z) \quad (63)$$

Finally, the kinetic energy of the fluid (32) is defined on Ω_F^*, and can be written using the potential displacement as:

$$\frac{\partial\left[\delta\left[\,E_C^F\,\right](\boldsymbol{\delta u}^{\boldsymbol{F}})\right]^*}{\partial \boldsymbol{U}^{\boldsymbol{F}}}(\boldsymbol{u}^{\boldsymbol{F}}) = \int_{\Omega_F^*}\nabla\dot{\varphi}\cdot\nabla\delta\dot{\varphi}\mathrm{d}\Omega_F^*$$

6 Linearized fluid dynamic formulation

Relation (a) of the problem (60) gives two formulations corresponding to the two variables used to describe the fluid movement $(\boldsymbol{u}^{\boldsymbol{F}},\varphi)$. Introducing $\mathcal{C}_\varphi$, the admissible space of functions $\delta\varphi$, φ is in $\mathcal{C}_\varphi^*$ defined by $\mathcal{C}_\varphi^* = \left\{\varphi\in\mathcal{C}_\varphi\,/\,l(\varphi)=0,\ l(1)\neq 0\right\}$ and, for all $\delta\varphi$ in $\mathcal{C}_\varphi$:

$$\underbrace{\int_{\Omega_F^*}\nabla\varphi\cdot\nabla\delta\ddot{\varphi}\,\mathrm{d}\Omega_F}_{\boldsymbol{F}(\varphi,\delta\ddot{\varphi})} \underbrace{-\int_{\Gamma^*}\boldsymbol{u}^{\boldsymbol{F}}\cdot\boldsymbol{n}_\Gamma^*\,\delta\ddot{\varphi}\,\mathrm{d}\Gamma}_{\boldsymbol{B}(\boldsymbol{u}^{\boldsymbol{F}},\delta\ddot{\varphi})} - \cdots$$
$$\cdots \underbrace{\int_{\Sigma^*}\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^*\,\delta\ddot{\varphi}\,\mathrm{d}\Sigma}_{\boldsymbol{C}(\boldsymbol{u}^{\boldsymbol{S}},\delta\ddot{\varphi})} = 0 \quad (64)$$

$\boldsymbol{F}$ is the operator associated to the kinetic energy and $\boldsymbol{B}$ represents the coupling between φ and $\boldsymbol{u}^{\boldsymbol{F}}$ on the free surface Γ^*. $\boldsymbol{C}$ is the fluid structure coupling operator at the interface Σ^*. We notice that only the second derivative of the test function $\delta\varphi$ appears in (64), it is then possible to replace $\delta\ddot{\varphi}$ by $\delta\varphi$, this operation corresponds to a double time-integration with zero initial conditions. Then considering the terms in $\boldsymbol{\delta u}^{\boldsymbol{F}}$ we write the second variational equation of the fluid using the operators previously introduced and introducing the two following operators:

$$\boldsymbol{K}_\gamma^\theta(\boldsymbol{u}^{\boldsymbol{F}},\boldsymbol{\delta u}^{\boldsymbol{F}}) = \sigma\oint_{\gamma^*}\boldsymbol{\delta u}^{\boldsymbol{F}}\cdot\delta\left[\boldsymbol{\nu}_\Gamma\,\mathrm{d}\gamma\right]^*(\boldsymbol{u}^{\boldsymbol{F}}) \quad (65)$$
$$\boldsymbol{R}_\Gamma(\boldsymbol{\delta u}^{\boldsymbol{F}}) = \int_{\Gamma^*}\boldsymbol{\delta u}^{\boldsymbol{F}}\cdot\boldsymbol{n}_\Gamma^*\,\mathrm{d}\Gamma$$

where $\boldsymbol{K}_\gamma^\theta(\boldsymbol{u}^{\boldsymbol{F}},\boldsymbol{\delta u}^{\boldsymbol{F}})$ is a stiffness contact angle defined on the triple line. For all test functions $\boldsymbol{\delta u}^{\boldsymbol{F}}$ in $\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{F}}}^*$ where $\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{F}}}^* = \{\boldsymbol{u}^{\boldsymbol{F}}\in\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{F}}}\,/\,\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^* = \boldsymbol{u}^{\boldsymbol{F}}\cdot\boldsymbol{n}_\Sigma^*\ \text{on}\ \Sigma^*\}$:

$$\left(\boldsymbol{K}_\Gamma^g + \boldsymbol{K}_\Gamma^\sigma - \boldsymbol{K}_\Gamma^\Lambda\right)(\boldsymbol{u}^{\boldsymbol{F}},\boldsymbol{\delta u}^{\boldsymbol{F}})$$
$$-\boldsymbol{K}_\gamma^\theta(\boldsymbol{u}^{\boldsymbol{F}},\boldsymbol{\delta u}^{\boldsymbol{F}}) + \boldsymbol{B}(\ddot{\varphi},\boldsymbol{\delta u}^{\boldsymbol{F}}) - \pi\,\boldsymbol{R}_\Gamma(\boldsymbol{\delta u}^{\boldsymbol{F}}) = 0 \quad (66)$$

7 Linearized structure dynamic formulation

The structure equation is given by the relation (b) of (60) and can be written using (33), (36), (40) and (48) (M^*, M_γ^* and μ^S are given by (24) and (63)). For all test functions $\boldsymbol{\delta u}^{\boldsymbol{S}}$ in $\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{S}}}^*$ where $\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{S}}}^* = \{\boldsymbol{u}^{\boldsymbol{S}}\in\mathcal{C}_{\boldsymbol{U}^{\boldsymbol{S}}}\,/\,\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^* = \boldsymbol{u}^{\boldsymbol{F}}\cdot\boldsymbol{n}_\Sigma^*\ \text{on}\ \Sigma^*\}$:

$$\boldsymbol{M}_S(\ddot{\boldsymbol{u}^{\boldsymbol{S}}},\boldsymbol{\delta u}^{\boldsymbol{S}})$$
$$+\left(\boldsymbol{K}_E+\boldsymbol{K}_G+\boldsymbol{K}_\Sigma^{\sigma_L}+\boldsymbol{K}_{\Sigma_G}^{\sigma_G}+\boldsymbol{K}_\Sigma^g-\boldsymbol{K}_\Sigma^\Lambda+\boldsymbol{K}_\gamma^\theta\right)(\boldsymbol{u}^{\boldsymbol{S}},\boldsymbol{\delta u}^{\boldsymbol{S}})$$
$$+\boldsymbol{C}(\ddot{\varphi},\boldsymbol{\delta u}^{\boldsymbol{S}}) - \pi\,\boldsymbol{R}_\Sigma(\boldsymbol{\delta u}^{\boldsymbol{S}}) = \boldsymbol{f}_{ext}(\boldsymbol{\delta u}^{\boldsymbol{S}}) \quad (67)$$

where $\boldsymbol{f}_{ext}$ is the external work of the force $\boldsymbol{f}$ for a displacement $\boldsymbol{\delta u}^{\boldsymbol{S}}$ and we have:

$$\boldsymbol{f}_{ext}(\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Sigma_f^*}\boldsymbol{f}\cdot\boldsymbol{\delta u}^{\boldsymbol{S}}\mathrm{d}\Sigma_f \quad (68)$$

$$\boldsymbol{M}_S(\ddot{\boldsymbol{u}^{\boldsymbol{S}}},\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Omega_S^*}\ddot{\boldsymbol{u}^{\boldsymbol{S}}}\cdot\boldsymbol{\delta u}^{\boldsymbol{S}}\mathrm{d}\Omega_S \quad (69)$$

$$\boldsymbol{K}_\Sigma^g(\ddot{\boldsymbol{u}^{\boldsymbol{S}}},\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Sigma^*}\rho^F g\,Z^*\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\boldsymbol{\tau}_\Sigma^*\mathrm{d}\Sigma$$
$$+\int_{\Sigma^*}\rho^F g\,(\boldsymbol{u}^{\boldsymbol{S}}\cdot\boldsymbol{i}_z)\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^*\mathrm{d}\Sigma$$

$$\boldsymbol{K}_\Sigma^\Lambda(\ddot{\boldsymbol{u}^{\boldsymbol{S}}},\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Sigma^*}\Lambda^*\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\boldsymbol{\tau}_\Sigma^*\mathrm{d}\Sigma$$

$$\boldsymbol{R}_\Sigma(\boldsymbol{\delta u}^{\boldsymbol{S}}) = \int_{\Sigma^*}\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\boldsymbol{n}_\Sigma^*\mathrm{d}\Sigma$$

$$\boldsymbol{K}_\Sigma^{\sigma_L}(\boldsymbol{u}^{\boldsymbol{S}},\boldsymbol{\delta u}^{\boldsymbol{S}}) = \sigma_L\int_{\Sigma^*}\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\delta\left[2H_\Sigma\boldsymbol{n}_\Sigma\mathrm{d}\Sigma\right]^*(\boldsymbol{u}^{\boldsymbol{S}})$$
$$+\sigma_L\oint_{\gamma^*}\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\delta\left[\boldsymbol{\nu}_\Sigma\,\mathrm{d}\gamma\right]^*(\boldsymbol{u}^{\boldsymbol{S}})$$

$$\boldsymbol{K}_{\Sigma_G}^{\sigma_G}(\boldsymbol{u}^{\boldsymbol{S}},\boldsymbol{\delta u}^{\boldsymbol{S}}) = \sigma_G\int_{\Sigma_G^*}\boldsymbol{\delta u}^{\boldsymbol{S}}\cdot\delta\left[2H_{\Sigma_G}\boldsymbol{n}_\Sigma\,\mathrm{d}\Gamma\right]^*(\boldsymbol{u}^{\boldsymbol{S}})$$

$$-\sigma_G \oint_{\gamma^*} \boldsymbol{\delta u}^S \cdot \delta\left[\boldsymbol{v}_\Sigma \, \mathrm{d}\gamma\right]^* (\boldsymbol{u}^S)$$

A function φ in $\mathcal{C}_\varphi$ can be uniquely written as $\varphi = \varphi^* + c$ where φ^* is in $\mathcal{C}_\varphi^*$ and c is a constant defined by $c = l(\varphi)/l(1)$. Then we have: $\mathcal{C}_\varphi = \mathcal{C}_\varphi^* \oplus \mathbb{R}$ and from the formulation (64) it is possible to write two separated equations considering test functions $\delta\varphi$ in $\mathcal{C}_\varphi^*$:

$$\boldsymbol{F}(\varphi, \delta\varphi) - \boldsymbol{B}(\boldsymbol{u}^F, \delta\varphi) - \boldsymbol{C}(\boldsymbol{u}^S, \delta\varphi) = 0 \quad \forall \delta\varphi \in \mathcal{C}_\varphi^* \tag{70}$$

and in $\mathbb{R}$ (writing them $\delta\pi$):

$$\delta\pi \left(\boldsymbol{R}_\Gamma(\boldsymbol{\delta u}^F) + \boldsymbol{R}_\Sigma(\boldsymbol{\delta u}^S) \right) = 0 \quad \forall \delta\pi \in \mathbb{R} \tag{71}$$

Let $\boldsymbol{K}_F$ and $\boldsymbol{K}_S$ be the stiffnesses associated to the fluid and the structure, we have $\boldsymbol{K}_F = \boldsymbol{K}_\Gamma^g + \boldsymbol{K}_\Gamma^\sigma - \boldsymbol{K}_\Gamma^\Lambda - \boldsymbol{K}_\gamma^\theta$ and $\boldsymbol{K}_S = \boldsymbol{K}_E + \boldsymbol{K}_G + \boldsymbol{K}_\Sigma^{\sigma_L} + \boldsymbol{K}_{\Sigma_G}^{\sigma_G} + \boldsymbol{K}_\Sigma^g - \boldsymbol{K}_\Sigma^\Lambda + \boldsymbol{K}_\gamma^\theta$. Finally, the system to solve is given by (66), (67), (70) and (71):

$$\begin{aligned}
&\exists (\boldsymbol{u}^F, \boldsymbol{u}^S, \varphi, \pi) \in \mathcal{C}^*_{U^F} \times \mathcal{C}^*_{U^S} \times \mathcal{C}^*_\varphi \times \mathbb{R}, \\
&\forall (\delta\varphi, \boldsymbol{\delta u}^F, \boldsymbol{\delta u}^S, \delta\pi) \in \mathcal{C}^*_{U^F} \times \mathcal{C}^*_{U^S} \times \mathcal{C}^*_\varphi \times \mathbb{R} \\
&\quad \boldsymbol{F}(\varphi, \delta\varphi) - \boldsymbol{B}(\boldsymbol{u}^F, \delta\varphi) - \boldsymbol{C}(\boldsymbol{u}^S, \delta\varphi) = 0 \\
&\quad \boldsymbol{K}_F(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) + \boldsymbol{B}(\ddot{\varphi}, \boldsymbol{\delta u}^F) - \pi \boldsymbol{R}_\Gamma(\boldsymbol{\delta u}^F) = 0 \\
&\quad \boldsymbol{K}_S(\boldsymbol{u}^S, \boldsymbol{\delta u}^S) + \boldsymbol{M}_S(\ddot{\boldsymbol{u}^S}, \boldsymbol{\delta u}^S) + \cdots \\
&\quad \cdots C(\ddot{\varphi}, \boldsymbol{\delta u}^S) - \pi \boldsymbol{R}_\Sigma(\boldsymbol{\delta u}^S) = \boldsymbol{f}_{ext} \\
&\quad \delta\pi \left(\boldsymbol{R}_\Gamma(\boldsymbol{\delta u}^F) + \boldsymbol{R}_\Sigma(\boldsymbol{\delta u}^S) \right) = 0
\end{aligned} \tag{72}$$

From (69), it is obvious that $\boldsymbol{M}_S$ is symmetric. To show the symmetry of $\boldsymbol{K}^F$ and $\boldsymbol{K}^S$ let us first consider the generic bilinear operator $\boldsymbol{A}_S^k(\boldsymbol{v}, \boldsymbol{w})$ defined on a surface S and for any scalar function k:

$$\boldsymbol{A}_S^k(\boldsymbol{v}, \boldsymbol{w}) = \int_S (\boldsymbol{v} \cdot \nabla k)\boldsymbol{w} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_S k \, \boldsymbol{w} \cdot \boldsymbol{\tau}_S(\boldsymbol{v}) \, \mathrm{d}S \tag{73}$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vectors defined on S. Let ∂S be the boundary of S, we will use the Darboux frame $(\boldsymbol{n}, \boldsymbol{\nu}_S, \boldsymbol{t}_S)$ introduced by Fig. 2 where $\boldsymbol{t}_S$ is tangent to ∂S and oriented with the right hand rule. It can be shown that [9]:

$$\boldsymbol{A}_S^k(\boldsymbol{v}, \boldsymbol{w}) - \boldsymbol{A}_S^k(\boldsymbol{w}, \boldsymbol{v}) = \oint_\gamma k \, (\boldsymbol{w} \wedge \boldsymbol{v}) \cdot \boldsymbol{t}_S \, \mathrm{d}S \tag{74}$$

The operator $\mathbf{K}_F$ is given for vector fields $\boldsymbol{\delta u}^F$ and $\boldsymbol{u}^F$ defined on Γ^* by $\boldsymbol{K}_F(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = \left(\boldsymbol{K}_\Gamma^g + \boldsymbol{K}_\Gamma^\sigma - \boldsymbol{K}_\Gamma^\Lambda - \boldsymbol{K}_\gamma^\theta \right)(\boldsymbol{u}^F, \boldsymbol{\delta u}^F)$. Operators $\boldsymbol{K}_\Gamma^g$, $\boldsymbol{K}_\Gamma^\sigma$ and $\boldsymbol{K}_\Gamma^\Lambda$ are given by (35a), (39), (42) and (65). We introduce the operator $\widehat{\boldsymbol{K}_\Gamma^\sigma}$ such that $\widehat{\boldsymbol{K}_\Gamma^\sigma}(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = (\boldsymbol{K}_\Gamma^\sigma - \boldsymbol{K}_\gamma^\theta)(\boldsymbol{u}^F, \boldsymbol{\delta u}^F)$:

$$\widehat{\boldsymbol{K}_\Gamma^\sigma}(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = \sigma \int_{\Gamma^*} \boldsymbol{\delta u} \cdot \delta\left[2H_\Gamma \boldsymbol{n}_\Gamma \, \mathrm{d}\Gamma\right]^* (\boldsymbol{u}) \tag{75}$$

$\boldsymbol{K}_\Gamma^g$, $\boldsymbol{K}_\Gamma^\sigma$ and $\boldsymbol{K}_\Gamma^\Lambda$, and thus $\boldsymbol{K}_F$, can be written using

$$\boldsymbol{A}_S^k : \boldsymbol{K}_\Gamma^g = \boldsymbol{A}_{\Gamma^*}^{\rho^F g Z^*}, \ \widehat{\boldsymbol{K}_\Gamma^\sigma} = \boldsymbol{K}_{\Gamma^*}^{\sigma 2 H_\Gamma^*} \text{ and } \boldsymbol{K}_\Gamma^\Lambda = \boldsymbol{A}_{\Gamma^*}^{\Lambda^*}.$$

Then:

$$\boldsymbol{K}_F(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) = \boldsymbol{A}_S^k \Big|_{k=\rho^F g Z + \sigma 2 H_\Gamma - \Lambda}^{S \equiv \Gamma} (\boldsymbol{u}^F, \boldsymbol{\delta u}^F) \tag{76}$$

and from (74) we have:

$$\begin{aligned}
&\boldsymbol{K}_F(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) - \boldsymbol{K}_F(\boldsymbol{u}^F, \boldsymbol{\delta u}^F) \\
&= \oint_{\gamma^*} \left(\rho^F g Z^* + \sigma 2 H_\Gamma^* - \Lambda^* \right) \left(\boldsymbol{\delta u}^F \wedge \boldsymbol{u}^F \right) \cdot \boldsymbol{t}_\gamma \mathrm{d}\gamma
\end{aligned} \tag{77}$$

Using relation (24), expression (77) is null which shows that $\boldsymbol{K}_F$ is symmetric. We notice in [9], that the authors give an expression of $\boldsymbol{K}_\Gamma^\sigma$ which is symmetric, this operator is proportional to the second variation of the surface Γ.

For vector fields $\boldsymbol{\delta u}$ and $\boldsymbol{u}$ defined on $\Omega_F^* \cup \Omega_S^*$, the operator $\mathbf{K}_S$ is given by $\boldsymbol{K}_S(\boldsymbol{u}, \boldsymbol{\delta u}) = \left(\boldsymbol{K}_E + \boldsymbol{K}_G + \boldsymbol{K}_\Sigma^{\sigma_L} + \boldsymbol{K}_{\Sigma_G}^{\sigma_G} + \boldsymbol{K}_\Sigma^g - \boldsymbol{K}_\Sigma^\Lambda + \boldsymbol{K}_\gamma^\theta \right)(\boldsymbol{u}, \boldsymbol{\delta u})$. We first notice that considering (37) and (38), $\boldsymbol{K}_E$ and $\boldsymbol{K}_G$ have symmetrical forms. Moreover, $\boldsymbol{K}_\Sigma^{\sigma_L}$ and $\boldsymbol{K}_{\Sigma_G}^{\sigma_G}$ are proportional to the second variation of surfaces Σ and Σ_G and are symmetric. We now need to show the symmetry of the operator $\boldsymbol{K}_I$ given by $\boldsymbol{K}_I = \boldsymbol{K}_\Sigma^g - \boldsymbol{K}_\Sigma^\Lambda + \boldsymbol{K}_\gamma^\theta$ for vector fields $\boldsymbol{\delta u}$ and $\boldsymbol{u}$ defined on $\Omega_F^* \cup \Omega_S^*$. Then we have:

$$\boldsymbol{K}_I(\boldsymbol{u}, \boldsymbol{\delta u}) = \Big(\underbrace{\boldsymbol{K}_\Sigma^g - \boldsymbol{K}_\Sigma^\Lambda - \widehat{\boldsymbol{K}_\Gamma^\sigma}}_{\boldsymbol{K}_I^1} + \underbrace{\widehat{\boldsymbol{K}_\Gamma^\sigma} + \boldsymbol{K}_\gamma^\theta}_{\boldsymbol{K}_I^2} \Big)(\boldsymbol{u}, \boldsymbol{\delta u})$$

$\boldsymbol{K}_I^2$ is equal to $\boldsymbol{K}_\Gamma^\sigma$ and is symmetric. Considering Σ^* its boundary γ^* oriented along $-\boldsymbol{t}_\gamma$, we have:

$$\begin{aligned}
&\boldsymbol{K}_I(\boldsymbol{u}, \boldsymbol{\delta u}) - \boldsymbol{K}_I(\boldsymbol{\delta u}, \boldsymbol{u}) \\
&= \oint_{\gamma^*} \left(-\rho^F g Z^* - \sigma 2 H_\Gamma^* + \Lambda^* \right) (\boldsymbol{\delta u} \wedge \boldsymbol{u}) \cdot \boldsymbol{t}_\gamma \mathrm{d}\gamma = 0
\end{aligned}$$

The operator $\boldsymbol{K}_I$ is symmetric, and because in (67), $\boldsymbol{K}_S(\boldsymbol{u}, \boldsymbol{\delta u})$ is defined on Ω_S^* we take $\boldsymbol{\delta u} = \boldsymbol{\delta u}^S$ and $\boldsymbol{u} = \boldsymbol{u}^S$. $\boldsymbol{K}_S(\boldsymbol{u}^F, \boldsymbol{\delta u}^F)$ is symmetric as a sum of symmetric operators.

8 Numerical resolution

To numerically solve this variational problem we first consider the discretization of each operators by the classical Finite Element Method, $\mathbf{u}^S$, $\mathbf{u}^F$ and $\boldsymbol{\Phi}$ being the nodal unknowns of $\boldsymbol{u}^S$, $\boldsymbol{u}^F$ and φ. The same notations is used for the matrices associated to these operators. The matrix equation associated with (72) writes:

$$\begin{bmatrix} \mathbf{K}_S & \mathbf{0} & \mathbf{0} & -\mathbf{R}_\Sigma \\ \mathbf{0} & \mathbf{K}_F & \mathbf{0} & -\mathbf{R}_\Gamma \\ -\mathbf{C}^T & -\mathbf{B}^T & \mathbf{F} & \mathbf{0} \\ -\mathbf{R}_\Sigma{}^T & -\mathbf{R}_\Gamma{}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}^S \\ \mathbf{u}^F \\ \boldsymbol{\Phi} \\ \pi \end{pmatrix} + \begin{bmatrix} \mathbf{M}_S & \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}^S \\ \ddot{\mathbf{u}}^F \\ \ddot{\boldsymbol{\Phi}} \\ \ddot{\pi} \end{pmatrix} = \mathbf{F}_e \tag{78}$$

Where $\mathbf{F}_e = (\mathbf{f}_{ext}^T, \mathbf{0}, \mathbf{0}, \mathbf{0})^T$. Because φ and $\delta\varphi$ are in $\mathcal{C}_\varphi^*$, we now need to impose a linear arbitrary relation between the elements of $\boldsymbol{\Phi}$. We chose to suppress the first degree of freedom and we write $\boldsymbol{\Phi}_2$, $\mathbf{C}_2$, $\mathbf{F}_{22}$ and $\mathbf{B}_2$ the truncated vector and matrices [17]. Introducing $\mathbf{K_G}$, $\mathbf{M_G}$, $\mathbf{C_G}$, $\mathbf{R_G}$, $\mathbf{u}^G$ and $\mathbf{f}^G$ defined by:

$$\mathbf{K}_G = \begin{bmatrix} \mathbf{K}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_F \end{bmatrix} \quad \mathbf{M}_G = \begin{bmatrix} \mathbf{M}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \mathbf{C}_G = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{C}_2 \end{bmatrix}$$
$$\mathbf{R}_G = \begin{bmatrix} \mathbf{R}_\Sigma \\ \mathbf{R}_\Gamma \end{bmatrix} \quad \mathbf{u}^G = \begin{bmatrix} \mathbf{u}^S \\ \mathbf{u}^F \end{bmatrix} \quad \mathbf{f}^G = \begin{bmatrix} \mathbf{f}_{ext} \\ \mathbf{0} \end{bmatrix}$$

the problem (78) writes

$$\begin{bmatrix} \mathbf{K}_G & \mathbf{0} & -\mathbf{R}_G \\ -\mathbf{C}_G^T & \mathbf{F}_{22} & \mathbf{0} \\ \mathbf{R}_G^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}^G \\ \boldsymbol{\Phi}_2 \\ \pi \end{pmatrix} + \begin{bmatrix} \mathbf{M}_G & \mathbf{C}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}^G \\ \ddot{\boldsymbol{\Phi}}_2 \\ \ddot{\pi} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^G \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{79}$$

The condition $\boldsymbol{u}^F \cdot \boldsymbol{n}_\Sigma^* = \boldsymbol{u}^S \cdot \boldsymbol{n}_\Sigma^*$ on γ^* still needs to be taken into account (the field $\boldsymbol{u}^F$ doesn't appear on Σ^*). This operation constraints the degrees of freedom of $\mathbf{u}^G$. It is then possible to build a matrix $\mathbf{Q}$ such that $\mathbf{u}^G = \mathbf{Q}\mathbf{u}_\diamond$ and $\mathbf{u}_\diamond$ is the vector of unknowns taking into account that constraint. The problem (79) writes:

$$\begin{bmatrix} \mathbf{K}_\diamond & \mathbf{0} & -\mathbf{R}_\diamond \\ -\mathbf{C}_\diamond^T & \mathbf{F}_{22} & \mathbf{0} \\ \mathbf{R}_\diamond^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_\diamond \\ \boldsymbol{\Phi}_2 \\ \pi \end{pmatrix} + \begin{bmatrix} \mathbf{M}_\diamond & \mathbf{C}_\diamond & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_\diamond \\ \ddot{\boldsymbol{\Phi}}_2 \\ \ddot{\pi} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\diamond \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

with $\mathbf{K}_\diamond = \mathbf{Q}^T\mathbf{K}_G\mathbf{Q}, \mathbf{M}_\diamond = \mathbf{Q}^T\mathbf{M}_G\mathbf{Q}, \mathbf{C}_\diamond = \mathbf{Q}^T\mathbf{C}_G, \mathbf{R}_\diamond = \mathbf{Q}^T\mathbf{R}_G$ and $\mathbf{f}_\diamond = \mathbf{Q}^T\mathbf{f}^G$. Finally, the particular choice of the unicity condition allows us to eliminate the variable $\boldsymbol{\Phi}_2$ as $\mathbf{F}_{22}$ is inversible: $\boldsymbol{\Phi}_2 = \mathbf{F}_{22}^{-1}\mathbf{C}_\diamond{}^T\mathbf{u}_\diamond$. This operation leads to the final system to solve:

$$\begin{bmatrix} \mathbf{K}_\diamond & -\mathbf{R}_\diamond \\ -\mathbf{R}_\diamond^T & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_\diamond \\ \pi \end{pmatrix} + \begin{bmatrix} \mathbf{M}_\diamond + \mathbf{C}_\diamond\mathbf{F}_{22}^{-1}\mathbf{C}_\diamond{}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_\diamond \\ \ddot{\pi} \end{pmatrix} = \begin{pmatrix} \ddot{\mathbf{f}}_\diamond \\ \mathbf{0} \end{pmatrix}$$

$\mathbf{K}_\diamond$ and $\mathbf{M}_\diamond$ are associated to symmetric operators and because $\mathbf{F}_{22}$ is symmetric and inversible, $\mathbf{F}_{22}^{-1}$ is symmetric. Then, we can easily deduce from this expression the eigenvalue problem that gives the real eigenmodes and eigenvalues of the coupled system.

9 Conclusion

An energy approach has been used to establish the non linear variational formulation for an elastic structure containing an incompressible and inviscid liquid. The originality of the present work is to take into account the surface tension phenomenon and its effect on the structure deformations and on the fluid sloshing. The linearization of this formulation, around the (nonlinear) equilibrium position of the system, is presented. We notice that this formulation is coherent with the ones presented in [18] and [9], respectively when the surface tensions and the structure deformations are not taken into account. A numerical resolution strategy using a finite element discretization yields the symmetrical matrix equation of the linearized coupled problem. Applications to satellite tanks and bladder tanks are in progress and will be presented in future papers. Following the initiated works to introduce the viscous dissipation in our conservative models [15], this new formulation is a promising framework to take into account the damping sources involving both the viscosity and the capillarity at the liquid free surface [11]. A non constant dynamic contact angle between the liquid free surface and the tank wall can be naturally represented by considering the dynamic evolution μ_γ^F of the Lagrange multiplier M_γ: $\mu_\gamma^F = \delta[\sin\theta](\boldsymbol{u}^F)$, where θ is a function of $\boldsymbol{u}^F$ that can be sometimes highly nonlinear [5]. Moreover, the presence of the lineic capillary force on the structure wall creates a singularity in the structure equations. Its modelization in the case of a three dimensional structure model is an open problem that will be the subject of future investigations.

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