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Embedded discontinuity finite element method for modeling of localized failure in heterogeneous materials with structured mesh: an alternative to extended finite element method

Adnan Ibrahimbegovic · Sergiy Melnyk

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Abstract In this work we discuss the finite element model using the embedded discontinuity of the strain and displacement field, for dealing with a problem of localized failure in heterogeneous materials by using a structured finite element mesh. On the chosen 1D model problem we develop all the pertinent details of such a finite element approximation. We demonstrate the presented model capabilities for representing not only failure states typical of a slender structure, with crack-induced failure in an elastic structure, but also the failure state of a massive structure, with combined diffuse (process zone) and localized cracking. A robust operator split solution procedure is developed for the present model taking into account the subtle difference between the types of discontinuities, where the strain discontinuity iteration is handled within global loop for computing the nodal displacement, while the displacement discontinuity iteration is carried out within a local, element-wise computation, carried out in parallel with the Gauss-point computations of the plastic strains and hardening variables. The robust performance of the proposed solution procedure is illustrated by a couple of numerical examples. Concluding remarks are stated regarding the class of problems where embedded discontinuity finite element method (ED-FEM) can be used as a favorite choice with respect to extended FEM (X-FEM).

A. Ibrahimbegovic (✉) · S. Melnyk
Ecole Normale Supérieure de Cachan, LMT, 61,
avenue de Président Wilson, 94235 Cachan, France
e-mail: ai@lmt.ens-cachan.fr

1 Introduction

The computation of limit load analysis of a complex structure, which is required in order to grasp any possible weakness of proposed design, often imposes the need to deal with a problem of localized failure. The latter is represented with the elastic-post-peak-softening constitutive model where, once passed the peak resistance, the stress will decrease with increasing strains (e.g. see [1,2]). An even more general failure pattern must be represented for massive structures with a structural crack accompanied by large process zone (e.g. see [3]), which requires that both hardening and softening phenomena be combined in representing two active mechanisms of inelastic behavior.

The main motivation for this work stems from the need to further extend the applicability of the proposed model for localized failure to heterogeneous materials (under heterogeneous stress field). A convient use of a structured mesh for such a case will imply that the elements are crossed by phase-interfaces (see [4]), leading to discontinuous material properties within the same element and a strain profile which does not remain the same on both sides of the discontinuity (see [5]). We show in Sect. 2 that both displacement and strain discontinuities ought to be introduced in order to properly accommodate the resulting strain field. An illustration for such a strain field approximation is presented in Sect. 3 for a simple 1D model of a 2-node truss-bar element with discontinuous displacement and strain fields. Between two different possibilities for constructing the discrete approximation of this kind, either by extended finite element method (X-FEM) (see [6–8]) or by an element with embedded discontinuity FEM (ED-FEM) (e.g. see [1–3]), we prefer the latter since it leads to a very robust solution scheme. Namely, as shown in Sect. 4, the choice of ED-FEM allows us to exploit the operator split methodology when treating each discontinuity parameter computation accordingly, either within a global loop for strain discontinuities or within a local loop for displacement discontinuities. A couple of numerical examples in Sect. 5 further illustrate the robust performance of the proposed solution scheme.

2 Model problem of localized failure

We consider a 1D model problem of a bar built-in on the left end and loaded by a traction force on the right end (see Fig. 1). The bar is built of two materials, the first occupying the sub-domain Ω_1^e and the second Ω_2^e, which are connected at the interface placed at $\overline{x}$, such that

$$\Omega^e = \Omega_1^e \cup \Omega_2^e;\;\; \Omega^e = [0, l^e];\;\; \Omega_1^e = [0, \overline{x}];\;\; \Omega_2^e = [\overline{x}, l^e]. \quad (1)$$

We consider herein a general case of heterogeneous material of this kind where each sub-domain can have different inelastic behavior (an equivalent problem occurs for the same inelastic material with different strain field on each side of the interface). Moreover, we consider the possibility of localized failure at the interface with the softening behavior driving the stress to zero. In particular we choose the plasticity constitutive model in order to describe the different phenomena, with the yield criteria in two sub-domains

$$\overline{\phi}_1(\sigma, \overline{q}_1) = |\sigma| - (\sigma_{y_1} - \overline{q}_1), \quad \forall x \in \Omega_1^e, \quad (2)$$

$$\overline{\phi}_2(\sigma, \overline{q}_2) = |\sigma| - (\sigma_{y_2} - \overline{q}_2), \quad \forall x \in \Omega_2^e, \quad (3)$$

where σ_{y_1} and σ_{y_2} are, respectively, yield stress values in both sub-domains, whereas $\overline{q}_1$ and $\overline{q}_2$ are hardening variables. We describe the cohesive behavior at the interface by also using the plasticity criterion

$$\overline{\overline{\phi}}(t, \overline{\overline{q}}) = |t| - (\sigma_u - \overline{\overline{q}}), \quad x = \overline{x}, \quad (4)$$

where σ_u is the ultimate stress value where the localized failure is initiated, $\overline{\overline{q}}(\overline{\overline{\xi}})$ is the softening variable which drives the current yield stress to zero, and t is the traction force at the interface. In thermodynamics interpretation of this failure process one admits the additive decomposition of the strain energy density accounting for both sub-domains and the discontinuity

$$\psi(\overline{\varepsilon}_1^e, \overline{\varepsilon}_2^e, \overline{\xi}_1, \overline{\xi}_2, \overline{\overline{\xi}}) = \overline{\psi}_1(\overline{\varepsilon}_1^e, \overline{\xi}_1) + \overline{\psi}_2(\overline{\varepsilon}_2^e, \overline{\xi}_2) + \delta_{\overline{x}} \overline{\overline{\psi}}(\overline{\overline{\xi}}), \quad (5)$$

$$\delta_{\overline{x}} = \begin{cases} \infty, & x = \overline{x}, \\ 0, & \text{otherwise}, \end{cases} \quad (6)$$

where ε is the total strain field. In the case of localized failure, the latter ought to be split into a regular

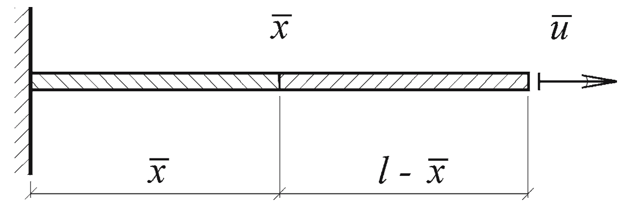

Fig. 1 Heterogeneous two-phase material for a truss-bar, with phase-interface placed at $\overline{x}$

(smooth) part, containing both elastic and plastic strain $\overline{\varepsilon}_i = \overline{\varepsilon}_i^e + \overline{\varepsilon}_i^p$, and irregular part concentrated upon the discontinuity, which is written as

$$\overline{\overline{\varepsilon}}(x, t) = \left[\frac{\mathrm{d}}{\mathrm{d}x} \chi(x) + \delta_{\overline{x}}(x) \right] \alpha(t); \quad \forall x \in \widetilde{\Omega}, \quad (7)$$

where $\chi(x)$ is the chosen function which should limit the influence of the discontinuity to a given domain $\widetilde{\Omega}$.

With such model ingredients on hand, we can write the dissipation in the region of influence of the discontinuity according to

$$\begin{aligned} 0 \le D_{\widetilde{\Omega}} := & \int_{\widetilde{\Omega}} \left[\sigma \cdot \dot{\varepsilon} - \frac{\mathrm{d}}{\mathrm{d}t} \overline{\psi}_1(\overline{\varepsilon}^e, \overline{\xi}_1) - \frac{\mathrm{d}}{\mathrm{d}t} \overline{\psi}_2(\overline{\varepsilon}^e, \overline{\xi}_2) \right. \\ & \left. - \delta_{\overline{x}} \frac{\mathrm{d}}{\mathrm{d}t} \overline{\overline{\psi}}(\overline{\overline{\xi}}) \right] \mathrm{d}x \\ = & \int_{\widetilde{\Omega}_1} \left[\sigma - \frac{\partial \overline{\psi}_1}{\partial \overline{\varepsilon}_1^e} \right] \cdot \dot{\overline{\varepsilon}}_1^e \mathrm{d}x + \int_{\widetilde{\Omega}_2} \left[\sigma - \frac{\partial \overline{\psi}_2}{\partial \overline{\varepsilon}_2^e} \right] \cdot \dot{\overline{\varepsilon}}_2^e \mathrm{d}x \\ & + \left[\int_{\widetilde{\Omega}_1} \sigma \cdot \frac{\mathrm{d}\chi}{\mathrm{d}x} \mathrm{d}x + \int_{\widetilde{\Omega}_2} \sigma \cdot \frac{\mathrm{d}\chi}{\mathrm{d}x} \mathrm{d}x + t_{\overline{x}} \right] \dot{\alpha} \\ & + \int_{\widetilde{\Omega}_1} \sigma \cdot \dot{\overline{\varepsilon}}_1^p \mathrm{d}x + \int_{\widetilde{\Omega}_2} \sigma \cdot \dot{\overline{\varepsilon}}_2^p \mathrm{d}x + \int_{\widetilde{\Omega}_1} \overline{q}_1 \cdot \dot{\overline{\xi}}_1 \mathrm{d}x \\ & + \int_{\widetilde{\Omega}_2} \overline{q}_2 \cdot \dot{\overline{\xi}}_2 \mathrm{d}x + \overline{\overline{q}} \cdot \dot{\overline{\overline{\xi}}}. \end{aligned} \quad (8)$$

In the case of an elastic process we obtain from (8) above the constitutive equation for the stress on each side of discontinuity

$$\sigma_i = \frac{\partial \psi_i}{\partial \overline{\varepsilon}_i^e}; \quad \forall x \in \widetilde{\Omega}_i; \;\; i = 1, 2. \quad (9)$$

By assuming that these equations remain valid for the case of an inelastic process, as well as imposing the stress orthogonality condition

$$\int_{\widetilde{\Omega}_1} \sigma_1 \cdot \frac{\mathrm{d}\chi}{\mathrm{d}x} \mathrm{d}x + \int_{\widetilde{\Omega}_2} \sigma_2 \cdot \frac{\mathrm{d}\chi}{\mathrm{d}x} \mathrm{d}x + t_{\overline{x}} = 0, \quad (10)$$

we obtain an additive decomposition of the total inelastic dissipation into the regular and discontinuity related part

$$0 \leq D^p_{\widetilde{\Omega}} = \int_{\widetilde{\Omega}_1} \left[\sigma_1 \cdot \dot{\bar{\varepsilon}}^p_1 + \bar{q}_1 \cdot \dot{\bar{\xi}}_1 \right] dx + \int_{\widetilde{\Omega}_2} \left[\sigma_2 \cdot \dot{\bar{\varepsilon}}^p_2 + \bar{q}_2 \cdot \dot{\bar{\xi}}_2 \right] dx + \bar{\bar{q}} \cdot \dot{\bar{\bar{\xi}}}. \tag{11}$$

The principle of maximum plastic dissipation is then the only ingredient which is needed in order to obtain evolution equations for internal variables

$$\dot{\bar{\varepsilon}}^p_i = \dot{\bar{\gamma}}_i \frac{\partial \bar{\phi}_i}{\partial \sigma}; \; \dot{\bar{\xi}}^p_i = \dot{\bar{\gamma}}_i \frac{\partial \bar{\phi}_i}{\partial \bar{q}_i}; \; \forall x \in \widetilde{\Omega}_i, \tag{12}$$

$$\dot{\bar{\bar{\xi}}} = \dot{\bar{\bar{\gamma}}} \frac{\partial \bar{\bar{\phi}}}{\partial \bar{\bar{q}}}; \; x = \bar{x}. \tag{13}$$

3 Finite element discretization by ED-FEM

The main novelty of the present development with respect to the previous works which consider only one discontinuity, either for strain (e.g. see [1]) or for displacement (e.g. [2]), is the need to account for two discontinuities simultaneously. The latter is imposed by the eventual heterogeneity of the strain field in the presence of localized failure, where the displacement discontinuity would also trigger the strain discontinuity by changing the level of strains on both sides of the discontinuity. The strain field for such a case can be constructed by using a 2-node bar element with embedded both strain and displacement discontinuities (see Fig. 2a).

With the choice of the discontinuity influence domain corresponding to a single element $\widetilde{\Omega} = \Omega^e$, the displacement field interpolation can then be written as

$$u(x,t)\Big|_{\Omega^e_i} = \sum_{a=1}^{2} N_a(x) d_a(t) + M_1(x)\alpha_1(t) + M_2(x)\alpha_2(t); \quad x \in \Omega^e_i, \tag{14}$$

where

$$N_a(x) = \begin{cases} 1 - \frac{x}{l}, & a = 1 \\ \frac{x}{l}, & a = 2 \end{cases}; \; M_1(x) = \begin{cases} -\frac{x}{l}, & x \in [0, \bar{x}] \\ 1 - \frac{x}{l}, & x \in [\bar{x}, l] \end{cases};$$

$$M_2(x) = \begin{cases} -\frac{x}{\bar{x}}, & x \in [0, \bar{x}] \\ \frac{x - l}{(l - \bar{x})}, & x \in [\bar{x}, l] \end{cases}. \tag{15}$$

The total strain field can then be computed as the space derivative of this displacement interpolation leading to

$$\varepsilon_i(x,t)\Big|_{\Omega^e_i} = \sum_{a=1}^{2} B_a(x) d_a(t) + \widetilde{G}_1(x)\alpha_1(t) + \delta_{\bar{x}}(x)\alpha_1(t) + G_2(x)\alpha_2(t), \tag{16}$$

where

$$B_a(x) = \begin{cases} -\frac{1}{l}, & a = 1 \\ \frac{1}{l}, & a = 2 \end{cases}; \; \widetilde{G}_1(x) = -\frac{1}{l}, \; x \in [0, l];$$

$$G_2(x) = \begin{cases} -\frac{1}{\bar{x}}, & x \in [0, \bar{x}] \\ +\frac{1}{(l - \bar{x})}, & x \in [\bar{x}, l] \end{cases}. \tag{17}$$

The last term in the strain approximation represents the discontinuity jump, with the corresponding interpretation as the localized inelastic strain. Therefore, the regular total strain field approximation is written as

$$\bar{\varepsilon}_i(x,t)\Big|_{\Omega^e} = \sum_{a=1}^{2} B_a(x) d_a(t) + \widetilde{G}_1(x)\alpha_1(t) + G_2(x)\alpha_2(t); \quad x \in \Omega^e_i; \; \forall x \neq \bar{x} \tag{18}$$

and the stress approximation at any point outside the discontinuity reads

$$\sigma_i(x,t)\Big|_{\Omega^e} = E_i\Big(\bar{\varepsilon}_i(x,t) - \bar{\varepsilon}^p_i(x,t)\Big); \quad \forall x \in \Omega^e_i. \tag{19}$$

Introducing these discrete approximations in the weak form of equilibrium equations, we obtain a set of global equations for nodal equilibrium by the FE assembling operator A, accompanied by local element-wise equilibrium equations

$$\begin{cases} A^{n_{el}}_{e=1}\left[f^{int,e} - f^{ext,e} \right] = 0 \\ h^e_1 = 0 \\ h^e_2 = 0 \end{cases} \quad \forall e \in [1, n_{el}], \tag{20}$$

where

$$\begin{cases} f^{int,e}_a = \int_{\Omega^e_1} B^T_a \sigma_1 \, dx + \int_{\Omega^e_2} B^T_a \sigma_2 \, dx, \\ h^e_1 = \int_{\Omega^e_1} \widetilde{G}^{e,T}_1 \sigma_1 dx + \int_{\Omega^e_2} \widetilde{G}^{e,T}_1 \sigma_2 dx + t_{\bar{x}}, \\ h^e_2 = \int_{\Omega^e_1} G^T_2 \sigma_1 dx + \int_{\Omega^e_2} G^T_2 \sigma_2 dx. \end{cases} \tag{21}$$

Remark 1 We note in passing that the same kind of approximation for total strain and displacement fields can also be constructed by X-FEM (e.g. [6–8]) by using the identical number of additional degrees of freedom as for proposed ED-FEM;

$$u(x,t)\Big|_{\Omega^e} = \sum_{a=1}^{2} N_a(x)\left[d_a(t) + H_{\bar{x}}(x)\beta_a(t) \right] = \sum_{a=1}^{2} N_a(x) d_a(t) + \sum_{a=1}^{2} N_a(x) H_{\bar{x}}(x)\beta_a(t). \tag{22}$$

It is easy to see (Fig. 2b) that the X-FEM interpolation parameters $\beta_a(t)$ are now placed at element nodes, and they can thus be shared by all the neighboring elements attached to that node. The physical interpretation of each parameter, regarding its role in controlling

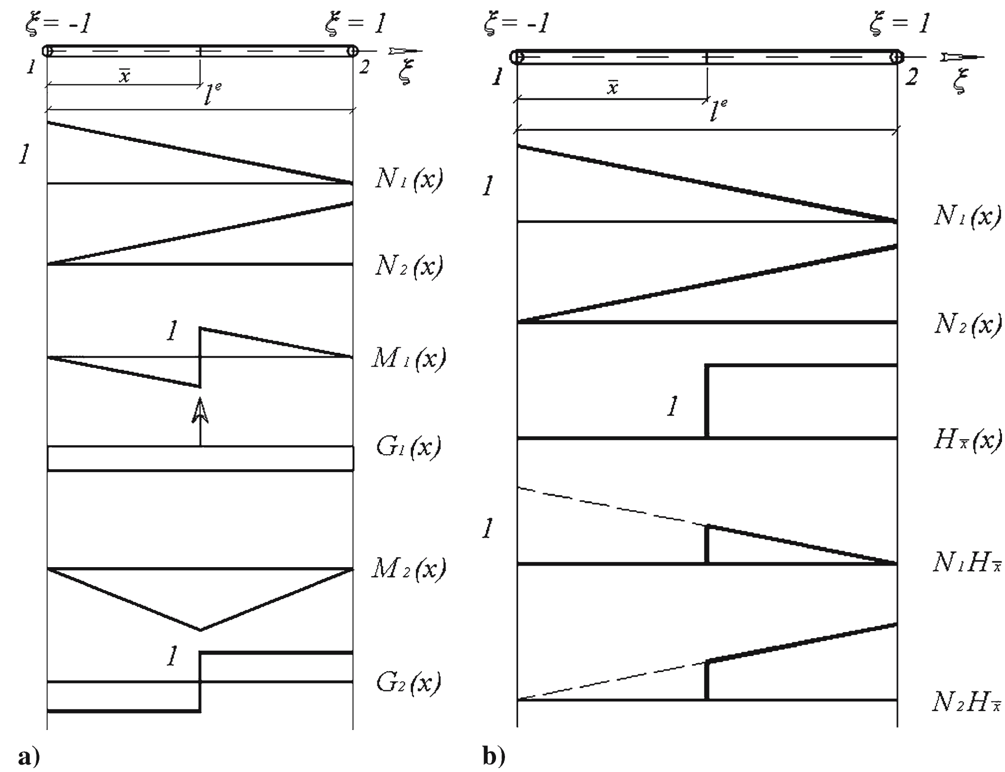

Fig. 2 Two node bar element with : **a** embedded strain and displacement discontinuities, where α_1 and α_2 are element parameters scaling $M_1(x)$ and $M_2(x)$, **b** equivalent X-FEM displacement interpolation, where β_1, β_2 are nodal parameters scaling $N_1(x)H_{\bar{x}}(x)$ and $N_2(x)H_{\bar{x}}(x)$

displacement versus strain discontinuity, becomes less obvious and one can no longer implement an efficient and robust operator split resolution scheme for the local-global system in (20), which is described next.

4 Operator split solution procedure

The key idea leading towards the proposed operator split solution procedure for the nonlinear equilibrium equations posed by the localized failure problem in (20), pertains to the crucial difference of two types of discontinuity parameters. The strain discontinuity, controlled by parameter $\alpha_2(t)$, contributes to the total strain field, whereas the displacement discontinuity, controlled by parameter $\alpha_1(t)$, counts as plastic strain contribution. For that reason, the element residual h_2^e is solved along the set of global equations for computing the nodal displacement values. In this manner we obtain the best iterative value of displacements $d^{(i)}_{a,n+1}$ and strain discontinuity parameter $\alpha^{(i)}_{2,n+1}$, which allows to carry on with the local computation phase. The latter can formally be written as :

Iterate $(i) = 1, 2, \ldots$ – on global equilibrium equations

Start the local computation of internal variables according to

(a) Local phase – computing plastic strain

given: $d^{(i)}_{a,n+1}, \alpha^{(i)}_{2,n+1}, \bar{\varepsilon}^p_{1,n}, \bar{\xi}_{1,n}\bar{\varepsilon}^p_{2,n}, \bar{\xi}_{2,n}, \alpha_{1,n}, \bar{\bar{\xi}}_n$

find: $\bar{\varepsilon}^p_{1,n+1}, \bar{\xi}_{1,n+1}\bar{\varepsilon}^p_{2,n+1}, \bar{\xi}_{2,n+1}, \alpha_{1,n+1}, \bar{\bar{\xi}}_{n+1}$

such that:

$$\begin{aligned}
&\bar{\phi}_i(\sigma_{n+1}, \bar{q}_{i,n+1}) \le 0,\ \bar{\gamma}_{i,n+1} \ge 0,\\
&\bar{\gamma}_{i,n+1}\bar{\phi}_{i,n+1} = 0; \qquad i = 1,2\\
&\bar{\bar{\phi}}(t_{n+1}, \bar{\bar{q}}_{n+1}) \le 0,\ \bar{\bar{\gamma}}_{n+1} \ge 0,\ \bar{\bar{\gamma}}_{n+1}\bar{\bar{\phi}}_{n+1} = 0.
\end{aligned} \tag{23}$$

This local computation can be carried out as follows :

Iterate $(j) = 1, 2, \ldots$ – At discontinuity $x = \bar{x}$

Iterate $(k) = 1, 2, \ldots$ – Outside discontinuity $x \in \Omega_i^e$

IF (elastic step in each sub-domain) THEN

$$\begin{aligned}
\bar{\varepsilon}_{i,n+1} &= \sum_{i=1}^{2} B_a d^{(i)}_{a,n+1} + G_2\alpha^{(i)}_{2,n+1} + \widetilde{G}_1\alpha^{(j)}_{1,n+1}\\
\Rightarrow \sigma^{\text{trial}}_{i,n+1} &= E_i\big(\bar{\varepsilon}_{i,n+1} - \varepsilon^p_{i,n}\big)
\end{aligned} \tag{24}$$

$$\begin{aligned}
\bar{\phi}^{\text{trial}}_{i,n+1} &:= |\sigma^{\text{trial}}_{i,n+1}| - (\sigma_{y_i} - \bar{q}_i) \le 0\\
\Rightarrow \sigma_{i,n+1} &= \sigma^{\text{trial}}_{i,n+1};\ C^{\text{ep}}_{i,n+1} = E_i.
\end{aligned} \tag{25}$$

ELSE (plastic step in each sub-domain)

$$0 = \widehat{\phi}_i(\overline{\gamma}_{i,n+1}^{(k)}) := \overline{\phi}_{i,n+1}^{\text{trial}} - \overline{\gamma}_{i,n+1}^{(k)} E_i + \widehat{q}_i\big(\underbrace{\xi_{i,n} + \overline{\gamma}_{i,n+1}^{(k)}}_{\xi_{i,n+1}^{(k)}}\big) - \widehat{q}_i(\xi_{i,n}) \quad (26)$$

WHILE $|\widehat{\phi}(\overline{\gamma}_{i,n+1}^{(k)})| > \text{tol.} \Rightarrow \overline{\gamma}_{i,n+1}^{(k+1)} = \overline{\gamma}_{i,n+1}^{(k)} - \widehat{\phi}_i(\overline{\gamma}_{i,n+1}^{(k)}) \Big/ \frac{\mathrm{d}\widehat{\phi}(\overline{\gamma}_{i,n+1}^{(k)})}{\mathrm{d}\overline{\gamma}_{i,n+1}}$; $(k) \leftarrow (k+1)$

$$\sigma_{i,n+1} = \sigma_{i,n+1}^{\text{trial}} - E_i \overline{\gamma}_{i,n+1}^{(k)} \frac{\partial \overline{\phi}_i(\sigma_{i,n+1}^{\text{trial}}, q_{i,n})}{\partial \sigma_i}$$

$$\Rightarrow C_{i,n+1}^{\text{ep}} = \frac{E_i \bar{K}_{i,n+1}^{(k)}}{E_i + \bar{K}_{i,n+1}^{(k)}}; \; \bar{K}_{i,n+1}^{(k)} = -\frac{\partial \bar{q}_i}{\partial \bar{\xi}_i^{(k)}} \quad (27)$$

IF (elastic step at discontinuity) THEN

$$t_{n+1}^{\text{trial}} = -\Big[\int_{\Omega_1^e} \widetilde{G}_1^T \sigma_{1,n+1} dx + \int_{\Omega_2^e} \widetilde{G}_1^T \sigma_{2,n+1} \mathrm{d}x\Big]$$

$$\overline{\overline{\phi}}_{n+1}^{\text{trial}} := |t_{n+1}^{\text{trial}}| - (\sigma_u - \overline{\overline{q}}_n) \le 0$$

$$\Rightarrow t_{n+1}^{(j)} = t_{n+1}^{\text{trial}}; \; \bar{\bar{K}}_{n+1}^{(j)} = 0 \quad (28)$$

ELSE (plastic step at discontinuity)

$$\overline{\overline{\gamma}}_{n+1}^{(j)} = \overline{\overline{\phi}}_{n+1}^{\text{trial}} \Big/ \Big[\int_{\Omega_1^e} \widetilde{G}_1^T C_{1,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x + \int_{\Omega_2^e} \widetilde{G}_1^T C_{2,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x - \frac{\partial \overline{\overline{q}}_{n+1}}{\partial \overline{\overline{\xi}}_{n+1}}\Big] \quad (29)$$

$$\alpha_{1,n+1}^{(j+1)} = \alpha_{1,n+1}^{(j)} + \overline{\overline{\gamma}}_{n+1}^{(j)} \,\text{sign}(t_{n+1}^{\text{trial}})$$
$$\overline{\overline{\xi}}_{n+1}^{(j+1)} = \overline{\overline{\xi}}_{n+1}^{(j)} + \overline{\overline{\gamma}}_{n+1}^{(j)} \quad (30)$$
$$\Rightarrow \bar{\bar{K}}_{n+1}^{(j)} = -\frac{\partial \overline{\overline{q}}}{\partial \overline{\overline{\xi}}}$$

WHILE $|\overline{\overline{\phi}}(\overline{\overline{\gamma}}_{n+1}^{(j)})| > \text{tol.}$; $(j) \leftarrow (j+1)$

(b) Global phase – computing displacement and total strain

Once we converged in this local computation, we turn to the global iterative loop in order to provide, if so needed, new iterative values of displacement and strain discontinuity parameter. First, the set of equilibrium equations in (20) is checked for convergence (with newly computed values of plastic strains);

WHILE $\| f_{n+1}^{\text{int},(i)} - f_{n+1}^{\text{ext}} \| >$ tol. a new iterative sweep is then performed, accounting for each element contribution

$$\begin{bmatrix} K^e & F_2^e \\ F_2^{e^T} & H_2^e \end{bmatrix}_{n+1}^{(i)} \begin{pmatrix} \Delta d_{n+1}^{(i)} \\ \Delta \alpha_{2,n+1}^{(i)} \end{pmatrix} = \begin{pmatrix} f_{n+1}^{e,\text{ext},(i)} - f_{n+1}^{e,\text{int},(i)} \\ -h_{2,n+1}^{e,(i)} \end{pmatrix} \quad (31)$$

where

$$\begin{aligned}
K_{n+1}^e &= \int_{\Omega_1^e} B^T C_{1,n+1}^{\text{ep}} B \mathrm{d}x + \int_{\Omega_2^e} B^T C_{2,n+1}^{\text{ep}} B \mathrm{d}x - [F_{1,n+1}^e][H_{1,n+1}^e]^{-1}[F_{1,n+1}^{e,T}] \\
F_{2,n+1}^e &= \int_{\Omega_1^e} B^T C_{1,n+1}^{\text{ep}} G_2 \mathrm{d}x + \int_{\Omega_2^e} B^T C_{2,n+1}^{\text{ep}} G_2 \mathrm{d}x - [F_{1,n+1}^e][H_{1,n+1}^e]^{-1}[P_{1,n+1}^{e,T}] \\
H_{2,n+1}^e &= \int_{\Omega_1^e} G_2^T C_{1,n+1}^{\text{ep}} G_2 \mathrm{d}x + \int_{\Omega_2^e} G_2^T C_{2,n+1}^{\text{ep}} G_2 \mathrm{d}x - [P_{1,n+1}^e][H_{1,n+1}^e]^{-1}[P_{1,n+1}^{e,T}] \\
H_{1,n+1}^e &= \int_{\Omega_1^e} \widetilde{G}_1^T C_{1,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x + \int_{\Omega_2^e} \widetilde{G}_1^T C_{2,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x - \frac{\partial \overline{\overline{q}}_{n+1}}{\partial \overline{\overline{\xi}}_{n+1}} \\
F_{1,n+1}^e &= \int_{\Omega_1^e} B^T C_{1,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x + \int_{\Omega_2^e} B^T C_{2,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x \\
P_{1,n+1}^e &= \int_{\Omega_1^e} G_2^T C_{1,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x + \int_{\Omega_2^e} G_2^T C_{2,n+1}^{\text{ep}} \widetilde{G}_1 \mathrm{d}x
\end{aligned} \quad (32)$$

The static condensation of this matrix allows us to form the final stiffness matrix for this element contribution to FE assembly

$$A_{e=1}^{n_{\text{el}}} [\widehat{K}_{n+1}^e] \Delta d_{n+1}^{(i)} = A_{e=1}^{n_{\text{el}}} \Big[f_{n+1}^{e,\text{ext}} - f_{n+1}^{e,\text{int}} - \widehat{h}_{2,n+1}^e\Big], \quad (33)$$

$$\widehat{K}_{n+1}^e = K_{n+1}^e - F_{2,n+1}^e [H_{2,n+1}^e]^{-1} F_{2,n+1}^{e^T},$$
$$\widehat{h}_{2,n+1}^e = F_{2,n+1}^e (H_{2,n+1}^e)^{-1} h_{2,n+1}^e. \quad (34)$$

5 Numerical examples

In the numerical example we consider a 1D model problem of a bar built-in on the left end and loaded by an imposed displacement $u = 0.75$ on the right end (see Fig. 1). In fact, the finite element model of the bar consists of two elements. The first element simulates the stiffness of the testing machine used to control the imposed displacement, with the length equal to $0.2l$ and constitutive behavior which remains always elastic with Young's modulus $E_1 = 100{,}000$. The second element is

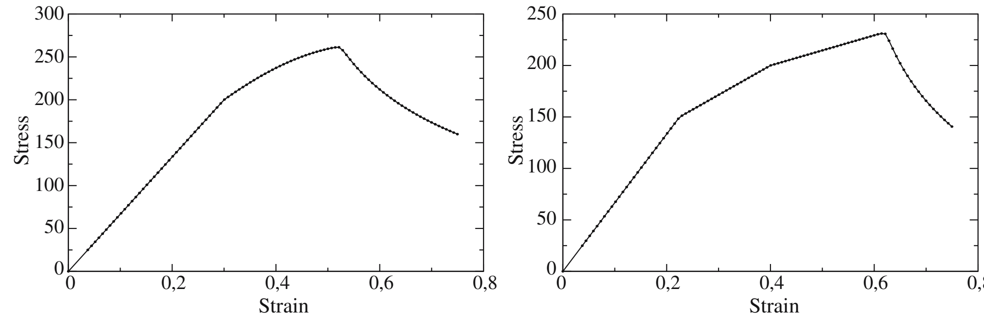

Fig. 3 Stress-strain graphics for: **a** linear hardening/softening law and **b** non-linear exponential hardening/softening law

Table 1 The convergence results of global-local equilibrium problem

Load step	Tot. iter.	Local eqs. – mat. 1		Global eqs. : $f^{\text{int}} - f^{\text{ext}}$		Tot. Iter.	Local eqs. – mat. 1		Global eqs. : $f^{\text{int}} - f^{\text{ext}}$	
		h_1^e	h_2^e	Initial	Final		h_1^e	h_2^e	Initial	Final
1	2	0.00E+000	−3.55E−015	2.53E+003	6.26E−029	2	0.00E+000	0.00E+000	7.02E+002	2.47E−031
2	2	0.00E+000	1.42E−014	2.53E+003	2.52E−030	2	0.00E+000	−3.55E−015	2.53E+003	6.26E−029
3	2	0.00E+000	1.42E−014	2.53E+003	3.52E−028	2	0.00E+000	1.42E−014	2.53E+003	2.52E−030
4	3	0.00E+000	0.00E+000	2.53E+003	3.37E−028	2	0.00E+000	1.42E−014	2.53E+003	2.22E−028
5	3	0.00E+000	2.84E−014	2.53E+003	2.94E−015	3	0.00E+000	−2.84E−014	2.53E+003	8.31E−021
6	4	1.24E−014	−2.84E−014	2.54E+003	5.93E−016	3	0.00E+000	−2.24E−011	2.53E+003	3.90E−018
7	4	2.49E−014	0.00E+000	2.54E+003	3.05E−014	4	0.00E+000	−2.19E−012	2.53E+003	2.90E−015
8	4	1.42E−014	0.00E+000	2.54E+003	9.32E−015	4	1.42E−014	0.00E+000	2.54E+003	2.28E−014
9	4	−1.42E−014	0.00E+000	2.54E+003	2.37E−015	4	7.11E−015	−2.84E−014	2.54E+003	4.78E−015
10	4	−1.42E−014	−2.84E−014	2.54E+003	8.23E−016	4	1.42E−014	−2.84E−014	2.54E+003	1.56E−015
11	4	0.00E+000	4.26E−014	2.54E+000	4.26E−014	4	−1.42E−014	−2.84E−014	2.54E+003	4.07E−016

the model of the specimen, with length equal to l. The specimen is assumed to be built of two materials, with Young's modulus of the first and second given respectively as $E_1 = 1,000$ and $E_1 = 500$, and the material interface placed at $\bar{x} = 0.5l$. The calculation is carried out for two sets of model parameters:

1. Different yield stress and linear hardening modulus for each material phase with $\sigma_{y1} = 150$, $\sigma_{y2} = 200$, $\bar{K}_1 = 250$, $\bar{K}_2 = 150$; along with the linear softening in the discontinuity: $\sigma_u = 220$, $\bar{\bar{K}} = 200$.
2. For non linear saturation-type hardening chosen the same for the whole bar along with non linear exponential softening in the discontinuity with: $\sigma_y = 200$, $\sigma_u = 220$, $b = 2$.

$$\bar{\bar{q}} = \overline{K} \cdot \bar{\bar{\xi}} \quad \text{or} \quad \bar{\bar{q}} = (\sigma_u - \sigma_y) \cdot \left[1 - \mathrm{e}^{-b\bar{\bar{\xi}}}\right]. \tag{35}$$

The representative result of force-displacement diagram are given in (see Fig. 3). The solution scheme is very robust, using the Newton method at both local and global level. The typical quadratic convergence rates are noted in our computation leading to a few iterations at each time step (see Table 1). The X-FEM solution procedure in this example can provide the same convergence rates while we are dealing with either elastic or plastic hardening response. However, once the softening regime starts, we could no longer assure convergence of the simultaneous iteration procedure, typical of X-FEM.

6 Concluding remarks

The solution method proposed herein is facilitated by the clear physical interpretation which can be given for each discontinuity mode in ED-FEM displacement interpolation. Namely, the strain discontinuity is treated as an integral part of the total strain field and solved for along with the displacement, whereas the displacement discontinuity is treated as the localized plastic strain field and solved for within the local element-wise loop. Similar physical interpretation cannot be given to the X-FEM parameters, so that one can not have the same robustness of the solution procedure in the present setting of localized failure of massive structures. This observation does not diminish the great efficiency of the X-FEM for the class of problems for which it was developed, such as those of linear fracture mechanics where one disregards any contribution to the dissipation from the fracture process zone.

The proposed ED-FEM should be replaced by X-FEM approximation in a problem where the chosen representation of a given field must remain global. One such case is the modelling of bond-slip in reinforced concrete structures (see [9]), where all the bond-slip degrees of freedom along the particular steel-bar must remain connected in order to capture the bar pull-out failure with the bar loosing any slip resistance. Another case where X-FEM can be of direct interest concerns providing the global field representation in shape optimization problems, simplifying the choice of global design variables for interface representation in a multi-phase material (e.g. see [10]).

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