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ORIGINAL PAPER

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Meshless techniques for convection dominated problems

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Abstract In this paper, the stability problem in the analysis of the convection dominated problems using meshfree methods is first discussed through an example problem of steady state convection-diffusion. Several techniques are then developed to overcome the instability issues in convection dominated phenomenon simulated using meshfree collocation methods. These techniques include: the enlargement of the local support domain, the upwind support domain, the adaptive upwind support domain, the biased support domain, the nodal refinement, and the adaptive analysis. These techniques are then demonstrated in one- and two-dimensional problems. Numerical results for example problems demonstrate the techniques developed in this paper are effective to solve convection dominated problems, and in these techniques, using the adaptive local support domain is the most effective method. Comparing with the conventional finite difference method (FDM) and the finite element method (FEM), the meshfree method has found some attractive advantages in solving the convection dominated problems, because it easily overcomes the instability issues.

Keywords Meshless method · Meshfree method · Convection-diffusion · Convection dominated · Numerical analysis

1 Introduction

Convection-diffusion problems are of significant importance and challenging in computational mechanics. Many practical problems in engineering are governed by the so-called convection-diffusion equations, in which, there are both convective and diffusive terms. To analyze the convection-diffusion problems, the traditional finite element method (FEM), the finite difference method (FDM), or the finite volume method (FVM) has been widely used.

However, there is a well-known technical issue in the analysis for the convection-diffusion problem using the numerical methods: the solution becomes instable or oscillatory when the problem becomes convection dominated if the standard FEM or FDM procedure is followed without special treatments. It is because that, in these convection dominated problems, a thin boundary layer is usually formed. In the thin boundary layer, it exists a very high gradient. Using the standard numerical techniques, this thin boundary layer is very difficult to be simulated, and it will result in an oscillatory (unstable) solution. A lot of studies have been performed to solve the instability problem in FEM and FDM, and an excellent documentation on this topic for FEM and FDM can be found in the book by Zienkiewicz and Taylor [27]. To stabilize the numerical approximation for these problems, schemes related to upwinding are the most general techniques in FEM, FDM and FVM. In addition, the Petrov-Galerkin forms are also used in FEM [27]. However, the mesh or regular grid used in FEM or FDM makes difficulty to totally overcome this instability problem. For example, the adaptive interpolation is difficult to fulfill in FEM and FDM because of the limitation of the mesh or regular grid that is pre-defined. In addition, the adaptive analysis for the convection dominated problems is also difficult to be performed in FEM and FDM.

In recent years, meshless or meshfree methods have attracted more and more attention from researchers, and are regarded as promising numerical methods for computational mechanics. These meshfree methods do not require a mesh to discretize the problem domain, because the approximate solution is constructed entirely based on a set of scattered nodes. A group of meshfree methods has been proposed. Some of these methods based on the collocation techniques and the meshfree shape functions (the moving least squares, the radial basis function interpolation, etc.), for example, the finite point method (FPM) [18], the *hp*-meshless cloud method [12], the meshfree collocation method [10, 17, 22, 23]. Some other meshfree methods are based on global or

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weak forms and meshfree shape functions, e.g., the element-free Galerkin (EFG) method [4] the radial point interpolation method (RPIM) [15,20] the meshless local Petrov-Galerkin method (MLPG) [1,2,7,8,25,26] the local point interpolation method [9,11,21] the method of finite spheres [6], and so on.

The meshfree methods based on the collocation techniques have been found to possess the following attractive advantages:

- They are truly meshless methods. No mesh is required in the whole processes including the function approximation and numerical integrations.
- The procedure is basically straightforward, and hence the algorithms and coding are very simple.
- They are computationally efficient, and the solution is accurate when there are only Dirichlet boundary conditions.

Owing to the above advantages, meshfree collocation methods have been studied and used in computational mechanics, especially in problems of the computational fluid mechanics. However, the major shortcoming of these methods is that the derivative (Neumann) boundary conditions may lead to large computational error, such as for solid mechanics problems with stress (natural) boundary conditions. Some techniques have been developed to avoid this problem, and they are summarized by Liu and Gu [16].

Only very few works was reported to solve convection dominated problems using the meshfree methods. Oñate et al. [18,19] applied the finite point method to the convection dominated problem with upwinding for the first derivative or with characteristic approximation. Atluri et al. [1,13] used the MLPG method to solve the convection-diffusion problems. They used the local upwinding weight and trial functions in MLPG to overcome the instability in the convection dominated problem. In this paper, techniques to stabilize the convection dominated problems will by developed and investigated for meshfree collocation methods. The stability problem in the analysis of the convection-diffusion problem using meshfree methods is first discussed through an example problem. Several techniques are then developed to overcome the instability issues in convection dominated problems. These techniques include: the enlargement of the local support domain, the upwind support domain, the adaptive upwind support domain, the biased support domain, the nodal refinement, and the adaptive analysis. Most of these techniques are developed and discussed in the first time for the analysis of convection dominated problems by the meshfree method. These techniques are then demonstrated in one- and two-dimensional problems simulated by the meshfree collocation method. Numerical results demonstrate that using these techniques the meshfree method is very effective to solve convection dominated problems. Comparing with the conventional FDM and the FEM, the meshfree method has found some attractive advantages in solving the convection dominated problems to overcome the instability problems.

2 Techniques for Meshfree Methods to Overcome the Instability Issues

In this section, several techniques for meshfree methods are developed to overcome the instability issues for the analysis of convection dominated problems. To unveil the stability issue, a one-dimensional (1-D) steady-state convection-diffusion problem governed by the following equation is first considered [27]

$$V\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q = 0, \quad x \in (0, 1) \tag{1}$$

where u is a scalar field variable, V, k and q are all given constants, and they carry different physical meanings for different engineering problems.

The following Dirichlet boundary conditions are considered.

$$u|_{x=1} = 1, \qquad u|_{x=0} = 0 \tag{2}$$

Equation (1) is an ordinary differential equation (ODE) of second order with constant coefficients. The exact solution for this problem can be easily obtained by solving ODE analytically with boundary conditions. The Peclet number is defined as follows and it often controls the stability of the numerical solution of this problem.

$$Pe = \frac{V d_c}{2k} \tag{3}$$

where d_c is the nodal spacing for two neighbor field nodes. For example, in this example, if the problem domain is represented by 21 regularly distributed nodes, $d_c = 0.05$.

The meshfree polynomial point collocation method (PPCM) [16] that uses shape functions created by the polynomial interpolation [14,15] is employed to solve this problem. The function u is approximated by

$$u(x) = \sum_{i=1}^{n} \Phi_i(x) u_i \tag{4}$$

where Φ_i is the meshfree shape functions that are constructed using the polynomial interpolations, and n is number of field nodes used in the local interpolation (support) domain. The local interpolation domain is usually defined to select several closest nodes for computing the meshfree shape functions.

It can be found that the accuracy of solutions deteriorates as Pe increases, if no special treatment is performed. When Pe is very large, Eq. (1) becomes convection dominated, and the accuracy of the standard numerical result becomes oscillatory. In the case of convection dominated (the Peclet number is large but finite), the effect of the second term in Eq. (1) becomes negligible resulting in the down stream boundary condition, $u|_{x=1} = 1$, to affect only a very narrow region to form a thin boundary layer (the boundary layer can be seen from following figures). The thin boundary layer is very difficult to be reproduced by a standard numerical method and results in an oscillatory (unstable) solution. This type of instability can occur in many numerical methods including FEM, FDM, FVM and the meshfree method if no special treatment is implemented. The key to overcoming this problem is to effectively capture the upstream information in the approximation of the field variables. To stabilize the solution for the convection dominated problem mentioned, several strategies for meshfree methods are newly developed in the following.

Technique 1: **Nodal refinement**

It is known that the instability is directly related to the Peclet number given in Eq. (3). Therefore, a natural and simple way to stabilize the solution is to reduce the Peclet number by reducing the nodal spacing d_c for given V and k. To confirm this argument, two models of 21 and 41 regularly distributed nodes are used to solve the same problem by the meshfree method. When 21 nodes and 41 nodes are used, the nodal spacings are $d_c = 0.05$ and $d_c = 0.025$, respectively. For example, when $V = 100$ and $k = 1$, $Pe = 2.5$ and $Pe = 1.25$ for 21 nodal model and 41 nodal model. Because the Peclet number becomes smaller, the instability of solution is naturally alleviated (it is quite easy to understand, hence the detailed results are not presented here). Using finer field nodes is a very simple way to alleviate the instability problem, and applicable for all methods of the domain discretization, e.g., FEM, FDM, and the meshfree methods. Note that an increase of nodes leads to an increase in computational time. Increasing the nodal density only in the boundary layer can certainly be more efficient.

Technique 2: **Enlargement of the local support domain**

The instability is caused by the failure to capture the upstream information by the discretion scheme used in the numerical methods. The simplest way to capture the upstream information is naturally to use more nodes in the local support domain for interpolations. This may not be done easily in FEM and FDM as the interpolation in them is mesh based, and it is limited in the pre-defined elements or grids. However, this technique can be easily used without any difficulty in meshfree methods by simply enlarging the local support domain of the interpolation node near the boundary layer, because, in the meshfree method, no mesh is used, and the local interpolation (support) domain can be selected freely based on the requirement for the problem.

To demonstrate this technique, three types of local support domains of selecting 3, 5 and 7 closest field nodes are used to solve the same problem, and results obtained using the meshfree method are plotted in Fig. 1. From this figure, it can be found that the accuracy and stability of solutions are significantly improved by using more nodes in the interpolation domains. It proves that the enlargement of the local interpolation (support) domain can effectively stabilize the numerical solution. Because a big local interpolation domain can capture more information from both upstream and downstream, it is quite straightforward manner in meshfree methods to improve the accuracy and stability of solutions without refining the nodes.

It should be mentioned again that this technique is very easy to implement in the meshfree methods because of the freedom in construction the meshfree shape functions. In addition, the meshfree interpolation domains often overlap with each other. The "overlap" feature also helps to stabilize the solution. In the other hand, the enlargement of the local interpolation (support) domain needs to be done only for the interpolation points that are in and near the boundary layer.

Technique 3: **Fully upwind support domain**

The upwind difference scheme has been often used in the FDM to solve the convection dominated problems. Similar to the upwind difference scheme used in FDM, the local upwind support domain that is fully biased on the upwind side, as shown in Fig. 2b, is proposed here and implemented in the meshfree method to stabilize the solution. Results for $Pe = 2.5$ are obtained and plotted in Fig. 3. It is observed that the upwind support domain can improve the accuracy and stability for problems with large Peclet numbers because it can fully capture the information from upstream. However, results for $Pe = 0.25$ are also obtained and plotted in Fig. 3. It can be found that it gives very poor results for cases of smaller Peclet numbers because of the fully "asymmetric" interpolation using the upwind support domain.

Technique 4: **Adaptive upwind support domain**

Comparing with the fully upwind support domain, when using the normal local support domain that is symmetric, it gives good results for small Peclet numbers but unstable results for large Peclet numbers. Hence, the ideal local support domain should be updated with Peclet number. We term such a local support domain as adaptive upwind support domain. The freedom to construct local interpolation domains in the meshfree method provides the possibility to fulfill the adaptive upwind domains. The adaptive upwind support domain can be defined using the following formula.

$$d_u = \alpha_u \cdot r_s, \quad where\ \alpha_u = \coth|Pe| - 1/|Pe| \tag{5}$$

In above equations, d_u is the central offset distance as shown in Fig. 2(c), and r_s is the size of the support domain, and $r_s = \alpha_s \cdot d_c$.

Figure 4 plots the change of α_u with *Pe*. It can be found that α_u satisfies

$$\alpha_u = \begin{cases} 0, & Pe \to 0 \\ 1, & Pe \to +\infty \end{cases} \tag{6}$$

Therefore, Eq. (5) satisfies the following two conditions.

- When $Pe \to 0$, the normal support domain is used and $d_u = 0$.
- When $Pe \to \infty$, fully upwind support domain is used and $d_u = r_s$.

Equation (5) works well for arbitrary *Pe*. It performs better when the number of the nodes used in the support domain is large, and is easy to use in the meshfree methods. Figure 5 shows that Eq. (5) works well for both large and small Peclet numbers. It is one of the most effective methods to ensure the stability of convection dominated problems.

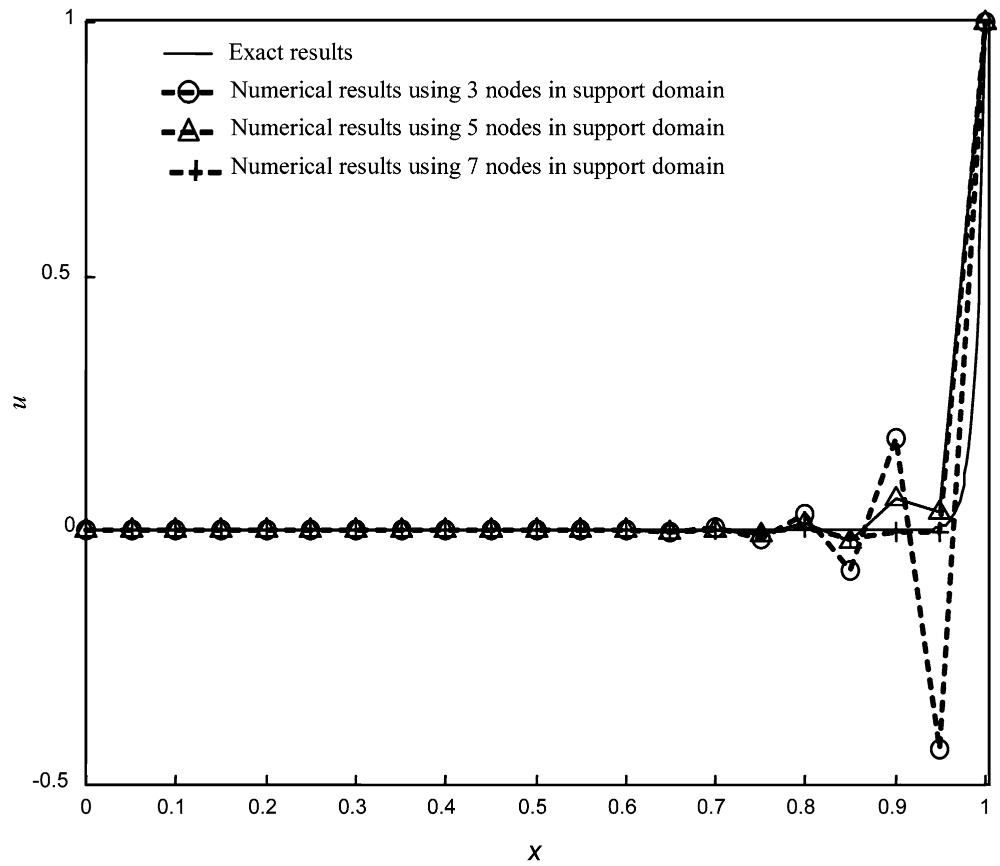

Fig. 1 Results of different support domains (Pe = 2.5)

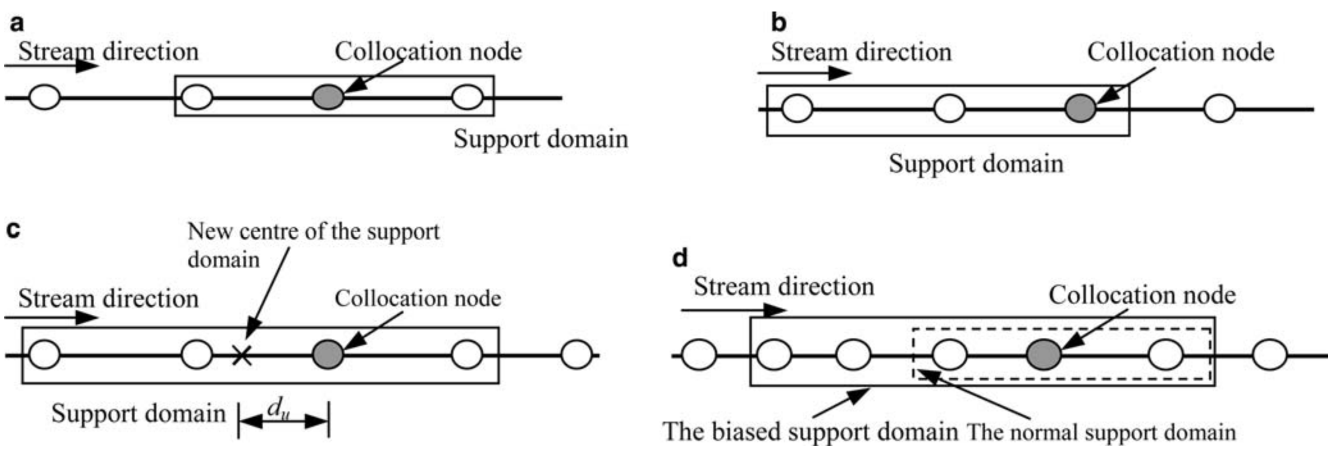

Fig. 2 Different types of local support domains **(a)** The normal support domain, **(b)** The fully upwind support domain, **(c)** An adaptive upwind local support domain, **(d)** A biased support domain by deliberately adding two more nodes

Another effective and simple way to establish a biased support domain is deliberately selecting more nodes in the upstream direction when constructing the local support domain for a interpolation node [5]. Fig. 2d shows a biased support domain constructed based on a normal support domain by adding two more nodes in the upstream direction. Due to the freedom in selecting the support domain in meshfree methods, the method of using the biased support domain is also very effective and easy to use in the practical applications.

3 Two-dimensional convection-diffusion problem

3.1 Governing equations

Some techniques to overcome the instability issue in the convection dominated problems solved by the meshfree methods have been developed and discussed in Sect. 2. We will use these developed techniques to solve 2-D convection-diffusion problems. Let us consider the following

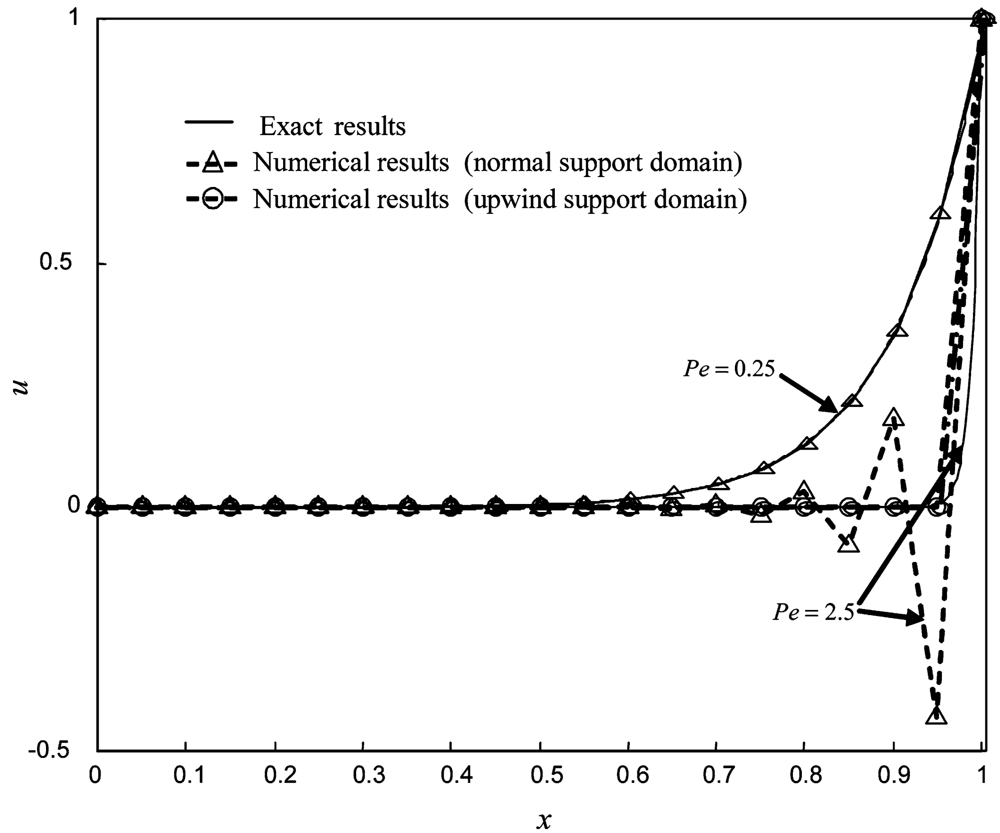

Fig. 3 Results obtained using normal and upwind support domain for $Pe = 0.25$ and $Pe = 2.5$

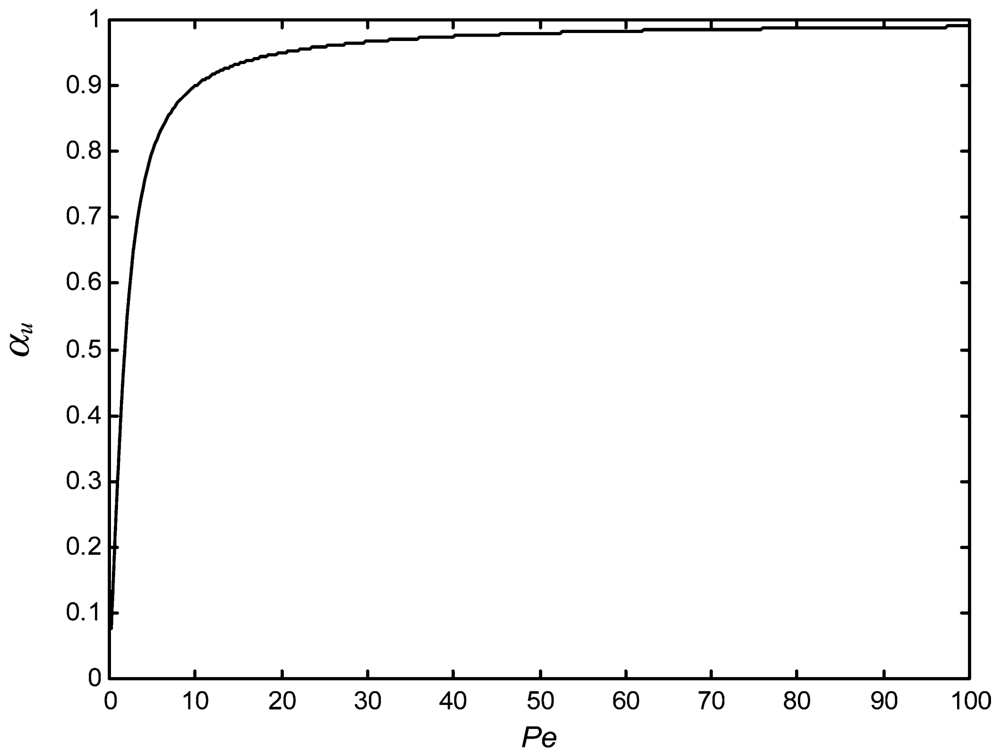

Fig. 4 α_u for different Pe

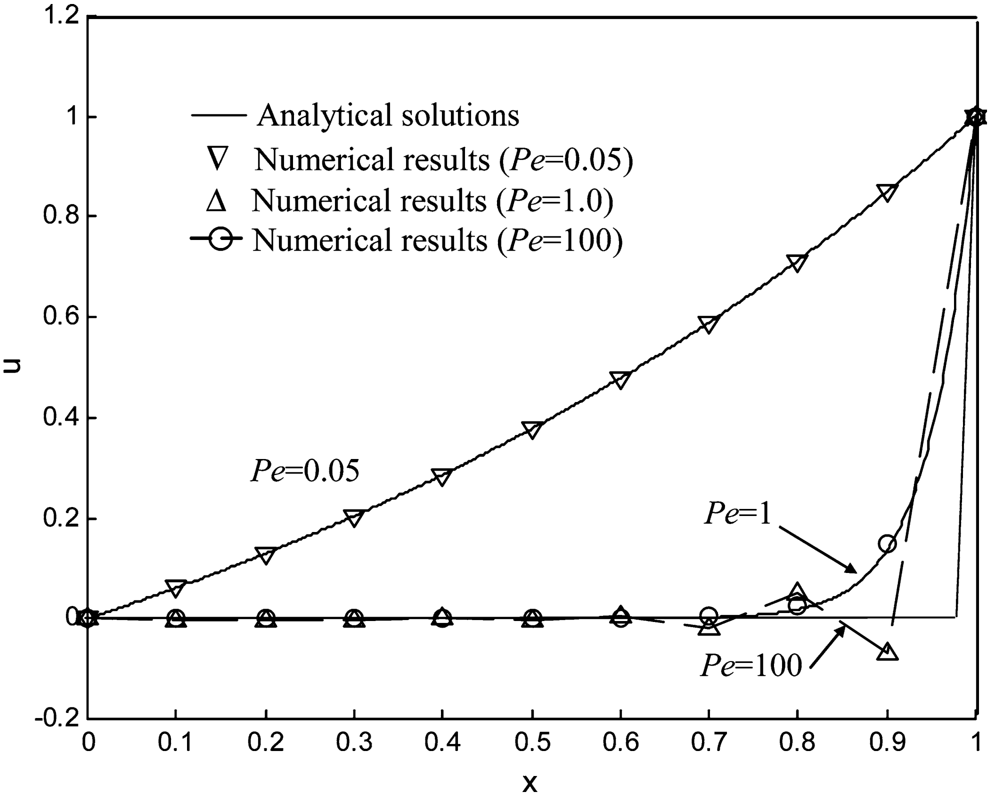

Fig. 5 Results using adaptive upwind support domains

convection-diffusion equation, given by

$$L(u) = \mathbf{v}^{\mathrm{T}} \cdot \nabla u - \nabla^{\mathrm{T}}(\mathbf{D}\nabla u) + \beta u - q(\mathbf{x}) = 0, \quad \text{in } \Omega \quad (7)$$

together with the general boundary conditions:

Neumann boundary condition:

$$L_{b1}(u) = \mathbf{n}^T \mathbf{D} \nabla u + \bar{q}_n = 0 \quad \text{on } \Gamma_{b1} \quad (8)$$

Dirichlet boundary condition:

$$u - \bar{u} = 0 \quad \text{on } \Gamma_{b2} \quad (9)$$

The following equations are satisfied in the internal nodes:

$$R_i = \mathbf{v}^T \cdot \nabla \hat{u}_i^{+} \nabla^T (\mathbf{D}\nabla \hat{u}_i) + \beta \hat{u}_i - q(\mathbf{x}_i) = 0 \quad (10)$$

The following equations are satisfied on Neumann boundary Γ_{b1}:

$$\mathbf{n}^T \mathbf{D} \nabla \hat{u}_i + \bar{q}_n = 0 \quad (11)$$

The following equations are satisfied on Dirichlet boundary Γ_{b2}:

$$\hat{u}_i - \bar{u} = 0 \quad (12)$$

where $\hat{u}_i$ is the approximation u at ith collocation point, and it can be obtained using the radial basis function (RBF) interpolation:

$$\hat{u}_i(\mathbf{x}) = \sum_{i=1}^{n} \Phi_i(\mathbf{x}) u_i \quad (13)$$

where Φ_i is the meshfree shape function that can be constructed by the radial basis function interpolation [14,15], n is number of field nodes used in the local support domain to construct meshfree shape functions.

Using Eqs. (10–13), we can obtain the discretized system equations for the meshfree method.

In the following computing, we consider the problem domain of $(x, y) \in \Omega = [0, 1] \times [0, 1]$, and the coefficients in Eq. (7) are

$$\mathbf{D} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad \mathbf{v} = \left\{ 3 - x,\ 4 - y \right\}, \text{ and } \beta = 1 \quad (14)$$

in which ε is a given constant of diffusion coefficient. The boundary conditions are considered as

$$u|_{\substack{x=0 \\ x=1 \\ y=0 \\ y=1}} = 0 \quad (15)$$

The exact solutions for this problem is given by

$$u^{\text{exact}} = \sin(x) \left(1 - e^{-\frac{2(1-x)}{\varepsilon}}\right) y^2 \left(1 - e^{-\frac{3(1-y)}{\varepsilon}}\right) \quad (16)$$

The velocity distribution is plotted in Fig. 6.

For error analyses, the following error indicators are defined.

$$e = \sqrt{\frac{\sum_{i=1}^{N} \left(u_i^{exact} - u_i^{num}\right)^2}{\sum_{i=1}^{N} (u_i^{exact})^2}}, \quad e_x = \sqrt{\frac{\sum_{i=1}^{N} \left(\frac{\partial u_i^{exact}}{\partial x} - \frac{\partial u_i^{num}}{\partial x}\right)^2}{\sum_{i=1}^{N} \left(\frac{\partial u_i^{exact}}{\partial x}\right)^2}} \quad (17)$$

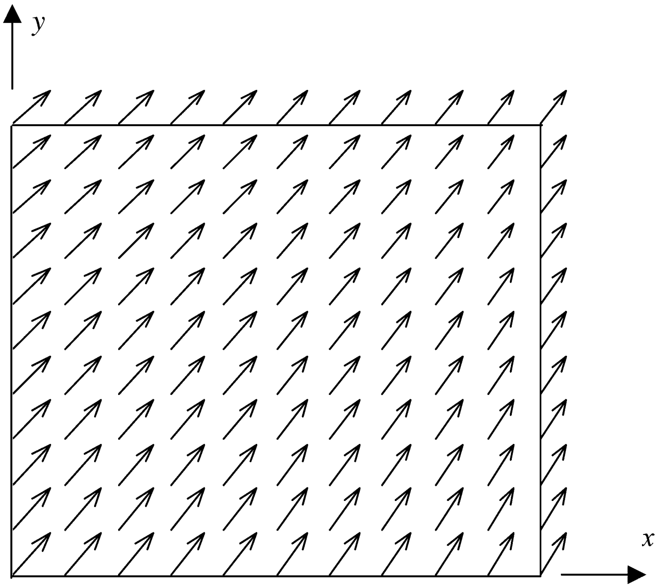

Fig. 6 Velocity distributions

3.2 Numerical results

Two nodal distribution models are used: 21 × 21 (441 nodes) regular nodes and 383 irregularly distributed nodes. The irregular nodes are shown in Fig. 7. The Multi-quadrics (MQ) RBF [14] is used to construct meshfree shape functions. The dimensionless size of support domain is chosen as $\alpha_s = 2.5$. The results obtained by the meshfree collocation method are listed in Table 1. It can be found that the meshfree collocation method obtains very good results using both regular and irregular nodes when ε is large enough (e.g. $\varepsilon > 0.1$) or the Peclet number is small enough, see also Fig. 8(a). However, the error becomes very large when ε is small (i.e. $\varepsilon = 0.01$) or the Peclet number is large, see also Fig. 8(b).

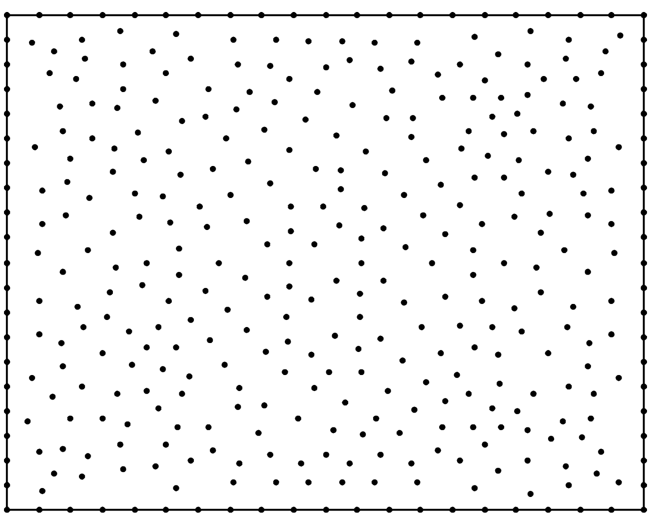

Fig. 7 383 irregular nodes

Table 1 Errors in the numerical results for different ε

ε	441 regular nodes		383 irregular nodes	
	$e_0(\%)$	$e_x(\%)$	$e_0(\%)$	$e_x(\%)$
100.0	0.245	0.966	0.532	1.061
10.0	0.255	0.995	0.546	1.654
1.0	0.346	1.722	1.122	2.476
0.1	1.276	20.069	2.023	26.13
0.01	15.832	80.021	38.17	82.58
0.001	195.345	196.271	243.64	341.87

Note that when ε is very small(e.g., $\varepsilon \leq 0.01$), the problem is convection dominated, for which the instability in the solution (see also Fig. 8(b)) has been well known for many numerical methods including the FDM and FEM, as already discussed in Sect. 2. Figures 8(b) and 9 plot the numerical and exact results for $\varepsilon = 0.01$. It can be found that there is a thin boundary layer near the right-up corner of the problem

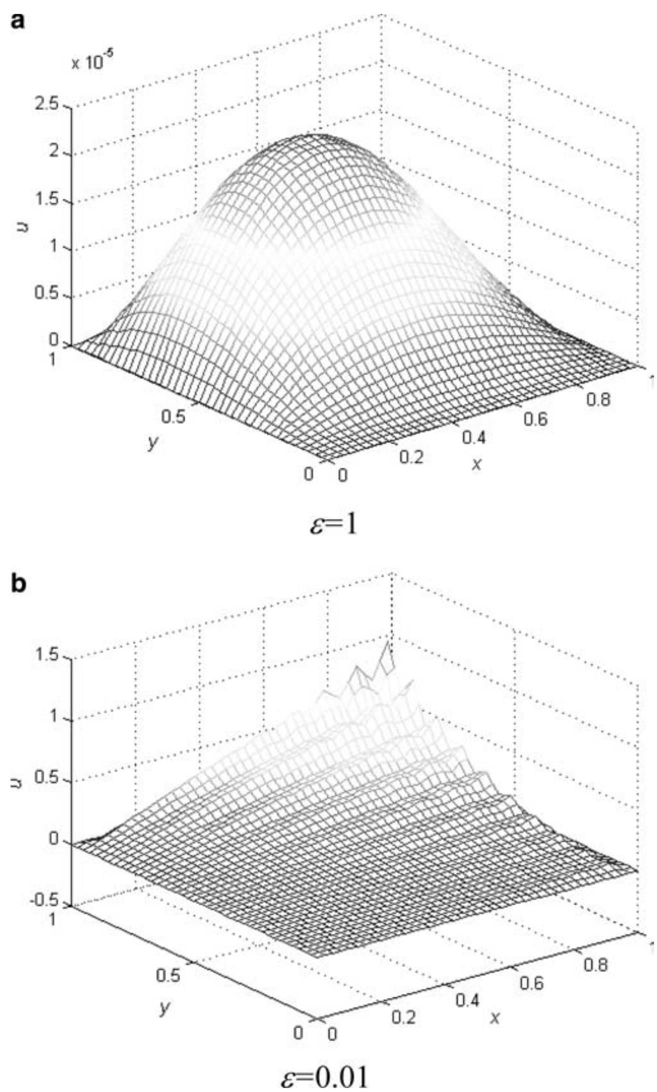

Fig. 8 Numerical results for different ε (a) $\varepsilon = 1$ (b) $\varepsilon = 0.01$

domain when ε is small. The pre.sence of the boundary layer is the major reason for the instability. These techniques developed in Sect. 2 will be used for this two-dimensional problem.

(a) Using enlarged local support domains Table 2 lists the results of different sizes of support domains α_s for the case of $\varepsilon = 0.01$ that is a highly convection dominated case. This table clearly shows that the accuracy of solution improves with the enlargement of the local support domain. It confirms that the enlargement of the support domain can help to stabilize the solution of a two-dimensional convection dominated problem. However, this technique cannot totally solve the instable issue, e.g., when $\alpha_s = 3.0$, the error, e_0, is still 13.55%.

(b) Using adaptive support domains As shown in Fig. 10, the adaptive upwind support domain is defined by

$$d_u = \alpha_u r_s \tag{18}$$

where d_u is the central offset distance against the stream direction from the collocation node as shown in Fig. 10, α_u is the dimensionless coefficient, and r_s is the size of the local support domain. Following the same formulation in Eq. (5), the adaptive upwind support domain for a 2-D problem is defined by assuming the following formula

$$\alpha_u = \coth|\mathbf{Pe}| - \frac{1}{\mathbf{Pe}} \tag{19}$$

where **Pe** is the vector of the local Peclet numbers. In a two-dimensional problem, **Pe** is a vector that has the following form

$$\mathbf{Pe}_i = \begin{Bmatrix} P_{ei}^x \\ P_{ei}^y \end{Bmatrix} = \begin{Bmatrix} \frac{V_x(\mathbf{x}_i)d_c}{2\varepsilon} \\ \frac{V_y(\mathbf{x}_i)d_c}{2\varepsilon} \end{Bmatrix} \tag{20}$$

where V_x and V_y are velocity components in x and y directions, respectively.

The vector of velocity can be expressed as

$$\vec{V}(\mathbf{x}_i) = V_x(\mathbf{x}_i)\vec{i} + V_y(\mathbf{x}_i)\vec{j} \tag{21}$$

Hence, the offset direction for the adaptive support domain can be determined as

$$\vec{n} = \frac{-V_x}{\sqrt{V_x^2 + V_y^2}}\vec{i} + \frac{-V_y}{\sqrt{V_x^2 + V_y^2}}\vec{j} \tag{22}$$

Using Eqs. (18) and (22), the adaptive support domain can be determined.

Table 2 Errors in the numerical results for $\varepsilon = 0.01$ using different local support domains

α_s	$e_0(\%)$	$e_x(\%)$
1.5	22.55	91.90
2.0	18.34	83.70
2.5	15.83	80.02
3.0	13.55	68.87

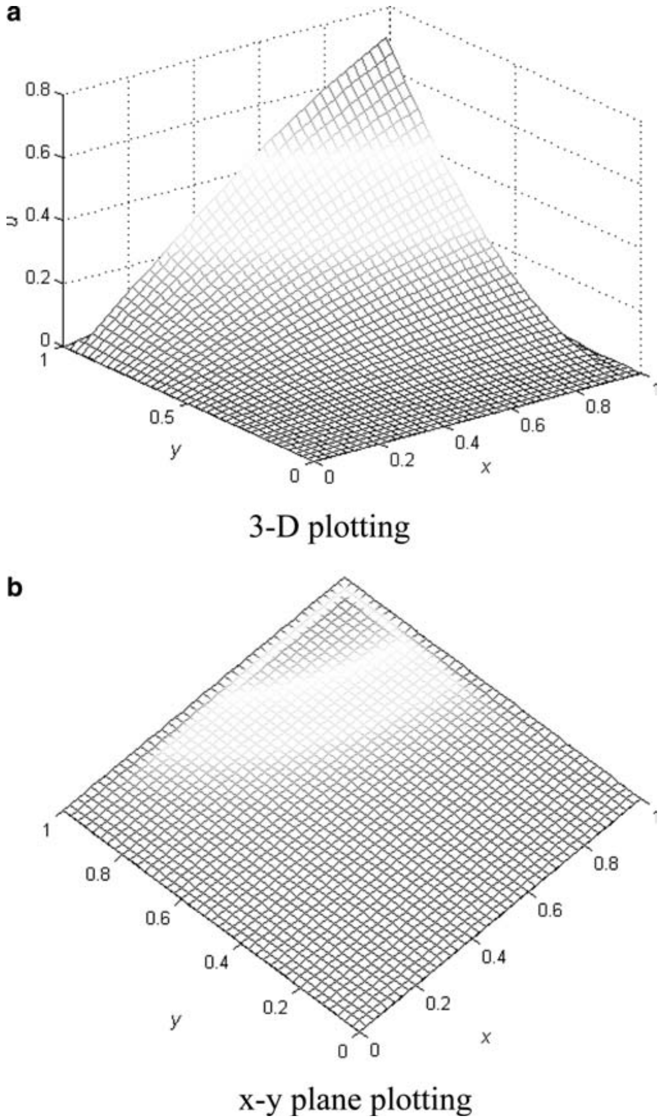

Fig. 9 Exact solution for $\varepsilon = 0.01$ **(a)** 3-D plotting, **(b)** x-y plane plotting

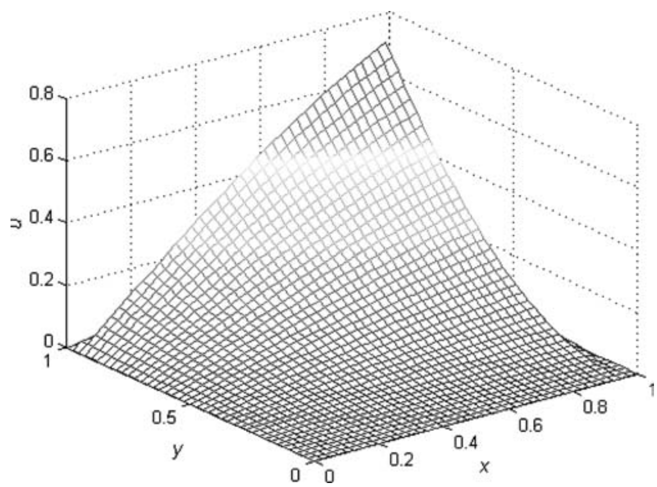

Fig. 10 Construction of an adaptive upwind local support domain using offset distance d_u

Table 3 Errors in the numerical results for different ε

ε	Conventional support domain		Adaptive support domain	
	$e_0(\%)$	$e_x(\%)$	$e_0(\%)$	$e_x(\%)$
100.0	0.245	0.966	0.245	0.966
10.0	0.255	0.995	0.255	0.995
1.0	0.346	1.722	0.345	1.692
0.1	1.276	20.069	1.242	14.956
0.01	15.832	80.021	4.833	28.633
0.001	195.345	196.271	5.923	32.378

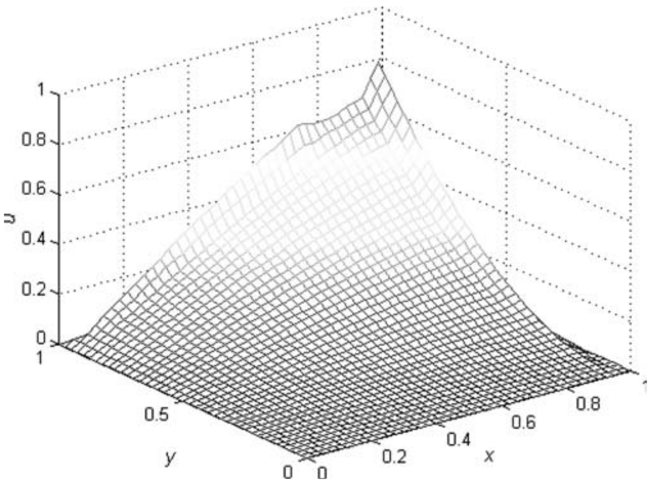

Fig. 11 Numerical solution for ε=0.01

The results of this problem are obtained using the above presented adaptive local support domain and listed in Table 3. The errors using the conventional support domain are also listed in the same table. It can be found from this table that the adaptive upwind support domains can stabilize the solution, and lead to good results for both small (large ε) and large (small ε) *Pe*. Figure 11 also plots the results for ε = 0.01, and it shows better results compared with Fig. 8b. These results prove that using the adaptive local support domain in the meshfree method is a very effective method to solve the convection dominated problems.

(c) Using more nodes near the area with boundary layer We already mentioned that the instable problem of convection-dominated problem is because the presence of the boundary layer. Therefore, we can deliberately distribute more nodes near the area with the boundary layer. Figure 12 presents two nodal distribution models: one uses more nodes on the up-right corner (Fig. 12a), and the other uses more nodes on the down-left corner (Fig. 12b). Using the conventional support domain and $\varepsilon = 0.001$, $e_0 = 18.345\%$ and $e_0 = 48.345\%$ for nodal distribution models I (shown Fig. 12a) and II (shown Fig. 12b), respectively. The model I leads to better results than the model II. From the exact solution, Fig. 9, we can find that the boundary layer locates on the up-right corner of the problem domain. Hence, the model I obtains better results because more nodes are distributed in the boundary layer area.

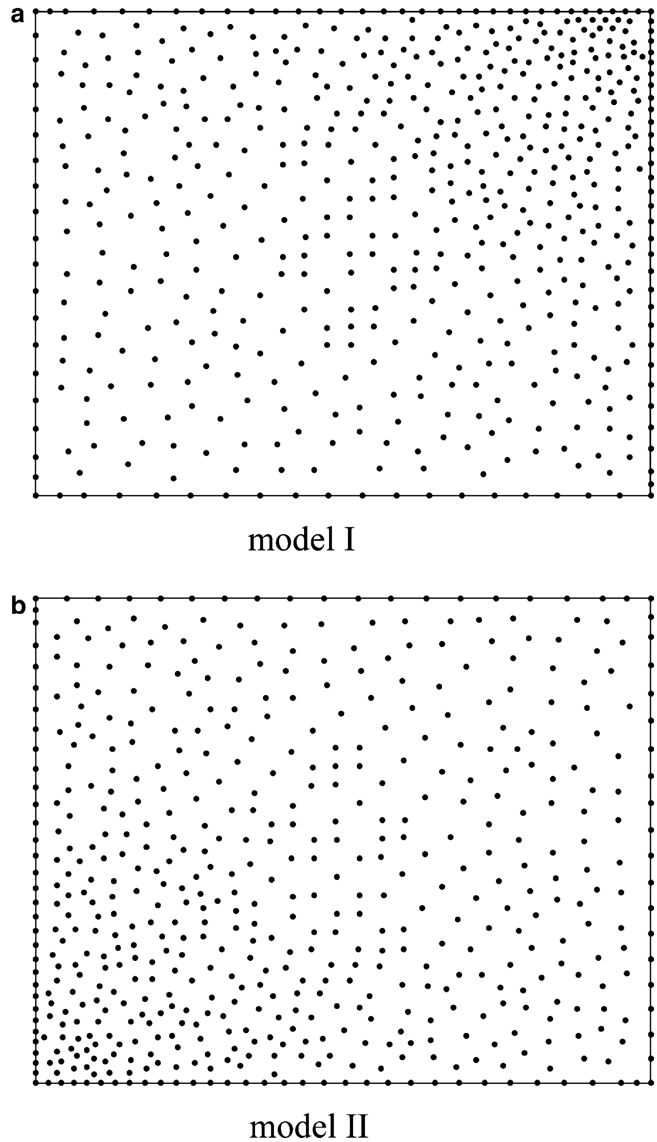

Fig. 12 Two nodal distributions with 552 nodes **(a)** model I, **(b)** model II

(d) Using adaptive analysis Comparing with the conventional FEM or FDM, one of very attractive advantages for the meshfree method is that it is easy to do the adaptive analysis, because no mesh is used in the meshfree method. Hence,

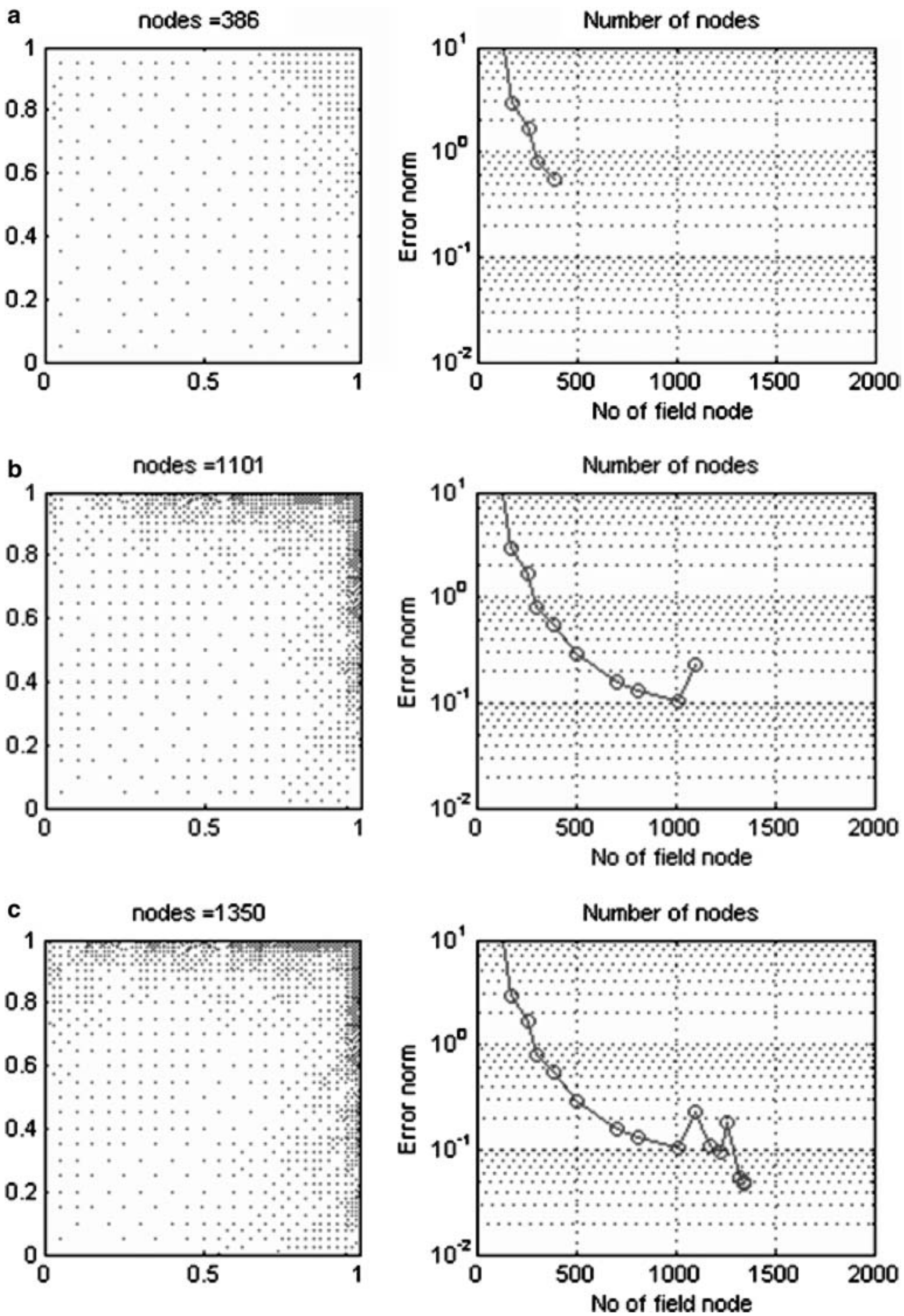

Fig. 13 Nodal distributions and the error norms of three steps in the adaptive analysis **(a)** step 1, **(b)** step 2, **(c)** step 3

the adaptive analysis technique is developed in this paper to solve the convection dominated problem.

In the adaptive analysis, one of the most important issues is to estimate the error and determine the local area to add or reduce nodes. Rather than estimate the error using a higher interpolation scheme, we use the posteriori error estimates proposed by Behrens [3]. This error estimates is a good scheme for the detection of discontinuities of a surface from scattered data, which was appeared in Wu [24]. The posteriori error estimates based on solution interpolation is

$$\eta(\mathbf{x}) = \left| s^{\bar{N}}(\mathbf{x}) - u(\mathbf{x}) \right| \tag{23}$$

where $u(\mathbf{x})$ is the value at node $\mathbf{x}$, and $s(\mathbf{x})$ is the value obtained by an interpolation for value at node $\mathbf{x}$, using the neighbouring set $\bar{N} \equiv N \backslash \{\mathbf{x}\}$ (using the current interpolation domain, but not included the current node). We note that, if $\bar{N} \equiv N$, then $\eta(x) \equiv 0$.

The posteriori error estimates provide an estimation to determine the region of the domain need to be added/reduced nodes. The value $\eta(\mathbf{x})$ is small whenever the reproduction quality of u around $\mathbf{x}$ is good, and in contrast, a high value of $\eta(\mathbf{x})$ indicates that u is subject to strong variation locally around $\mathbf{x}$.

In order to balance the accuracy of the solution and the computational cost, we will add new nodes into regions that have higher $\eta(\mathbf{x})$ values (refinement), or remove nodes from regions that have the small value of $\eta(\mathbf{x})$(coarsening). Let

$$\eta^* = \max_{\mathbf{x}\in\Omega} \eta(\mathbf{x}) \qquad (24)$$

Consider two tolerance values to be satisfied $0 < \theta_c < \theta_r < 1$. This two parameters: θ_c is known as the coarsening parameters and θ_r is known as the refinement parameters.

- A node $\xi \in \Omega$ will be refined, if and only if $\eta(\xi) > \theta_r \cdot \eta^*$, and
- A node $\xi \in \Omega$ will be coarsened, if and only if $\eta(\xi) < \theta_c \cdot \eta^*$.

By the fact of $\theta_c < \theta_r$, we can make sure that a node can only be either coarsened or refined at a time.

Using above discussed adaptive analysis algorithm, the convection dominated problem with $\varepsilon = 0.01$ is adaptively analyzed. Figure 13 shows results of nodal distribution and error e_0 in three steps for the adaptive analysis. The initial nodal distribution is 11×11 regular nodes. It can be found that more nodes are automatically added in the area with the thin boundary layer and the results converge although the convergent procedure is not always monotonous. It demonstrates that the adaptive analysis for the convection dominated problems can be effectively fulfilled using the meshfree methods and leads to very good results.

4 Conclusions

In this paper, the meshfree method is used to solve convection dominated problem. Several techniques are newly developed to overcome the instability when the convection dominated problems are solved by the meshfree method. These techniques are applied and demonstrated in both one- and two-dimensional problems. The following conclusions can be drawn through the studies in this paper.

(1) Comparing with the conventional FDM and FEM, the meshfree method has a very attractive advantage in solving the convection dominated problems because it can easily overcome the instability problem by using proper techniques.
(2) The techniques developed in this paper can overcome the instability issues in convection dominated problems analyzed by the meshfree method. These techniques include: the adaptive local support domain, the enlargement of the local support domain, using more nodes in the area with the boundary layer, and using the adaptive analysis.
(3) In these techniques, using the adaptive local support domain is the most effective method and it is very easy to use because of the freedom of selecting the support domain in a meshfree method.
(4) The adaptive analysis for the convection dominated problems can be effectively fulfilled using the meshfree methods and leads to very good results.
(5) In some cases, one technique may not solve the instability issue totally. Several techniques can be combined together to solve it. For example, if the Peclet number is very large, we can use the adaptive analysis, and, in each analysis step, the adaptive local support domains is used to construct the meshfree shape functions.
(6) The meshfree collocation methods are used in the above studies. Nearly all meshfree methods have the same feature in terms of determining the local support domains. Therefore, in solving a convection dominated problem, the similar conclusions can be drawn for other meshfree methods. The techniques developed in this paper can also be used in other meshfree methods.
(7) Numerical examples are used to demonstrate the developed techniques in this paper. More further research is required to improve these techniques and apply them to other practical convection dominated problems in engineering.

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