Computational Mechanics 30 (2002) 48–64 © Springer-Verlag 2002
DOI 10.1007/s00466-002-0366-3

A simple orthotropic finite elasto–plasticity model based on generalized stress–strain measures

J. Schröder, F. Gruttmann, J. Löblein

Abstract In this paper we present a formulation of orthotropic elasto-plasticity at finite strains based on generalized stress–strain measures, which reduces for one special case to the so-called Green–Naghdi theory. The main goal is the representation of the governing constitutive equations within the invariant theory. Introducing additional argument tensors, the so-called structural tensors, the anisotropic constitutive equations, especially the free energy function, the yield criterion, the stress-response and the flow rule, are represented by scalar-valued and tensor-valued isotropic tensor functions. The proposed model is formulated in terms of generalized stress–strain measures in order to maintain the simple additive structure of the infinitesimal elasto-plasticity theory. The tensor generators for the stresses and moduli are derived in detail and some representative numerical examples are discussed.

Keywords Anisotropy, Finite elasto–plasticity, Generalized measures

1 Introduction

The complex mechanical behaviour of elasto-plastic materials at large strains with an oriented internal structure can be described with tensor-valued functions in terms of several tensor variables, usually deformation tensors and additional structural tensors. General invariant forms of the constitutive equations lead to rational strategies for the modelling of the complex anisotropic response functions. Based on representation theorems for tensor functions the general forms can be derived and the type and minimal number of the scalar variables entering the constitutive equations can be given. For an introduction to the invariant formulation of anisotropic constitutive equations based on the concept of structural tensors and their representations as isotropic tensor functions see Spencer [26], Boehler [3], Betten [2] and for some specific model problems see also Schröder [19]. These invariant forms of the constitutive equations satisfy automatically the symmetry relations of the considered body. Thus, they are automatically invariant under coordinate transformations with elements of the material symmetry group. For the representation of the scalar-valued and tensor-valued functions the set of scalar invariants, the integrity bases, and the generating set of tensors are required. For detailed representations of scalar- and tensor-valued functions we refer to Wang [31, 32], Smith [23, 24]. The integrity bases for polynomial isotropic scalar-valued functions are given by Smith [23] and the generating sets for the tensor functions are derived by Spencer [26]. In this work we formulate a model for anisotropic elasto-plasticity at large strains following the line of Papadopoulos and Lu [16]. Here we use a representation of the free energy function and the flow rule which fulfill the material symmetry conditions with respect to the reference configuration a priori.

Papadopoulos and Lu [16] proposed a rate-independent finite elasto-plasticity model within the framework of a Green–Naghdi type theory, see e.g. Green and Naghdi [5], using a family of generalized stress–strain measures. An extension of this work to anisotropy effects is given in Papadopoulos and Lu [17], where the algorithmic treatment deals with nine of the twelve common material symmetry groups. Furthermore, the authors develop special return algorithms for some anisotropy classes. Generalized measures have been used e.g. by Doyle and Ericksen [4], Seth [20], Hill [8], Ogden [15] and Miehe and Lambrecht [13] in the case of nonlinear elasticity.

For an overview of the developments in the theory and numerics of anisotropic materials at finite strains we refer to the papers published in the special issue of the International Journal of Solids and Structures Vol. 38 (2001), EUROMECH Colloquium 394, and the references therein. In the following we discuss only a few contributions in this field. A yield criterion which describes the plastic flow of orthotropic metals has been first proposed by Hill [7]. A numerical study on integration algorithms for the latter model at small strains and especially the evaluation of iso-error maps is given in De Borst and Feenstra [14]. A constitutive frame for the formulation of large strain anisotropic elasto-plasticity based on the notion of a plastic metric is proposed by Miehe [11]. A consistent Eulerian-type constitutive elasto-plasticity theory with general isotropic and kinematic hardening has been developed by Xiao et al. [35], combining the additive and multiplicative decomposition of the stretch tensor and the deformation gradient. An anisotropic plasticity model at large strains

Received: 2 April 2002 / Accepted: 11 September 2002

J. Schröder (✉), J. Löblein
Institut für Mechanik, FB 10, Universität Essen,
Universitätsstr. 15, 45117 Essen, Germany
e-mail: j.schroeder@uni-essen.de

F. Gruttmann
Institut für Statik, FB 13, Technische Universität Darmstadt,
Alexanderstr. 7, 64283 Darmstadt, Germany

taking into account the postulate of Il'iushin is proposed by Tsakmakis [29] and specialized for transverse isotropy in Häusler et al. [6]. The main ingredient is the introduction of an evolution equation for the rotation of the preferred anisotropy directions. A further approach to describe the anisotropic elasto-plastic material behaviour is based on the homogenization of polycrystalline mesostructures, in this context see Miehe et al. [12] and references therein.

In this paper a theory of finite elasto-plastic strains using the notion of generalized stress and strain measures is presented. With the essential assumption of the additive decomposition of the generalized strains the structure of the constitutive equations corresponds to the linear theory, in this context see also [16, 17]. The model has been implemented in a brick-type shell element of Klinkel [9], see also Wagner et al. [30] and references therein. The element provides an interface to three-dimensional material laws; it is implemented in the program FEAP [36]. Due to special interpolation techniques based on mixed variational principles, the element is applicable for the numerical analysis of thin structures. For a comparison of three dimensional continuum elements with shell elements for finite plasticity problems see Wriggers et al. [33]. Furthermore, the interface of the program ABAQUS [1] is used for an implementation. We discuss three representative numerical examples: the necking of a circular bar; the load carrying behaviour of a conical shell; and the deep drawing process of a sheet metal plate.

The new aspects and essential features of the formulation are summarized as follows:

(i) In our formulation we exploit the fact that the generalized strains and the Green–Lagrange strains are characterized by the same eigenvectors. Using this feature one can extend results of Ogden [15], which have been derived in the context of isotropic elasticity. In [15] elastic stored energies as functions of the principal stretches are considered and first and second derivatives are derived.

(ii) Based on these features the components of the projection tensors with respect to the principal axes are derived. Using the fourth-order and sixth-order transformation tensors one can evaluate the second Piola–Kirchhoff stress tensor and associated linearization. Furthermore, explicit matrix representations are given. The expressions are simple and thus allow a very efficient finite element implementation.

(iii) The constitutive equations for orthotropy are formulated in an invariant setting. So-called structural tensors describe the privileged directions of the material. For the yield condition with isotropic and kinematic hardening we introduce a simple representation in terms of the invariants of the deviatoric part of the relative stress tensor and of the structural tensors.

(iv) The set of constitutive equations is solved applying a so-called general return method, see e.g. Simo and Hughes [22] and Taylor [27]. Additionally, the condition of plastic incompressibility is fulfilled by a correction of the inelastic part of the generalized strains.

(v) For the numerical examples, where we compare different stress–strain measures, an identification of all models is performed with respect to a given reference uniaxial tension test. Thus all formulations in the generalized measures reflect the same (given) macroscopic stress–strain characteristics, like linear isotropic hardening. This can be seen as a (minimum) essential feature in order to get physically comparable results for the phenomenological quantities. In this context see e.g. the discussions in Hill [8].

2 Kinematics and generalized stress–strain measures

The body of interest in the reference configuration is denoted with $\mathcal{B} \subset \mathbb{R}^3$, parametrized in $\mathbf{X}$ and the current configuration with $\mathcal{S} \subset \mathbb{R}^3$, parametrized in $\mathbf{x}$. The nonlinear deformation map $\boldsymbol{\varphi}_t : \mathcal{B} \to \mathcal{S}$ at time $t \in \mathbb{R}_+$ maps points $\mathbf{X} \in \mathcal{B}$ onto points $\mathbf{x} \in \mathcal{S}$. The deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F}(\mathbf{X}) := \operatorname{Grad} \boldsymbol{\varphi}_t(\mathbf{X}) \tag{1}$$

with the Jacobian $J(\mathbf{X}) := \det \mathbf{F}(\mathbf{X}) > 0$. The index notation of $\mathbf{F}$ is $F^a_A := \partial x^a / \partial X^A$. An important strain measure, the right Cauchy–Green tensor, is defined by

$$\mathbf{C} := \mathbf{F}^T\mathbf{F} \quad \text{with } C_{AB} = F^a_A F^b_B g_{ab} \ , \tag{2}$$

where g_{ab} denotes the coefficients of the covariant metric tensor $\mathbf{g}$ in the current configuration.

2.1 Generalized stress–strain measures

Following e.g. Doyle and Ericksen [4], Seth [20], Hill [8] and Ogden [15] we define the generalized strain measures

$$\mathbf{E}^{(m)} := \begin{cases} \frac{1}{2m}(\mathbf{C}^m - \mathbf{1}) & \text{for } m \neq 0 \\ \frac{1}{2}\ln[\mathbf{C}] & \text{for } m = 0 \end{cases} \ , \tag{3}$$

where $\mathbf{1}$ denotes the second-order unit tensor. Let $\lambda_A, A = 1,2,3$ be the eigenvalues and $\mathbf{N}^A$, $A = 1,2,3$ the eigenvectors of $\mathbf{C}$, then we arrive at

$$\begin{aligned} \mathbf{C}^m &= \sum_{A=1}^{3} \lambda_A^m \, \mathbf{N}^A \otimes \mathbf{N}^A \\ \ln[\mathbf{C}] &= \sum_{A=1}^{3} \ln \lambda_A \, \mathbf{N}^A \otimes \mathbf{N}^A \ . \end{aligned} \tag{4}$$

In this context we remark that formulations of isotropic metal elasto–plasticity based on logarithmic strains are successfully used e.g. in Peric et al. [18]. Furthermore, let the generalized stress measure $\mathbf{S}^{(m)}$ be the work conjugate to $\mathbf{E}^{(m)}$. To simplify the notation we write $\mathbf{S} := \mathbf{S}^{(1)}$ for the symmetric Second Piola–Kirchhoff stress tensor and $\mathbf{E} := \mathbf{E}^{(1)}$ for the Green–Lagrangian strain tensor. The related work conjugate stress measures are defined by the stress power density

$$\begin{aligned} \dot{w} &= \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S}^{(m)} : \dot{\mathbf{E}}^{(m)} = \mathbf{S}^{(m)} : 2\partial_C(\mathbf{E}^{(m)}) : \dot{\mathbf{E}} \\ &= \mathbf{S}^{(m)} : \mathbb{P}_E : \dot{\mathbf{E}} \ . \end{aligned} \tag{5}$$

This leads to the transformation rule for the second Piola–Kirchhoff stress tensor

$$\mathbf{S} = \mathbf{S}^{(m)} : \mathbb{P}_E \quad \text{with } \mathbb{P}_E = 2\partial_C \mathbf{E}^{(m)} \ . \tag{6}$$

For the following derivations we exploit the fact that $\mathbf{E}^{(m)}$ and $\mathbf{E}$ have the same eigenvectors. An illustration of this is given in Fig. 1. Now, following Ogden [15] we arrive at the explicit expression for the fourth-order transformation tensor

$$\mathbb{P}_E = \sum_{A=1}^{3}\sum_{B=1}^{3} P_{AABB}\, \mathbf{N}^A \otimes \mathbf{N}^A \otimes \mathbf{N}^B \otimes \mathbf{N}^B + \sum_{A=1}^{3}\sum_{B \neq A}^{3} P_{ABAB}\, (\mathbf{N}^A \otimes \mathbf{N}^B) \otimes (\mathbf{N}^A \otimes \mathbf{N}^B + \mathbf{N}^B \otimes \mathbf{N}^A) \ . \tag{7}$$

With the eigenvalues $E_A = \frac{1}{2}(\lambda_A - 1)$ of $\mathbf{E}$ and the eigenvalues of the generalized strain measures

$$E_A^{(m)} := \begin{cases} \frac{1}{2m}(\lambda_A^m - 1) & \text{for } m \neq 0 \\ \frac{1}{2}\ln[\lambda_A] & \text{for } m = 0 \end{cases} , \tag{8}$$

we derive the non-zero components of $\mathbb{P}_E$:

$$P_{AABB} = \partial_{E_B} E_A^{(m)} = \lambda_A^{m-1}\delta_{AB}$$

$$P_{ABAB} = \gamma_{AB}^{(m)} = \begin{cases} \frac{E_A^{(m)} - E_B^{(m)}}{2(E_A - E_B)} & \text{for } \lambda_A \neq \lambda_B \\ \frac{1}{2}\lambda_A^{m-1} & \text{for } \lambda_A = \lambda_B \end{cases} . \tag{9}$$

The result for two equal eigenvalues is obtained applying the rule of L'Hospital, in this context see e.g. Miehe and Lambrecht [13].

2.2 Matrix representation

Let $\bar{\mathbf{S}} = [S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23}]^{\mathrm{T}}$ and $\mathbf{P}$ be the matrix representations of $\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbb{P}_E$, respectively. Thus $(6)_1$ and (7) lead to

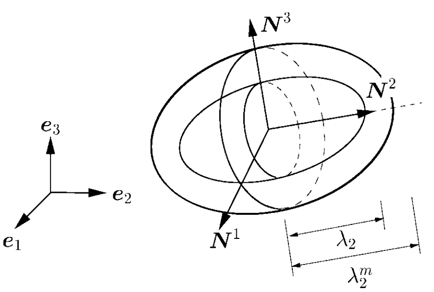

Fig. 1. Visualization of the coaxiality of $\mathbf{E}$ and $\mathbf{E}^{(m)}$

$$\bar{\bar{\mathbf{S}}} = \mathbf{P}^{\mathrm{T}}\bar{\mathbf{S}} \quad \text{and} \quad \mathbf{P} = \mathbf{T}^{\mathrm{T}}\mathbf{L}_1\mathbf{T} \ , \tag{10}$$

with the Cartesian components of the generalized stress tensor $\bar{S}_{ij} = \mathbf{e}_i \cdot \mathbf{S}^{(m)} \cdot \mathbf{e}_j$ organized in a vector $\bar{\mathbf{S}} = [\bar{S}_{11}, \bar{S}_{22}, \bar{S}_{33}, 2\bar{S}_{12}, 2\bar{S}_{13}, 2\bar{S}_{23}]^{\mathrm{T}}$. The $\mathbf{e}_i | i = 1,2,3$ denote the fixed Cartesian basis with respect to the reference configuration. The matrix $\mathbf{T}$ contains the components $N_j^A = \mathbf{N}^A \cdot \mathbf{e}_j$ of the eigenvectors $\mathbf{N}^A$

$$\mathbf{T} = \begin{bmatrix} (N_1^1)^2 & (N_2^1)^2 & (N_3^1)^2 & N_1^1N_2^1 & N_1^1N_3^1 & N_2^1N_3^1 \\ (N_1^2)^2 & (N_2^2)^2 & (N_3^2)^2 & N_1^2N_2^2 & N_1^2N_3^2 & N_2^2N_3^2 \\ (N_1^3)^2 & (N_2^3)^2 & (N_3^3)^2 & N_1^3N_2^3 & N_1^3N_3^3 & N_2^3N_3^3 \\ 2N_1^1N_1^2 & 2N_2^1N_2^2 & 2N_3^1N_3^2 & N_1^1N_2^2 + N_2^1N_1^2 & N_1^1N_3^2 + N_3^1N_1^2 & N_2^1N_3^2 + N_3^1N_2^2 \\ 2N_1^1N_1^3 & 2N_2^1N_2^3 & 2N_3^1N_3^3 & N_1^1N_2^3 + N_2^1N_1^3 & N_1^1N_3^3 + N_3^1N_1^3 & N_2^1N_3^3 + N_3^1N_2^3 \\ 2N_1^2N_1^3 & 2N_2^2N_2^3 & 2N_3^2N_3^3 & N_1^2N_2^3 + N_2^2N_1^3 & N_1^2N_3^3 + N_3^2N_1^3 & N_2^2N_3^3 + N_3^2N_2^3 \end{bmatrix} . \tag{11}$$

The matrix $\mathbf{L}_1$ is of diagonal form

$$\mathbf{L}_1 = \mathrm{diag}\left[\lambda_1^{m-1}, \lambda_2^{m-1}, \lambda_3^{m-1}, \gamma_{12}^{(m)}, \gamma_{13}^{(m)}, \gamma_{23}^{(m)}\right] , \tag{12}$$

where the components $\gamma_{AB}^{(m)}$ are defined in (9). The compact matrix formulation of $\mathbb{P}_E$, see $(10)_2$, (11) and (12), provides a very efficient finite element implementation.

3 Constitutive framework

In this section, we point out the main components for a simple finite anisotropic plasticity model, see also Papadopoulos and Lu [16, 17]. It consists of an additive decomposition of the generalized strain tensor in "elastic" and plastic parts, with $\mathbf{E}^{e(m)} := \mathbf{E}^{(m)} - \mathbf{E}^{p(m)}$. For the calculation of the generalized stresses, the back stresses $\boldsymbol{\beta}$ and the stress-like isotropic hardening variable ξ we assume the existence of a free energy function ψ, which is decoupled additively in an elastic part ψ^e, a plastic part $\psi^{p,i}$ due to isotropic hardening and $\psi^{p,k}$ due to kinematic hardening. The yield criterion Φ is formulated in terms of the relative stresses $\boldsymbol{\Sigma} := \mathbf{S}^{(m)} - \boldsymbol{\beta}$ and the isotropic hardening stress ξ. For the evolution of the plastic strains $\mathbf{E}^{p(m)}$ and of the internal variables $\boldsymbol{\alpha}, e^{p(m)}$ we use Φ as a plastic potential. The loading-unloading conditions in Kuhn–Tucker form complete the model. The constitutive equations are summarized as follows:

Additive split	$\mathbf{E}^{(m)} = \mathbf{E}^{e(m)} + \mathbf{E}^{p(m)}$
Free energy	$\psi = \psi^e(J_1,\ldots,J_7) + \psi^{p,i}(e^{p(m)}) + \psi^{p,k}(\boldsymbol{\alpha})$
Generalized stresses	$\mathbf{S}^{(m)} = \partial_{E^{e(m)}}\psi^e$
Back stress	$\boldsymbol{\beta} = \partial_{\alpha}\psi^{p,k}$
Isotropic hardening	$\xi = \partial_{e^{p(m)}}\psi^{p,i}$
Relative stresses	$\boldsymbol{\Sigma} = \mathbf{S}^{(m)} - \boldsymbol{\beta}$
Yield criterion	$\Phi = \hat{\Phi}(\boldsymbol{\Sigma}, {}^i\mathbf{M}, \xi) = \hat{\Phi}(I_1,\ldots,I_6,\xi)$
Associative flow rule	$\dot{\mathbf{E}}^{p(m)} = \lambda\partial_{S^{(m)}}\Phi$
Evolution of $\boldsymbol{\alpha}$	$\dot{\boldsymbol{\alpha}} = -\lambda\partial_{\beta}\Phi$
Evolution of $e^{p(m)}$	$\dot{e}^{p(m)} = \sqrt{\frac{2}{3}}\|\|\dot{\mathbf{E}}^{p(m)}\|\|$
Optimization condition	$\lambda \geq 0, \Phi \leq 0, \lambda\Phi = 0$

(13)

For the explicit formulation of invariant constitutive equations the representation theorems of tensor functions are used. The governing constitutive equations have to represent the material symmetries of the body of interest a priori. Furthermore, the minimal number of independent scalar variables which has to enter the constitutive expression is required. For a detailed discussion of this topic we refer to Boehler [3]. The structural tensors ${}^i\mathbf{M}$, the invariants $J_i|i = 1,\ldots,7$ used in free energy and $I_i|i = 1,\ldots,6$ entering the yield criterion are defined in the following sections.

3.1 Invariance conditions

Following $(13)_2$ we now focus on hyperelastic materials, i.e. we assume the existence of a so-called Helmholtz free energy function ψ^e in terms of the generalized strain measures. Assume ψ^e to be a function solely in the deformation gradient, i.e. $\psi^e = \psi^e(\mathbf{F},\bullet)$. The argument $(\bullet)$ in the free energy function denotes additional tensor arguments. They characterize the class of anisotropy of the material; we discuss this topic in the following sections. We consider perfect elastic materials, that means that the internal dissipation $\mathcal{D}_{\text{int}}$ is zero for every admissible process. The principle of material frame indifference requires the invariance of the constitutive equation under superimposed rigid body motions onto the current configuration, i.e. under the mapping $\mathbf{x} \to \mathbf{Qx}$ the condition $\psi^e(\mathbf{F}) = \psi^e(\mathbf{QF})$ holds $\forall \mathbf{Q} \in \mathrm{SO}(3)$. For the stress response this principle leads to the well known reduced constitutive equations $\psi^e = \hat{\psi}^e(\mathbf{C})$ which fulfill a priori the principle of material objectivity. In the case of anisotropy we introduce a material symmetry group $\mathcal{G}_k$ with respect to a local reference configuration, which characterizes the anisotropy class of the material. The elements of $\mathcal{G}_k$ are denoted by the unimodular tensors ${}^i\mathbf{Q}|i = 1,\ldots,n$. The concept of material symmetry requires that the response be invariant under transformations on the reference configuration with elements of the symmetry group

$$\hat{\psi}^e(\mathbf{FQ}) = \hat{\psi}^e(\mathbf{F}) \quad \forall \mathbf{Q} \in \mathcal{G}_k, \mathbf{F} \ . \tag{14}$$

We say that the function ψ is a $\mathcal{G}_k$-invariant function. Without any restrictions we set $\mathcal{G}_k \subset \mathrm{SO}(3)$, where $\mathrm{SO}(3)$ characterizes the special orthogonal group, because only invariants of (absolute) second order tensors appear in (13), see e.g. Spencer [26]. Based on the mapping $\mathbf{X} \to \mathbf{Q}^{\mathrm{T}}\mathbf{X}$ for arbitrary rotation tensors $\mathbf{Q} \in \mathrm{SO}(3)$ we have, in view of a coordinate free representation to fulfill the transformation rule $\mathbf{Q}^{\mathrm{T}}\mathbf{S}(\mathbf{F},\bullet)\mathbf{Q} = \mathbf{S}(\mathbf{FQ},\bullet)\ \forall\, \mathbf{Q} \in \mathrm{SO}(3)$. If we assume the free energy function to be a function of the generalized strain tensor $\hat{\psi}^e(\mathbf{E}^{e(m)})$, we obtain

$$\begin{aligned}\mathbf{S} &= 2\partial_C \psi^e(\mathbf{E}^{e(m)}) \\ &= \mathbf{S}^{(m)} : 2\partial_C \mathbf{E}^{e(m)} \quad \text{with } \mathbf{S}^{(m)} := \partial_{E^{e(m)}} \psi(\mathbf{E}^{e(m)}) \ .\end{aligned} \tag{15}$$

The invariance requirement with respect to the material symmetry group is then given by

$$\hat{\psi}^e(\mathbf{Q}^{\mathrm{T}}\mathbf{E}^{e(m)}\mathbf{Q}) = \hat{\psi}^e(\mathbf{E}^{e(m)}) \quad \forall \mathbf{Q} \in \mathcal{G}_k, \mathbf{E}^{e(m)} \ . \tag{16}$$

Thus it is clear that material symmetries impose several restrictions on the form of the constitutive functions of the anisotropic material. In order to work out the explicit restrictions for the individual symmetry groups or more reasonably to point out general forms of the functions which fulfill these restrictions it is necessary to use representation theorems for anisotropic tensor functions. Similar arguments are used for the construction of the coordinate invariant yield condition $\Phi = \hat{\Phi}(\boldsymbol{\Sigma}, \xi)$, see below.

3.2 Representation of anisotropic tensor functions

In order to construct an isotropic tensor function for the anisotropic constitutive behaviour we have to extend the $\mathcal{G}_k$-invariant functions to functions which are invariant under the special orthogonal group. For this purpose we introduce the so-called structural tensors, which reflect the symmetry group of the considered material. The symmetry group of a material is defined by (14). Here, we consider orthotropic material which can be characterized by three symmetry planes, where the anisotropy can be described by some second-order tensors ${}^i\mathbf{M}|i = 1,2,3$ defined with respect to the reference configuration. Let $\mathcal{G}_k$ be the invariance group of the structural tensors, i.e.

$$\mathcal{G}_k = \{\mathbf{Q} \in \mathrm{SO}(3),\ \mathbf{Q}^{\mathrm{T}\,i}\mathbf{MQ} = {}^i\mathbf{M} \quad \text{for } i = 1,2,3\} \ . \tag{17}$$

Thus the invariance group of the structural tensors characterizes the class of anisotropy. Figure 2 illustrates the preferred directions ${}^i\mathbf{a}$ and ${}^i\tilde{\mathbf{a}} := \mathbf{F}\,{}^i\mathbf{a}$ with respect to the reference and current configuration, respectively.

With this definition we arrive at a further reduction of the constitutive equation of the form

$$\begin{aligned}\psi &= \hat{\psi}(\mathbf{E}^{e(m)}, {}^i\mathbf{M}|i = 1,2,3) \\ &= \hat{\psi}(\mathbf{Q}^{\mathrm{T}}\mathbf{E}^{e(m)}\mathbf{Q}, \mathbf{Q}^{\mathrm{T}\,i}\mathbf{MQ}|i = 1,2,3) \quad \forall \mathbf{Q} \in \mathrm{SO}(3) \ .\end{aligned} \tag{18}$$

This is the definition of an isotropic scalar-valued tensor function in the arguments $(\mathbf{E}^{e(m)}, {}^1\mathbf{M}, {}^2\mathbf{M}, {}^3\mathbf{M})$, which fulfills the above postulated transformation rule for the stresses. It should be noted that the function is anisotropic with respect to $\mathbf{E}^{e(m)}$.

Remark There are further material symmetries which are finite sub-groups of SO(3), for the different crystal classes,

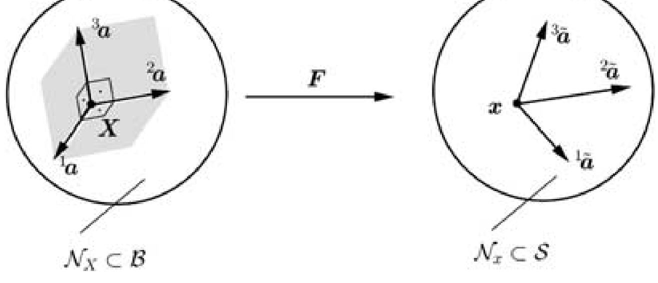

Fig. 2. Preferred directions ${}^i\mathbf{a}$ and ${}^i\tilde{\mathbf{a}}$ in a neighborhood $\mathcal{N}$ of the material point $\mathbf{X}$ and $\mathbf{x}$ defined with respect to the reference $\mathcal{B}$ and the actual configuration $\mathcal{S}$, respectively

see e.g. Smith et al. [25], Spencer [26] and the references therein.

3.3
Free energy function and related polynomial basis

The material symmetry group of the considered orthotropic material is defined by

$$\mathcal{G}_o := \{\pm\mathbf{I}; \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\} \ , \tag{19}$$

where $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ are the reflections with respect to the basis planes $({}^2\mathbf{a}, {}^3\mathbf{a})$, $({}^3\mathbf{a}, {}^1\mathbf{a})$ and $({}^1\mathbf{a}, {}^2\mathbf{a})$, respectively. Here, $({}^1\mathbf{a}, {}^2\mathbf{a}, {}^3\mathbf{a})$ represents an orthonormal privileged frame. Based on this, we obtain for this symmetry group the three structural tensors

$${}^1\mathbf{M} := {}^1\mathbf{a} \otimes {}^1\mathbf{a}, \quad {}^2\mathbf{M} := {}^2\mathbf{a} \otimes {}^2\mathbf{a} \quad \text{and} \quad {}^3\mathbf{M} := {}^3\mathbf{a} \otimes {}^3\mathbf{a} \ , \tag{20}$$

which represent the orthotropic material symmetry. Due to the fact that the sum of the three structural tensors yields $\sum_{i=1}^{3} {}^i\mathbf{M} = \mathbf{1}$ we may discard ${}^3\mathbf{M}$ from the set of structural tensors (20). So the integrity basis is given by

$$\mathcal{P} := \{J_1, \ldots, J_7\} \ . \tag{21}$$

The invariants (J_1, J_2, J_3) are defined by the traces of powers of $\mathbf{E}^{e(m)}$, i.e.

$$J_1 := \operatorname{tr} \mathbf{E}^{e(m)}, \quad J_2 := \operatorname{tr}[(\mathbf{E}^{e(m)})^2], \quad J_3 := \operatorname{tr}[(\mathbf{E}^{e(m)})^3] \ . \tag{22}$$

The irreducible mixed invariants are given by

$$\begin{aligned} J_4 &:= \operatorname{tr}[{}^1\mathbf{M}\mathbf{E}^{e(m)}], \quad J_5 := \operatorname{tr}[{}^1\mathbf{M}(\mathbf{E}^{e(m)})^2] \\ J_6 &:= \operatorname{tr}[{}^2\mathbf{M}\mathbf{E}^{e(m)}], \quad J_7 := \operatorname{tr}[{}^2\mathbf{M}(\mathbf{E}^{e(m)})^2] \end{aligned} \tag{23}$$

see e.g. Spencer [26]. For the free energy function we assume a quadratic form:

$$\begin{aligned} \psi^e = &\frac{1}{2}\lambda J_1^2 + \mu J_2 + \frac{1}{2}\alpha_1 J_4^2 + \frac{1}{2}\alpha_2 J_6^2 + 2\alpha_3 J_5 + 2\alpha_4 J_7 \\ &+ \alpha_5 J_4 J_1 + \alpha_6 J_6 J_1 + \alpha_7 J_4 J_6 \ . \end{aligned} \tag{24}$$

The generalized stresses appear with $(15)_2$ in the form

$$\begin{aligned} \mathbf{S}^{(m)} = &\lambda J_1 \mathbf{1} + 2\mu \mathbf{E}^{e(m)} + \alpha_1 J_4 {}^1\mathbf{M} + \alpha_2 J_6 {}^2\mathbf{M} \\ &+ 2\alpha_3(\mathbf{E}^{e(m)} {}^1\mathbf{M} + {}^1\mathbf{M}\mathbf{E}^{e(m)}) \\ &+ 2\alpha_4(\mathbf{E}^{e(m)} {}^2\mathbf{M} + {}^2\mathbf{M}\mathbf{E}^{e(m)}) + \alpha_5(J_1 {}^1\mathbf{M} + J_4 \mathbf{1}) \\ &+ \alpha_6(J_1 {}^2\mathbf{M} + J_6 \mathbf{1}) + \alpha_7(J_4 {}^2\mathbf{M} + J_6 {}^1\mathbf{M}) \ . \end{aligned} \tag{25}$$

The second derivative of ψ^e yields in this special case the constant generalized fourth-order elasticity tensor

$$\begin{aligned} \mathbb{C}^{e(m)} = &\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbb{I} + \alpha_1 {}^1\mathbf{M} \otimes {}^1\mathbf{M} + \alpha_2 {}^2\mathbf{M} \otimes {}^2\mathbf{M} \\ &+ 2\alpha_3 \mathbb{K}_1 + 2\alpha_4 \mathbb{K}_2 + \alpha_5({}^1\mathbf{M} \otimes \mathbf{1} + \mathbf{1} \otimes {}^1\mathbf{M}) \\ &+ \alpha_6({}^2\mathbf{M} \otimes \mathbf{1} + \mathbf{1} \otimes {}^2\mathbf{M}) \\ &+ \alpha_7({}^2\mathbf{M} \otimes {}^1\mathbf{M} + {}^1\mathbf{M} \otimes {}^2\mathbf{M}) \end{aligned} \tag{26}$$

with $\mathbb{I}_{IJKL} = \delta_{IK}\delta_{JL}$, $\mathbb{K}^1_{IJKL} = \delta_{IK} {}^1M_{JL} + \delta_{JL} {}^1M_{IK}$ and $\mathbb{K}^2_{IJKL} = \delta_{IK} {}^2M_{JL} + \delta_{JL} {}^2M_{IK}$. The elasticity parameters $(\lambda, \mu, \alpha_i | i = 1, \ldots, 7)$ can be identified using the matrix notation

$$\begin{bmatrix} S_{11}^{(m)} \\ S_{22}^{(m)} \\ S_{33}^{(m)} \\ S_{12}^{(m)} \\ S_{13}^{(m)} \\ S_{23}^{(m)} \end{bmatrix} = \begin{bmatrix} \mathbb{C}_{11} & \mathbb{C}_{12} & \mathbb{C}_{13} & 0 & 0 & 0 \\ \mathbb{C}_{12} & \mathbb{C}_{22} & \mathbb{C}_{23} & 0 & 0 & 0 \\ \mathbb{C}_{13} & \mathbb{C}_{23} & \mathbb{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}_{66} \end{bmatrix} \begin{bmatrix} E_{11}^{e(m)} \\ E_{22}^{e(m)} \\ E_{33}^{e(m)} \\ 2E_{12}^{e(m)} \\ 2E_{13}^{e(m)} \\ 2E_{23}^{e(m)} \end{bmatrix} , \tag{27}$$

with the elasticity constants $\mathbb{C}_{ij}$. Choosing the preferred directions as ${}^1\mathbf{a} = (1, 0, 0)^{\mathrm{T}}$ and ${}^2\mathbf{a} = (0, 1, 0)^{\mathrm{T}}$ we obtain the material parameters

$$\begin{aligned} \lambda &= \mathbb{C}_{33} + 2(\mathbb{C}_{44} - \mathbb{C}_{55} - \mathbb{C}_{66}) \\ \mu &= \mathbb{C}_{55} + \mathbb{C}_{66} - \mathbb{C}_{44} \\ \alpha_1 &= \mathbb{C}_{11} + \mathbb{C}_{33} - 4\mathbb{C}_{55} - 2\mathbb{C}_{13} \\ \alpha_2 &= \mathbb{C}_{22} + \mathbb{C}_{33} - 4\mathbb{C}_{66} - 2\mathbb{C}_{23} \\ \alpha_3 &= \mathbb{C}_{44} - \mathbb{C}_{66} \\ \alpha_4 &= \mathbb{C}_{44} - \mathbb{C}_{55} \\ \alpha_5 &= \mathbb{C}_{13} - \mathbb{C}_{33} - 2(\mathbb{C}_{44} - \mathbb{C}_{55} - \mathbb{C}_{66}) \\ \alpha_6 &= \mathbb{C}_{23} - \mathbb{C}_{33} - 2(\mathbb{C}_{44} - \mathbb{C}_{55} - \mathbb{C}_{66}) \\ \alpha_7 &= \mathbb{C}_{12} - \mathbb{C}_{13} - \mathbb{C}_{23} + \mathbb{C}_{33} + 2(\mathbb{C}_{44} - \mathbb{C}_{55} - \mathbb{C}_{66}) \end{aligned} \tag{28}$$

in the invariant setting. In case of isotropy the only remaining constants are λ and μ, which can be directly determined from Young's modulus E and Poisson's ratio ν.

3.4
Yield criterion and related polynomial basis

In the following, we consider an orthotropic yield condition using isotropic tensor functions. We assume that the plastic yield condition should not be influenced by the hydrostatic pressure. Thus, the integrity basis for the argument tensor $\operatorname{dev} \mathbf{\Sigma}$ and the structural tensors ${}^1\mathbf{M}$ and ${}^2\mathbf{M}$ are given by

$$\begin{aligned} &I_1 := \operatorname{tr}[(\operatorname{dev} \mathbf{\Sigma})^2], \ \ I_2 := \operatorname{tr}[{}^1\mathbf{M}\,(\operatorname{dev} \mathbf{\Sigma})^2], \\ &I_3 := \operatorname{tr}[{}^2\mathbf{M}(\operatorname{dev} \mathbf{\Sigma})^2] \ \ I_4 := \operatorname{tr}[{}^1\mathbf{M} \operatorname{dev} \mathbf{\Sigma}], \\ &I_5 := \operatorname{tr}[{}^2\mathbf{M} \operatorname{dev} \mathbf{\Sigma}], \ \ I_6 := \operatorname{tr}[(\operatorname{dev} \mathbf{\Sigma})^3] \ . \end{aligned} \tag{29}$$

The anisotropic flow criterion is formulated as an isotropic tensor function, i.e.

$$\begin{aligned} &\hat{\Phi}(\operatorname{dev} \mathbf{\Sigma}, {}^1\mathbf{M}, {}^2\mathbf{M}) \\ &\quad = \hat{\Phi}(\mathbf{Q}^{\mathrm{T}} \operatorname{dev} \mathbf{\Sigma} \mathbf{Q}, \mathbf{Q}^{\mathrm{T}1}\mathbf{M}\mathbf{Q}, \mathbf{Q}^{\mathrm{T}2}\mathbf{M}\mathbf{Q}) \quad \forall \mathbf{Q} \in \mathrm{SO}(3) \ . \end{aligned} \tag{30}$$

The quadratic flow criterion function $\Phi = \hat{\Phi}(I_1, I_2, I_3, I_4, I_5, \xi) \leq 0$ is given as follows:

$$\begin{aligned} \Phi = &\eta_1 I_1 + \eta_2 I_2 + \eta_3 I_3 + \eta_4 I_4^2 + \eta_5 I_5^2 + \eta_6 I_4 I_5 \\ &- \left(1 + \frac{\hat{\xi}(e^{p(m)})}{Y_{11}^0}\right)^2 \ . \end{aligned} \tag{31}$$

The material parameters $\eta_i | i = 1, \ldots, 6$ can be identified by six independent tests. Assume the tests are relative to the fixed orientation of the specimen ${}^1\mathbf{a} = (1, 0, 0)^{\mathrm{T}}$ and

$^2\mathbf{a} = (0,1,0)^T$. Let Y^0_{ij} be the yield stress in ij-direction, with respect to $^i\mathbf{a}$ and $^j\mathbf{a}$. The linear independent numerical tests with $\boldsymbol{\beta} = \mathbf{0}$ are:

1. uniaxial tension in $^1\mathbf{a}$-direction

$$\mathbf{S} = \begin{pmatrix} Y^0_{11} & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \begin{matrix} I_1 = 2/3(Y^0_{11})^2 & I_4 = 2/3Y^0_{11} \\ I_2 = 4/9(Y^0_{11})^2 & I_5 = -1/3Y^0_{11} \\ I_3 = 1/9(Y^0_{11})^2 & I_4 I_5 = -2/9(Y^0_{11})^2 \end{matrix} \tag{32}$$

2. uniaxial tension in $^2\mathbf{a}$-direction

$$\mathbf{S} = \begin{pmatrix} 0 & & \\ & Y^0_{22} & \\ & & 0 \end{pmatrix} \quad \begin{matrix} I_1 = 2/3(Y^0_{22})^2 & I_4 = -1/3Y^0_{22} \\ I_2 = 1/9(Y^0_{22})^2 & I_5 = 2/3Y^0_{22} \\ I_3 = 4/9(Y^0_{22})^2 & I_4 I_5 = -2/9(Y^0_{22})^2 \end{matrix} \tag{33}$$

3. uniaxial tension in $^3\mathbf{a}$-direction

$$\mathbf{S} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & Y^0_{33} \end{pmatrix} \quad \begin{matrix} I_1 = 2/3(Y^0_{33})^2 & I_4 = -1/3Y^0_{33} \\ I_2 = 1/9(Y^0_{33})^2 & I_5 = -1/3Y^0_{33} \\ I_3 = 1/9(Y^0_{33})^2 & I_4 I_5 = 1/9(Y^0_{33})^2 \end{matrix} \tag{34}$$

4. shear test in $^1\mathbf{a}$–$^2\mathbf{a}$ plane

$$\mathbf{S} = \begin{pmatrix} 0 & Y^0_{12} & \\ Y^0_{12} & 0 & \\ & & 0 \end{pmatrix} \quad \begin{matrix} I_1 = 2(Y^0_{12})^2 & I_4 = 0 \\ I_2 = (Y^0_{12})^2 & I_5 = 0 \\ I_3 = (Y^0_{12})^2 & I_4 I_5 = 0 \end{matrix} \tag{35}$$

5. shear test in $^1\mathbf{a}$–$^3\mathbf{a}$ plane

$$\mathbf{S} = \begin{pmatrix} 0 & & Y^0_{13} \\ & 0 & \\ Y^0_{13} & & 0 \end{pmatrix} \quad \begin{matrix} I_1 = 2(Y^0_{13})^2 & I_4 = 0 \\ I_2 = (Y^0_{13})^2 & I_5 = 0 \\ I_3 = 0 & I_4 I_5 = 0 \end{matrix} \tag{36}$$

6. shear test in $^2\mathbf{a}$–$^3\mathbf{a}$ plane

$$\mathbf{S} = \begin{pmatrix} 0 & & \\ & 0 & Y^0_{23} \\ & Y^0_{23} & 0 \end{pmatrix} \quad \begin{matrix} I_1 = 2(Y^0_{23})^2 & I_4 = 0 \\ I_2 = 0 & I_5 = 0 \\ I_3 = (Y^0_{23})^2 & I_4 I_5 = 0 \end{matrix} \tag{37}$$

This leads after evaluation of the flow criterion to the parameters

$$\begin{aligned}
\eta_1 &= \frac{1}{2}\left(\frac{-1}{(Y^0_{12})^2} + \frac{1}{(Y^0_{13})^2} + \frac{1}{(Y^0_{23})^2}\right) \\
\eta_2 &= \frac{1}{(Y^0_{12})^2} - \frac{1}{(Y^0_{23})^2} \\
\eta_3 &= \frac{1}{(Y^0_{12})^2} - \frac{1}{(Y^0_{13})^2} \\
\eta_4 &= \frac{2}{(Y^0_{11})^2} - \frac{1}{(Y^0_{22})^2} + \frac{2}{(Y^0_{33})^2} - \frac{1}{(Y^0_{13})^2} \\
\eta_5 &= \frac{-1}{(Y^0_{11})^2} + \frac{2}{(Y^0_{22})^2} + \frac{2}{(Y^0_{33})^2} - \frac{1}{(Y^0_{23})^2} \\
\eta_6 &= \frac{-1}{(Y^0_{11})^2} - \frac{1}{(Y^0_{22})^2} + \frac{5}{(Y^0_{33})^2} + \frac{1}{(Y^0_{12})^2} - \frac{1}{(Y^0_{13})^2} - \frac{1}{(Y^0_{23})^2} \ .
\end{aligned} \tag{38}$$

Remark If we set $Y^0_{ii} = Y^0$ for $i = 1,2,3$ and $Y^0_{ij} = Y^0/\sqrt{3}$ for $i \neq j$ with $i,j = 1,2,3$ we arrive at the well known von Mises criterion $\Phi = \frac{3\|\mathrm{dev}\,\boldsymbol{\Sigma}\|^2}{2(Y^0)^2} - \left(1 + \frac{\hat{\xi}(e^{p(m)})}{Y^0}\right)^2 \leq 0$.

3.5 Solution algorithm for the set of constitutive equations

To solve the set of Eqs. (13) we apply a so-called operator split along with a general return method. The procedure is given here for a typical time step $[t_n, t_{n+1}]$ with the time increment $\Delta t := t_{n+1} - t_n$. We denote the solution at time t_n by $\{\mathbf{S}^{(m)}_n, \boldsymbol{\alpha}_n, \mathbf{E}^{p(m)}_n, e^{p(m)}_n\}$. Furthermore, at time t_{n+1} the strain tensor $\mathbf{E}^{(m)}_{n+1}$ is given. The procedure is based on the introduction of the so-called trial state by

$$\begin{aligned} \mathbf{E}^{p(m),\mathrm{trial}}_{n+1} &= \mathbf{E}^{p(m)}_n, \quad \boldsymbol{\alpha}^{\mathrm{trial}}_{n+1} = \boldsymbol{\alpha}_n, \\ e^{p(m),\mathrm{trial}}_{n+1} &= e^{p(m)}_n, \quad \lambda^{\mathrm{trial}}_{n+1} = 0 \ . \end{aligned} \tag{39}$$

Hence, the trial stresses are evaluated according to (13)

$$\begin{aligned} \mathbf{S}^{\mathrm{trial}}_{n+1} &= \partial_{E^{\mathrm{trial}}} \hat{\psi}(\mathbf{E}^{\mathrm{trial}}_{n+1}) \text{ with } \mathbf{E}^{\mathrm{trial}}_{n+1} := \mathbf{E}^{(m)}_{n+1} - \mathbf{E}^{p(m),\mathrm{trial}}_{n+1} \\ \boldsymbol{\beta}^{\mathrm{trial}}_{n+1} &= \partial_{\alpha^{\mathrm{trial}}} \hat{\psi}^{p,k}(\alpha^{\mathrm{trial}}_{n+1}) \ . \end{aligned} \tag{40}$$

An elastic step is present if

$$\Phi^{\mathrm{trial}}_{n+1} = \hat{\Phi}(\mathbf{S}^{\mathrm{trial}}_{n+1}, \boldsymbol{\beta}^{\mathrm{trial}}_{n+1}, e^{p(m),\mathrm{trial}}_{n+1}) \leq 0 \tag{41}$$

and the state variables at time t_{n+1} are given by the elastic predictor step. In case of $\Phi^{\mathrm{trial}}_{n+1} > 0$ one has to perform the corrector step, see Appendix A. This leads to a solution at time t_{n+1} which is denoted by $\{\mathbf{S}^{(m)}_{n+1}, \boldsymbol{\alpha}_{n+1}, \mathbf{E}^{p(m)}_{n+1}, e^{p(m)}_{n+1}\}$. Furthermore, we obtain the consistent tangent tensor in the generalized stress–strain space.

Remark It is well known that the general return algorithm according to Appendix A is not volume preserving for all generalized measures, except for the case $m = 0$. In case of $m \neq 0$ we perform the following post-processing algorithm in order to fulfill the plastic incompressibility condition $\det[\mathbf{C}^p] = 1$ at the beginning of each time interval. Starting from $(3)_1$ and the additive decomposition $(13)_1$ we consider $\mathbf{C}^{p(m)}$, which is implicitly defined by

$$\mathbf{E}^{p(m)} = \frac{1}{2m}(\mathbf{C}^{p(m)} - \mathbf{1}) \ . \tag{42}$$

It should be noted that $\mathbf{E}^{p(m)}$ is a deviatoric tensor. The latter equation yields

$$\mathbf{C}^{p(m)} = 2m\mathbf{E}^{p(m)} + \mathbf{1} \ , \tag{43}$$

with $\det[\mathbf{C}^{p(m)}] \neq 1$ in general. The correction is performed as follows:

$$\mathbf{C}^{p(m),*} := \mathbf{C}^{p(m)}/(\det \mathbf{C}^{p(m)})^{1/3} \ , \tag{44}$$

thus simply constructing the unimodular tensor $\mathbf{C}^{p(m),*}$ by scaling the initial tensor. Using this procedure, we arrive at the corrected plastic part of the generalized strain tensor

$$\mathbf{E}^{p(m),*} = \frac{1}{2m}(\mathbf{C}^{p(m),*} - \mathbf{1}) \ . \tag{45}$$

In order to set $\mathbf{E}^{p(m)}$ equal to $\mathbf{E}^{p(m),*}$ in the next time step the latter quantities are stored in the history array. As mentioned above, for $m = 0$ the plastic incompressibility condition is fulfilled automatically. This statement can be shown using the well known identity $\det[\exp \mathbf{A}] = \exp[\mathrm{tr}\,\mathbf{A}]$, where $\mathbf{A}$ is a symmetric second-order tensor.

4 Variational formulation and finite element discretization

In the following we give a brief summary of the corresponding boundary value problem and finite element formulation in the material description. Let $\mathcal{B}$ be the reference body of interest which is bounded by the surface $\partial\mathcal{B}$. The surface is partitioned into two disjoined parts $\partial\mathcal{B} = \partial\mathcal{B}_u \bigcup \partial\mathcal{B}_t$ with $\partial\mathcal{B}_u \bigcap \partial\mathcal{B}_t = \emptyset$. The equation of balance of linear momentum for the static case is governed by the first Piola–Kirchhoff stresses $\mathbf{P} = \mathbf{FS}$ and the body force $\bar{\mathbf{f}}$ in the reference configuration

$$\mathrm{Div}[\mathbf{FS}] + \bar{\mathbf{f}} = \mathbf{0} \ . \tag{46}$$

The Dirichlet boundary conditions and the Neumann boundary conditions are given by

$$\mathbf{u} = \bar{\mathbf{u}} \ \text{on} \ \partial\mathcal{B}_u \quad \text{and} \quad \mathbf{t} = \bar{\mathbf{t}} = \mathbf{PN} \ \text{on} \ \partial\mathcal{B}_t \ , \tag{47}$$

respectively. Here $\mathbf{N}$ represents the unit exterior normal to the boundary surface $\partial\mathcal{B}_t$. With standard arguments of variational calculus we arrive at the variational problem

$$G(\mathbf{u}, \delta\mathbf{u}) = \int_{\mathcal{B}} \mathbf{S} : \delta\mathbf{E}\,\mathrm{d}V + G^{\mathrm{ext}} \quad \text{with}$$
$$G^{\mathrm{ext}}(\delta\mathbf{u}) := -\int_{\mathcal{B}} \bar{\mathbf{f}} \cdot \delta\mathbf{u}\,\mathrm{d}V - \int_{\partial\mathcal{B}_t} \bar{\mathbf{t}} \cdot \delta\mathbf{u}\,\mathrm{d}A \ , \tag{48}$$

where $\delta\mathbf{E} := \frac{1}{2}(\delta\mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\delta\mathbf{F})$ characterizes the virtual Green–Lagrangian strain tensor in terms of the virtual deformation gradient $\delta\mathbf{F} := \mathrm{Grad}\,\delta\mathbf{u}$. The principle of virtual work (48) for a static equilibrium state of the considered body requires $G = 0$. For the solution of this nonlinear equation we apply a standard Newton iteration scheme which requires the consistent linearization of (48) in order to guarantee the quadratic convergence rate near the solution. Since the stress tensor $\mathbf{S}$ is symmetric, the linear increment of G denoted by ΔG is given by

$$\Delta G(\mathbf{u}, \delta\mathbf{u}, \Delta\mathbf{u}) := \int_{\mathcal{B}} (\delta\mathbf{E} : \Delta\mathbf{S} + \Delta\delta\mathbf{E} : \mathbf{S})\mathrm{d}V \ , \tag{49}$$

where $\Delta\delta\mathbf{E} := \frac{1}{2}(\Delta\mathbf{F}^{\mathrm{T}}\delta\mathbf{F} + \delta\mathbf{F}^{\mathrm{T}}\Delta\mathbf{F})$ denotes the linearized virtual Green–Lagrange strain tensor as a function of the incremental deformation gradient $\Delta\mathbf{F} := \mathrm{Grad}\,\Delta\mathbf{u}$. The incremental second Piola–Kirchhoff stress tensor $\Delta\mathbf{S}$ can be derived using (6) as

$$\Delta\mathbf{S} = \Delta\mathbf{S}^{(m)} : \mathbb{P}_E + \mathbf{S}^{(m)} : \Delta\mathbb{P}_E = \mathbb{C} : \Delta\mathbf{E} \ , \tag{50}$$

with $\Delta\mathbf{E} := \frac{1}{2}(\Delta\mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\Delta\mathbf{F})$. The moduli appear in the form

$$\mathbb{C} = \mathbb{P}_E : \mathbb{C}_{ep}^{(m)} : \mathbb{P}_E + \mathbf{S}^{(m)} : \mathbb{K} \ , \tag{51}$$

with the consistent tangent tensor $\mathbb{C}_{ep}^{(m)} = \partial_{E^{(m)}}\mathbf{S}^{(m)}$, in this context see Appendix A, Eq. (A.71).

The sixth-order tensor $\mathbb{K} = 2\partial_{C}\mathbb{P}$ is derived following Ogden [15]

$$\begin{aligned}
\mathbb{K} = \sum_{A=1}^{3}\sum_{B=1}^{3}\sum_{C=1}^{3}[&K_{AABBCC}\,\mathbf{N}^A \otimes \mathbf{N}^A \otimes \mathbf{N}^B \otimes \mathbf{N}^B \otimes \mathbf{N}^C \otimes \mathbf{N}^C \\
&+ K_{ABABCC}\,\mathbf{N}^A \otimes \mathbf{N}^B \otimes (\mathbf{N}^A \otimes \mathbf{N}^B + \mathbf{N}^B \otimes \mathbf{N}^A) \otimes \mathbf{N}^C \otimes \mathbf{N}^C \\
&+ K_{ABCCAB}\,\mathbf{N}^A \otimes \mathbf{N}^B \otimes \mathbf{N}^C \otimes \mathbf{N}^C \otimes (\mathbf{N}^A \otimes \mathbf{N}^B + \mathbf{N}^B \otimes \mathbf{N}^A) \\
&+ K_{AABCBC}\,\mathbf{N}^A \otimes \mathbf{N}^A \otimes \mathbf{N}^B \otimes \mathbf{N}^C \otimes (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B) \\
&+ K_{ABBCCA}\,\mathbf{N}^A \otimes \mathbf{N}^B \otimes (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B) \otimes (\mathbf{N}^C \otimes \mathbf{N}^A + \mathbf{N}^A \otimes \mathbf{N}^C) \\
&+ K_{ABCABC}\,\mathbf{N}^A \otimes \mathbf{N}^B \otimes (\mathbf{N}^C \otimes \mathbf{N}^A + \mathbf{N}^A \otimes \mathbf{N}^C) \otimes (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B) \ .
\end{aligned} \tag{52}$$

The non-zero components of $\mathbb{K}$ are found to be

$$\begin{aligned}
K_{AABBCC} &= \partial_{E_C} P_{AABB} = 2(m-1)\lambda_A^{m-2}\delta_{AB}\delta_{AC} \\
K_{ABABCC} &= K_{ABCCAB} = \partial_{E_C} P_{ABAB} \qquad \text{where } A \neq B \\
&= \begin{cases} \gamma_{AAB}^{(m)}\delta_{AC} + \gamma_{BBA}^{(m)}\delta_{BC} & \text{for } \lambda_A \neq \lambda_B \\ \frac{1}{2}(m-1)\lambda_A^{m-2} & \text{for } \lambda_A = \lambda_B \\ 0 & \text{for } A \neq B \neq C \neq A \end{cases} \\
K_{AABCBC} &= \frac{1}{2}\frac{P_{AABB} - P_{AACC}}{E_B - E_C} - P_{BCBC}\frac{\delta_{AB} - \delta_{AC}}{E_B - E_C} \qquad \text{where } B \neq C \\
&= \begin{cases} \gamma_{AAC}^{(m)}\delta_{AB} + \gamma_{AAB}^{(m)}\delta_{AC} & \text{for } \lambda_B \neq \lambda_C \\ \frac{1}{2}(m-1)\lambda_B^{m-2} & \text{for } \lambda_B = \lambda_C \\ 0 & \text{for } A \neq B \neq C \neq A \end{cases} \\
K_{ABBCCA} &= K_{ABCABC} \\
&= \frac{1}{4}\frac{P_{ABAB} - P_{CACA}}{E_B - E_C} + \frac{1}{4}\frac{P_{BCBC} - P_{ABAB}}{E_C - E_A} \qquad \text{where } A \neq B \neq C \neq A \\
&= \begin{cases} \gamma^{(m)} & \text{for } \lambda_A \neq \lambda_B \neq \lambda_C \\ \frac{1}{2}\gamma_{AAC}^{(m)} & \text{for } \lambda_A = \lambda_B \neq \lambda_C \\ \frac{1}{4}(m-1)\lambda_A^{m-2} & \text{for } \lambda_A = \lambda_B = \lambda_C \ . \end{cases}
\end{aligned} \tag{53}$$

Here, we use the abbreviations

$$\gamma^{(m)}_{AAB} = \frac{\lambda_A^{m-1}(\lambda_A - \lambda_B) - 2\left(E_A^{(m)} - E_B^{(m)}\right)}{(\lambda_A - \lambda_B)^2},$$
$$\gamma^{(m)} = \frac{\lambda_1\left(E_2^{(m)} - E_3^{(m)}\right) + \lambda_2\left(E_3^{(m)} - E_1^{(m)}\right) + \lambda_3\left(E_1^{(m)} - E_2^{(m)}\right)}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)} . \tag{54}$$

The contraction $\mathbf{S}^{(m)}$ with $\mathbb{K}$ in Eq. (51) yields

$$\begin{aligned}\mathbf{S}^{(m)} : \mathbb{K} = \sum_{A=1}^{3}\sum_{B=1}^{3}\sum_{C=1}^{3} & \bar{K}^{(AA)}_{BBCC}\, \mathbf{N}^B \otimes \mathbf{N}^B \otimes \mathbf{N}^C \otimes \mathbf{N}^C \\ & + \bar{K}^{(AB)}_{ABCC}\, (\mathbf{N}^A \otimes \mathbf{N}^B + \mathbf{N}^B \otimes \mathbf{N}^A) \otimes \mathbf{N}^C \otimes \mathbf{N}^C \\ & + \bar{K}^{(AB)}_{CCAB}\, \mathbf{N}^C \otimes \mathbf{N}^C \otimes (\mathbf{N}^A \otimes \mathbf{N}^B + \mathbf{N}^B \otimes \mathbf{N}^A) \\ & + \bar{K}^{(AA)}_{BCBC}\, \mathbf{N}^B \otimes \mathbf{N}^C \otimes (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B) \\ & + \bar{K}^{(AB)}_{BCCA}\, (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B) \\ & \otimes (\mathbf{N}^C \otimes \mathbf{N}^A + \mathbf{N}^A \otimes \mathbf{N}^C) \\ & + \bar{K}^{(AB)}_{CABC}\, (\mathbf{N}^C \otimes \mathbf{N}^A + \mathbf{N}^A \otimes \mathbf{N}^C) \\ & \otimes (\mathbf{N}^B \otimes \mathbf{N}^C + \mathbf{N}^C \otimes \mathbf{N}^B)\end{aligned} , \tag{55}$$

with the components

$$\bar{K}^{(AB)}_{CDEF} = S^{(m)}_{(AB)} K_{(AB)CDEF} . \tag{56}$$

In (56) no summation over the indices (AB) takes place. Furthermore, the projection of the generalized stress tensors into the space of principal directions yields

$$S^{(m)}_{AB} = \mathbf{S}^{(m)} : \mathbf{N}^A \otimes \mathbf{N}^B . \tag{57}$$

The matrix representation of (51) considering (55) is given in Appendix B. The expressions are very simple and allow an efficient computer implementation. It should be noted, that in case of isotropic material behaviour the eigenvectors of $\mathbf{S}^{(m)}$ and $\mathbf{E}^{(m)}$ coincide, thus $S^{(m)}_{AB} = 0$ holds for $A \neq B$.

The spatial discretization of the considered body $\mathcal{B} \approx \bigcup_{e=1}^{n_{\mathrm{ele}}} \mathcal{B}^e$ with n_{ele} finite elements $\mathcal{B}^e$ leads within a standard displacement approximation $\mathbf{u} = \sum_{I=1}^{\mathrm{nel}} N^I \mathbf{d}_I$, $\delta\mathbf{u} = \sum_{I=1}^{\mathrm{nel}} N^I \delta\mathbf{d}_I$, and $\Delta\mathbf{u} = \sum_{I=1}^{\mathrm{nel}} N^I \Delta\mathbf{d}_I$, of the actual-, virtual-, and incremental-displacement fields, respectively, to a set of algebraic equations of the form which can be solved for the solution point $\mathbf{d}$. For a detailed discussion of this point we refer to the standard text book Zienkiewicz and Taylor [36] or others.

5 Numerical examples

The constitutive model described in the previous sections has been implemented in the finite element programs FEAP [28] and ABAQUS [1]. In FEAP we use an 8-noded brick-type shell element as is documented in Klinkel [9] or Wagner et al. [30] and references therein. The basis of the element formulation is given with a standard isoparametric displacement approach. Based on so-called ANS-methods the transverse shear strains and the thickness normal strains are independently interpolated using special shape functions. Furthermore, the membrane behaviour of the element is essentially improved applying the enhanced strain method with 5 additional parameters, [30]. Due to the different interpolation techniques the element orientation has to be considered when discretizing the mesh. An interface to arbitrary three-dimensional material laws is available. In ABAQUS the material model is programmed via the user interface `umat`.

In this section three representative numerical examples with finite plastic deformations are presented. The computations of the first two examples are performed with FEAP. First we investigate necking of a circular bar subjected to uniaxial tension. Our solution for $m = 0$ is compared with a reference model. In the second example we simulate the mechanical behaviour of a conical shell considering different material parameters. For a given linear isotropic hardening law we adjust the hardening functions of the generalized stress–strain models with different parameters m for a simple tension test. The results for nonlinear isotropic hardening are compared with a reference solution. Furthermore, we consider orthotropic elastic and plastic material properties. Finally, the deep drawing process of a circular blank is simulated using the 3-D hybrid solid element C3D8H of the program ABAQUS/Standard. The material is assumed to be isotropic in the elastic range and orthotropic within plastic deformations. The earing effect caused by the anisotropy of the blank is investigated.

5.1 Necking of a circular bar

The three dimensional necking of a circular bar is an example widely investigated in the literature, see e.g. Simo and Armero [21] or Klinkel [9]. The geometrical data are $R = 6.413$ mm, $R_0 = 0.982R$ and $L = 26.667$ mm.

To initialize the necking process we introduce an imperfection with the reduced radius R_0 at $z = L$. The material data for isotropy are given as follows. The hardening function consists of a linear part and an exponential part. It is approximated by piecewise linear functions.

Elasticity constants:	$E = 206.9\,\mathrm{GPa}$ $\quad \nu = 0.29$ $\quad m = 0$
Yield parameter:	$Y^0 = 0.45\,\mathrm{GPa}$ $\quad Y^\infty = 0.715\,\mathrm{GPa}$
	$h = 0.12924\,\mathrm{GPa}$ $\quad \delta = 16.93$
	$\eta_1 = \frac{3}{2}\frac{1}{(Y^0)^2}$ $\quad \eta_i = 0\vert_{i=2,\ldots,6}$
Hardening function:	$\hat{\xi}(e^{p(m)}) = h e^{p(m)} + (Y^\infty - Y^0) \times \left(1 - \exp(-\delta e^{p(m)})\right)$ (58)

Figure 3 shows a finite element discretization of half the bar. At $z = L$ we impose the symmetry boundary conditions, whereas in a displacement-controlled computation the axial displacements $u(z = 0)$ are given. Furthermore, we consider symmetry conditions in the cross-section of the plane. Thus, one quarter is discretized with 960 elements, where the thickness direction of the shell elements corresponds to the global z-axis. We only consider the parameter $m = 0$. Figure 4b displays the deformed structure at $u = 7$ and the equivalent plastic strains. As can be

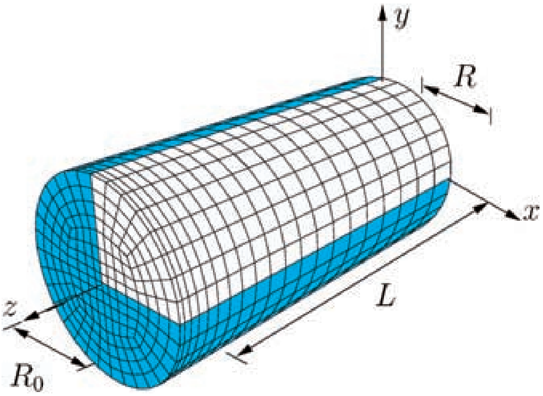

Fig. 3. Geometry of circular bar

seen large plastic strains occur in the necking range. The results are in very good agreement with the reference solution of Klinkel [9].

5.2
Conical shell

The second example is a conical shell subjected to a constant ring load $\bar{\lambda} p$ with $p = 1$ GN/m, see Wagner et al. [30] and references therein. The problem and the finite element discretization with 8-node shell elements are depicted in Fig. 5. The geometrical data are $r = 1$ m, $R = 2$ m, $L = 1$ m, $t = 0.1$ m. For isotropic material behaviour one quarter is discretized with $8 \times 8 \times 1$ elements, whereas for orthotropy the whole cone has to be considered. In our computations the vertical displacement w is controlled. At $w = 2.25$ the structure is completely unloaded. It should be remarked that we have used a nine-point Gaussian quadrature.

5.2.1
Isotropic material response with nonlinear hardening

Here, the parameter m of the generalized stress–strain model is set to $m = 0$. The material data are given in (58). The nonlinear hardening function is approximated by piecewise linear functions. The results are depicted in Figure 6a. The load factor $\bar{\lambda}$ is plotted versus the vertical deflection w. There is exact agreement with the reference solution of Klinkel [9]. The equivalent plastic strains for the deformed structure ($w = 2.25$) are shown in Figure 6b.

5.2.2
Isotropic material response with linear hardening

Next we compare the structural response of the conical shell considering the parameters $m = 0$, $m = 0.5$, $m = 1$, $m = -1$. The material response is restricted to isotropy with material data according to (58). Now the hardening

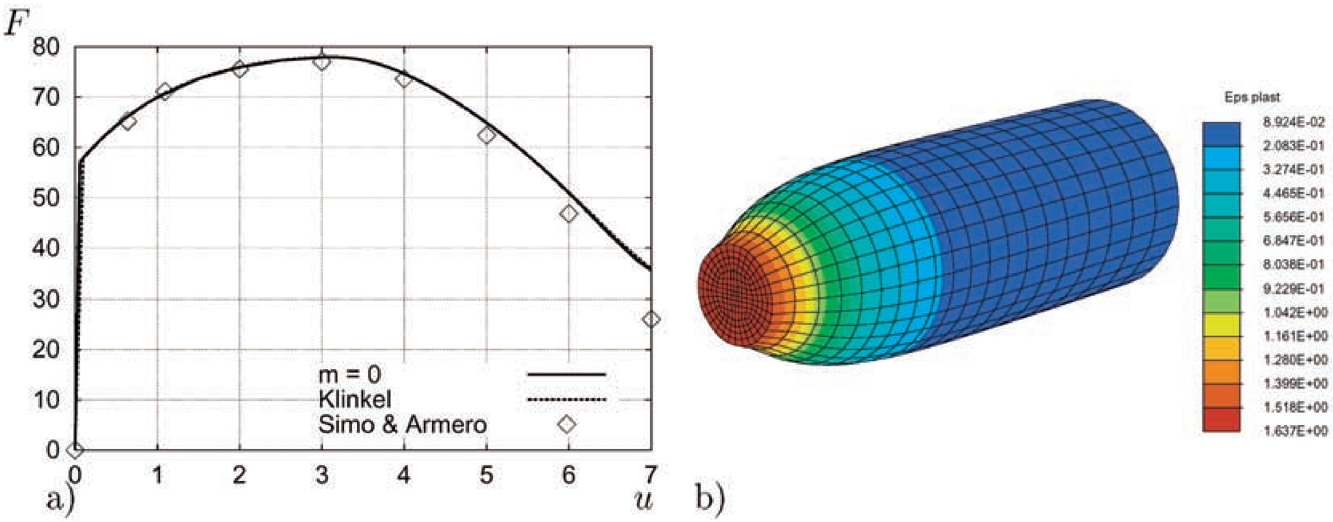

Fig. 4a, b. Necking of circular bar: **a** Load-displacement curve, **b** deformed structure for $u = 7$ and equivalent plastic strains

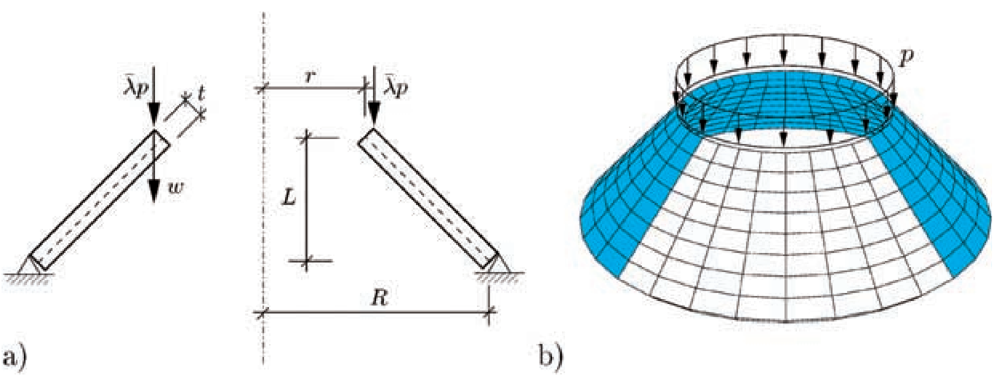

Fig. 5a, b. Conical shell. **a** Geometry and boundary conditions, **b** finite element discretization

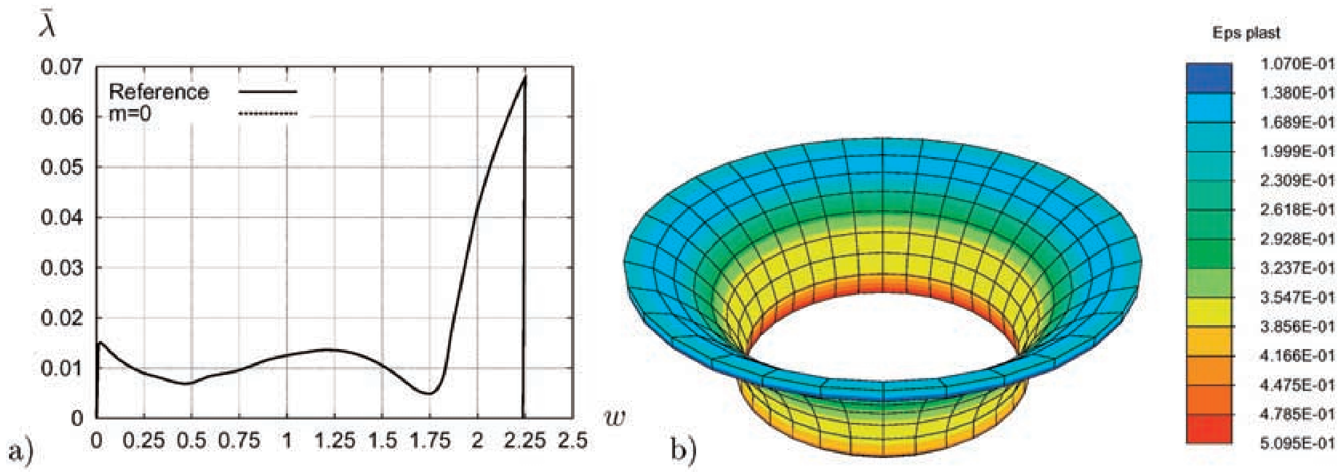

Fig. 6a, b. Conical shell, isotropy and nonlinear hardening. **a** Load displacement curves using the reference model (Klinkel) [9] and for the present model with $m = 0$, **b** equivalent plastic strains at $w = 2.25$

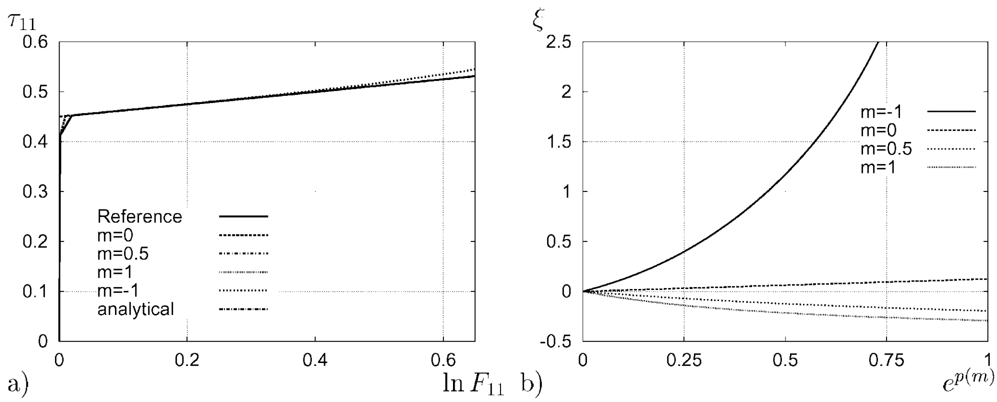

Fig. 7a, b. Tension test with linear hardening. **a** $\tau_{11}- \ln F_{11}$ curve for reference model and for $m = 1, 0.5, 0, -1$. **b** $\xi - e^{p(m)}$ curve for the different models

function for the model with $m = 0$ contains only the linear part with $h = 0.125$ GPa. For each parameter m except $m = 0$ the function $\hat{\xi}(e^{p(m)})$ is adjusted in such a way that all five models lead to the same results for a simple tension test. The test is performed with one element subjected to a homogeneous stress state.

The solutions are depicted in Fig. 7a, where the uniaxial Kirchhoff stress τ_{11} is plotted versus $\ln F_{11}$. As a result of the fitting process one can see nearly the same response for the parameters $m = 0$, $m = 0.5$, $m = 1$, $m = -1$ and the reference model of Klinkel [9] in comparison to the given linear hardening law. The corresponding hardening curves $\xi - e^{p(m)}$ are shown in Fig. 7b.

Using the adjusted functions $\hat{\xi}(e^{p(m)})$ we now calculate the load displacement behaviour of the conical shell for each parameter m. Figure 8 shows that the results obtained with the model in [9] are identical to $m = 0$. For a negative parameter m we obtain a stiffer behaviour, whereas for positive m the computed load factors are below the reference solution.

Plots of the equivalent plastic strains at a displacement $w = 2.25$ are depicted in Fig. 9. A comparison of the contour values is not possible, since $e^{p(m)}$ is defined differently for each parameter m.

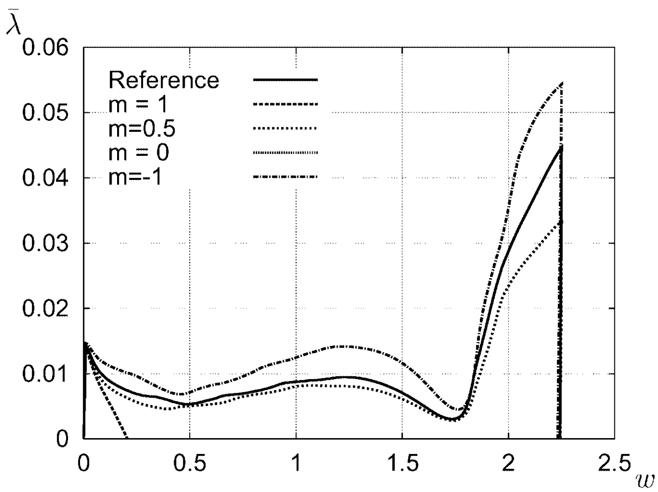

Fig. 8. Load displacement curves for the reference model and for the generalized models $m = 1, 0.5, 0$ and -1

For $m = 1$ a solution can not be obtained, because numerical instabilities due to the mixed element formulation occur, see the load deflection curve for $m = 1$ in Fig. 8. In this context, we refer to investigations of the

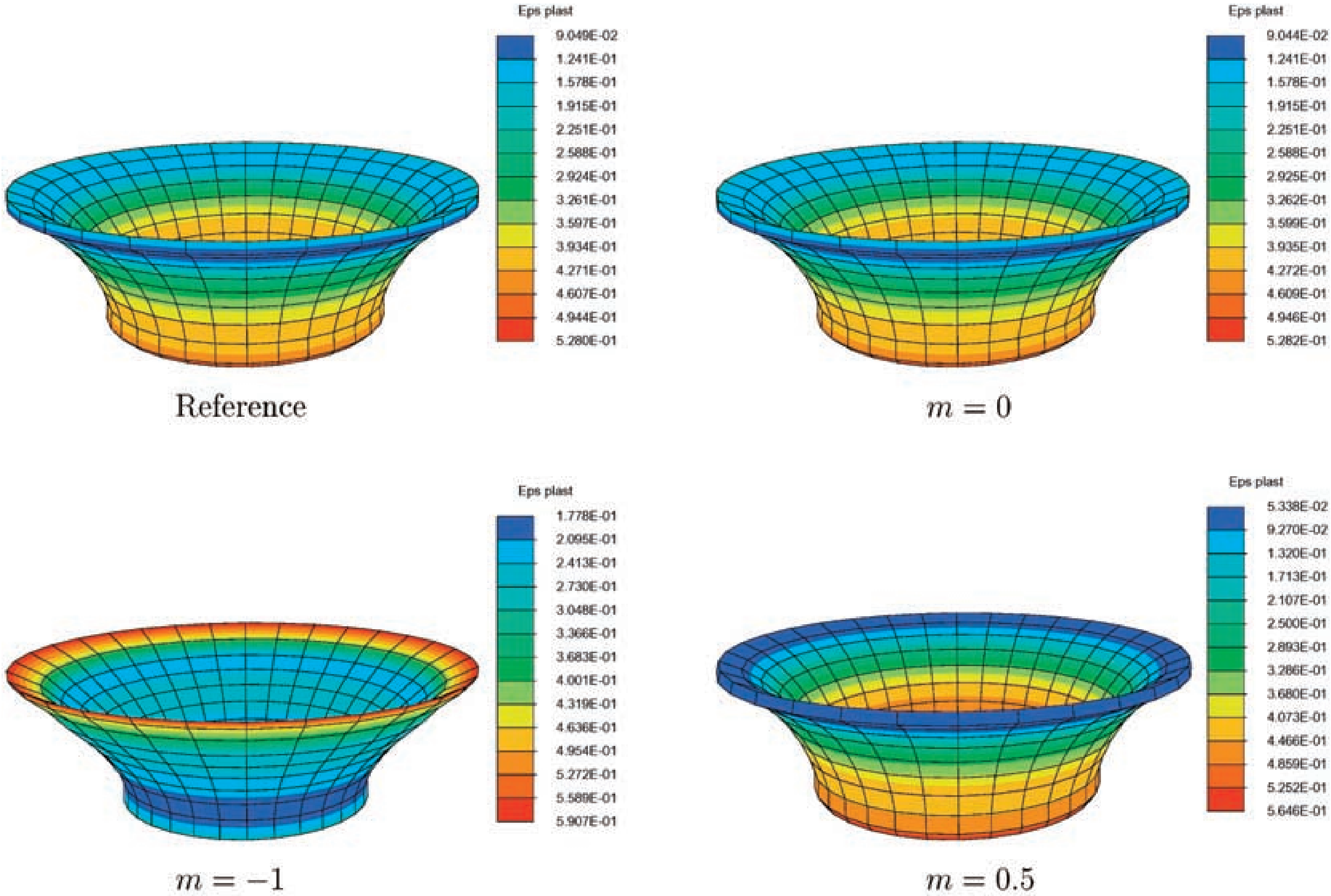

Fig. 9. Equivalent plastic strains for the reference model (Klinkel) and the generalized models $m = 0, -1$ and 0.5

numerical stability of enhanced strain formulations in Wriggers and Reese [34]. Due to these difficulties we finally analyze the load deflection behaviour for the cases $m = 1$ and $m = 0$ without activating the five enhanced assumed strain parameters. The corresponding results are depicted in Fig. 10. It can be seen that the structure shows now a relatively stiff behaviour.

5.2.3
Orthotropic material response with nonlinear hardening

Next we assume orthotropic elastic and plastic material behaviour. Here, we only consider the case $m = 0$. The material parameters for elastic and plastic orthotropy are given below. According to Eq. (20) the privileged directions are denoted by $^1\mathbf{a}$ and $^2\mathbf{a}$.

Elasticity constants:	$\mathbb{C}_{11} = 240.71$	$\mathbb{C}_{12} = 62.68$	$\mathbb{C}_{13} = 69.25$
	$\mathbb{C}_{22} = 211.05$	$\mathbb{C}_{23} = 59.70$	$\mathbb{C}_{33} = 229.25$
	$\mathbb{C}_{44} = 66.00$	$\mathbb{C}_{55} = 66.00$	$\mathbb{C}_{66} = 81.00$
Yield parameter:	$Y_{11} = 0.585$	$Y_{22} = 0.81$	$Y_{33} = 0.36$
	$Y_{12} = 0.286$	$Y_{13} = 0.234$	$Y_{23} = 0.260$
Orientation:	$^1\mathbf{a} = [1\,,0\,,0]^T$	$^2\mathbf{a} = [0\,,1\,,0]^T$	

The yield parameters and the elasticity constants are given in GPa. The nonlinear hardening curve is shown in Fig. 11a. Figure 11b depicts the computed load deflection curve.

Finally, Fig. 12 shows the equivalent plastic strains for a sequence of deformed configurations. The orthotropy of the material is reflected by the distribution of the equivalent plastic strains. It can be seen that four regions with

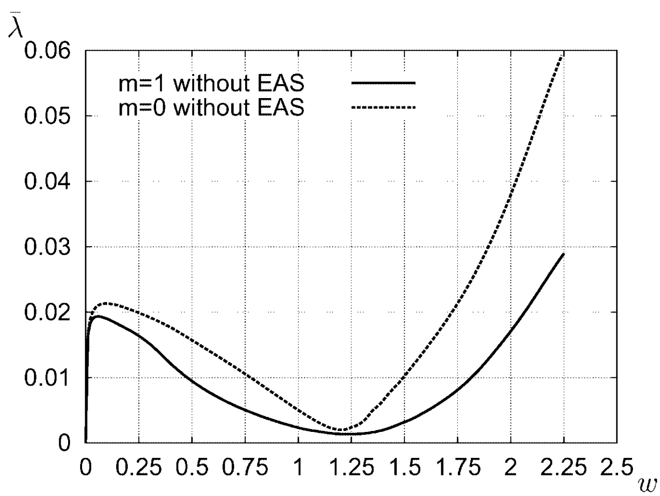

Fig. 10. Load displacement curves for the generalized models $m = 0$ and 1 without considering the EAS-parameters

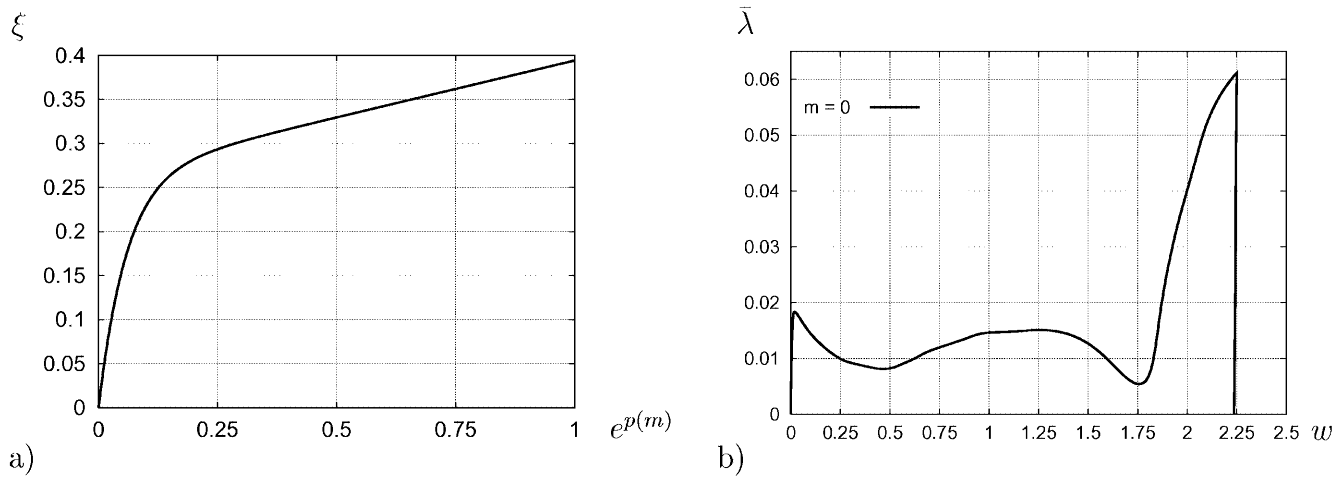

Fig. 11a, b. **a** Nonlinear hardening function, **b** load displacement curve

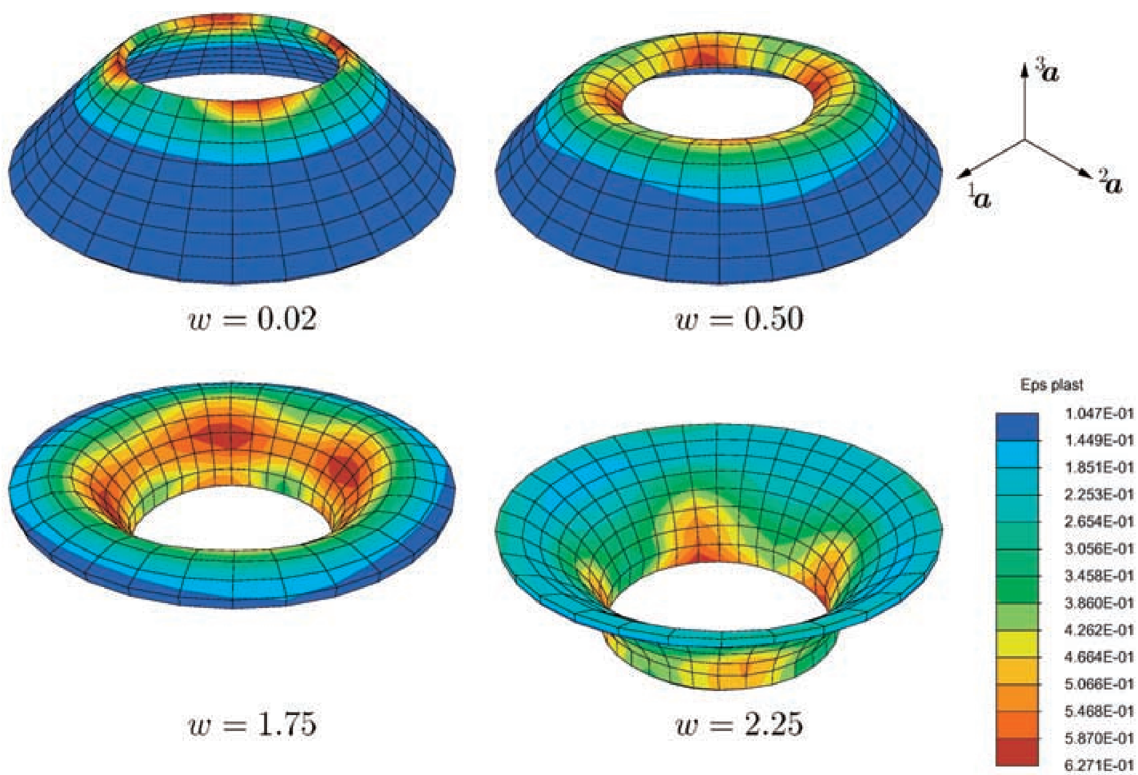

Fig. 12. Equivalent plastic strains for anisotropic material behaviour

higher values of $e^{p(m)}$ evolve with respect to the orientation of the preferred directions.

5.3
Cylindrical cup drawing from a circular blank

In this section, the implemented material model is applied for the simulation of a cylindrical cup drawing. Deep drawing is an important process in the domain of the forming technique. As an example we mention sheet metal forming in the automobile industry. For an overview we refer to e.g. Lange [10]. The tools of the deep drawing process include die, holder and punch, see Fig. 13. The geometrical data for punch, holder and die are given in cm in Fig. 14a. The blank radius is 3.95 cm and the thickness is 0.081 cm. In our simulation the blank is assumed to behave isotropic in the elastic range and orthotropic in the plastic range. Again, we choose $m = 0$ in the material model.

The nonlinear isotropic hardening law is depicted in Fig. 14. It is again approximated by piecewise linear functions. The privileged directions of the material are described by $^1\mathbf{a}$ and $^2\mathbf{a}$. The constitutive parameters are summarized as follows:

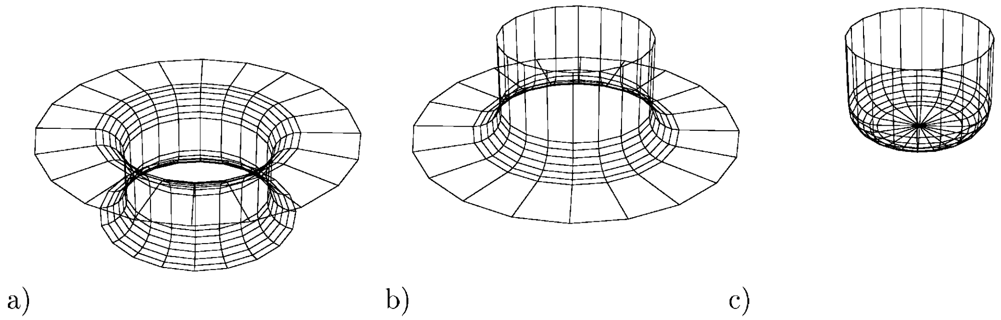

Fig. 13a–c. Tools for deep drawing process. **a** Die, **b** holder and **c** punch

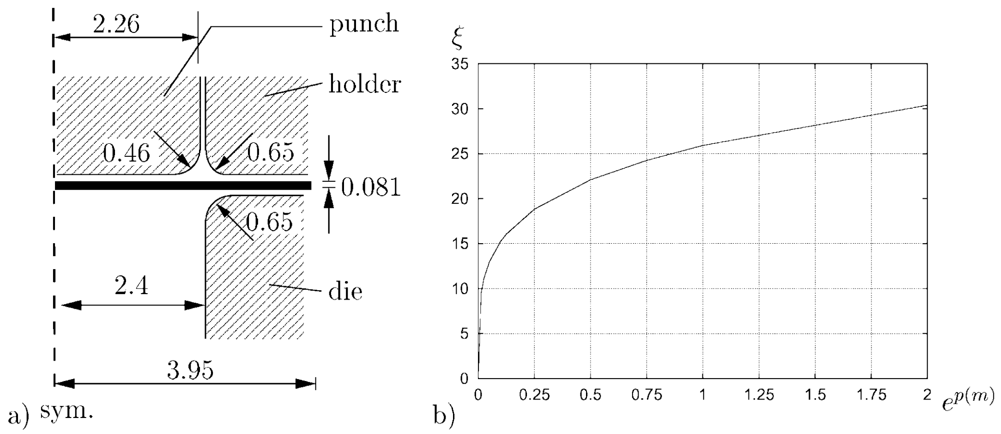

Fig. 14a, b. Deep drawing process. **a** Geometry, **b** nonlinear hardening curve

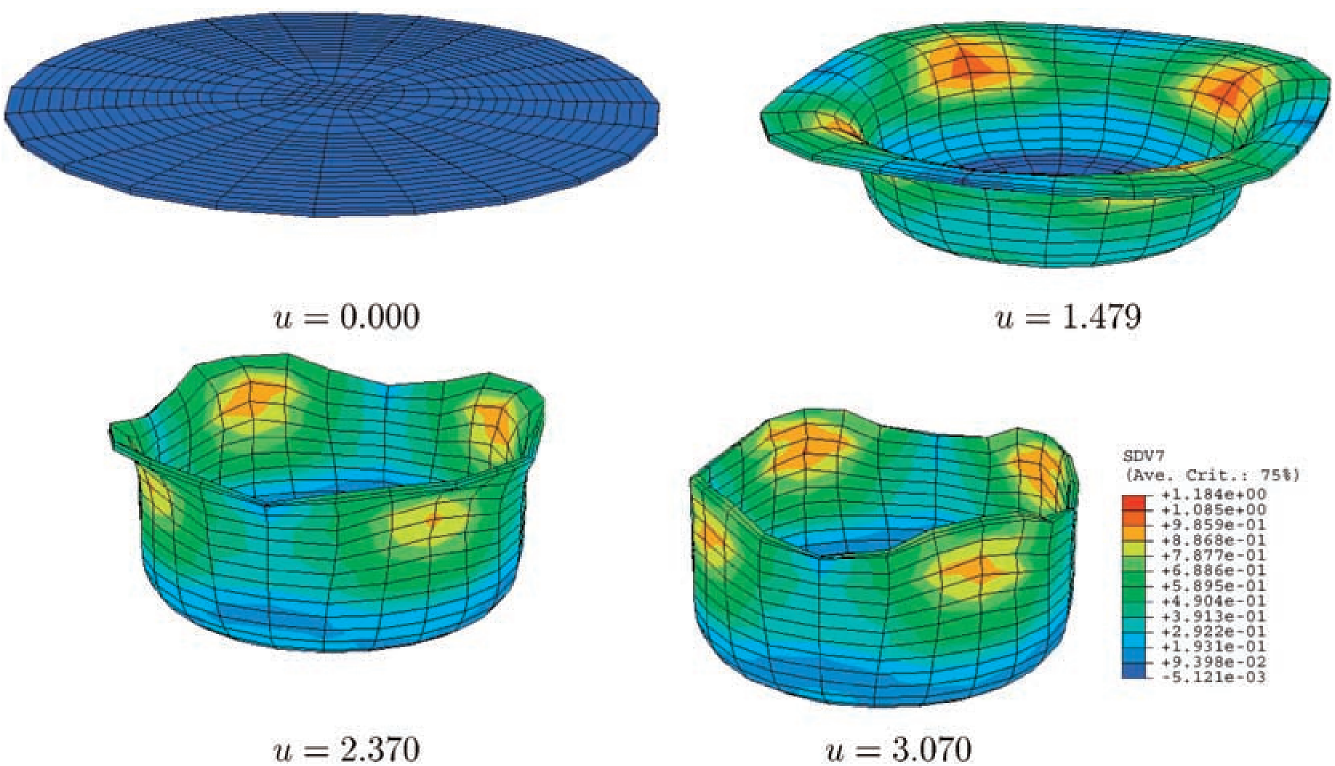

Fig. 15. Equivalent plastic strains at different punch displacements

$$E = 19600\,\mathrm{kN/cm^2}, \quad \nu = 0,3, \quad m = 0,$$
$$Y_{11} = 20.00\,\mathrm{kN/cm^2}, \quad Y_{22} = 25.00\,\mathrm{kN/cm^2},$$
$$Y_{33} = 18.00\,\mathrm{kN/cm^2}, \quad Y_{12} = 6.61\,\mathrm{kN/cm^2},$$
$$Y_{13} = 6.61\,\mathrm{kN/cm^2}, \quad Y_{23} = 6.61\,\mathrm{kN/cm^2},$$
$${}^1\mathbf{a} = [1\ ,\ 0\ ,\ 0]^{\mathrm{T}}, \quad {}^2\mathbf{a} = [0\ ,\ 1\ ,\ 0]^{\mathrm{T}} \ .$$

The blank is discretized with 936 solid elements C3D8H using ABAQUS, where 2 elements are positioned in thickness direction. Die, punch and holder are modelled as rigid bodies. Frictionless contact is considered along the interfaces. The present calculation is carried out to investigate the effect of the planar anisotropy of the blank and the earing phenomenon due to the anisotropic material behaviour.

Figure 15 shows the equivalent plastic strains of the blank for different punch displacements u. The earing phenomenon is visible. In total four ears are formed. The maximum equivalent tensile plastic strains are located at the ears.

6 Conclusions

In this paper a finite element model for orthotropic elasto–plastic material behaviour at finite strains has been presented. The theory is based on so-called generalized stress and strain measures, which allow an adaptation of constitutive models of the infinitesimal theory. The governing constitutive equations have been written in an invariant setting, where the privileged directions of the material are described by so-called structural tensors. The additive decomposition of the generalized strains is the essential kinematic assumption. As a consequence standard return algorithms can be applied to solve the set of material equations. The condition of plastic incompressibility has been fulfilled by a correction of the inelastic strains for the case $m \neq 0$. As an essential contribution we have derived explicit matrix representations for the transformation relations of the generalized stress tensors and the associated linearized expressions. This allows a simple and effective finite element implementation. The examples show the robustness of the developed formulation. The computed results show for the parameter $m = 0$ good agreement with available solutions from the literature. With the simulation of a deep drawing process the earing phenomenon of an anisotropic metal sheet has been demonstrated. To sum up, the generalized model with $m = 0$ seems to be the most suitable one within this additive framework for the analysis of anisotropic finite plasticity, in this context see also Xiao et al. [35].

Appendix A General return algorithm

The evolution laws for the plastic strains and the internal variables according to (13) are integrated approximately in time using an implicit backward Euler integration procedure

$$\begin{aligned} \mathbf{E}^p_{n+1} &= \mathbf{E}^p_n + \Delta\mathbf{E}^p_{n+1} \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \Delta\boldsymbol{\alpha}_{n+1} \\ e^p_{n+1} &= e^p_n + \gamma_{n+1}\sqrt{\frac{2}{3}\partial_{S^{(m)}}\Phi : \partial_{S^{(m)}}\Phi}\,\Bigg|_{n+1} \ , \end{aligned} \tag{A1}$$

with $\gamma_{n+1} = \Delta t \lambda_{n+1}$. The incremental plastic parameters are given with $\partial_{S^{(m)}}\Phi = \partial_\Sigma\Phi : \partial_{S^{(m)}}\boldsymbol{\Sigma} = \partial_\Sigma\Phi$ and $\partial_\beta\Phi = \partial_\Sigma\Phi : \partial_\beta\boldsymbol{\Sigma} = -\partial_\Sigma\Phi$ by

$$\begin{aligned} \Delta\mathbf{E}^p_{n+1} &= \gamma_{n+1}\partial_{S^{(m)}}\Phi_{n+1} = \gamma_{n+1}\partial_\Sigma\Phi_{n+1} \\ \Delta\boldsymbol{\alpha}_{n+1} &= -\gamma_{n+1}\partial_\beta\Phi_{n+1} = \gamma_{n+1}\partial_\Sigma\Phi_{n+1} \ . \end{aligned} \tag{A2}$$

We introduce the residual vectors

$$\begin{aligned} \mathbf{R}^p_{n+1} &= -\mathbf{E}^p_{n+1} + \mathbf{E}^p_n + \Delta\mathbf{E}^p_{n+1} = \mathbf{0} \\ \mathbf{R}^\alpha_{n+1} &= \boldsymbol{\alpha}_{n+1} - (\boldsymbol{\alpha}_n + \Delta\boldsymbol{\alpha}_{n+1}) = \mathbf{0} \\ R^\Phi_{n+1} &= \hat{\Phi}_{n+1}\left(\mathbf{S}^{(m)}_{n+1}, \boldsymbol{\beta}_{n+1}, e^p_{n+1}\right) = 0 \ , \end{aligned} \tag{A3}$$

which are solved with respect to the variables

$$\left\{\mathbf{S}^{(m)}_{n+1}, \boldsymbol{\beta}_{n+1}, \gamma_{n+1}\right\} \ . \tag{A4}$$

For this purpose a Newton iteration scheme is applied. Linearization of (A3) yields a system of linear equations

$$\begin{aligned} &\mathbf{R}^p_{n+1} + \partial_{S^{(m)}}\mathbf{R}^p_{n+1} : \Delta\mathbf{S}^{(m)} + \partial_\beta\mathbf{R}^p_{n+1} : \Delta\boldsymbol{\beta} + \partial_\gamma\mathbf{R}^p_{n+1}\Delta\gamma_{n+1} = \mathbf{0} \\ &\mathbf{R}^\alpha_{n+1} + \partial_{S^{(m)}}\mathbf{R}^\alpha_{n+1} : \Delta\mathbf{S}^{(m)} + \partial_\beta\mathbf{R}^\alpha_{n+1} : \Delta\boldsymbol{\beta} + \partial_\gamma\mathbf{R}^\alpha_{n+1}\Delta\gamma_{n+1} = \mathbf{0} \\ &R^\Phi_{n+1} + \partial_{S^{(m)}}R^\Phi_{n+1} : \Delta\mathbf{S}^{(m)} + \partial_\beta R^\Phi_{n+1} : \Delta\boldsymbol{\beta} + \partial_\gamma R^\Phi_{n+1}\Delta\gamma_{n+1} = 0 \end{aligned} \tag{A5}$$

for the stress and strain increments. Equation (A5) can be specified more precisely as

$$\begin{aligned} &\mathbf{R}^p + \boldsymbol{\Xi}^{-1} : \Delta\mathbf{S}^{(m)} + \gamma\partial^2_{\beta S^{(m)}}\Phi : \Delta\boldsymbol{\beta} + \partial_{S^{(m)}}\Phi\Delta\gamma = \mathbf{0} \\ &\mathbf{R}^\alpha + \gamma\partial^2_{S^{(m)}\beta}\Phi : \Delta\mathbf{S}^{(m)} + \boldsymbol{\Omega}^{-1} : \Delta\boldsymbol{\beta} + \partial_\beta\Phi\Delta\gamma = \mathbf{0} \\ &R^\Phi + \partial_{S^{(m)}}\Phi : \Delta\mathbf{S}^{(m)} + \partial_\beta\Phi : \Delta\boldsymbol{\beta} + \partial_\gamma\Phi\Delta\gamma = 0 \end{aligned} \tag{A6}$$

with

$$\begin{aligned} \boldsymbol{\Xi}^{-1} &:= (\mathbb{C}^{e(m)})^{-1} + \gamma_{n+1}\partial^2_{S^{(m)}S^{(m)}}\Phi_{n+1} \quad \text{and} \\ \boldsymbol{\Omega}^{-1} &:= \mathbf{H}^{-1} + \gamma_{n+1}\partial^2_{\beta\beta}\Phi_{n+1} \end{aligned} \tag{A7}$$

and the abbreviation $\mathbf{H} := \partial^2_{\alpha\alpha}\psi^{p,k}$. Formally the solution of (A6) can be written as

$$\begin{bmatrix} \Delta\mathbf{S}^{(m)}_{n+1} \\ \Delta\boldsymbol{\beta}_{n+1} \\ \Delta\gamma_{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Xi}^{-1} & \gamma_{n+1}\partial^2_{\beta S^{(m)}}\Phi_{n+1} & \partial_{S^{(m)}}\Phi_{n+1} \\ \gamma_{n+1}\partial^2_{S^{(m)}\beta}\Phi_{n+1} & \boldsymbol{\Omega}^{-1} & \partial_\beta\Phi_{n+1} \\ \partial_{S^{(m)}}\Phi_{n+1} & \partial_\beta\Phi_{n+1} & \partial_\gamma\Phi_{n+1} \end{bmatrix}^{-1} \times \begin{bmatrix} -\mathbf{R}^p \\ -\mathbf{R}^\alpha \\ -R^\Phi \end{bmatrix} \ . \tag{A8}$$

The derivative $\partial_\gamma \Phi_{n+1}$ is given by

$$\partial_\gamma \Phi = \partial_\xi \Phi \partial^2_{e^p e^p} \psi^{p,i} \sqrt{\frac{2}{3}(\partial_{S^{(m)}}\Phi : \partial_{S^{(m)}}\Phi)}\Bigg|_{n+1} \quad . \tag{A9}$$

Using submatrices we can write (A8) as follows:

$$\begin{bmatrix} \Delta \mathbf{S}^{(m)}_{n+1} \\ \Delta \boldsymbol{\beta}_{n+1} \\ \Delta \gamma_{n+1} \end{bmatrix} := \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & A_{33} \end{bmatrix} \begin{bmatrix} -\mathbf{R}^p \\ -\mathbf{R}^\alpha \\ -R^\Phi \end{bmatrix} \quad . \tag{A10}$$

Considering $\Delta \mathbf{E}^{p(l)}_{n+1} = -(\mathbb{C}^{e(m)})^{-1} \Delta \mathbf{S}^{(m)}_{n+1}$, $\Delta \boldsymbol{\alpha}^{(l)}_{n+1} = \mathbf{H}^{-1}\Delta \boldsymbol{\beta}_{n+1}$ and $\Delta \gamma^{(l)}_{n+1} = \Delta \gamma_{n+1}$ the update of the internal variables is performed as follows:

$$\begin{aligned} \mathbf{E}^{p(l+1)}_{n+1} &= \mathbf{E}^{p(l)}_{n+1} + \Delta \mathbf{E}^{p(l)}_{n+1} \\ \boldsymbol{\alpha}^{(l+1)}_{n+1} &= \boldsymbol{\alpha}^{(l)}_{n+1} + \Delta \boldsymbol{\alpha}^{(l)}_{n+1} \\ \gamma^{(l+1)}_{n+1} &= \gamma^{(l)}_{n+1} + \Delta \gamma^{(l)}_{n+1} \end{aligned} \tag{A11}$$

where l denotes the local iteration index and $\mathbf{E}^{p(0)}_{n+1} = \mathbf{E}^p_n$, $\boldsymbol{\alpha}^{(0)}_{n+1} = \boldsymbol{\alpha}_n$ and $\gamma^{(0)}_{n+1} = \gamma_n$. The next iteration step starts with the evaluation of the generalized stresses $(13)_3$ considering $(13)_1$ and the back stresses with $(13)_4$. The iteration is aborted if the components of the residual vectors

$$||R^\Phi_{n+1}|| < \text{tol}, \quad ||\mathbf{R}^p_{n+1}|| < \text{tol}, \quad ||\mathbf{R}^\alpha_{n+1}|| < \text{tol} \tag{A12}$$

vanish at the solution point.

Linearization of (A10) with respect to $\Delta \mathbf{E}^{(m)}_{n+1}$ yields the increment of the generalized stresses as

$$\Delta \mathbf{S}^{(m)}_{n+1} = \mathbb{C}^{(m)}_{ep} \Delta \mathbf{E}^{(m)}_{n+1} \quad \text{with } \mathbb{C}^{(m)}_{ep} := \mathbf{A}_{11} \tag{A13}$$

where $\mathbb{C}^{(m)}_{ep}$ denotes the so-called consistent tangent tensor.

Appendix B
Matrix representation of tangent moduli

The matrix representation of (51) considering (55) yields

$$\mathbb{C} = \mathbf{P}^T \mathbb{C}^{(m)}_{ep} \mathbf{P} + \mathbf{T}^{\mathrm{T}} \mathbf{L}_2 \mathbf{T} \tag{B1}$$

with the consistent tangent matrix according to (A13)

$$\mathbb{C}^{(m)}_{ep} = \begin{bmatrix} \mathbb{C}^{11}_{ep} & \mathbb{C}^{12}_{ep} & \mathbb{C}^{13}_{ep} & 2\mathbb{C}^{14}_{ep} & 2\mathbb{C}^{15}_{ep} & 2\mathbb{C}^{16}_{ep} \\ & \mathbb{C}^{22}_{ep} & \mathbb{C}^{23}_{ep} & 2\mathbb{C}^{24}_{ep} & 2\mathbb{C}^{25}_{ep} & 2\mathbb{C}^{26}_{ep} \\ & & \mathbb{C}^{33}_{ep} & 2\mathbb{C}^{34}_{ep} & 2\mathbb{C}^{35}_{ep} & 2\mathbb{C}^{36}_{ep} \\ & & & 4\mathbb{C}^{44}_{ep} & 4\mathbb{C}^{45}_{ep} & 4\mathbb{C}^{46}_{ep} \\ & \text{sym.} & & & 4\mathbb{C}^{55}_{ep} & 4\mathbb{C}^{56}_{ep} \\ & & & & & 4\mathbb{C}^{66}_{ep} \end{bmatrix} \tag{B2}$$

and the symmetric matrix

$$\mathbf{L}_2 = \begin{bmatrix} L_{1111} & 0 & 0 & L_{1112} & L_{1113} & 0 \\ & L_{2222} & 0 & L_{2212} & 0 & L_{2223} \\ & & L_{3333} & 0 & L_{3313} & L_{3323} \\ & & & L_{1212} & L_{1213} & L_{1223} \\ & \text{sym.} & & & L_{1313} & L_{1323} \\ & & & & & L_{2323} \end{bmatrix} . \tag{B3}$$

In case of $\lambda_1 \neq \lambda_2 \neq \lambda_3$ the components read with $\gamma^{(m)}_{AAB}$ and $\gamma^{(m)}$ according to (54)

$$\begin{aligned} L_{1111} &= 2S^{(m)}_{11}(m-1)\lambda_1^{m-2} \\ L_{2222} &= 2S^{(m)}_{22}(m-1)\lambda_2^{m-2} \\ L_{3333} &= 2S^{(m)}_{33}(m-1)\lambda_3^{m-2} \\ L_{1212} &= S^{(m)}_{11}\gamma^{(m)}_{112} + S^{(m)}_{22}\gamma^{(m)}_{221} \\ L_{1313} &= S^{(m)}_{11}\gamma^{(m)}_{113} + S^{(m)}_{33}\gamma^{(m)}_{331} \\ L_{2323} &= S^{(m)}_{22}\gamma^{(m)}_{223} + S^{(m)}_{33}\gamma^{(m)}_{332} \\ L_{1112} &= 2S^{(m)}_{12}\gamma^{(m)}_{112} \\ L_{2212} &= 2S^{(m)}_{12}\gamma^{(m)}_{221} \\ L_{1113} &= 2S^{(m)}_{13}\gamma^{(m)}_{113} \\ L_{3313} &= 2S^{(m)}_{13}\gamma^{(m)}_{331} \\ L_{2223} &= 2S^{(m)}_{23}\gamma^{(m)}_{223} \\ L_{3323} &= 2S^{(m)}_{23}\gamma^{(m)}_{332} \\ L_{1223} &= 2S^{(m)}_{13}\gamma^{(m)} \\ L_{1323} &= 2S^{(m)}_{12}\gamma^{(m)} \\ L_{1213} &= 2S^{(m)}_{23}\gamma^{(m)} \quad . \end{aligned} \tag{B4}$$

In the case of two equal eigenvalues we only specify the components which distinguish it from (B4):

$$\lambda_1 = \lambda_2 \neq \lambda_3 \quad \begin{aligned} L_{1212} &= \frac{1}{2}\left(S^{(m)}_{11} + S^{(m)}_{22}\right)(m-1)\lambda_1^{m-2} \\ L_{1112} &= S^{(m)}_{12}(m-1)\lambda_1^{m-2} \\ L_{2212} &= S^{(m)}_{12}(m-1)\lambda_1^{m-2} \\ L_{1223} &= S^{(m)}_{13}\gamma^{(m)}_{113} \\ L_{1323} &= S^{(m)}_{12}\gamma^{(m)}_{113} \\ L_{1213} &= S^{(m)}_{23}\gamma^{(m)}_{113} \end{aligned} \tag{B5}$$

$$\lambda_2 = \lambda_3 \neq \lambda_1 \quad \begin{aligned} L_{2323} &= \frac{1}{2}\left(S^{(m)}_{22} + S^{(m)}_{33}\right)(m-1)\lambda_2^{m-2} \\ L_{2223} &= S^{(m)}_{23}(m-1)\lambda_2^{m-2} \\ L_{3323} &= S^{(m)}_{23}(m-1)\lambda_2^{m-2} \\ L_{1223} &= S^{(m)}_{13}\gamma^{(m)}_{221} \\ L_{1323} &= S^{(m)}_{12}\gamma^{(m)}_{221} \\ L_{1213} &= S^{(m)}_{23}\gamma^{(m)}_{221} \end{aligned} \tag{B6}$$

$$\lambda_3 = \lambda_1 \neq \lambda_2 \quad L_{1313} = \frac{1}{2}\left(S_{11}^{(m)} + S_{33}^{(m)}\right)(m-1)\lambda_3^{m-2}$$
$$L_{1113} = S_{13}^{(m)}(m-1)\lambda_3^{m-2}$$
$$L_{3313} = S_{13}^{(m)}(m-1)\lambda_3^{m-2}$$
$$L_{1223} = S_{13}^{(m)}\gamma_{332}^{(m)}$$
$$L_{1323} = S_{12}^{(m)}\gamma_{332}^{(m)}$$
$$L_{1213} = S_{23}^{(m)}\gamma_{332}^{(m)} \ . \qquad \text{(B7)}$$

In the case of three equal eigenvalues we only specify the components which distinguish it from (B5)–(B7):

$$\lambda_1 = \lambda_2 = \lambda_3 \quad L_{1223} = \frac{1}{2}S_{13}^{(m)}(m-1)\lambda_1^{m-2}$$
$$L_{1323} = \frac{1}{2}S_{12}^{(m)}(m-1)\lambda_1^{m-2} \qquad \text{(B8)}$$
$$L_{1213} = \frac{1}{2}S_{23}^{(m)}(m-1)\lambda_1^{m-2}$$

The components $S_{AB}^{(m)}$ of the generalized stress tensor with respect to the eigenvector basis $\mathbf{N}^A$ considering (57) are evaluated as

$$\hat{\mathbf{S}} = \mathbf{T}\bar{\mathbf{S}}$$
$$\hat{\mathbf{S}} = \left[S_{11}^{(m)}, S_{22}^{(m)}, S_{33}^{(m)}, 2S_{12}^{(m)}, 2S_{13}^{(m)}, 2S_{23}^{(m)}\right]^{\mathrm{T}} \qquad \text{(B9)}$$

where $\mathbf{T}$ is given in (11). The Cartesian components of the generalized stress tensor $\bar{S}_{ij} = \mathbf{e}_i \cdot \mathbf{S}^{(m)} \cdot \mathbf{e}_j$ are organized in a vector $\bar{\mathbf{S}} = [\bar{S}_{11}, \bar{S}_{22}, \bar{S}_{33}, 2\bar{S}_{12}, 2\bar{S}_{13}, 2\bar{S}_{23}]^{\mathrm{T}}$.

References

1. **ABAQUS/Standard User's Manual 6.2** (2001) Hibbitt, Karlson & Sorensen Inc.
2. **Betten J** (1987) Formulation of anisotropic constitutive equations. In: Boehler JP (ed) Applications of Tensor Functions in Solid Mechanics. CISM Course No. 292, Springer-Verlag
3. **Boehler JP** (1987) Introduction to the invariant formulation of anisotropic constitutive equations. In: Boehler JP (ed) Applications of Tensor Functions in Solid Mechanics. CISM Course No. 292, Springer-Verlag
4. **Doyle TC, Ericksen JL** (1956) Non-linear elasticity. Adv. Appl. Mech. 4: 53–115
5. **Green AE, Naghdi PM** (1965) A general theory of an elasto–plastic continuum. Arch. Rational Mech. Anal. 18: 251–281
6. **Häusler O, Schick D, Tsakmakis Ch** (2002) Description of plastic anisotropy effects at large deformations. Part II: The case of transverse isotropy. Int. J. Plasticity (accepted for publication)
7. **Hill R** (1948) A theory of the yielding and plastic flow of anisotropic metals. Proc. Royal Soc. London A 193: 281–297
8. **Hill R** (1978) Aspects of invariance in solid mechanics. Adv. Appl. Mech. 18: 1–75
9. **Klinkel S** (2000) Theorie und Numerik eines Volumen-Schalen-Elementes bei finiten elastischen und plastischen Verzerrungen. Dissertation, Universität Karlsruhe (TH), Institut für Baustatik
10. **Lange K** (1984) Umformtechnik. Handbuch für Industrie und Wissenschaft, Band 3: Blechbearbeitung, Springer-Verlag
11. **Miehe C** (1998) A constitutive frame of elastoplasticity at large strains based on the notion of a plastic metric. Int. J. Solids Struct. 35: 3859–3897
12. **Miehe C, Schröder J, Schotte J** (1999) Computational homogenization analysis in finite plasticity: simulation of texture development in polycrystalline materials. Comp. Meth. Appl. Mech. Eng. 171: 387–418
13. **Miehe C, Lambrecht M** (2001) Algorithms for computation of stresses and elasticity moduli in terms of Seth–Hills family of generalized strain tensors. Commun. Numer. Meth. Eng. 17: 337–353
14. **De Borst R, Feenstra PH** (1990) Studies in anisotropic plasticity with reference to the Hill criterion. Int. J. Numer. Meth. Eng. 29: 315–336

15. **Ogden RW** (1984) Non-linear elastic deformations. Ellis Horwood: Chichester
16. **Papadopoulos P, Lu J** (1998) A general framework for the numerical solution of problems in finite elasto–plasticity. Comput. Meth. Appl. Mech. Eng. 159: 1–18
17. **Papadopoulos P, Lu J** (2001) On the formulation and numerical solution of problems in anisotropic finite plasticity. Comput. Meth. Appl. Mech. Eng. 190(37, 38): 4889–4910
18. **Peric D, Owen DRJ, Honnor ME** (1992) A model for finite strain elasto–plasticity based on logarithmic strains: computational issues. Comput. Meth. Appl. Mech. Eng. 94: 35–61
19. **Schröder J** (1996) Theoretische und algorithmische Konzepte zur phänomenologischen Beschreibung anisotropen Materialverhaltens. Bericht Nr.: I-1, Institut für Mechanik (Bauwesen), Lehrstuhl I, Universität Stuttgart
20. **Seth BR** (1964) Generalized strain measure with application to physical problems. In: Rainer M, Abir D (eds) Second–order Effects in Elasticity, Plasticity and Fluid Dynamics. Pergamon Press, Oxford, pp. 162–172
21. **Simo JC, Armero F** (1992) Geometrically nonlinear enhanced strain mixed methods and the method of incompatible modes. Int. J. Numer. Meth. Eng. 33: 1413–1449
22. **Simo JC, Hughes TJR** (1998) Computational Inelasticity. Springer, NY, Berlin, Heidelberg
23. **Smith GF** (1965) On isotropic integrity bases. Arch. Rat. Mech. An. 18: 282–292
24. **Smith GF** (1971) On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors. Int. J. Eng. Sci. 19: 899–916
25. **Smith GF, Smith MM, Rivlin RS** (1963) Integrity basis for a symmetric tensor and a vector. The crystal classes. Arch. Rational Mech. Anal. 12: 93–133
26. **Spencer AJM** (1971) Theory of invariants. In: Eringen AC (ed) Continuum Physics Vol. 1, Academic Press, New York, pp. 239–353
27. **Taylor RL** (2001) FEAP – A Finite Element Analysis Program: Theory Manual, University of California, Berkeley, http://www.ce.berkeley.edu/~rlt
28. **Taylor RL** (2001) FEAP – A Finite Element Analysis Program: Users Manual, University of California, Berkeley, http://www.ce.berkeley.edu/~rlt
29. **Tsakmakis Ch** (2002) Description of plastic anisotropy effects at large deformations. Part I: Restrictions imposed by the second law and the postulate of Il'iushin. Int. J. Plasticity (accepted for publication)
30. **Wagner W, Klinkel S, Gruttmann F** (2002) Elastic and plastic analysis of thin-walled structures using improved hexahedral elements. Comput. Struct. 80: 857–869
31. **Wang CC** (1969) On representations for isotropic functions: Part I. Isotropic functions of symmetric tensors and vectors. Arch. Rational Mech. Anal. 33: 249–267
32. **Wang CC** (1971) Corrigendum to my recent papers on: Representations for isotropic functions. Arch. Rational Mech. Anal. 43: 392–395

33. **Wriggers P, Eberlein R, Reese S** (1996) A comparison of three-dimensional continuum and shell elements for finite plasticity. Int. J. Solids Struct. 33: 3309–3326
34. **Wriggers P, Reese S** (1996) A note on enhanced strain methods for large deformations. Comput. Meth. Appl. Mech. Eng. 135: 201–209
35. **Xiao H, Bruhns OT, Meyers A** (2000) A consistent finite elasto–plasticity theory combining additive and multiplicative decomposition of the stretching and the deformation gradient. Int. J. Plasticity 16: 143–177
36. **Zienkiewicz OC, Taylor RL** (2000) The finite element method, 2: Solid and Fluid Mechanics, Dynamics and Nonlinearity. 4th edition, McGraw-Hill, NY