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Convex Polytopes, Dihedral Angles, Mean Curvature and Scalar Curvature

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Abstract
We approximate boundaries of convex polytopes $X \subset \mathbb{R}^n$ by smooth hypersurfaces $Y = Y_\varepsilon$ with *positive mean curvatures* and, by using basic geometric relations between the scalar curvatures of Riemannian manifolds and the mean curvatures of their boundaries, establish *lower bound on the dihedral angles* of X.

Keywords Geometric measure theory · Approximation theorems · Polytopes · Riemannian manifolds

Mathematics Subject Classification 52A20 · 52B70 · 53C20 · 53C23

1 Combinatorial Spread, $\square_\rangle$-Spread and $\square_\rangle$-Inequality

Let $X \subset \mathbb{R}^n$ be a compact convex polytope let ∂X denote its (topologically spherical) boundary and let $X^{\circledcirc}$ be *the dual convex tessellation* of the sphere S^{n-1}, i.e. where $(n-k-1)$-cells are the sets of the (unit normal to the) supporting hyperplanes to X along the interiors of the k-faces of X.

Let $E = E(X) \subset S^{n-1}$ be the edge graph of $X^{\circledcirc}$. Combinatorially, this is the $(n-2)$-*adjacency graph*, where the set of the $(n-1)$-*faces* F of X is taken for the set vertices and where the edges e in E correspond to the pairs of $(n-2)$-*adjacent faces*:

vertices v_1 and v_2 are joined by an edge $e = e_{12}$, whenever the corresponding *closed faces* $\bar{F}_1, \bar{F}_2 \subset X$ meet over a closed $(n-2)$-face, namely $\bar{F}_{12} = F_1 \cap F_2 \subset X$.

Dedicated to the memory of Eli Goodman.

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Remark on Adjacency and on Simple Polytopes. Recall that a convex polytope X is *simple* if

$$\text{adjacent} \implies (n-2)\text{-adjacent},$$

where "adjacent" signifies that the intersection $F_1 \cap F_2$ is *non-empty*, i.e. F_1 and F_2 meet at a vertex in X.

The *combinatorial distance* $dist_{comb}(F_1, F_2)$ is the length of the shortest path in E between the corresponding vertices corresponding to F_1 and F_2.

For instance, these distances between opposite faces in the n-cube $[-1,1]^n$ are equal to 2.

Let $\angle_{1,2} = \angle(F_1, F_2)$ denote the dihedral angle between $(n-2)$-*adjacent faces* and let $\rangle$ stands for the complementary angle,

$$\rangle_{1,2} = \pi - \angle_{1,2},$$

that is the spherical arc length of the edge $e_{12} \subset S^{n-1}$ dual to the $(n-2)$-face $F_{12} = F_1 \cap F_2$.

$\rangle$-Angular Distance. The *angular distance* or $\rangle$-*distance* $dist_{\rangle}(F_1, F_2)$ between (not necessarily $(n-2)$-adjacent) $(n-1)$-faces F_1 and F_2 in X is the minimum of the spherical lengths of edge paths in E between the vertices of $X^{\circledcirc}$ dual to these faces.

Accordingly, the $\rangle$- (*angular*) *distance* between (unions of) sets of faces, say $\mathcal{F}_1, \mathcal{F}_2 \subset V$, is the minimum of the $\rangle$-distances between the faces in these sets,

$$dist_{\rangle}(\mathcal{F}_1, \mathcal{F}_2) = \min_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} dist_{\rangle}(F_1, F_2).$$

Cubical Example. The $\rangle$-distances between opposite faces of the n-cube $\square^n = [-1,1]^n$ are equal to π.

Combinatorial and Angular Spreads. Let $\square^k_{comb}(X)$ be the *maximum of the numbers* $d \geq 0$, such that X admits a continuous map to the k-cube,

$$\Phi : X \to \square^k = [-1,1]^k,$$

with the following properties.[1]

$\bullet_{comb}$ The Φ-pullbacks of the $(k-1)$-faces from $\square^k$ are unions of $(n-1)$-faces in X.

$\bullet_{dist}$ The combinatorial distances between the pullbacks of the opposite cubical faces $\underline{F}_{i\mp} \subset \square^k$ are $\geq d$,

$$dist_{comb}(\Phi(\underline{F}_{i-}), \Phi(\underline{F}_{i+})) \geq d, \ i = 1, \ldots, k.$$

$\bullet_{deg}$ The the induced relative homology homomorphism

[1] If no such map exists, then let $\square^k_{comb}(X) = 0$.

$$\Phi_* : H_k(X, \Phi^{-1}(\partial\Box^k)) \to H_k(\Box^k, \partial\Box^k) = \mathbb{Z},$$

doesn't vanish.

(If $k = n$, this is equivalent to $\Phi^{-1}(\partial\Box^n) = \partial X$ and to non-vanishing of the degree of the map $\Phi : \partial X \to \partial\Box^n$. For instance, homeomorphisms $\Phi : X \to \Box^n$ satisfy tis condition.[2])

Similarly define the angular spread $\Box^k_{\rangle}(X)$ with the $dist_{\rangle}$ inequality instead of $dist_{comb}$:

$$\bullet_{dist_\rangle} \qquad dist_\rangle(\Phi(\underline{F}_{i-}), \Phi(\underline{F}_{i+})) \geq d, \ i = 1, \ldots, k.$$

Observe that the combinatorial and the angular spreads satisfy

$$\Box^n(X) \leq \Box^{n-1}(X) \leq \ldots \leq \Box^1(X) = diam(X),$$

where the diameter refers to the combinatorial and to the angular distances correspondingly,

and that

$$\Box^n(X) \geq \frac{\Box^{n-1}(X)}{2n+2} \text{ for all convex } n\text{-polytopes } X.$$

$\Box^3_N$***-Example.*** Let $\Box^3_N$ be the subdivision of the 3-cube $\Box^3 = \Box^3_1$, where each 2-face is subdivided into N^2 equal squares in an obvious way. (If you wish it to became simple, ε-perturb with $\varepsilon << 1/N$ the edges of) these small squares, such that the resulting subdivision $\Box^3_{N,\varepsilon}$ has three squares at each vertex.) Then the combinatorial $\Box^3$-spread of the so subdivided cube is $N + 1$.

"Random" Example. Apparently, the combinatorial $\Box$-spread of a suitably defined *random n-polytope* with M faces (see Sect. 8) grows, roughly, as $\sqrt[n-1]{M}$.

1.A. Angular Spread Theorem. *The top-dimensional $\rangle$-spreads, of all compact convex n-polytopes $X \subset \mathbb{R}^n$ are bounded by a universal constant,*

$$\Box^n_{\rangle}(X) \leq D = D_n \leq 2(n-1)\sqrt{n}.$$

We shall prove this in Sect. 5 by reduction to the *normalized mean curvature mapping theorem* (see Sect. 2) the proof of which (see Sect. 2.1) depends on the index theory for Dirac operators on Riemannian spin manifolds with positive scalar curvatures (see [13, Sects. 3.1.2 and 3.5]).

1.B. Corollary. The minimum of the complementary angles of X is bounded by the combinatorial spread $\Box^n_{comb}(X)$ as follows,

$$\rangle_{min}(X) \leq D\frac{1}{\Box^n_{comb}(X)},$$

[2] The topological degree is defined for all *continuous equidimensional* maps f between *oriented manifolds*, e.g. such as our spherical ∂X and $\partial\Box^n$, where the non-vanishing condition $deg(f) \neq 0$ doesn't depend on the orientation for connected (orientable) manifolds. Also the degree is defined for the boundary respecting maps between manifolds with boundaries.

for the above constant D.

1.C. Conjecture. The above $D = D_n$ is equal to+ π.[3]

Remark Probably 1.A, 1.B and 1.C generalize to all convex tessellation of S^{n-1}.[4] (See next section for more about it.)

1.1 Combinatorially Large Polytopes with Large Complementary Angles

Dirac operators notwithstanding, evaluation of the ranges of possible values of the dihedral angles of polytopes depending on their combinatorial types and/or determination of the combinatorial and metric geometries of polytopes with all complementary angles $\rangle(X)$ bounded from below remains problematic.

It is known here (Steinitz?) that if $\rangle_{min}(X) \geq \pi/2$, then X is the product of simplices. But – this was pointed out to me by Karim Adiprasito three years ago – there is no bound on the number of faces of X for $\rangle_{min}(X) \geq \alpha$ for small $\alpha > 0$. Later, I found the following on the web.

$\pi/4$-*Example.* Chop off the corners from the prism $\bigcirc_i \times [0, \delta] \subset \mathbb{R}^3$, where $\bigcirc_N$ is the regular N-gon, $N = 3, 4, \ldots$ and $\delta = 10^{-N}$, such that this "chopping" fully consumes the δ-edges of the prisms, and such that all complementary dihedral angles of the resulting polytopes X_N are mutually equal and satisfy $\alpha_N \to \pi/4$ for $N \to \infty$. see [21].

Recently, Karim informed me [1] that infinity of combinatorial types of convex polytopes with $\rangle_{min}(X) \geq \alpha$ exists if an only if $\alpha < \pi/3$.

Below, in a similar spirit, we construct n-polytopes, which

"infinitely stretch" in $n-2$ directions, while having all complementary dihedral angles bounded from below.

1.1.A. Skyscrapers. Given convex polytopes $\mathbf{0} \in X_1 \subset \ldots \subset X_N \subset \mathbb{R}^k$ and numbers $h_1 > \ldots > h_N \geq 0$ let

$$\overset{N}{\bigwedge\!\!\!\bigwedge}\{X_i\} = \overset{N}{\underset{\{h_i\}}{\bigwedge\!\!\!\bigwedge}}\{X_i\} \subset \mathbb{R}^k \times \mathbb{R}_+ \subset \mathbb{R}^{k+1},$$

denote the intersection of the cones of heights h_i over X_i, where the top vertices of these cones lie on the "vertical" axes $\mathbf{0} \times \mathbb{R}_+ \subset \mathbb{R}^k \times \mathbb{R}_+$,

$$\overset{N}{\bigwedge\!\!\!\bigwedge}\{X_i\} = \bigcap_i^N cone_{h_i}(X_i).$$

Such a $\bigwedge\!\!\!\bigwedge = \bigwedge\!\!\!\bigwedge^N = \bigwedge\!\!\!\bigwedge\{X_i\}$ is called *a skyscraper* with the *bottom* X_1 and the *top* X_N if the following holds:

[3] In view of what Karim Adiprasito recently told me, this D is better to be the one from 1.C rather than from 1.B.

[4] According to a remark by the referee (if I properly understood it) there exist convex tessellations, which don't come from convex polytopes, as it follows by dualization of classical examples of non-regular triangulations, see [6], [2], [25] and appendix in [3] with references therein.

$\bullet_\cap$ the bounary of $\bigwedge\!\!\!\bigwedge\{X_i\}$ has *non-empty intersections* with *all* open $(n-1)$-faces of the cones $cone_{h_i}(X_i^{h_i})$ and

two closed side faces $F_1, F_2 \subset \bigwedge\!\!\!\bigwedge\{X_i\}$ *do not intersect unless they contained in faces of the cone over some* X_i,

$$F_1, F_2 \subset \partial cone_{h_i}(X_i),$$

or in faces of two consecutive cones.

$$F_1 \subset \partial cone_{h_i}(X_i) \text{ and } F_2 \subset \partial cone_{h_{i+1}}(X_{i+1}).$$

Notice that $\bullet_\cap$ implies the following:

$\bullet_\#$ the number of the k-faces of $\bigwedge\!\!\!\bigwedge\{X_i\}$ satisfies

$$\#_k \bigwedge\!\!\!\bigwedge\{X_i\} = 1 + \sum_{i=1}^{N} \#_{k-1} X_i.$$

Observe that $\bullet_\cap$ can be achieved with suitable h_i and homothetically scaled X_i.

$\bullet_{\{\lambda_i\}}$ *Given* X_i *there exist* $h_1 > \ldots > h_i > \ldots > h_N$ *and* $0 < \lambda_1 < \ldots < \lambda_i < \ldots < \lambda_N$, *such that* $\bullet_\cap$ *is satisfied by* $\bigwedge\!\!\!\bigwedge_{\{h_i\}}^{N}\{\lambda_i X_i\}$.[5]

The usefulness of his for our purpose is due to the following obvious property of skyscrapers.

1.1.B. Large* $\rangle$ *Lemma. Let $\bigwedge\!\!\!\bigwedge_{\{h_i\}}\{X_i\}$ be a *a skyscraper* (with the bottom X_1 and the top X_N), such that the complementary angles of all X_i as well as (by definition acute) angles between the pairs of hyperplanes, which define the faces of consecutive $\underline{X}_i$ and of $\underline{X}_{i+1}$, are strictly bounded from below by $\alpha > 0$.

Then there exist a (large) positive number C such that vertically C-stretched $\bigwedge\!\!\!\bigwedge$, that is $C \uparrow \bigwedge\!\!\!\bigwedge = \bigwedge\!\!\!\bigwedge\{C \cdot h_i\}\{X_i\}$ (a true skyscraper) has the complementary dihedral angles between the side faces bounded from below by α

$$\rangle_{side}\left(C \uparrow \bigwedge\!\!\!\bigwedge\right) > \min(\alpha, \pi/2),$$

while these angles at the bottom face are $> \pi/2$.

$\pi/3$-*Example.* Let $X_i \subset \mathbb{R}^2$ be regular triangles, where $X_i = \underline{X}_{i+2}$ and $X_2 = -X_1$. Then the complementary side dihedral angles $\rangle$ of the corresponding skyscraper

$$X_{\hexagon} = X_{\hexagon}(N, C) = C \uparrow \bigwedge\!\!\!\bigwedge = \bigwedge\!\!\!\bigwedge_{\{C\cdot(N-i+1)\}} \{(N+1)^i X_i\},$$

satisfy

$$\rangle_{side}(X_{\hexagon}(C, N)) \to \pi/3 \text{ for } C \to \infty,$$

[5] $\lambda X = \{\lambda x\}_{x\in X} \subset \mathbb{R}^k$.

while

$$\rangle_{bottom}(X_{\hexagon}(C, N)) \to \pi/2.$$

(This, I guess, must be exactly Adiprasito's example.)

Remarks (a) The essential difference of $X_{\hexagon}(C, N))$ from the above "pruned" prism $\hexagon_N \times [0, \delta]$ is that

the combinatorial diameters of $X_{\hexagon}(C, N)$ tend to infinity for $N \to \infty$.
In fact,

$$diam_{comb}(X_{\hexagon}(C, N)) = N + 1.$$

(b) The Cartesian products of m copies of $X_{\hexagon}$ provide examples of $3m$-polytopes, $m = 1, 2....$, with all complementary angles $\geq \frac{\pi}{3} - \varepsilon$ for all $\varepsilon > 0$ and with arbitrarily large $\square^m_{comb}$-spreads.

(c) *The directional limit set of the faces* of the 3-polytops $X_{\hexagon}(C, N))$ for $N, C \to \infty$, that is the Hausdorff limit of the sets of vertices of the dual tessellations $X^{\circledcirc}_{\hexagon}(C, N))$ of S^2, is a 7-point set: a regular hexagon on the equator plus the south pole, while similar X_N with suitably rotated triangles $\underline{X}_i$ may have arbitrary limit sets on the equator.

Question. Is this limit set always *discrete* away from an equatorial circle $S^1 \subset S^2$?
(Adiprasito bound $\rangle_{min} \leq \pi/3$ makes this plausible for $\rangle_{min} \underset{N\to\infty}{\to} \pi/3$.)

1.1.C. Skyscrapers on Skyscrapers. Finiteness of the directional limit sets of Skyscraper $\bigwedge\!\!\!\bigwedge\{X_i\}$ allows a lower bound on the complementary dihedral angles of double skyscraper $\bigwedge\!\!\!\bigwedge\{\bigwedge\!\!\!\bigwedge_j\{X_i\}\}$, etc.

$\pi/3(2n-5)$*-Example.* Let $\rho_{n,k}(\triangle) \subset \mathbb{R}^2$, $n = 3, 4, ..., k = 0, ...2n-5$, be the regular triangle rotated by $\rho_{n,k} = k\pi/3(2n-5)$ and define by induction on m polytopes $\mathcal{X}_m = \mathcal{X}_m(n, N_{n,m-2}, C_{n,m-2}) \subset \mathbb{R}^m = \mathbb{R}^2 \times \mathbb{R}^{m-2}_+$, $m = 3.4, ...n$, as follows.

Let

$$\mathcal{X}_3 = \bigwedge\!\!\!\bigwedge\{\rho_{n,0}(\triangle), \rho_{n,1}(\triangle)|N_{n,1}\} = C_{n,1} \uparrow \bigwedge\!\!\!\bigwedge_{\{h_i\}}^{2Nn,1} \{\lambda_i|\rho_{n,0}(\triangle), \rho_{n,1}(\triangle)|N_{n,1}\},$$

where $\{\lambda_i|A, B|N\}$ stands for $\{\lambda_1 A, \lambda_2 B, ...\lambda_{2N-1}A, \lambda_{2N}B\}$ and where the constants h_i and λ_i are chosen as in 1.1.A and where eventually $C_{n,1} \to \infty$ as earlier.

Then we slightly modify $\mathcal{X}_3$ by turning the base 2-face $F_{base} = \triangle = \mathcal{X}_3 \cap \mathbb{R}^2 \times 0$ by $\pi/4$, call the result $\mathcal{X}'_3$ and inductively define

$$\mathcal{X}'_{m+1} = \bigwedge\!\!\!\bigwedge\{\rho_{n,2(m-3)}(\mathcal{X}'_m), \rho_{n,2(m-3)+1}(\mathcal{X}'_m)|N_{n,m+1}\},$$

where the rotations ρ apply to the $\mathbb{R}^2$-factor in $\mathbb{R}^2 \times \mathbb{R}^{m-2}_+ \supset \mathcal{X}'_m$ and where the implicit h, λ and C- constants are adjusted as earlier.

It is easy to show – we leave checking this to the reader that
★□ *the* $\square^{n-2}_{comb}$ *stretch* of $\mathcal{X}'_n$ can be made *arbitrarily large* with all $N_{n,m} \to \infty$[6]
and that
★⟩ *the complementary dihedral angles* of $\mathcal{X}'_n$ satisfy

$$\rangle(\mathcal{X}'_n) \geq \frac{\pi}{3(2n-5)} - \varepsilon,$$

where $\varepsilon > 0$ can be made *arbitrarily small* with $C_{n,m} \to \infty$.

Probbaly a skyscraper pattern is present in all polytopes X with $\square^{n-2}_{comb} >> \frac{1}{\rangle_{min}}$. We partly justify this (conclusively only for $dim(X) = n = 3$) by looking at the dual spherical tessellations $X^{\circledcirc}$ as follows.

Given a cellular tessellation $T^{\circledcirc}$, e.g. a triangulation, of an $(n-1)$-manifold Y, define the combinatorial distance between cells, as earlier, by the lengths of minimal chains of cells, denote this by $dist_{\circledcirc}$ and define the combinatorial $\square^k_{\circledcirc}(Y) = \square^k_{comb}(T^{\circledcirc})$, including $diam_{\circledcirc} = \square^1_{\circledcirc}$, via continuous maps $\Phi : Y \to \square^k$ by just saying "cell" instead of "face".

1.1.D. Large Subdomain Lemma. Let $T^{\circledcirc}$ be a convex tessellation of the unit sphere S^{n-1}, where the cells are called $\triangle$, and let $B^{\circledcirc}_s \subset S^{n-1}$, $s \in S^{n-1}$, denote the union of closed cells which contain s.

Then, given a (small) number $v > 0$,

there exists a connected cellular (i.e. a union of cells) domain $U^{\circledcirc}$ *in the sphere* S^{n-1}, *such that the spherical volumes of the "*$\circledcirc$*-balls"* $B^{\circledcirc}_s$ *around all points in* U *are bounded by*

$$vol(B^{\circledcirc}_s) \leq v, s \in U^{\circledcirc},$$

and such that the $\square_{comb}$*-spreads of* $U^{\circledcirc}$ *are bounded from below by these of* $T^{\circledcirc}$ *as follows:*

$$\square^k_{\circledcirc}(U^{\circledcirc}) \geq const \cdot v \cdot \square^k_{\circledcirc}(U^{\circledcirc}) - 1,\ const = const_n > (10n)^{-n}.$$

Indeed, the cardinalities $N = N(v)$ of subsets $S \subset S^{n-1}$ of "v-*thick*" points $s \in S^{n-1}$, i.e. with $vol(B^{\circledcirc}_s) \geq v$, such that no pair of different points from S is contained in the same closed cell of $T^{\circledcirc}$, are bounded by $N = \frac{vol(S^{n-1})}{v}$, while the combinatorial diameters of all $\circledcirc$-balls, are at most 2,

$$diam_{comb}(B^{\circledcirc}_s) \leq 2 \text{ for al } s \in S^{n-1}.$$

Therefore, given a map $\Phi = \{\Phi_1, ..., \Phi_k\} : S^{n-1} \to [-1, 1]^k$ from the definition of $\square^k_{comb}$, there exist gaps between pairs of *neighbouring images*, say $t_i = \Phi_i(s_i)$, $t'_i =$

[6] The boundary of $\mathcal{X}'_n$ (as well as that of $\mathcal{X}_n$) with the $dist_{comb}$-geometry is shaped roughly the same as (*properly coarsely homotopy equivalent to*) the rectangular solid $[0, N_{n,1}] \times ... \times [0, N_{n,n-2}]$.

$\Phi_i(s'_i) \in [-1, 1]$, $i = 1, ...k$, of pairs of "v-thick" vertices s_i and s'_i, such that

$$dist_{comb}(\Phi_i^{-1}[-1, t_i], \Phi_i^{-1}[t'_i, 1]) \geq \frac{1}{N} dist_{comb}(\Phi_i^{-1}(-1), \Phi_i^{-1}(1)) - 2(N+1),$$

and the "$B^{\circledcirc}$-enlargement" of the intersection $U_\cap$ of the pullbacks $\Phi^{-1}[t_i, t'_i] \subset S^{n-1}$ is taken for the required $U^{\circledcirc}$

$$U_\cap = \bigcap_{i=1}^{k} \Phi^{-1}[t_i, t'_i] \text{ and } U^{\circledcirc} = \bigcup_{s \in U_\cap} B_s^{\circledcirc}.$$

Here is another obvious observation.

1.1.E. Narrow Band Lemma. If the edges (1-cells) from $T^{\circledcirc}$, adjacent to a vertex $s \in S^{n-1}$, have lengths $\geq l$ and if $vol(B_s^{\circledcirc}) \leq v$, then $B_s^{\circledcirc} \subset S^{n-1}$ is *contained in the δ-neighbourhood of an equatorial sphere* $S^{n-2} \subset S^{n-1}$,

$$B_s^{\circledcirc} \subset U_\delta(S^{n-2}),$$

where this $\delta = \delta_n(l, v) > 0$ satisfies for all n and $l > 0$,

$$\delta_n(l, v) \to 0 \text{ for } v \to 0.$$

Moreover, if all $(n-2)$-cells $\triangle^{n-2}$ adjacent to s have

$$vol_{n-2}(\triangle^{n-2}) \geq a > 0,$$

then this equatorial $S^{n-2} \subset S^{n-1}$ is *unique up to an ε-perturbation*, i.e. all equators for which $U_\delta(S^{n-2}) \supset B^{\circledcirc}$ lie within distance ε one from another, where

$$\varepsilon = \varepsilon_n(v, \delta, a) \to 0 \text{ for } v \to 0.$$

1.1.F. Corollary: Elementary Bound on $\square^2_{comb}$. A lower bound by $a > 0$ on the $(n-2)$-volumes of $(n-2)$-cells in a convex tessellation $T^{\circledcirc}$ of S^{n-1} implies an upper bound on the combinatorial $\square^2$-spread of $T^{\circledcirc}$,

$$vol_{n-2}(F^{n-2}) \geq a > 0 \implies \square^2(T^{\circledcirc}) \leq \Theta_n(a).$$

where Θ_n is a (bounded monotone decreasing) function in $a > 0$.

(If $n = 2$, this is just a qualitative version of 1.B.)

Proof It follows from 1.1.F. and 1.1.E that the above $U^{\circledcirc} \subset S^{n-1}$ It follows from 1.1.F. and 1is contained in a δ'-neighbourhood of an equator $S^{n-2} \subset S^{n-1}$, where, for a fixed $a > 0$,

$$\square^k_{comb}(T^{\circledcirc}) \to \infty \implies \square^k_{comb}(U^{\circledcirc}) \to \infty \text{ for all } k = 1.2....$$

Then one sees that, for δ' much smaller than a, this $U^{\circledcirc}$ admits a cellular map of *degree one* from the cylinder $S^{n-2} \times [0, 1]$, which is decomposed into $m \times M$ cells, which are products of cells of some triangulation of S^{n-2} into m-simplices and a decompositions of $[0, 1]$ into M segments, where m is bounded by a constant depending on a.

It follows that $\square^2(U^{\circledcirc})$ is also $\leq m$, hence, it is bounded in terms of $a > 0$. □

Remark The above shows that if $\square^1_{comb}(T^{\circledcirc}) \to \infty$ with the $(n-2)$ volumes of all $(n-2)$-cells bounded from below by a, then the unit sphere S^{n-2} acquires several limit tessellations with the same bound on the volumes of their $(n-2)$-cells and some cells spanned by vertices of different tessellations.

Then, for instance, by looking on pairs of such tessellations, one recovers a special case of Adiprasito's result for $n = 3$.

1.1.G. Conjecture. For all $k = 1, ...n-2$, a lower bound on the k-volumes of the k-cells in a convex tessellation $T^{\circledcirc}$ of S^{n-1} implies an upper bound on the combinatorial $\square^{n-k}$-spread of $T^{\circledcirc}$.

Conversely,

there exist convex tessellations $T^{\circledcirc}$ of S^{n-1} with arbitrary large $\square^{n-k-1}_{comb}(T^{\circledcirc})$ and with the volumes of all k-cells bounded away from zero.

Moreover, there are such $T^{\circledcirc}$, which are dual of convex polytopes $X \subset \mathbb{R}^n$. (A quantitative form of a special case of this is suggested in 6.B.)

2 Manifolds with Corners, Mean Convexity and Distance $dist^{\natural}_{\rangle}$

Let X be a smooth n-manifold with corners, i.e. locally, at all $x \in X$, it is diffeomorphic to a convex polytope $Q = Q_x \subset \mathbb{R}^n$.

For instance, diffeomorphic images of convex polytopes are manifolds with corners.

Also recall that the *mean curvature* of a cooriented hypersurface in a Riemannian manifold is the sum of the principal curvatures.

Example The R-sphere $S^{n-1}(R) \subset \mathbb{R}^n$ and the round cylinder $S^{n-2} \times \mathbb{R}^1 \subset \mathbb{R}^n$ satisfy

$$mean.curv(S^n) = \frac{n-1}{R} \text{ and } mean.curv(S^{n-2}(R) \times \mathbb{R}^1) = \frac{n-2}{R}.$$

A Riemannian manifold with corners is called *mean convex* if all its $(n-1)$ -faces $F \subset \partial X$ have non-negative mean curvatures.

For instance, convex domains in $\mathbb{R}^n$ with corners are mean convex.

Given a smooth curve in the boundary of a manifold with corners, say $\gamma \subset \partial X$, which doesn't intersect $(n-2)$-faces of X and which meets all $(n-2)$-faces of X transversally, say at $x_i \in \partial X$, $i = 1, ..., j$, let

$$length^{\natural}_{\rangle}(\gamma) = length^{\natural}(\gamma) + \sum_{i=1}^{j} \rangle_{x_i},$$

where $length^{\natural}(\gamma) = \int_\gamma mean.curv(\partial X))d\gamma$, where $\rangle_{x_i}$ are the complementary dihedral angles, $\rangle_{x_i} = \pi - \angle_{x_i}$ and where the dihedral angle $\angle_{x_i}$ of X at the point x_i is the angle between the (naturally cooriented) $(n-1)$-dimensional tangent spaces $T_i, T_i' \subset T_{x_i}(X)$ to the two $(n-1)$-faces adjacent to the (n-2)-face, which contains x_i.

Next, assuming X is mean convex and $x_1, x_2 \in \partial X$ are contained *inside* $(n-1)$-faces, let

$$dist^{\natural}_{\rangle}(x_1, x_2) = \inf_{\gamma_{1,2}} length^{\natural}_{\rangle}(\gamma_{1,2}),$$

where the infimum is taken over the above kind of curves $\gamma_{1,2} \subset \partial X$ between x_1 and x_2.

Although this $dist^{\natural}_{\rangle}$ is defined not for all points and it may vanish at some pairs of non-equal points, we treat it as a true distance; in particular, we define the corresponding distance between $(n-1)$-faces[7] $F_1, F_2 \subset \partial X$ in the usual way:

$$dist^{\natural}_{\rangle}(F_1, F_2) = \inf_{x_1, x_2} dist^{\natural}_{\rangle}(x_1, x_2) \text{ for } x_1 \in F_1 \text{ and } x_2 \in F_2.$$

If the boundary of X contains no corners, i.e. it is smooth, then the corresponding distance is denoted $dist^{\natural}$. This is a true positive distance if X is *strictly mean convex*, i.e. $mean.curv(\partial X) > 0$.

Semi(in)stability of $dist^{\natural}$. An arbitrarily C^1-small perturbation of a smooth convex hypersurface $Y \subset \mathbb{R}^n$, $n \geq 3$, may significantly diminish the metric $dist^{\natural}$ on Y.

For instance,

the unit sphere $S^{n-1} \subset \mathbb{R}^n$, which has $diam^{\natural}(Y) = \frac{(n-1)\pi}{2}$, can be C^1-approximated by smooth convex hypersurfaces Y_ε with $diam^{\natural}(Y_\varepsilon) = \pi + \varepsilon$ for all $\varepsilon > 0$ as follows.[8]

Let $A = A_{N,\delta} \subset S^{n-1}$, where $N > \delta^{-2n}$, be the union of regular equatorial N-gons in general position, such for all pairs of points in S^{n-1}, there are our N-gons passing ε close to both points. Let $B_A \subset \mathbb{R}^n$ be the intersection of subspaces bounded by the hyperplanes tangent to S^{n-1} at the vertices of the N-gons and let $Y(N, \delta, \epsilon)$ be the boundary of the ϵ-neighbourhood of B_A for $0 < \epsilon \leq N^{-2n}$.

Then

$$diam^{\natural}(Y(N, \delta, \epsilon)) \to \pi \text{ for } \delta \to 0,$$

and, for the same reason,

all $\natural$-spreads of $Y(N, \delta, \epsilon)$ *converge to the ordinary spreads of the unit sphere*,

$$\square^k_{\natural}(Y(N, \delta, \epsilon) \to \square^k(S^{n-1}) = \frac{1}{n-1}\square^k_{\natural}(S^{n-1}),$$

[7] An open k-face in X is understood as a maximal *connected* subset where X is locally diffeomorphic to a polyhedral k-face.

[8] This can't be done with $\varepsilon = 0$.

where, observe for instance, $\square^n(S^{n-1}) = 2\arcsin\frac{1}{\sqrt{n}}$.[9]

But the metric $dist^\natural$ of a compact convex hypersurface Y can't *everywhere* significantly increase under small C^0-perturbations of Y.

In fact, if $Y = S^{n-1}$, this follows from theorem 2.A below, which is a special case of the *normalized mean curvature mapping theorem* from [13, Sect. 3.5] and which makes the key ingredient of the proof of 1.A.

2.A. Euclidean $dist^\natural$-Non-contraction Theorem. Let X be a compact oriented *mean convex* Riemannian n-manifold with smooth boundary, let $B \subset \mathbb{R}^n$ be a smooth compact *convex* domain. e.g the unit ball, and let $f : \partial X \to \partial B$ be a smooth map, which which has *non-zero degree*.

If X has *non-negative* scalar curvature, $Sc(X) \geq 0$, and if X is *spin*,[10] then f *can't be strictly $dist^\natural$-decreasing*: *there exists a pair of points $x_1, x_2 \in \partial X$, such that*

$$dist^\natural(x_1, x_2) \leq dist^\natural(f(x_1), f(x_2)).$$

Remarks (a) As far as the proof of 1.A is concerned, one needs only a very special case of this theorem, namely, where X is also a smooth convex domain in $\mathbb{R}^n$ and Y is the unit ball in $\mathbb{R}^n$. Amazingly, however, even in this case, the only available proof of 2.A relies on the spin geometry and Dirac operators (see below).

(b) The assumption $Sc(X) \geq 0$ is, obviously, essential: there is no curvature constrains on the boundaries of general Riemannian manifolds.

But what is non-obvious, is how sensitive the geometry of ∂X may be to the sign of the scalar curvature of X.

For instance, in agreement with the *positive mass theorem* in general relativity, there is no Riemannian metric g on the unit ball $B^n \subset \mathbb{R}^n$ with $Sc(g) > 0$ and with $dist_g^\natural$ (non-strictly) greater than the original $dist^\natural$ on the unit sphere $S^{n-1} = \partial B^n \subset \mathbb{R}^n$,

(c) It is unknown if the spin condition is essential.

The second components of the proof of 1.A - this is an actual contribution of the present paper, is the following.

2.B. $dist^\natural$-Approximation **Theorem.**[11] Let X be a *compact mean convex* Riemannian n-manifold with corners. Then, for all $\varepsilon > 0$, there exists a smooth mean convex hypersurface $\mathcal{Y} = \mathcal{Y}_\varepsilon \subset X$ and a homeomorphism $\psi = \psi_\varepsilon : \partial X \to \mathcal{Y}$ with the following properties.

$\bullet_1$ The map ψ is *ε-close to the identity,* $dist(\psi_\varepsilon(x), x) \leq \varepsilon$ for all $x \in \partial X$.

[9] Here $\square^n(S^{n-1})$ means $\square^n(S^n_+)$, i.e. this is defined via maps $S^{n-1} \to \partial\square^n$ of positive degrees.

[10] An oriented vector bundle $T \to X$ is *spin* if the associated principle bundle $G \to X$ with the fibres $G_x = SL(n, \mathbb{R})$, $n = rank(T)$, admits a double covering $s : \tilde{G} \to G$, such that the pullbacks of the fibers $s^{-1}(G_x)$ are *connected*; an orientable manifold X is *spin* if the tangent bundle $T(X)$ is spin. A necessary and sufficient condition for spin is *vanishing of the second Stiefel-Whitney class* $w_2(T) \in H^2(X; \mathbb{Z}_2)$; for instance, if $\pi_2(X) = 0$, then the universal covering of X is spin. It is also known that all 3-manifolds are spin, while the complex projective plane $\mathbb{C}P^2$ is non-spin.

[11] Smoothing the corners + "Dirac with boundary" is also used by Brendle [5] in the proof of his polyhedral scalar curvature rigidity theorem.

$\bullet_2$ The $dist^\natural$ in $\mathcal{Y}$ is *greater than* $dist^\natural_\rangle$ *in* ∂X *up to an* ε*-error*,

$$dist^\natural_\mathcal{Y}(\psi(x_1), \psi(x_2)) \geq (1-\varepsilon) dist^\natural_\rangle(x_1, x_2)$$

for all pairs of points positioned within distances $\geq \varepsilon$ from the corners of X.

We shall proof this in Sect. 4, where we also show that, in the case of convex domains $X \subset \mathbb{R}^n$, the approximation is possible with *strictly convex* $\mathcal{Y}$.

Then we shall see in Sect. 5 that 2.A and 2.B (trivially) imply the following generalization of 1.A.

2.C. Riemannian Angular Spread Theorem. Let X be a compact orientable *mean convex* Riemannian n-manifold with corners and with *non-negative scalar curvature*, $Sc(X) \geq 0$. If X is *spin*, then the cubical $\rangle$-spread of X is universally bounded as follows,

$$\square^n_\rangle(X) \leq D = D_n \leq 2(n-1)\sqrt{n}.$$

Technical Strictness Remark. Non-strictness of mean convexity may create inconvenience, e.g. a terminological one in dealing with vanishing "metrics". But this is mainly irrelevant, since, in the cases of our immediate interest, e.g. for compact smooth hypersurfaces in $\mathbb{R}^n$, strictness of mean convexity, $mean.curv \geq 0 \rightsquigarrow mean.curv > 0$, can is easily achieved by arbitrarily C^∞-small perturbations.

In general, with a minor analytic effort, one can C^∞-approximate a compact connected mean boundary ∂X of a Riemannian manifold X corners by a *strictly convex* hypersurfaces $Y \subset X$, unless this ∂X is smooth (no corners) with zero mean curvature.

Thus, one may assume strictness of mean convexity in the present paper whenever this helps to simplify understanding.

2.1 *Sc*-Normalized Metric g°, Derivation of 2.A from the LGSL Theorem and $\rangle$-Capillary Problem

The counterpart of $g^\natural$ for Riemannian manifolds $X = (X, g)$ with positive scalar curvatures

$$Sc(X, x) = Sc_g(x) > 0$$

is the Sc-normalized Riemannian metric

$$g^\circ = g^\circ(x) = Sc_g(x) g(x)$$

on X.

The basic geometric property of this g° is the following special case of the Llarull -Goette-Semmelmann-Listing theorem (see 3.1.2 in [13] and references therein)[12]

[12] The essential ingredients of the proofs in [14, 17, 18] is a sharp evaluation of eigenvalues of certain operators $\mathcal{R}$ in moduli over Clifford algebras, where these $\mathcal{R}$ are algebraically associated with the curvature operators R of the underlying Riemannian manifolds X. This suggests a direct Clifford algebraic approach

2.1.A. Euclidean $dist^\circ$***-Area Non-contraction Theorem.*** Let X be a connected orientable n-dimensional Riemannian manifold with $Sc(X) > 0$ and let $\underline{X} \subset \mathbb{R}^{n+1}$ a closed convex hypersurface.

Let $f : X \to \underline{X}$ be a smooth g°-area decreasing map, that is

$$area_{\underline{g}^\circ} f(S) < area_{g^\circ}(S)$$

for all smooth surfaces $S \subset X$, where $\underline{g}$ is the induced Riemannian metric in $X \subset \mathbb{R}^{n+1}$.

If X is spin, then the map f has degree zero (hence, f is contractible).

Remarks (a) It is unknown, not even for $n = 4$, if the spin condition is essential.

(b) The proof of 2.1.A depends on the index and vanishing theorems for the Dirac operator on X with coefficients in the vector bundle induced by f from a unitary bundle on $\underline{X}$.

The simplest kind of result of this kind, where the proof is technically very simple (see [7], says that

for *no Riemannian metric g* on S^n the corresponding g° *can be significantly greater* than the spherical metric:

2.1.B. *If* $dist_{g^\circ} \geq C dist_{S^n}$, *then* $C \leq C_n$ *for a universal constant* C_n.

(In fact, $C_n = \sqrt{n(n-1)}$ by Llarull's theorem [18].)

(c) If g has constant scalar curvature, then 2.1.B (but not 2.1.A) can be proven by the technique of the geometric measure theory following ideas from [22].

Moreover:

2.1.C. *If a metric g on the unit ball* $B^n \subset \mathbb{R}^n$ *satisfies* $Sc(g) \geq C_n$, *then the identity map* $id : (B^n, g) \to (B^n, g_{Eucl})$ *can't be distance decreasing.*

This is proven in [9] for $n \leq 7$ and extended to all n in [12] modulo [23], and directly in [19].

(d) The obvious counterpart of 2.1.A for open manifolds fails to be true.

2.1.D. Example. The Euclidean space $\mathbb{R}^n$, $n \geq 2$, admits a Riemannian metric g with $Sc(g) > 1$ and such that g° is greater than the Euclidean metric.

$$g^\circ \geq g_{Eucl}.$$

(Notice that g° for such a g is complete, but, (see [9]), g can't be complete.)

Proof Recall that the scalar curvature of the metric $g_\phi = dx^2 + \phi^2(x)dy^2$ on the (x, y)-plane is

$$Sc(g_\phi(x)) = -2\frac{\phi''(x)}{\phi(x)}.$$

to the geometry of convex polytopes, where the complementary dihedral angle play the role of R (compare with [24]).

Thus, if $\phi(x)$ is a strictly concave positive function on the open interval (0, 1), such that the integrals $\int_0^{1/2} \frac{\phi''(x)}{\phi(x)} dx$ and $\int_{1/2}^{1} \frac{\phi''(x)}{\phi(x)} dx$ diverge, then the metric g_ϕ° on the band $U = (0, 1) \times (-\infty \times \infty)$ is complete. Moreover, for all $\varepsilon > 0$ there obviously exists a distance decreasing diffeomorphism from (U, g_ϕ°) onto $\mathbb{R}^2$.

Now let $\phi(x)$ be equal x^α near $x = 0$ and to $(1 - x)^\alpha$ near $x = 1$ for $0 < \alpha < 1$, observe that these integral diverge and make our example with the obvious distance decreasing diffeomorphism $U \times \mathbb{R}^{n-2} \to \mathbb{R}^n$.

On Reduction of 2.A to 2.1.A. This is achieved for a manifold X with a (mean convex) boundary by applying 2.1.A to the double $\mathbb{D}(X)$ with a a suitably smoothed metric on it (see [13, Sect. 3.5]). A more direct but analytically more involved proof of 2.A based on the the index theorem for manifolds with boundaries was given in [20].

Then, on the next level of sophistication, the index theory directly applies to manifolds with corners [24].

This, formally speaking, delivers a two line proof of 1.A, but my unsatisfactory understanding of the techniques developed in [24] makes me reluctant to make such a shortcut in the proof. □

On Capillary Geometry of X. The above example highlights the difficulty of applying the geometric measure theory to g° and $g^\natural$, but it doesn't fully rule out such applications.

Here is an instance of what one may expect of such an application.

Let X be a mean convex Riemannian n-manifold with corners and with positive scalar curvature and let $F_\mp^{n-1} \subset \partial X$ be two faces positioned *far away one from another in a suitable sense*, where the weakest condition (which may fail to be sufficient) would be a lower bound on the distance $dist_\rangle^\natural$ between them: $dist_\rangle^\natural(F_-^{n-1}, F_+^{n-1}) \geq const_n$, where, ideally, $const_n = \pi$.

Then one wants to have a smooth hypersurface $Y \subset X$ with $\partial Y \subset \partial X$ transversal to the faces of X and a smooth positive function $\psi(y)$ on Y, such that the the ψ-warped product of Y with the circle, $X_\rtimes = (Y \times \mathbb{T}^1, g_\rtimes)$, for $g_\rtimes = g_Y + \psi^2 dt^2$, where g_Y is the induced Riemannian metric in Y, such that the following conditions are satisfied:

$\bullet_{Sc}$ the metric $g_\rtimes$ has positive scalar curvature,

$\bullet_{mean}$ the (boundary of the) manifold $X_\rtimes$ is mean convex,

$\bullet_{dist}$ the $dist_\rangle^\natural$-distances between $(n-1)$-faces in $X_\rtimes$ are bounded from below, possibly, times a controlled $(1 + \delta_n)$-factor, by the $dist_\rangle^\natural$-distances between the corresponding faces in X.

This would allow an inductive proof of (a sharp version?) of 1.A, where, observe, the expected $Y \subset X$, say for $dim(X) = 3$ is a minimal surface (or something of this kind), which, even for 3-polytopes $X \subset \mathbb{R}^3$ is by no means flat. (Compare with [11, 12, 16] and [13, Sect. 5.81]).

3 Rounding the Corners and *dist*$^\natural$-Approximation of Simple Polytopes

Let $X \subset \mathbb{R}^n$ be a convex polytope and $\nu : \mathbb{R}^n \to X$ be the *normal projection*, that is $\nu(x) \in X$ is the nearest point point to X, i.e.

$$dist(x, \nu(x)) = dist(x, X), x \in \mathbb{R}^n,$$

and let $X_\circ = X_{\circ_\varepsilon} \supset X$, $\varepsilon > 0$, be the ε-neighbourhood of X that is the set of points $x \in \mathbb{R}^n$ with $dist(x, X) \leq \varepsilon$.

Observe the following (compare with [10, Sect. 5.7] and 11.3 in [12]).

$\bullet_{\cup G_k}$ The boundary $\partial X_\circ$ is equal to the union of *closures of the pullbacks of the open k-faces* $F^k \subset X$, $k = 0, 1, ..., n-1$ intersected with ∂X_ε, denoted

$$G_k = \nu^{-1}(F^k) \cap \partial X_\circ \subset \partial X_\varepsilon,$$

where such a $G_k \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ is isometric to the product of the corresponding face $F^k \subset \mathbb{R}^k$ by a convex ε-spherical polyhedron (dual to the normal section of F^k) denoted

$$F^k_\perp \subset S^{n-k-1}(\varepsilon) \subset \mathbb{R}^{n-k}.$$

Thus, *the principal curvatures of* $G_k \subset \mathbb{R}^n$ are

$$\bullet_{curv} \qquad \underbrace{0, ...0}_{k}, \underbrace{\frac{1}{\varepsilon}, ..., \frac{1}{\varepsilon}}_{n-1-k}$$

and their mean curvatures satisfy

$$\bullet_{mean} \qquad mean.curv(G_k)) = \frac{1}{\varepsilon^{n-k-1}}.$$

$\bullet_{C^1}$ Different G_k, which intersect across parts of their boundaries, have equal tangent spaces at their common points; thus the boundary $\partial X_\circ \subset \mathbb{R}^n$ is C^1- actually $C^{1,1}$-smooth.

Quadratic Form $g^\natural_{\circ_\varepsilon}$ *and Definition of* $dist^\natural_\circ = g^\natural_{\circ_\varepsilon}$. Let $g^\natural_{\circ_\varepsilon}$ be the product of the induced Riemannian metric on the hypersurface $\partial X_\varepsilon \subset \mathbb{R}^n$ by the squared mean curvature of this hypersurface,

$$g\natural_{\circ_\varepsilon} = (mean.curv)^2 g_{Eucl}|\partial X_{\circ_\varepsilon}$$

and observe that the metric defined with this Riemannian form $g^\natural_\varepsilon$ is exactly our $dist^\natural_{\partial X_{\circ_\varepsilon}}$, which is denoted here $dist^\natural_{\circ_\varepsilon}$.

3.A. $dist^{\natural}_{\circ}$**-Convergence Theorem.** *If X is a simple polytope, then the $g^{\natural}_{\varepsilon}$-distance converges to the $\rangle$-distance,*

$$dist^{\natural}_{\circ_\varepsilon}(K_1, K_2) \underset{\varepsilon\to 0}{\to} dist_{\rangle}(F_1, F_2)$$

for all pairs of compact subsets in open $(n-1)$-faces $F_1, F_2 \subset X$.

$$K_1 \subset F_1, K_2 \subset F_2 \subset X.$$

$\square$-*Example.* If $X = \square^n = [-1, 1]^n$ is the n-cube, where, as we know, $dist_{\rangle}$ between opposite $(n-1)$-faces is π, the $g^{\natural}_{\circ_\varepsilon}$-distance between the corresponding faces in $\square^n$ is only $\pi/2$. To get the full π, one needs to go ε away from the boundaries of these faces.

Proof [13] Let $Q \subset \mathbb{R}^n$ be a convex polyhedral n-dimensional cone and $R \subset \partial Q_\circ = \partial Q_{\circ_1}$ be the complement to the flat part of $\partial Q_\circ$, that is the union of all G_k with $k \neq n-1$.

Observe that this R is a connected $(n-1)$-manifold with a boundary, where the connected components of this boundary are equal to the boundaries of the $(n-1)$-faces of Q. $\square$

3.B. Minimal Path Lemma. The shortest paths $\gamma \subset R$ between different connected components $\partial_1, \partial_2 \subset \partial R$ are geodesic segments contained in the subsets $G_k = F^k \times F^k_\perp \subset R \subset \partial Q_{\varepsilon=1}$, or in the intersection of several such subsets.

Consequently,

the Riemannian distance between ∂_1 and ∂_2 is equal to the spherical distance between the intersection of ∂_1 and ∂_2 with the spherical polytope $G_0 = Q_{\varepsilon=1} \cap S^{n-1}$.

$$dist(\partial_1, \partial_2) = dist_{S^{n-1}}(\partial_1 \cap S^{n-1}, \partial_1 \cap S^{n-1}).$$

Proof A priori, γ (which is a C^1-smooth curve) is composed of several geodesic segments contained in different G_k But since *all* geodesic segments in all G_k are *distance minimizing*, γ is equal to the geodesic continuation of its initial segment, say γ_1 in some G_k; thus γ stays in this very G_k all along. $\square$

Now, let a path $\gamma^{\natural} \in \partial X_\circ$ implement the distance $dist^{\natural}$ between two flat cells in $\partial X_\circ$, say between $G_{n-1,1}$ and $G_{n-1,2}$ The length of this path is equal to the sum of $dist^{\natural}$ between components, say ∂_1 and ∂_1, of the boundary of the non-flat part $R \subset \partial X_\circ$ crossed by $\gamma\natural$.

If X is simple and all $G_k = F_k \times \Delta^{n-1-k}$, where Δ^{n-1-k} are spherical simplices, these distances, because of 3.B, can implemented by geodesic segments in G_k with $k = n-2$ and 3.A follows.

***About Non-simple** X*. Examples show that 3.A fails to be true for non-simple polyhedra X, but, due to 3.B it allows a a modification applicable to non-simple X.

[13] Compare with [10, Sect. 5.7] and [13, Sect. 3.1.2(8)].

Namely, the $(n-2)$-adjacency graph E must be replaced by the *full adjacency graph* $\mathcal{E}_+(X) \supset E(X)$, which, similarly to E, has the $(n-1)$-faces for vertices and where the edges correspond to pairs of $(n-1)$-faces which meet at 0-faces (vertices) of X and where the lengths of these edges are defined by the corresponding angles between these faces.

However the resulting version of 1.A for non-simple X doesn't bring anything new since it follows from the "simple" case by a generic perturbation of the $(n-2)$-faces of X.

4 Locally Conical Hypersurfaces and the *dist*$^\natural$-Approximation Theorem for Non-simple *X*

4.A. Conical Function Lemma. Let $Y \subset \mathbb{R}^m$ be a (possibly unbounded, e.g. conical) convex polytope. Then, for all $\varepsilon > 0$ there exists a *positive concave,* function $\phi = \phi_\varepsilon : Y \to \mathbb{R}_+$, which is piecewise smooth in the interior of Y, which *vanishes on the boundary* ∂Y and which satisfies the following four conditions.[14]

$\bullet_\varepsilon$ The *directional derivatives* of ϕ at all *boundary points* $y \in \partial Y$ are *bounded* in absolute values by ε, or equivalently ϕ is ε-*Lipschitz*:

$$|\phi(y_1) - \phi(y_2)| \leq \varepsilon \cdot dist(y_1, y_2) \text{ for all } y_1, y_2 \in X.$$

$\bullet_{curv}$ The *principal curvatures* of the graph $\Gamma_\phi \subset \mathbb{R}^m \times \mathbb{R}_+$ at the smooth points of ϕ are everywhere *strictly positive.*

$\bullet_{mean}$ The mean curvature of Γ_ϕ is uniformly positive on compact parts of Y at smooth points $(y, \phi(y)) \in \Gamma_\phi$,

$$mean.curv(\Gamma_\phi, (y, \phi(y)) \geq \epsilon(y) > 0.$$

for a positive continuous function $\epsilon(y)$ on Y.

Moreover, for all $(m-2)$-faces $F \subset \partial Y$,

$$\bullet_{1/d} \qquad mean.curv(\Gamma_\phi(y, \phi(y))) \geq const \frac{1}{dist(y, F)}$$

for some positive constant $const = const_{P,\varepsilon} > 0$ and all interior points $y \in Y$, where ϕ is smooth.

Proof The existence of ϕ is obvious for $m = 1$ and the general case follows by induction in m as follows.

Represent Y by the intersection of the wedges $W_i \subset \mathbb{R}^m$, $i = 1, ..., j$, which are based on the 1-faces $F_i^1 \subset Y$,

$$Y = \bigcap_{i=1}^{j} W_i, \text{ for } W_i = C_i \times L_i, \text{ and } C_i \subset \mathbb{R}_i^{m-1},$$

[14] With a little extra effort one can make ϕ smooth in the interior of Y.

where

$L_i \subset \mathbb{R}^m$ are the straight lines, which extend the 1-faces $F_i^1 \subset Y$;
$\mathbb{R}_i^{m-1} \subset \mathbb{R}^m$ are normal spaces to the faces F_i^1 at some points $y_i \subset F_i^1$;
$C_i \subset \mathbb{R}_i^{m-1}$ are the convex tangent cones to Y at the points y_i, that are the conical extensions of the intersections of $\mathbb{R}_i^{m-1}$ with small neighbourhoods of y_i in Y.

Let $\phi_i(c)$ be concave functions in the cones C_i, which satisfy the four conditions $\bullet_\varepsilon$, $\bullet_{curv}$, $\bullet_{mean}$, $\bullet_{1/d}$,
let $\bar\phi_i(c,l) = \phi_i(c)$ for $(c,l) \in W_i = C_i \times L_i \subset \mathbb{R}^m$
and let $\bar\phi$ be the minimum of generic λ_i-perturbations of the functions $\bar\phi_i$ on Y,

$$\bar\phi(y) = \min_i \lambda_i \phi_i(y),\ y \in Y.$$

for small *generic* $\lambda_i > 0$.[15]

Clearly, the function $\bar\phi$ satisfies $\bullet_\varepsilon$, $\bullet_{curv}$, $\bullet_{mean}$, but it may fail $\bullet_{1/d}$ at the vertices $y_\nu \in Y$.

To correct this, modify $\bar\phi$ at y_ν as follows. Let $U_\nu \subset Y$ be (very) small (pyramidal) neighbourhoods of $y_i \in Y$, which are bounded in Y by hyperplanes cutting y_ν away from Y, let

$$\bar Y = Y \setminus \bigcup_\nu U_\nu$$

be the correspondingly truncated Y and let $\phi(y)$ be the *smallest concave* function on Y, which is equal to $\bar\phi$ on $\bar Y$ and which vanishes on the boundary of Y.

In geometric term, the convex body $Y_\phi^+ \subset \mathbb{R}^m \times \mathbb{R}_+$ under the graph $\Gamma_\phi \subset \mathbb{R}^m \times \mathbb{R}_+$ is obtained by firstly cutting away y_ν from $Y_{\bar\phi}^+ \subset \mathbb{R}^m \times \mathbb{R}_+$ by vertical half-hyperplanes $H_\nu^+ \subset \mathbb{R}^m \times \mathbb{R}_+$ and then adding the cones from y_ν over the intersection $Y_{\bar\phi}^+ \cap H_\nu^+$ to the resulted truncated $Y_{\bar\phi}^+$.

Now, clearly, the mean curvature of Γ_ϕ does blow-up as $1/d$ for the distance d to the $(m-2)$-faces of Y and the proof of 4.A is concluded.

Proof of the** $dist^\natural$**-approximation theorem 2.B for convex polytopes. Let $X \subset \mathbb{R}^m$ be a compact convex polytope and let $X_\varepsilon^+ \subset \mathbb{R}^n$ be obtained by adding the subgraphs of the functions $\phi = \phi_\varepsilon$ on all $(n-1)$-faces Y of X to X.

The following five properties of X_ε^+ trivially follow from 4.A.

$\bullet_\delta$ The set X_ε^+ is *pinched* between X and a (small) δ-neighbourhood of X,

$$X \subset X_\varepsilon^+ \subset U_\delta(X) \subset \mathbb{R}^n,\ \text{where } \delta \to 0 \text{ for } \varepsilon \to 0.$$

$\bullet_{conv}$ If $\varepsilon > 0$ is sufficiently small, then X_ε^+ is *convex*.

[15] Generic λ_i are needed to assure piecewise smoothness of $\bar\phi(y)$ in the interior of Y.

$\bullet_{n-2}$ The intersection of the boundary of X^+ with X is equal to the *union of the closed* $(n-2)$*-faces* of X,

$$\partial X_\varepsilon^+ \cap X = \partial X_\varepsilon^+ \cap \partial X = \bigcup_{i=1,\ldots,k\leq n-2} F_i^k.$$

$\bullet_\angle$ The *dihedral angles* of X_ε^+ along $(n-2)$-faces of X (contained in ∂X_ε^+)[16] are bounded by the dihedral angles of X between these faces as follows,

$$\angle^+ \leq \angle + 2\varepsilon.$$

$\bullet_{1/d}$ The mean curvature of ∂X_ε^+ at smooth points $x \in \partial X_\varepsilon^+$ satisfies

$$mean.curv(\partial X_\varepsilon^+, x) \geq const \frac{1}{dist(x, F^{n-3})}$$

for some $const > 0$ and all $(n-3)$-faces F^{n-3} of X.

It follows, that paths $\gamma \subset \partial X_\varepsilon^+$, which approach F^{n-3} have *infinite* $g^\natural$-lengths; hence $g^\natural$-shortest paths cross $(n-2)$-faces away from $(n-3)$ faces.

Then an additionally C^2-smoothed boundary Y of the $\varepsilon_\circ$-neighbourhood $U = U_{\varepsilon_\circ}(X_\varepsilon^+)$ serves as the required approximation of X by the (trivial) argument from 3.A.

Generalization to Mean Convex Manifolds* X *with Corners. Think of X as a mean convex domain with corners in a larger Riemannian manifold, say $W \supset X$ and construct $X_\varepsilon^+ \subset W$ in three steps.

1. Make the $(n-1)$-faces F_i^{n-1} of X *strictly* mean convex by C^∞-perturbations, while keeping these faces unperturbed on the parts of their boundaries which are close to $(n-3)$-faces, i.e. on the intersections $\partial F_i^{n-1} \cap U_\varepsilon(F_j^{n-3})$.

This is done by linearizing the problem as it is done in the first proof of $(\bigstar_>)$ in [12, Sect. 11.2][17]

Warning. One can't, in general, achieve this while keeping the faces fixed everywhere on their boundaries as it was done for convex $X \subset \mathbb{R}^n$.

For instance, if X is a locally convex geodesic polygon in a Riemannian surface W, then an edge F^1 in X can be approximated by a strictly convex curve with the same ends as F^1, if and only if F^1, which itself is a geodesic segment, contains *no conjugate points*.

2. At the second step one make the mean curvature of the faces blow up at the $(n-3)$-faces with the rate $1/d$. as in the above convex case. In fact, since this blow-

[16] These are the angles between the pairs of extremal supporting hyperplanes to X_ε^+ at the points $x \in F^{n-2} \subset X \cap X_\varepsilon^+$.

[17] This argument is outlined in [12] for *simple* X, i.e. where the $(n-1)$-faces intersect transversally and thus the combinatorial structure of X remains stable under small perturbations, while in the present case, one needs to keep the perturbation fixed on ∂F_i^{n-1} near the $(n-3)$-faces to preserve the combinatorial structure of X. In any case, all this is a minor matter and one doesn't loose much by assuming that X is strictly mean convex to start with. Also notice that second "variational proof" of $(\bigstar_>)$ in [12] is invalid.

up property is invariant under diffeomorphisms, one can perform it locally in normal geodesic coordinates and then glue these together by a partition of unity argument.

Notice that this is unneeded if X is simple, where one goes directly to the third step.

3. Once 1 and 2 are done and one arrives at a *strictly* mean convex X_ε, which satisfy the above $\bullet_{1.d}$, then, as earlier, one takes the C^∞-smoothed boundary of a small $\varepsilon_\circ$-neighbourgood $U_{\varepsilon_\circ}(X_\varepsilon)$ for the required approximation $\mathcal{Y}$ of ∂X (compare with 5.7 in [10] and 11.4 in [12]). □

Remark It would take a couple of extra pages to explicitly write down the (quite boring) details of the above argument but it would add nothing new to what we have already seen in the convex case. ing

Convexly Stratified Manifolds. The step 2 in the above argument takes X out of the category of manifolds with corners, where the new manifolds are locally diffeomorphic not to convex polytopes but to certain smoothly stratified convex subsets $\underline{X} \subset \mathbb{R}^n$, such, for instance, as cones over smooth convex bodies in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$.

The most general class $\underline{\mathcal{X}}_{gen}$ of such $\underline{X}$, where the statement of theorem 2.A makes sense, consists of closed convex domains $\underline{X}$, such that the boundaries of X are piecewise smooth in the complements of closed $(n-3)$-dimensional subset $Z \subset \partial\underline{X}$.

Probably, the proof of theorem 2.A can be extended to the corresponding class $\mathcal{X}$ of mean convex Riemannian manifolds X locally diffeomorphic to such $\underline{X}$.

5 Lipschitz Maps and the Proof of Theorems 1.A. and 2.C

Here is an essential, albeit elementary (and trivial), geometric fact one needs.

5.A. Lipschitz Mapping Lemma. Let Y be a closed orientable Riemannian $(n-1)$-manifold and ϕ be a continuous map from Y to the boundary of the n-cube $\square^n = [-1,1]^n$,

$$\phi : Y \to \partial\square^n,$$

such that the distances between the pullbacks of the opposite faces $\square^{n-1}_{i\pm} \subset \square^n$, $i = 1, ..., n$, satisfy

$$dist(\phi^{-1}(\square^{n-1}_{i+}, \phi^{-1}(\square^{n-1}_{i-})) > D.$$

Then the composition of ϕ with the obvious radial homeomorphism from $\partial\square^n$ to the unit sphere

$$\partial\square^n \to S^{n-1} \subset \square^n$$

is *homotopic to a smooth map*,

$$f : Y \to S^n,$$

such that the differential of f satisfies

$$||df|| < \frac{2\sqrt{n}}{D}.$$

Proof Let $\delta_i(y)$ be the distance functions to $\phi^{-1}(\square^{n-1}_{i-}) \subset Y$ truncated by $D' > D$ such that

$$D < D' < \max_i dist(\phi^{-1}(\square^{n-1}_{i+}, \square^{n-1}_{i-}),$$

namely,

$$\delta_i(y) = \min(dist(y, \phi^{-1}(\square^{n-1}_{i-})), D')$$

and observe that the map

$$\Delta = \left(\frac{2}{D'}\delta_i(y) - 1, ..., \frac{2}{D'}\delta_n(y) - 1\right)$$

sends

$$Y \to \partial\square^n \text{ for } \square^n = [-1, 1]^n \subset \mathbb{R}^n,$$

that this map is *homotopic to* ϕ and that it is $\frac{2\sqrt{n}}{D'}$-*Lipschitz*. Since the radial map radial map $\partial\square^n \to S^{n-1} \to S^{n-1}$ is distance decreasing and $D' > D$, the composed map

$$Y \overset{\Delta}{\to} \square^n \to S^n$$

can be approximated by the required f.

5.B. Conclusion of the Proof of Theorem 2.C Let X be a Riemannian manifold with corners as in 2.C, let $\square^n_{\rangle}(X) > D$ and let $\Phi : X \to \square^n$ be a continuous combinatorial map, which satisfies $\bullet_{dist_{\rangle}}$ and $\bullet_{deg}$ from Sect. 1 and also $\bullet_{dist_{\rangle}}$, but now with D instead of d.

Then a smooth mean convex hypersurface $\mathcal{Y} \subset X$, which approximates ∂X according to 2.B, which we assume strictly mean convex and which we endow with the metric $dist^{\natural}$, and the map $\phi = \Phi_{|\mathcal{Y}} : \mathcal{Y} \to \partial\square^n$ satisfy the assumptions of 5.A. Hence, $(\mathcal{Y}, dist_{\natural})$ admits a λ-*Lipschitz* map to the unit sphere S^{n-1} for $\lambda < \frac{2\sqrt{n}}{D}$ as in 5.A.

Since the degree of this map doesn't vanish according to $\bullet_{deg}$, theorem 2.A says that $\lambda \geq \frac{1}{mean.curv(S^{n-1})} = \frac{1}{n-1}$, which implies that $\frac{2\sqrt{n}}{D} \leq \frac{1}{n-1}$ and

$$D < 2(n-1)\sqrt{n}.$$

□

6 Combinatorial Waists and $\rangle^{n-k-1}$-Angles

The F^k-overlaps of a map from a manifold X with corners, e.g. from a polytope, to some set, say $\alpha : X \to \Xi$, denoted

$$\overset{\smile k}{\#}_{\alpha}(X) \text{ and } \widehat{\#}^k_\alpha(X),$$

are the maxima of the numbers of open, respectively closed, l-faces in X the α-images of which in Ξ have a common point ξ,

For instance, generic linear maps α from n-polytopes $X \subset \mathbb{R}^n$ to $\mathbb{R}^{n-1}$ satisfy $\overset{\smile n-1}{\#}_{\alpha}(X) = 2$ and, if X simple, then $\widehat{\#}^{n-1}_\alpha(X) = n + 1$.

The spherical $(n-k-1)$-***(co)angle*** of a convex subset $X \subset \mathbb{R}^n$, e.g. a Euclidean n-polytope, at a point $x \in \partial X$, denoted

$$\rangle^{n-k-1}_x(X),$$

is the $(n-k-1)$-dimensional spherical volume (Hausdorff measure) of the set of the supporting planes to X at x, and where we denote

$$\rangle_{F^k} = \rangle^{n-k-1}_{x \in F^k_\circ}(X)$$

for open and closed k-faces F^k and for $F^k_\circ \subset F^k$ being the interior parts of these faces.

Then the "angle" $\rangle^{n-k-1}_x(X)$ at a point x in a Riemannian manifold X with corners is defined as the corresponding angle of the tangent cone of X at x

For instance, if $x \in F^{n-2}$, this is the complementary dihedral angle of the face F^{n-2} defined earlier.

Define $\overset{\smile}{\sum}\rangle^{n-k-1}_\alpha(X)$ and $\widehat{\sum}\rangle^{n-k-1}_\alpha(X)$ for Riemannian manifolds X with corners as the supremum over $\xi \in \Xi$ of the sums of these angles over the set of non-empty intersections of the k-faces F^k in X with the α-pullbacks of points ξ in Ξ,

$$\sup_{\xi \in \Xi} \sum \rangle_{F^k},\ F^k \cap \alpha^{-1}(\xi) \neq \emptyset,$$

i.e. the sum is taken over all open, respectively closed, k-faces $F^k \subset X$,which intersect $\alpha^{-1}(\xi)$.

Remark This definition makes sense for all weight functions w on the faces instead of $\rangle^{n-k-1}$, where, e.g. for $w(x) = 1$, one recaptures the numbers $\overset{\smile k}{\#}_{\alpha}(X)$ and $\widehat{\#}^k_\alpha(X)$.

PROBLEM. Given a class $\mathcal{A}$ of function α, evaluate possible values $\#^k_\alpha(X)$ and $\sum\rangle^{n-k-1}_\alpha(X)$ for convex polytopes and other "interesting" manifolds with corners in terms of other geometric invariants.

Example 6.A Let $X \subset \mathbb{R}^n$ be a convex polytope and $\alpha : X \to \mathbb{R}^{n-2}$ a *continuous* map. If X is simple,[18] then the number $\widehat{\#}_\alpha^{n-2}(X)$ is bounded from below by the combinatorial $\square$-spread of X as follows

$$\widehat{\#}_\alpha^{n-2}(X) \geq const_n \cdot \square_{comb}^n(X).$$

Sketch of the Proof. Let g_ε be a Riemannian metric on ∂X, which distance-wise ε-approximates $dist_{comb}$ on X. By the argument from the previous section, $(\partial X, g_\varepsilon)$ admits a 1-Lipschitz map Φ of non-zero degree to the sphere $S^{n-1}(R)$ of radius $R \geq const'_n \square_{comb}(X)$. It follows by the (quite elementary) 1-waist inequality for spheres (see [15] and references therein) the Φ-image of the pullback $\alpha^{-1}(\xi)$, $\xi \in \mathbb{R}^{n-2}$, has length $\geq 2\pi R$. Hence, the g_ε-length of $\alpha_{-1}(\xi)$ is also $\geq 2\pi R$, which, since X is simple, implies the required bound $\widehat{\#}_\alpha^{n-2}(X) \geq const'_n R$ for $\varepsilon \to 0$.

Probabaly, a similar argument applies to continuous maps $\alpha : X \to \mathbb{R}^k$ for all $k = 1, ..., n-2$, thus showing, at least for manifolds X with simple corners, that

$$\widehat{\#}_\alpha^k(X) \geq const_n \cdot \left(\square_{comb}^n(X)\right)^{n-k-1}$$

for all continuous α.

But it is unclear what happens to $\sum\rangle_\alpha^k$.

For instance, let $X \subset \mathbb{R}^n$ be a convex polytope and $k = 1, ..., n-2$.

Question 6.B Is

$$\inf_\alpha \breve{\sum\rangle}_\alpha^k(X) \leq const_n$$

the infimum is taken over all continuous (may be even linear?) maps $\alpha : X \to \mathbb{R}^k$?

Question 6.C Does there exist an $(n-k)$-dimensinal affine subspace $A \subset \mathbb{R}^n$, which transversally meets $N > 0$ (open) k-dimensional faces $F_i^k \subset X$, $i = 1, ...N$, such that

$$\frac{1}{N} \sum_{i=1}^{N} \rangle_{F_i^k}^{n-k-1}(X) \leq \frac{const_n}{(\square_{comb}(X))^{n-k-1}}?$$

The positive answers to this would yield the following generalization of corollary 1.B to $k \leq n-3$.

Conjecture 6.D If the combinatorial $\square^n$-spread of a convex polytope $X \subset \mathbb{R}^n$ is large, then there exists a k-dimensional face $F_{min}^k \subset X$ with small $\rangle^{n-k-1}$-angle:

$$\rangle_{F_{min}^k}^{n-k-1}(X) \leq const_n \left(\square_{comb}^n(X)\right)^{-(n-k-1)},$$

[18] This is probabaly redundant.

or, at least,

$$\rangle^{n-k-1}_{F^k_{min}}(X) \to 0 \text{ for } \square^n_{comb}(X) \to \infty.$$

for simple polytopes X.

7 Surgery with Corners and Related Problems

It is claimed in [11, Sect. 1.3] that the so called *staircase thin surgery* of mean convex manifolds with positive scalar curvatures[19] can be also applied to manifolds X with corners. However, I overlooked the difficulty in proving the following.

7.A**. $\angle$-**Shrinking Problem. Let $Y_0 \subset S^{n-1} \subset \mathbb{R}^n$ be a convex spherical polytope.

Does there exist a continuous deformation $Y_t \subset S^{n-1}$, $0 \le t \le 1$, of Y_0, where all Y_t for $t < 1$ are convex spherical polytopes combinatorially isomorphic to X_0 and having their dihedral angles bounded by the corresponding angles of Y_0 and where Y_1 is a single point?

It is easy to construct such a Y_t for $dim(Y_0) = 2$, and also for "sufficiently round" spherical polytopes of dimensions>2, where such shrinking can be achieved by projective transformations of Y_0, but I was unable to prove or disprove it for general $Y \subset S^{n-1}$ if $n \ge 4$.

And granted such a deformation for the spherical base Y_0 of the tangent cone $TC_{x_0}(X) \subset T_{x_0}(X) = \mathbb{R}^n$, at a vertex $x_0 \subset X$, say for a strictly mean convex domain $X \subset \mathbb{R}^n$ with corners, the staircase construction delivers another strictly mean convex $X_1 \subset \mathbb{R}^n$, such that

$\bullet_{cut}$ the domain X_1 is diffeomorphic to X with the vertex x_0 cut away by a hyperplane $H_0 \subset \mathbb{R}^n$ parallel to a supporting hyperplane of X at at x_0, where this diffeomorphism moves all points at most by a given $\varepsilon > 0$ and fixes the points ε-far from x_0;

$\bullet_{>\rangle}$ the dihedral angles at the (old) $(n-1)$-faces of X_1 away from the cut $X \cap H_0$ are bounded by the corresponding angles of X;

$\bullet_{\pi/2}$ the dihedral angles between the new $(n-1)$-face corresponding to $X \cap H_0$ with the old ones are equal to $\pi/2$.

Observe that if $n = 3$, this construction, when applied to all vertices of X, delivers a simple polytope and thus provides an alternative reduction of the general case of theorem 1.A to that for simple X.

However, since 7.A. remains problematic for $dim(Y) \ge 3$

the thin surgery at the corners remains problematic as well.

Also pondering over 7.A brings to one's mind the following more general problems.

7.B**. $\rangle$-**Variation Problem. Find the homotopy type of the space $\mathcal{X}(\mathcal{C}, \kappa)$ of (possible) dihedral angles of convex n-polytopes X of given combinatorial type $\mathcal{C}$ in the space of constant curvature κ and determine how this space varies depending on $\infty < \kappa < \infty$.

[19] See [4, 8, 13].

7.C. Scalar Curvature $\rangle$-Problem. Let X be a compact connected smooth manifold with corners, let $-\infty < \mu_i < \infty$ be numbers associated to the $(n-1)$-faces of X and $0 < \alpha_j < \pi$ be associated to the $(n-2)$-faces. Determine the homotopy type of the space $\mathcal{G}(X, \sigma, \mu_i, \alpha_j)$, $\sigma > 0$ of Riemannian metrics g on X such that

$\bullet_\sigma$ the scalar curvature of X satisfies:

$$Sc(X) > \sigma;$$

$\bullet_\mu$ the mean curvatures of the $(n-1)$-faces of X satisfy:

$$mean.curv_g(F_i^{n-1}) > \mu_i;$$

$\bullet_\alpha$ the complementary dihedral angles at the $(n-2)$-faces satisfy:

$$\rangle_g((F_j^{n-2}) \geq \alpha_j.$$

Also determine how this space varies depending on $(\sigma, \mu_i, \alpha_j)$.

8 On Random Polytopes

Let $\Sigma = \{\sigma_i\}_{i=1,\dots,N^{n-1}} \subset S^{n-1}$ be randomly chosen points on the unit sphere and $X_N = X(\Sigma)$ be the (necessarily simple) convex polyhedron defined by the tangent hyperplanes to the sphere at the points σ_i.

Let $dist_{comb,N}(s_1, s_2)$, $s_1, s_2 \in S^{n-1}$ be the combinatorial distance between the $(n-1)$-faces $F_1, F_2 \subset X$ of X_N the normal projections of which to S^{n-1} contains the points s_1 and s_2 respectively. (Never mind the distinction between open and closed faces.)

8.A. Spherical $dist_{comb}$-Conjecture. There exists a universal constant Δ_n such that

$$\frac{dist_{comb,N}(s_1, s_2)}{N \cdot dist_{S^{n-1}}} \to \Delta_n \text{ for } N \to \infty,$$

with probability 1 for all pairs of points $s_1, s_2 \in S^{n-1}$:

the probability of the inequality $\left|\frac{dist_{comb,N}(s_1,s_2)}{N \cdot dist_{S^{n-1}}} - \Delta_n\right| > \varepsilon$ tends to zero for $N \to \infty$ for all $\varepsilon > 0$.

Remark Probabaly, this follows by the results/arguments from [BDGHL2021] but I haven't looked at this closely.[20] in any case an elementary (Poisson) percolation argument shows that, with overwhelming probability,

$$\frac{dist_{comb,N}(s_1, s_2)}{N \cdot dist_{S^{n-1}}} \leq cost_n \text{ and } \frac{dist_{comb,N}(s_1, s_2)}{N \cdot dist_{S^{n-1}}} \geq \frac{cost'_n}{\log N} \text{ for } N \to \infty.$$

[20] There is an extensive literature on random polytopes, where much of known estimates of the sizes of random polytopes concern *upper* bounds on combinatorial edge-diameters, which are motivated by the *Hirsch conjecture,* while we are interested on lower bounds on the □-spreads.

8.A. Spherical $dist_{\rangle}$**-Conjecture**. Let $dist_{\rangle}(s_1, s_2) = dist_{\rangle}(F_1, F_2)$ for the above F_1, F_2. Then there exists a universal constant $\Delta_n^{\rangle}$ such that

$$\frac{dist_{\rangle}(s_1, s_2)}{dist_{S^{n-1}}} \to \Delta_n^{\rangle} \text{ for } N \to \infty,$$

with probability 1 for all pairs of points $s_1, s_2 \in S^{n-1}$.

Remark Exact evaluation of Δ_n and $\Delta_n^{\rangle}$ may be difficult but the ratio $\Delta_n / \Delta_n^{\rangle}$ seems computable.

There are other commonly used definition of "random polytope" (see [Schneider2008]); we single out the following.

Let $\mathcal{C}(n, M)$ be the set of combinatorial types of simple n-polyhedra X with M faces and observe that the cardinality of this set is pinched between two exponentials:

$$A^M \leq \#\mathcal{C}(n, M) \leq B^M.$$

Cutting X by hyperplanes in two parts suggests that $\log \#\mathcal{C}(n, M)$ is (essentially) super-additive, and the limit

$$\lim_{M \to \infty} \frac{\log \#\mathcal{C}(n, M)}{M},$$

which seems an interesting number, exists.

Then we assign equal probabilities to all points (combinatorial types) in $\mathcal{C}(n, M)$ and conjecture that

the graphs $E = E(X)$ of the so defined random n-polytopes X with M faces endowed with metrics $M^{\frac{-1}{n-1}} dist_{comb}$ *Hausdorff converge* to the sphere $S^{n-1}(R_n)$ of some radius R_n for $M \to \infty$.

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