



A Geometric Study of Circle Packings and Ideal Class Groups

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Abstract

A family of fractal arrangements of circles is introduced for each imaginary quadratic field K . Collectively, these arrangements contain (up to an affine transformation) every set of circles in the extended complex plane with integral curvatures and Zariski dense symmetry group. When that set is a circle packing, we show how the ambient structure of our arrangement gives a geometric criterion for satisfying the asymptotic local–global principle. Connections to the class group of K are also explored. Among them is a geometric property that guarantees certain ideal classes are group generators.

Keywords Imaginary quadratic field · Circle packing · Bianchi group · Class group

Mathematics Subject Classification Primary 52C26 · 11R11 · 30F40 · 11R29 ·
Secondary 20G30 · 20F65 · 11E39

1 Introduction

Let K be an imaginary quadratic field with ring of integers \mathcal{O} and discriminant Δ . To each K we associate a family of arrangements, meaning a set of oriented circles in the extended complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We denote a member of this family \mathcal{S}_D , where D is an integer whose allowable values depend on K . Briefly stated (see Sect. 2 for details), an oriented circle with curvature r , cocurvature \hat{r} , and curvature-center ζ belongs to \mathcal{S}_D for the field K if and only if $i\sqrt{D}\zeta \in \mathcal{O}$ and $i\sqrt{D}r, i\sqrt{D}\hat{r} \in \sqrt{\Delta}\mathbb{Z}$, where $i = \sqrt{-1}$. Figure 1 shows two examples for the same field.

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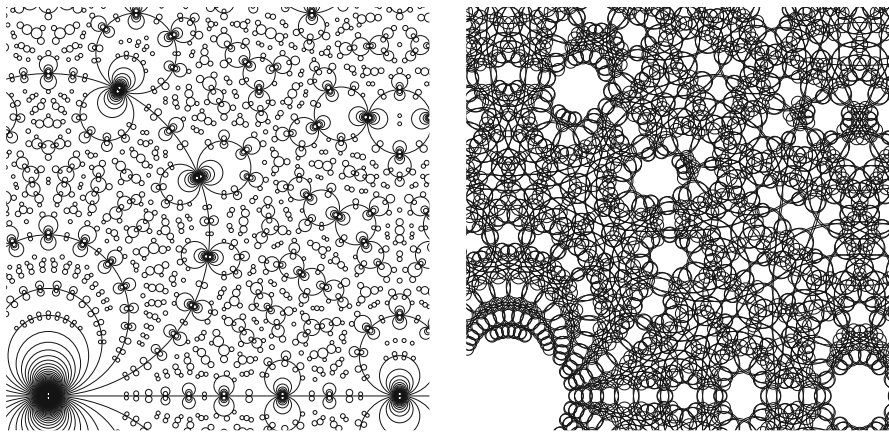


Fig. 1 \mathcal{S}_1 (left) and \mathcal{S}_{24} (right) for $\mathbb{Q}(i\sqrt{19})$, discussed in Sect. 3.2

1.1 History and Motivation

The origin of our arrangements traces back to Apollonian circle packings, which first appeared in a 17th century notebook of Leibniz. There are sufficiently many images, descriptions, and histories of Apollonian circle packings in the literature to warrant omitting them here [18]. An example of a non-Apollonian circle packing is given by the bold circles in Fig. 2. (The circles “pack” space because their interiors are pairwise disjoint and dense.)

Instead our timeline picks up with the Apollonian superpacking, first introduced by Graham et al. in [11]. The paragraph following Definition 4.4 defines superpackings formally, but for now the reader can imagine something like the background circles in Fig. 2—an organized union of infinitely many circle packings nested inside one another. The Apollonian superpacking is the first of our arrangements to appear in print; it is \mathcal{S}_1 for $\mathbb{Q}(i)$. Superpackings have been used in [2, 6, 11–13, 16, 17, 28] as a tool for studying and classifying the circle packings they contain. We employ \mathcal{S}_D for the same purpose here, especially in Sect. 4.3 to prove results on the so-called local–global principle for circle packings, which was also introduced by Graham, Lagarias, Mallows, Wilks, and Yan. Indeed, their series of papers [10–13] helped generate significant interest in the algebraic and number theoretic properties of arrangements like ours. The reader might see [8] or [25] for expositions of the mathematics that has since developed.

Several other appearances of \mathcal{S}_D followed shortly after. The next was that of \mathcal{S}_2 for $\mathbb{Q}(i\sqrt{2})$ in [14]. Following [11], Guettler and Mallows also treat the arrangement as a superpacking, but they build it from a different (non-Apollonian) structure of initial circles.

An infinite family of arrangements was then introduced by Stange [27], whose work motivated this paper. She defined the Schmidt arrangement of an imaginary quadratic field K and studied its relationship to the corresponding ring of integers \mathcal{O} . The Schmidt arrangement is always contained in \mathcal{S}_1 , and the two coincide whenever the

class group of K contains only 2-torsion (like for $\mathbb{Q}(i\sqrt{19})$)—the first image in Fig. 1 is the Schmidt arrangement). Stange also found additional arrangements that she called “ghost circles,” and she used them to prove the Schmidt arrangement is disconnected when \mathcal{O} is non-Euclidean. Ghost circles are \mathcal{S}_D for $D = (\Delta^2 + 14\Delta + 1)/16$ when $4 \nmid \Delta$ and $D = (\Delta^2 + 12\Delta)/16$ when $4 \mid \Delta$. Stange’s theorem on connectivity and Euclidean-ness is the first example of how the geometry of an arrangement provides arithmetic information about the underlying field. But the Schmidt arrangement is limited in that it does not manifest properties of the ideal class group. As Sect. 5 shows, however, our generalization creates a richer geometry-to-arithmetic connection with K , and the class group in particular.

While these are the only \mathcal{S}_D that have already appeared in the literature to the author’s knowledge, it is likely that some of the superpackings defined by Kontorovich and Nakamura in [17] and Kapovich and Kontorovich in [16] would also turn out to be among our arrangements. But their definition of a superpacking does not encompass our definition of \mathcal{S}_D , nor is the reverse containment true. Rather, our work here is complementary to theirs; results from [16] are used in Sect. 4.1 to prove a fundamental property of \mathcal{S}_D (Theorem 1.1), and we return the favor in Sects. 4.2 and 4.3 by proving properties of objects defined in [16].

Finally, many circle packings in print are subarrangements of some \mathcal{S}_D . Those in [4, 17, 20, 26] are contained in \mathcal{S}_D for $\mathbb{Q}(i\sqrt{D})$ with either $D = 2, 3$ or 6 . Each of the infinitely many circle packings discovered by Baragar and Lautzenheiser in [1] is also contained in \mathcal{S}_D for $\mathbb{Q}(i\sqrt{D})$ for some D ; we derive properties of these from \mathcal{S}_D in Sects. 4.2 and 4.3. And the circle packing in [16] is contained in \mathcal{S}_{24} for $\mathbb{Q}(i\sqrt{30})$.

One of the primary purposes of this work is to provide a common thread among the papers listed above. As we show, \mathcal{S}_D turns out to be a wonderful tool for investigating the algebra and number theory at play behind the aforementioned circle packings and superpackings.

1.2 Summary of Results

Section 2 is devoted to background, then \mathcal{S}_D is introduced formally in Definition 3.1. The main result of Sect. 3 is Theorem 3.11, which splits our arrangements into two very different categories. Figure 1 shows one arrangement from each. Those in the same category as \mathcal{S}_1 (first image) are “orbits” under subsets of $\text{PGL}_2(K)$ —a fact used throughout the rest of the paper to connect the geometry of \mathcal{S}_D to the number theory of K . All of the arrangements from literature belong to this category except the circle packing from [16] and certain ghost circles from [27].

Section 4 begins with the result below, the proof of which relies crucially on the “Subarithmeticity Theorem” of Kapovich and Kontorovich [16]. It shows the extent to which \mathcal{S}_D is universal, explaining why our paper connects to so many past works. *Integral* means that all curvatures are integers up to a single scaling factor.

Theorem 1.1 *Any integral arrangement that is fixed by a Zariski dense subgroup of $\text{PSL}_2(\mathbb{C})$ is contained in some \mathcal{S}_D after scaling, rotating, and translating.*

Consequences of Theorem 1.1 include a restriction on possible intersection angles within certain subarrangements called bugs from [16] (Corollary 4.3), a criterion for

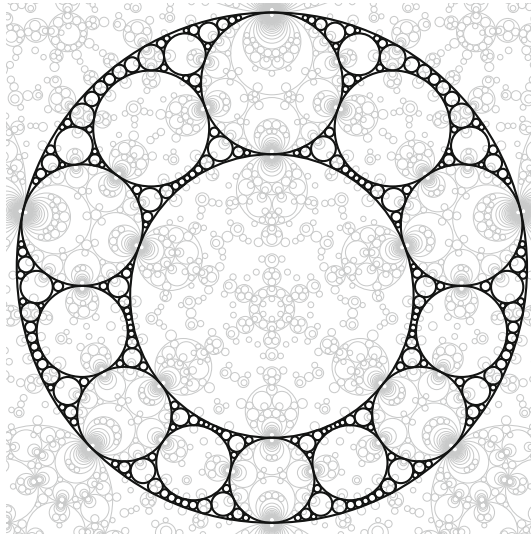


Fig. 2 New circle packing from \mathcal{S}_0 for $\mathbb{Q}(i\sqrt{39})$

determining superintegrality (Definition 4.4; Proposition 4.6), and a geometric connection between \mathcal{S}_D and the local–global principle for curvatures in an integral circle packing (Theorem 4.11 and Corollary 4.13).

Regarding the last result, the relative position of a circle packing in \mathcal{S}_D can give a sufficient condition for applying a theorem of Fuchs et al. [9], which states that curvatures in certain packings have asymptotic density 1 among integers that pass a set of local obstructions. Figure 2 displays our sufficient condition: at each intersection point on a given bold circle, it is always the largest exteriorly tangent circle from \mathcal{S}_0 that also belongs to the packing. Stange calls such circles *immediately tangent* [28], and we name the circle packing accordingly.

Corollary 1.2 *Immediate tangency packings (Definition 4.12) in \mathcal{S}_D satisfy the local–global principle up to density 1.*

Section 5 proves connections between \mathcal{S}_D and the class group of K . If $\alpha_0 \in K$ lies on a circle in \mathcal{S}_D , then so does each $\alpha \in K$ for which the fractional ideals $(\alpha, 1)$ and $(\alpha_0, 1)$ belong to the same ideal class. So it makes sense to ask whether or not an ideal class is *covered* (Definition 5.4) by \mathcal{S}_D . Those that are covered can often be distinguished by the geometry of \mathcal{S}_D . In Fig. 3, circles intersect non-tangentially at some $\alpha \in \mathbb{Q}(i\sqrt{39})$ if and only if the ideal class of $(\alpha, 1)$ generates the class group (a consequence of Proposition 5.3).

Also in Fig. 3, note that any two circles appear to be linked by a chain of circles. This arrangement turns out to be *finitely connected* (Definition 5.11), meaning there is an upper bound on the chain length needed to connect any circle of nonzero curvature to one at least twice as large.

Theorem 1.3 *If \mathcal{S}_D is finitely connected, then the class group of K is generated by ideal classes of primes with norm dividing $D\Delta$.*

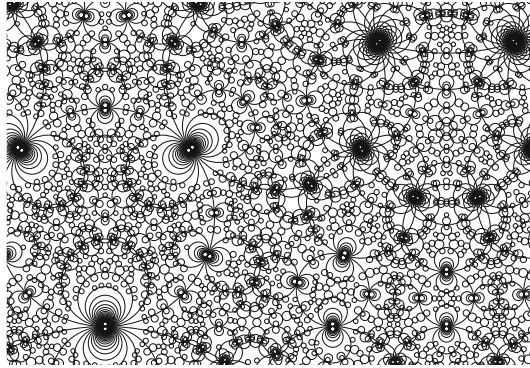


Fig. 3 Connectivity of S_4 for $\mathbb{Q}(i\sqrt{39})$ implies a cyclic class group

Primes dividing Δ generate the 2-torsion subgroup of the class group, so primes dividing D are responsible for each 2-torsion coset in Theorem 1.3. If we strengthen the connectivity hypothesis by assuming consecutive circles in a chain intersect at a point in K , then $D\Delta$ can be replaced by D in the theorem (Corollary 5.13). This is the case in Fig. 3. So S_4 is a kind of geometric assertion that the class group of $\mathbb{Q}(i\sqrt{39})$ is cyclic, generated by a prime over 2.

2 Background

As detailed in [27], an *oriented circle* C is the set of $\alpha/\beta \in \widehat{\mathbb{C}}$ solving

$$r|\alpha|^2 - 2\Re(\bar{\alpha}\beta\zeta) + \hat{r}|\beta|^2 = 0 \tag{1}$$

for some $\zeta \in \mathbb{C}$ and $r, \hat{r} \in \mathbb{R}$ satisfying $|\zeta|^2 - \hat{r}r = 1$. Given C , let $\mathbf{v}_C = [\zeta \ \bar{\zeta} \ \hat{r} \ r]$.

If $r \neq 0$, (1) rearranges to the equation of a circle with center ζ/r and radius $1/|r|$: $|\alpha/\beta - \zeta/r| = 1/|r|$. The sign of r indicates orientation. Positive defines the *interior* of C so as to contain its center, while this is the *exterior* when r is negative. If $r = 0$, then the point at infinity, $1/0$, is a solution, and C appears as a line orthogonal to ζ . In this case, *interior* is defined as the side to which ζ points.

The *curvature* of C is r , the *curvature-center* is ζ , and the *cocurvature* is \hat{r} .

$\text{PSL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{C}}$ as Möbius transformations, which map oriented circles to oriented circles and preserve the orientation just described. This action corresponds to a linear transformation on vectors \mathbf{v}_C . See [5] for a proof of the following.

Proposition 2.1 *If $M \in \text{PSL}_2(\mathbb{C})$, then*

$$M = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \alpha\bar{\delta} & \beta\bar{\gamma} & \alpha\bar{\gamma} & \beta\bar{\delta} \\ \beta\bar{\gamma} & \alpha\bar{\delta} & \alpha\bar{\alpha} & \beta\bar{\beta} \\ \alpha\bar{\beta} & \alpha\bar{\alpha} & |\alpha|^2 & |\beta|^2 \\ \gamma\bar{\delta} & \bar{\gamma}\delta & |\gamma|^2 & |\delta|^2 \end{bmatrix} \tag{2}$$

are such that $\mathbf{v}_{MC} = \mathbf{v}_CN$.

The map $M \mapsto N$ is called the *spin homomorphism*. Those M for which N has entries in \mathcal{O} form the *extended Bianchi group*.

Vectors of the form \mathbf{v}_C generate a four-dimensional subspace of $\mathbb{C}^2 \times \mathbb{R}^2$, which we identify with Minkowski space via the Hermitian form of signature $(1, 3)$,

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

With respect to Q , each \mathbf{v}_C is a unit vector: $\|\mathbf{v}_C\|_Q = \langle \mathbf{v}_C, \mathbf{v}_C \rangle_Q = |\zeta|^2 - r\hat{r} = 1$. In particular, N from (2) preserves unit vectors (a so-called Lorentz transformation).

A modern perspective on the following properties is to treat them as inheritance from the geometry of Minkowski space. But they can also be verified directly with Euclidean geometry.

Proposition 2.2 *C and C' intersect if and only if $\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q \in [-1, 1]$, in which case $\arccos(\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q)$ is the intersection angle.*

Proposition 2.3 *The vector associated to the image of C' reflected in C is $\mathbf{v}_{C'} - 2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q \mathbf{v}_C$. In terms of Möbius transformations, this image is*

$$\begin{bmatrix} \zeta & -\hat{r} \\ r & -\bar{\zeta} \end{bmatrix} \overline{C'},$$

where $[\zeta \ \bar{\zeta} \ \hat{r} \ r] = \mathbf{v}_C$ and $\overline{C'}$ is the image of C' under complex conjugation.

3 The Object of Study

Definition 3.1 Given a fixed imaginary quadratic field, for nonzero $D \in \mathbb{R}$ let \mathcal{S}_D denote the set of oriented circles C for which $i\sqrt{D}\mathbf{v}_C \in \mathcal{O}^2 \times (\sqrt{D}\mathbb{Z})^2$.

As an example, consider a circle with curvature-center $\zeta = \sqrt{5} + i\sqrt{11}$. Since $i\sqrt{11}\zeta$ is an integer in $\mathbb{Q}(i\sqrt{55})$, such a circle might possibly belong to \mathcal{S}_{11} for $\mathbb{Q}(i\sqrt{55})$. In order to be in \mathcal{S}_{11} , the curvature of our circle must satisfy $i\sqrt{11}r \in i\sqrt{55}\mathbb{Z}$. So let us try $r = 7\sqrt{5}$. Then the cocurvature is $\hat{r} = (|\zeta|^2 - 1)/r = 3\sqrt{5}/7$. Since $i\sqrt{11}\hat{r} \notin i\sqrt{55}\mathbb{Z}$, the circle of radius $1/7\sqrt{5}$ and center $(\sqrt{5} + i\sqrt{11})/7\sqrt{5}$ does not belong to \mathcal{S}_{11} (though it is in \mathcal{S}_{11d^2} for any nonzero $d \in 7\mathbb{Z}$). On the other hand, $r = 3\sqrt{5}$ gives $\hat{r} = \sqrt{5}$, and therefore $i\sqrt{11}\hat{r} \in i\sqrt{11}\mathbb{Z}$. Thus the circle of radius $1/3\sqrt{5}$ and center $(\sqrt{5} + i\sqrt{11})/3\sqrt{5}$ does belong to \mathcal{S}_{11} .

Proposition 3.2 *\mathcal{S}_D is nonempty if and only if D is a positive integer for which there exists $\alpha \in \mathcal{O}$ satisfying $|\alpha|^2 \equiv D \pmod{\Delta}$.*

Proof Suppose the congruence holds for some $\alpha \in \mathcal{O}$ and positive integer D . Then

$$\mathbf{v} = \begin{bmatrix} \frac{i\alpha}{\sqrt{D}} & \frac{\bar{i\alpha}}{\sqrt{D}} & \frac{|\alpha|^2 - D}{\sqrt{D|\Delta|}} & \frac{\sqrt{|\Delta|}}{\sqrt{D}} \end{bmatrix}$$

corresponds to a circle in \mathcal{S}_D since $\|\mathbf{v}\|_{\mathcal{O}} = 1$, and $i\sqrt{D}\mathbf{v} \in \mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$.

Conversely, let $C \in \mathcal{S}_D$ and let $\mathbf{v}_C = [\zeta \ \bar{\zeta} \ \hat{r} \ r]$. Set $\alpha = i\sqrt{D}\zeta \in \mathcal{O}$. Scaling $1 = \|\mathbf{v}_C\|_{\mathcal{O}}$ by D gives $D = |\alpha|^2 - D\hat{r}r \equiv |\alpha|^2 \pmod{\Delta}$, where the congruence is due to $i\sqrt{D}r, i\sqrt{D}\hat{r} \in \sqrt{\Delta}\mathbb{Z}$. This shows $D \in \mathbb{Z}$, and $i\sqrt{D}r \in \sqrt{\Delta}\mathbb{Z}$ shows $D > 0$. \square

As a consequence of D being a rational integer, we see that if the curvature-center ζ of an oriented circle in \mathcal{S}_D for some field K has nonzero real and imaginary parts, then $i\sqrt{D}\zeta \in K$ implies $K = \mathbb{Q}(i\Re(i\sqrt{D}\zeta)\Im(i\sqrt{D}\zeta)) = \mathbb{Q}(iD\Im(\zeta)\Re(\zeta)) = \mathbb{Q}(i\Im(\zeta)\Re(\zeta))$. Furthermore, $\Re(i\sqrt{D}\zeta) \in \mathbb{Q}$ implies $\mathbb{Q}(\Im(\zeta)) = \mathbb{Q}(\sqrt{D})$. So just from knowing the curvature-center $\sqrt{5} + i\sqrt{11}$ in the example above, Proposition 3.2 already forbids containment in anything but \mathcal{S}_{11d^2} for $\mathbb{Q}(i\sqrt{55})$ for some nonzero integer d .

Henceforth, all arrangements are assumed to be nonempty.

3.1 Basic Properties

Here we determine the symmetries of \mathcal{S}_D as well as its geometry at points where circles intersect.

Proposition 3.3 *The extended Bianchi group is the maximal subgroup of $\text{PSL}_2(\mathbb{C})$ that fixes \mathcal{S}_D .*

Proof Any subgroup of $\text{PSL}_2(\mathbb{C})$ that preserves \mathcal{S}_D is discrete since its image under the spin homomorphism preserves the lattice $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$. So it suffices to show the extended Bianchi group is such a subgroup, as it is the maximal discrete subgroup of $\text{PSL}_2(\mathbb{C})$ containing $\text{PSL}_2(\mathcal{O})$ [7].

Let $C \in \mathcal{S}_D$. Let M belong to the extended Bianchi group so that N from (2) has entries from \mathcal{O} . Then the first two entries of $i\sqrt{D}\mathbf{v}_C N$ also belong to \mathcal{O} . Next, the third entry of $i\sqrt{D}\mathbf{v}_C N$ is

$$2i\Im(\alpha\bar{\gamma}(i\sqrt{D}\zeta)) + (i\sqrt{D}\hat{r})|\alpha|^2 + (i\sqrt{D}r)|\gamma|^2 \in \sqrt{\Delta}\mathbb{Z}.$$

The fourth entry takes a similar form and is also in $\sqrt{\Delta}\mathbb{Z}$. Thus Proposition 2.1 gives $i\sqrt{D}\mathbf{v}_{MC} = i\sqrt{D}\mathbf{v}_C N \in \mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$, implying $MC \in \mathcal{S}_D$. \square

Affine transformations in the extended Bianchi group are translations by \mathcal{O} and rotations by the unit group \mathcal{O}^* . Translative symmetry can be seen in the first image of Fig. 4, which shows \mathcal{S}_8 centered on a fundamental region for the integers in $\mathbb{Q}(i\sqrt{31})$. Some reflective symmetries generated by complex conjugation (which always fixes \mathcal{S}_D), negation, and translation are also visible in Fig. 4.

Corollary 3.4 \mathcal{S}_D is dense in $\widehat{\mathbb{C}}$.

Proof The orbit of any point under the extended Bianchi group is dense. \square

Proposition 3.5 *The angle between intersecting circles in \mathcal{S}_D is $\theta = \arccos(n/2D)$ for some $n \in \mathbb{Z}$, where n is even if Δ is. Moreover, the point(s) of intersection is/are in $\widehat{K} = K \cup \{\infty\}$ if and only if $e^{i\theta} \in K$.*

Note that the point at infinity is considered rational: $1/0$. In particular, this proposition implies that the intersection angle θ between two lines in \mathcal{S}_D (should they exist) always satisfies $e^{i\theta} \in K$.

Proof If $i\sqrt{D}\mathbf{v}_C, i\sqrt{D}\mathbf{v}_{C'} \in \mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$ then $\langle i\sqrt{D}\mathbf{v}_C, i\sqrt{D}\mathbf{v}_{C'} \rangle_{\mathcal{O}} = n/2$ for some $n \in \mathbb{Z}$, where n is even if Δ is. By Proposition 2.2, C and C' intersect if and only if $\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_{\mathcal{O}} \in [-1, 1]$, in which case the angle of intersection is $\arccos(n/2D)$.

For the second claim, suppose C and C' intersect (but do not coincide). Since $\text{PSL}_2(\mathcal{O})$ preserves \mathcal{S}_D , intersection angles, and \widehat{K} , we may assume without loss of generality that C and C' have nonzero curvature r and r' —if not replace them by MC and MC' for almost any choice of $M \in \text{PSL}_2(\mathcal{O})$. Let ζ and ζ' denote their curvature-centers. Then, with $\theta = \arccos(\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_{\mathcal{O}}) \in [0, \pi]$, the point(s) of intersection is/are $(\zeta - \zeta'e^{\pm i\theta})/(r - r'e^{\pm i\theta})$. Scaling numerator and denominator by $i\sqrt{D}$ shows that intersections belongs to K if and only if $e^{i\theta}$ does. \square

In \mathcal{S}_8 for $\mathbb{Q}(i\sqrt{31})$, shown in the first image of Fig. 4, there are four possible values of $\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_{\mathcal{O}}$ that lie in $[-1, 1]$. They are ± 1 , which correspond to tangential intersections, and $\pm 15/16$, which correspond to non-tangential intersections. Since $\arccos(\pm 1) = \arg(\pm 1)$ and $\arccos(\pm 15/16) = \arg((15 \pm i\sqrt{31})/16)$, all points of intersection are in \widehat{K} by Proposition 3.5.

3.2 A Useful Dichotomy

This section splits our arrangements into two categories. The category to which \mathcal{S}_D belongs is determined by the Hilbert symbol (D, Δ) . We modify usual notation to avoid confusion with ideals.

Notation 3.6 For $a, b \in \mathbb{Q}$, let $H(a, b) = 1$ if $ax^2 + by^2 = z^2$ has a nonzero solution $x, y, z \in \mathbb{Q}$, and let $H(a, b) = -1$ otherwise.

Lemma 3.7 *If a circle in \mathcal{S}_D contains a point in $\widehat{K} = K \cup \{\infty\}$ then $H(D, \Delta) = 1$, and if $H(D, \Delta) = 1$ then every circle in \mathcal{S}_D contains a point in \widehat{K} .*

Proof Suppose $\alpha \in C \cap \widehat{K}$ for some $C \in \mathcal{S}_D$ with curvature-center ζ and curvature r . Since $\text{PSL}_2(\mathcal{O})$ fixes \widehat{K} and \mathcal{S}_D , we may assume $r \neq 0$ without loss of generality (as in the previous proof). Then $|\alpha - \zeta/r| = 1/|r|$, implying $i\sqrt{D}r\alpha - i\sqrt{D}\zeta$ is an element of K with magnitude \sqrt{D} . Denote this element $(z + y\sqrt{\Delta})/x$ to see that $Dx^2 + \Delta y^2 = z^2$ is solvable.

Now suppose $Dx^2 + \Delta y^2 = z^2$ for $x, y, z \in \mathbb{Q}$, not all zero. Then $\Delta < 0$ forces $x \neq 0$, so $\alpha = (z + y\sqrt{\Delta})/x \in K$ has magnitude \sqrt{D} . Thus if $C \in \mathcal{S}_D$ has curvature-center ζ and curvature $r \neq 0$, then $\zeta/r + \alpha/i\sqrt{Dr} \in C \cap \widehat{K}$. \square

The contrast between the two types of arrangements is displayed in Fig. 1, where $H(1, -19) = 1$ and $H(24, -19) = -1$. By the lemma, \mathcal{S}_{24} avoids all points in $\mathbb{Q}(i\sqrt{19})$. This makes it appear like the photographic negative of \mathcal{S}_1 , whose focal points are those in $\mathbb{Q}(i\sqrt{19})$ (a consequence of Theorem 5.3).

We use $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to denote the extended real line with positive orientation, so its interior is the upper half-space. This way $\mathbf{v}_{\widehat{\mathbb{R}}} = [i \ -i \ 0 \ 0]$. The following is proved in [27].

Lemma 3.8 (Stange [27]) *For $M \in \text{PSL}_2(\mathbb{C})$ with entries as in (2), $M\widehat{\mathbb{R}}$ has curvature-center $i(\alpha\bar{\delta} - \bar{\beta}\gamma)$, cocurvature $i(\alpha\bar{\gamma} - \bar{\alpha}\gamma)$, and curvature $i(\beta\bar{\delta} - \bar{\beta}\delta)$.*

Notation 3.9 For $M \in \text{PGL}_2(K)$ with entries $\alpha, \beta, \gamma, \delta \in K$, let (M) denote the fractional ideal $(\alpha, \beta, \gamma, \delta) := \alpha\mathcal{O} + \beta\mathcal{O} + \gamma\mathcal{O} + \delta\mathcal{O}$, and let $\|(M)\|$ denote its norm.

The notation $(\alpha, \beta) := \alpha\mathcal{O} + \beta\mathcal{O}$ for $\alpha, \beta \in K$ is also used throughout the remainder of the paper.

Lemma 3.10 *Let $M \in \text{PGL}_2(K)$ have entries as in (2). For any $\lambda, \mu \in K$ we have $(\lambda, \mu)(\det M)/(M) \subseteq (\alpha\lambda + \gamma\mu, \beta\lambda + \delta\mu) \subseteq (\lambda, \mu)(M)$. In particular, if $N \in \text{PSL}_2(\mathcal{O})$ then $(MN) = (M) = (NM)$.*

Proof Observe that

$$\begin{aligned} (\lambda \det M, \mu \det M) &= (\delta(\alpha\lambda + \gamma\mu) - \gamma(\beta\lambda + \delta\mu), \alpha(\beta\lambda + \delta\mu) - \beta(\alpha\lambda + \gamma\mu)) \\ &\subseteq (M)(\alpha\lambda + \gamma\mu, \beta\lambda + \delta\mu) \subseteq (\lambda, \mu)(M)^2. \end{aligned}$$

Divide all sides by (M) to get the desired containment.

To see that $(MN) = (M)$ for $N \in \text{PSL}_2(\mathcal{O})$, we will let N play the role of “ M ” from the previous argument. Let λ and μ denote the top-row entries of M . Since $(\det N)/(N) = \mathcal{O} = (N)$, we see that the ideal generated by the top-row entries of MN is still (λ, μ) . The ideal generated by bottom-row entries of M is similarly preserved, giving $(MN) = (M)$. For NM let λ and μ be a column of M instead. \square

Theorem 3.11 *If $H(D, \Delta) = 1$ then $C \in \mathcal{S}_D$ if and only if $C = M\widehat{\mathbb{R}}$ for some $M \in \text{PGL}_2(K)$ with $|\det M|/\|(M)\| = \sqrt{D}$.*

Proof Let $M \in \text{PGL}_2(K)$ have entries as in (2), and suppose $|\det M|/\|(M)\| = \sqrt{D}$. We claim $C = M\widehat{\mathbb{R}} \in \mathcal{S}_D$. Since $M/\sqrt{\det M} \in \text{PSL}_2(\mathbb{C})$, the formulas in Lemma 3.8 show that $i\sqrt{D}\mathbf{v}_C$ has entries

$$i\sqrt{D}\zeta = i\sqrt{D} \left(\frac{i(\alpha\bar{\delta} - \bar{\beta}\gamma)}{|\det M|} \right) = \frac{i|\det M|}{\|(M)\|} \left(\frac{i(\alpha\bar{\delta} - \bar{\beta}\gamma)}{|\det M|} \right) = \frac{\bar{\beta}\gamma - \alpha\bar{\delta}}{\|(M)\|} \in \mathcal{O}$$

(and by similar arithmetic),

$$i\sqrt{D}\hat{r} = \frac{\bar{\alpha}\gamma - \alpha\bar{\gamma}}{\|(M)\|} \in \sqrt{\Delta}\mathbb{Z}, \quad \text{and} \quad i\sqrt{D}r = \frac{\bar{\beta}\delta - \beta\bar{\delta}}{\|(M)\|} \in \sqrt{\Delta}\mathbb{Z}.$$

Thus $C \in \mathcal{S}_D$ as claimed.

Now assume $H(D, \Delta) = 1$. For $C \in \mathcal{S}_D$ we seek $M \in \text{PGL}_2(K)$ with $M\widehat{\mathbb{R}} = C$ and $|\det M|/\|(M)\| = \sqrt{D}$. By Lemma 3.7, there exists $\alpha \in C \cap K$. Pick any split prime ideal \mathfrak{p} that belongs to the same ideal class as $(\alpha, 1)$ and does not contain $2D$. Let $p = \|\mathfrak{p}\|$, and pick $\alpha' \in \mathcal{O}$ for which $(\alpha', p) = \mathfrak{p}$. Now take any $N \in \text{PSL}_2(\mathcal{O})$ such that $N(\alpha/1) = \alpha'/p$, which is possible since the ideal classes of $(\alpha, 1)$ and (α', p) are equal. Our strategy is to find $M' \in \text{PGL}_2(K)$ with $|\det M'|/\|(M')\| = \sqrt{D}$ and $M'\widehat{\mathbb{R}} = NC$. Then $M = N^{-1}M'$ is the desired matrix by Lemma 3.10.

Let ζ and r denote the curvature-center and curvature of NC . Fix any $\beta \in \mathfrak{p}$ with $\Im(\beta) = \sqrt{|\Delta|}/2$. Set $\gamma' = i\sqrt{D}\zeta + (i\sqrt{D}r/\sqrt{\Delta})(\alpha'\beta/p)$ and $\delta' = (i\sqrt{D}r/\sqrt{\Delta})\beta$, both of which are in \mathcal{O} . Define

$$M' = \begin{bmatrix} \alpha' & \gamma' \\ p & \delta' \end{bmatrix}.$$

Substituting in the formulas for γ' and δ' gives

$$|\det M'| = \left| \frac{i\sqrt{D}r\alpha'(\beta - \bar{\beta})}{\sqrt{\Delta}} - i\sqrt{D}\zeta p \right| = \sqrt{D}|r|p \left| \frac{\alpha'}{p} - \frac{\zeta}{r} \right| = \sqrt{D}p,$$

where the last equality uses $\alpha'/p \in NC$. From here it is straightforward to verify that $i(\alpha'\bar{\delta}' - p\gamma')/|\det M'| = \zeta$ and $i(p\bar{\delta}' - p\delta')/|\det M'| = r$. So $M'\widehat{\mathbb{R}} = NC$.

The proof will be complete if we show $(M') = \mathfrak{p}$, since then $|\det M'|/\|(M')\| = |\sqrt{D}p|/\|p\| = \sqrt{D}$. To this end, first observe that $\alpha' \notin \bar{\mathfrak{p}}$ because $\mathfrak{p} \neq \bar{\mathfrak{p}}$ and $p \nmid \alpha'$. Next we claim that $p \nmid i\sqrt{D}r$, which would imply $\delta' \notin \bar{\mathfrak{p}}$ by choice of β . Indeed, if $p \mid i\sqrt{D}r$ then $\alpha'/p \in NC$ gives

$$0 = i\sqrt{D}r|\alpha'|^2 + 2\Im(\bar{\alpha}'p(i\sqrt{D}\zeta)) + i\sqrt{D}\hat{r}p^2 \equiv 2\Im(\bar{\alpha}'p(i\sqrt{D}\zeta)) \pmod{p^2}.$$

But then $i\sqrt{D}\zeta \in \mathfrak{p}$ is forced, in turn showing p divides $|i\sqrt{D}\zeta|^2 + (i\sqrt{D}\hat{r})(i\sqrt{D}r) = D$, contradicting our choice of \mathfrak{p} . So $\delta' \notin \bar{\mathfrak{p}}$ as desired. Now, $\alpha'\delta' \notin \bar{\mathfrak{p}}$ gives $\det M' = \alpha'\delta' - p\gamma' \notin \bar{\mathfrak{p}}$. Combined with $p^2 \mid |\det M'|^2$, this implies $\det M' \in \mathfrak{p}^2$. But then $\alpha', \delta' \in \mathfrak{p}$ and $(p, \mathfrak{p}^2) = \mathfrak{p}$ forces $\gamma' \in \mathfrak{p}$, completing the proof. \square

The matrices used in Theorem 3.11, $M \in \text{PGL}_2(K)$ with $|\det M|/\|(M)\| = \sqrt{D}$, form the extended Bianchi group when $D = 1$ (though typically each matrix M is scaled by $1/\sqrt{\det M}$ so as to obtain a subgroup of $\text{PSL}_2(\mathbb{C})$). In particular, Theorem 3.11 asserts that the extended Bianchi group acts transitively on \mathcal{S}_1 . More generally, it can be shown as a corollary to the theorem that when $H(D, \Delta) = 1$, the extended Bianchi group acts transitively on \mathcal{S}_D if and only if D is square-free. The author believes the same is true when $H(D, \Delta) = -1$, but has not proved it.

4 Relation to Arrangements from the Literature

4.1 Universality

We would like to determine what kinds of arrangements can occur as subarrangements of some \mathcal{S}_D . Integrality of curvatures (or cocurvatures or curvature-centers) up to a single scaling factor is evidently necessary, but not sufficient to guarantee containment in some \mathcal{S}_D . A certain amount of symmetry is also required to avoid random collections of integral circles that are not found in any \mathcal{S}_D . The Subarithmicity Theorem of Kapovich and Kontorovich suggests what sufficient symmetry might be.

Theorem 4.1 (Special case of Kapovich–Kontorovich [16]) *If the orbit of a circle under a Zariski dense subgroup $\Gamma < \text{PSL}_2(\mathbb{C})$ is integral, then Γ is contained in a group weakly commensurable to some $\text{PSL}_2(\mathcal{O})$.*

So Zariski denseness of Γ is enough to associate an imaginary quadratic field to an integral orbit of a circle. Weak commensurability means there exists $M \in \text{PSL}_2(\mathbb{C})$ such that $M\Gamma M^{-1} \cap \text{PSL}_2(\mathcal{O})$ is a finite index subgroup of $M\Gamma M^{-1}$. Letting C be a circle from our integral orbit, if it happens that $MC \in \mathcal{S}_D$ for some D , then $(M\Gamma M^{-1} \cap \text{PSL}_2(\mathcal{O}))MC \subseteq \mathcal{S}_D$ by Proposition 3.3. But there seems no reason that this should be true, or that it should be true for any remaining circles in $M\Gamma C$. And even if all of $M\Gamma C$ is contained in some \mathcal{S}_D , why should an integral arrangement composed of multiple orbits be contained in a single \mathcal{S}_D ? The next theorem addresses these concerns.

Theorem 4.2 *Any integral arrangement that is fixed by a Zariski dense subgroup of $\text{PSL}_2(\mathbb{C})$ is contained in some \mathcal{S}_D after scaling, rotating, and translating.*

Proof Let \mathcal{A} denote the arrangement, scaled to have curvatures in \mathbb{Z} , and let Γ be the maximal Zariski dense subgroup of $\text{PSL}_2(\mathbb{C})$ that fixes \mathcal{A} . Theorem 4.1 gives a finite index subgroup $\Gamma_0 \leq \Gamma$ that is, up to conjugation, contained in some $\text{PSL}_2(\mathcal{O})$. Since conjugation preserves traces, all traces from Γ_0 belong to \mathcal{O} . Moreover, Γ_0 has finite index in a Zariski dense group and is thus Zariski dense itself. In particular, the invariant trace field of Γ_0 , which is generated over \mathbb{Q} by $\text{tr } M^2$ for $M \in \Gamma_0$, must be nonreal [19]. So fix some $M_0 \in \Gamma_0$ with $\Im(\text{tr } M_0^2) \neq 0$. Let $\tau = \text{tr } M_0 \in \mathcal{O}$ and set $t = (\tau^2 - \bar{\tau}^2)/\sqrt{\Delta} = 2\Im(\text{tr } M_0^2)/\sqrt{|\Delta|} \in \mathbb{Z}$.

Let α, β, γ , and δ be the entries of M_0 as in (2). Suppose for a contradiction that $\beta = 0$. Since $\Gamma\mathcal{A} = \mathcal{A}$ must be Zariski dense in $\text{PSL}_2(\mathbb{C})\mathcal{A}$, there is some $C \in \mathcal{A}$ with nonzero curvature r . From the fourth column of N in (2) with β set to 0, we see that the curvature of $M_0^n C$ for $n \in \mathbb{Z}$ is $|\delta|^{2n}r$, which is assumed to be integral for any n . Thus $|\delta| = 1$, which combines with $1 = \det M = \alpha\delta$ to give $\alpha = \bar{\delta}$. Finally, $\text{tr } M_0^2 = \alpha^2 + \delta^2 = \bar{\delta}^2 + \delta^2$ contradicts $\Im(\text{tr } M_0^2) \neq 0$. Thus $\beta \neq 0$.

We claim that after translating \mathcal{A} by $-\alpha/\beta$ and rotating/scaling by $t\beta$, it is contained in \mathcal{S}_D with

$$D = |t\beta\sqrt{\Delta}|^2. \tag{3}$$

First observe that the new copy of \mathcal{A} has curvatures in $\mathbb{Z}/|t\beta|$, and its symmetry group contains

$$\begin{bmatrix} t\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha/\beta \\ 0 & 1 \end{bmatrix} M_0 \begin{bmatrix} 1 & \alpha/\beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/t\beta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -t \\ 1/t & \tau \end{bmatrix}. \tag{4}$$

Relabel M_0 as the last matrix above.

Let R_0 denote the unique matrix satisfying $\mathbf{v}_C R_0 = [r_{-2} \ r_{-1} \ r_0 \ r_1]$ for any oriented circle C , where r_n denotes the curvature of $M_0^n C$. For the values $n = -2, -1, 0$, and 1, we use the bottom two entries of M_0^n to replace β and δ in the fourth

column of N from (2); this provides the corresponding column of R_0 . We get

$$R_0 = \begin{bmatrix} \tau/t & 0 & 0 & \bar{\tau}/t \\ \bar{\tau}/t & 0 & 0 & \tau/t \\ |\tau/t|^2 & 1/t^2 & 0 & 1/t^2 \\ 1 & 0 & 1 & |\tau|^2 \end{bmatrix}.$$

We have $\det R_0 = (\tau^2 - \bar{\tau}^2)/t^4 = \sqrt{\Delta}/t^3 \neq 0$. Letting $\sigma = \bar{\tau}(1 - \tau^2) \in \mathcal{O}$, the first and third columns of R_0^{-1} are

$$\begin{bmatrix} \tau/\sqrt{\Delta} \\ \sigma/\sqrt{\Delta} \\ -\bar{\sigma}/\sqrt{\Delta} \\ -\bar{\tau}/\sqrt{\Delta} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ t^2 \\ 0 \\ 0 \end{bmatrix}.$$

Recall that $r_n \in \mathbb{Z}/|t\beta| = \sqrt{\Delta}\mathbb{Z}/i\sqrt{D}$ for any $C \in \mathcal{A}$. So it follows from $\mathbf{v}_C = [r_{-2} \ r_{-1} \ r_0 \ r_1]R_0^{-1}$ that C has curvature-center $(\tau r_{-2} + \sigma r_{-1} - \bar{\sigma} r_0 - \bar{\tau} r_1)/\sqrt{\Delta} \in \mathcal{O}/i\sqrt{D}$ and cocurvature $t^2 r_{-1} \in \sqrt{\Delta}\mathbb{Z}/i\sqrt{D}$. Thus $i\sqrt{D}\mathbf{v}_C \in \mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$. \square

The conjugating matrix product in (4) works as “ M ” from the discussion immediately preceding the theorem. That is, our affine transformation M gives $M\Gamma M^{-1} \cap \text{PSL}_2(\mathcal{O})$ finite index in $M\Gamma M^{-1}$. We do not need this fact, so we do not prove it.

In [16], Kapovich and Kontorovich introduce *bugs* as a generalization of circle packings. A bug is a set of oriented circles in which overlapping interiors must intersect at angle π/n for n belonging to some finite subset of \mathbb{N} .

Corollary 4.3 *If \mathcal{A} is an integral arrangement with Zariski dense symmetry group, the angle θ between two intersecting circles of \mathcal{A} satisfies $\cos(\theta) \in \mathbb{Q}$. In particular, if \mathcal{A} is a bug then θ is either $0, \pm\pi/3, \text{ or } \pi/2$.*

Proof The first claim is a combination of Proposition 3.5 and Theorem 4.2. The claim about bugs is Niven’s theorem [23]: $\arccos(\pi/n) \in \mathbb{Q}$ only if $n \in \{1, 2, 3\}$. \square

As the remainder of this section shows, actually identifying an arrangement \mathcal{A} within some \mathcal{S}_D is more useful than just knowing it is possible. Note that the proof of Theorem 4.2 gives a formula for D in (3).

4.2 Superintegrality

Our first property of subarrangements of \mathcal{S}_D applies when D divides Δ . The reader may recall from the introduction that this happens to be true of many, perhaps most, arrangements already appearing in literature.

The Apollonian supergroup is introduced in [11] by Graham et al.. Building on their idea, Kontorovich and Nakamura consider the supergroup more generally. We use their definition [17].

Definition 4.4 The *supergroup* of an arrangement \mathcal{A} is generated by reflections in its circles as well as the matrices in $\text{PSL}_2(\mathbb{C})$ that fix \mathcal{A} . We call \mathcal{A} *superintegral* if the orbit under its supergroup is integral.

Remark that if \mathcal{A} in the definition above happens to be a circle packing, then its orbit under the supergroup is called a superpacking. As mentioned in the introduction, certain \mathcal{S}_D arise as superpackings in prior literature.

Superintegrality is used in [2, 6, 17] to help classify crystallographic circle packings, and again in [16] to help classify Kleinian circle packings and bugs.

Lemma 4.5 *If \mathcal{A} is integral with Zariski dense supergroup, then it is superintegral if and only if $2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q \in \mathbb{Z}$ for all $C, C' \in \mathcal{A}$. In this case, each intersection angle is either 0, $\pm\pi/3$, or $\pi/2$.*

Proof Given oriented circles C and C' , the reflection of C' in C corresponds to the vector $\mathbf{v}_{C'} - 2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q \mathbf{v}_C$ by Proposition 2.3. Thus if $2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q \in \mathbb{Z}$ for all $C, C' \in \mathcal{A}$, reflections in circles from \mathcal{A} preserve the lattice generated by their vectors. This lattice is thus preserved by the supergroup’s image under the spin homomorphism, implying curvatures in the orbit of the supergroup remain integral.

Conversely, suppose \mathcal{A} is superintegral. Let $M, M' \in \text{PSL}_2(\mathbb{C})$ be such that $C = M\widehat{\mathbb{R}}$ and $C' = M'\widehat{\mathbb{R}}$ belong to \mathcal{A} . Let $\mathbf{v}_C = [\zeta \ \bar{\zeta} \ \hat{r} \ r]$ and $\mathbf{v}_{C'} = [\zeta' \ \bar{\zeta}' \ \hat{r}' \ r']$. By Proposition 2.3, composing the reflection in C followed by the reflection in C' is given as a Möbius transformation by

$$\begin{bmatrix} \zeta' & -\hat{r}' \\ r' & -\bar{\zeta}' \end{bmatrix} \begin{bmatrix} \bar{\zeta} & -\hat{r} \\ r & -\zeta \end{bmatrix} = \begin{bmatrix} \bar{\zeta}\zeta' - r\hat{r}' & \hat{r}'\zeta - \hat{r}\zeta' \\ r'\bar{\zeta} - r\bar{\zeta}' & \zeta\bar{\zeta}' - \hat{r}r' \end{bmatrix}.$$

Call the last matrix N , and observe that $\text{tr } N = 2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q$. Theorem 4.1 says that some power of N , say N^n , is contained in $\text{PSL}_2(\mathcal{O})$ up to conjugation. Therefore $\text{tr } N^n \in \mathcal{O}$ because conjugation preserves traces. But $\text{tr } N^n$ can be expressed as a monic, integral polynomial in $\text{tr } N$, so we must have $\text{tr } N \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z}$.

The final assertion about intersection angles follows from Proposition 2.2. □

Proposition 4.6 *If $D \mid \Delta$, all subsets of \mathcal{S}_D are superintegral.*

Proof Let $C, C' \in \mathcal{S}_D$ and let \mathbf{v}_C and $\mathbf{v}_{C'}$ be as in the last proof. Since $D \mid \Delta$ and $\Delta \mid (i\sqrt{D}\hat{r})(i\sqrt{D}r)$,

$$D = |i\sqrt{D}\zeta|^2 - D\hat{r}r \equiv |i\sqrt{D}\zeta|^2 \pmod{D}.$$

Similarly, $D \mid |i\sqrt{D}\zeta'|^2$. As any prime ideal containing Δ equals its conjugate, we see that D also divides $(i\sqrt{D}\zeta)(i\sqrt{D}\zeta')$ and $(i\sqrt{D}\zeta)(i\sqrt{D}\zeta')$. Thus $2\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_Q = \zeta\bar{\zeta}' + \bar{\zeta}\zeta' - \hat{r}r' - r\hat{r}' \in \mathbb{Z}$, proving our claim by Lemma 4.5. □

Baragar and Lautzenheiser discovered a one-parameter family of circle packings that generalize the classical Apollonian strip packing [1]. They begin with four circles, say C_1, C_2, C_3 , and C_4 , oriented so as to have disjoint interiors. Two are parallel lines distanced 1 apart. Two are circles of radius $1/2$ with centers distanced \sqrt{D} apart for

some $D \in \mathbb{N}$, each circle tangent to both lines. It is then proved in [1] that circles C such that \mathbf{v}_C is in the lattice generated by $\mathbf{v}_{C_1}, \mathbf{v}_{C_2}, \mathbf{v}_{C_3}$, and \mathbf{v}_{C_4} (over \mathbb{Z}) are dense in \widehat{C} and only intersect tangentially. Among such circles, they keep those that are positively oriented, lie between the two lines, and are not properly contained in the interior of another such circle. This produces a circle packing. The same procedure is used to construct the circle packing in Fig. 4 from its background arrangement, so the reader can see the resulting effect.

Proposition 4.7 *The Baragar-Lautzenheiser packing of parameter D is contained in \mathcal{S}_D for $\mathbb{Q}(i\sqrt{D})$.*

Proof Let C_1 be the vertical line through 0 and C_2 the vertical line through 1. Orient them to have disjoint interiors, so

$$\mathbf{v}_{C_1} = [-1 \ -1 \ 0 \ 0], \quad \text{and} \quad \mathbf{v}_{C_2} = [1 \ 1 \ 2 \ 0]. \tag{5}$$

We also have two positively oriented circles of curvature 2, C_3 and C_4 :

$$\mathbf{v}_{C_3} = [1 \ 1 \ 0 \ 2], \quad \text{and} \quad \mathbf{v}_{C_4} = [1 + 2i\sqrt{D} \ 1 - 2i\sqrt{D} \ 2D \ 2]. \tag{6}$$

They are tangent to C_1 and C_2 with centers distanced \sqrt{D} apart. For $k = 1, 2, 3, 4$ we have $i\sqrt{D}\mathbf{v}_{C_k} \in \mathbb{Z}[i\sqrt{D}]^2 \times (i\sqrt{D}\mathbb{Z})^2$ implying $C_k \in \mathcal{S}_D$ for $K = \mathbb{Q}(i\sqrt{D})$. \square

Note that if D is divisible by a perfect square, say $f^2 \neq 1, 4$, then D does not divide the discriminant of $\mathbb{Q}(i\sqrt{D})$. In these cases Proposition 4.6 does not apply. Rather, \mathbf{v}_C for C in the Baragar-Lautzenheiser packing satisfies $i\sqrt{D}\mathbf{v}_C \in \mathcal{O}_f^2 \times (\sqrt{\Delta_f}\mathbb{Z})^2$, where \mathcal{O}_f is the order of discriminant $\Delta_f = f^2\Delta$. We avoid discussing non-maximal orders here, except to mention that the proof of Proposition 4.6 still works for $D \mid \Delta_f$. Alternatively, Lemma 4.5 can be applied directly to a Baragar-Lautzenheiser packing by checking that products of lattice generators in (5) and (6) are integers or half-integers. Either way, we see that Baragar-Lautzenheiser packings are superintegral.

4.3 The Asymptotic Local–Global Principle

Certain integral circle packings have congruence restrictions on what integers can appear as curvatures. For example, curvatures in classical Apollonian packings always lie in a proper subset of the congruence classes mod 24 [8]. Graham, Lagarias, Mallows, Wilkes, and Yan conjectured (for Apollonian packings) that every sufficiently large integer which might occur as a curvature does [10]. The conjecture in this form was shown to be false by Haag, Kertzer, Rickards, and Stange, who added the outputs of certain quadratic and quartic forms to the list of prohibited curvatures (which they conjecture to now be complete) [15]. Progress toward an affirmative proof was achieved by Bourgain and Kontorovich. They showed that curvatures in an Apollonian packing have asymptotic density 1 among integers passing local obstructions [3].

In each of the packings considered here, local obstructions to curvatures always form a set of forbidden congruence classes modulo some positive integer L_0 , as in the

Apollonian case where $L_0 = 24$. This is a consequence of strong approximation for Zariski dense subgroups of $\mathrm{PSL}_2(\mathcal{O})$ [24]. As such, we define the asymptotic local–global principle *with respect to* L_0 . A more general definition need not assume L_0 exists.

Definition 4.8 An arrangement \mathcal{A} satisfies the *asymptotic local–global principle with respect to* L_0 if the number of positive integers less than x that occur as curvatures in \mathcal{A} is asymptotic to kx/L_0 , where k is the number of congruence classes mod L_0 represented by curvatures in \mathcal{A} .

The Bourgain–Kontorovich result has been generalized by Fuchs, Stange, and Zhang.

Theorem 4.9 (Fuchs–Stange–Zhang [9]) *Suppose $\Gamma \leq \mathrm{PSL}_2(\mathcal{O})$ is finitely generated and Zariski dense, has infinite covolume, and contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. If $M, N \in \mathrm{PGL}_2(K)$ with $N\widehat{\mathbb{R}}$ tangent to $\widehat{\mathbb{R}}$, then $M\Gamma N\widehat{\mathbb{R}}$ satisfies the asymptotic local–global principle with respect to some L_0 that depends only on Γ , N , and $|\det M|/\|(M)\|$.*

Remark that [9] restricts attention to $M, N \in \mathrm{PSL}_2(K)$, not $\mathrm{PGL}_2(K)$ as stated above. Communication with the second- and third-named authors confirmed that their proof still applies. When $M \in \mathrm{PSL}_2(K)$, a common denominator for the entries of M is used in [9] as a scaling factor to achieve integrality of certain quadratic forms. When $M \in \mathrm{PGL}_2(K)$, the role of a common denominator squared can be assumed more generally by the integral ideal $(\det M)/(M)^2$.

The hypotheses of Theorem 4.9 can often be verified without ever computing the symmetry group of an arrangement. Let us show how the manner in which a subarrangement sits in \mathcal{S}_D can be used to conclude the asymptotic local–global principle. Recall that $H(D, \Delta)$ is the Hilbert symbol (Notation 3.6).

Lemma 4.10 *If $H(D, \Delta) = 1$, then \mathcal{S}_D is a finite union of $\mathrm{PSL}_2(\mathcal{O})$ orbits.*

Proof By Theorem 3.11, every element of \mathcal{S}_D is of the form $M\widehat{\mathbb{R}}$ for some $M \in \mathrm{PGL}_2(K)$ with $|\det M|/\|(M)\| = \sqrt{D}$. If $M, M' \in \mathrm{PGL}_2(K)$ are such that $(M) = (M')$, $\det M = \det M'$ and $M \equiv M' \pmod{(\det M)/(M)^2}$, then $M^{-1}M' \in \mathrm{PSL}_2(\mathcal{O})$. That is, $M\widehat{\mathbb{R}}$ and $M'\widehat{\mathbb{R}}$ are in the same $\mathrm{PSL}_2(\mathcal{O})$ orbit. There are finitely many possible ideals (M) up to scaling, as well as finitely many integral ideals $(\det M)/(M)^2$ of norm D , each with only finitely many congruence classes of matrices. \square

The proof above overcounts the number of orbits, but it will not matter.

Theorem 4.11 *Let $\mathcal{A} \subset \mathcal{S}_D$ be such that each circle in \mathcal{A} intersects at least one other tangentially, and $\mathcal{A} \cap M\mathcal{A}$ is either empty or \mathcal{A} for any M in some fixed congruence subgroup of $\mathrm{PSL}_2(\mathcal{O})$. Then \mathcal{A} satisfies the asymptotic local–global principle.*

Proof Let Γ_0 be the maximal subgroup of $\mathrm{PSL}_2(\mathcal{O})$ that preserves \mathcal{A} . Let $\Gamma(L) \leq \mathrm{PSL}_2(\mathcal{O})$ be the congruence subgroup (of level L) assumed in our hypothesis.

Proposition 3.5 tells us tangential intersection in \mathcal{S}_D occurs only at points in \widehat{K} , which means $H(D, \Delta) = 1$ by Lemma 3.7. So each $C \in \mathcal{A}$ is of the form $M\widehat{\mathbb{R}}$ for

some $M \in \text{PGL}_2(K)$ with $|\det M|/\|(M)\| = \sqrt{D}$. We claim for such a matrix M , that $MAM^{-1} \in \Gamma_0$ whenever $A \in \text{PSL}_2(\mathbb{Z})$ is in the principal congruence subgroup of level DL . To see this, first scale M so that $\|(M)\|$ and DL are coprime. Let $\text{adj } M = M^{-1} \det M$. We have

$$MA \text{ adj } M \equiv M \text{ adj } M = \begin{bmatrix} \det M & 0 \\ 0 & \det M \end{bmatrix} \pmod{DL}. \tag{7}$$

In particular, the entries of $MA \text{ adj } M$ are contained in $(\det M)/(M)^2$. But they are also contained in $(M)^2$, which is coprime to $(\det M)/(M)^2$. Thus entries of $MA \text{ adj } M$ are divisible by $\det M$. Now divide both sides of (7) by $\det M$, and recall that $\|(M)\|$ and L are coprime to see that $MAM^{-1} \in \Gamma(L)$. Since $(MAM^{-1})C = (MAM^{-1})M\widehat{\mathbb{R}} = M\widehat{\mathbb{R}} = C$, we have $C \in \mathcal{A} \cap (MAM^{-1})\mathcal{A}$. So our hypothesis gives $MAM^{-1} \in \Gamma_0$ as claimed. Note that C is the limit set of MAM^{-1} for $A \in \text{PSL}_2(\mathbb{Z})$ from the principal congruence subgroup of level DL . In particular, \mathcal{A} is in the limit set of Γ_0 , which is therefore Zariski dense.

Now, since $[\text{PSL}_2(\mathcal{O}) : \Gamma(L)]$ is finite, \mathcal{S}_D is a finite union of $\Gamma(L)$ orbits by Lemma 4.10. But if it happens for some $C \in \mathcal{A}$ and $M \in \Gamma(L)$ that $MC \in \mathcal{A}$, then $MC \in \mathcal{A} \cap M\mathcal{A}$ implies $M \in \Gamma_0$ by our hypothesis. Therefore \mathcal{A} is a finite union of orbits of Γ_0 . Fix one element from each orbit, call them C_1, \dots, C_n , as well as some $C'_k \in \mathcal{A}$ tangent to C_k for each $k = 1, \dots, n$. Write $C'_k = M_k\widehat{\mathbb{R}}$ for $M_k \in \text{PGL}_2(K)$ with $|\det M_k|/\|M_k\| = \sqrt{D}$. Since congruence subgroups of $\text{PSL}_2(\mathbb{Z})$ are finitely generated, for each $k = 1, \dots, n$ we can find some $\Gamma_k \leq M_k^{-1}\Gamma_0M_k$ that is finitely generated and of infinite covolume, while still being Zariski dense and containing the principal congruence subgroup of level DL in $\text{PSL}_2(\mathbb{Z})$. We may then write Γ_0C_k as a union of “orbits” of the form $(N_jM_k)\Gamma_k(M_k^{-1}C_k)$, where $N_j \in \text{PSL}_2(\mathcal{O})$ runs over a complete set of coset representatives for $\Gamma_0/(M_k\Gamma_kM_k^{-1})$.

Observe that $M_k^{-1}C_k$ is tangent to the real line by choice of C'_k , that Γ_k meets the criteria of Theorem 4.9, and that $|\det(N_jM_k)|/\|(N_jM_k)\| = |\det M_k|/\|(M_k)\| = \sqrt{D}$ for all j by Lemma 3.10. Thus $(N_jM_k)\Gamma_k(M_k^{-1}C_k)$ satisfies the asymptotic local–global principle with respect to some L_k that does not depend on j . By taking any finite set N_{j_1}, \dots, N_{j_i} for which the curvatures of the corresponding orbits represent every congruence class mod L_k represented by those in Γ_0C_k , we see that Γ_0C_k also satisfies the asymptotic local–global principle. Thus \mathcal{A} does as well, with respect to $L_0 = \text{lcm}(L_1, \dots, L_n)$. □

The argument in which we pass to the subgroup Γ_k and observe that multiplying by coset representatives N_j leaves $|\det M_k|/\|(M_k)\|$ unchanged can be used to show that the “finitely generated” and “infinite covolume” hypotheses are not needed in Theorem 4.9.

Theorem 4.11 has a nice geometric realization when we restrict attention to circle packings—an infinite arrangement in which circle interiors are dense and disjoint. In Fig. 2 we constructed a packing from a single, initial oriented circle by taking its largest exterior neighbor at each point of tangency. Continuing in this way, just one circle uniquely determines the rest of the packing. In particular, the packing and its

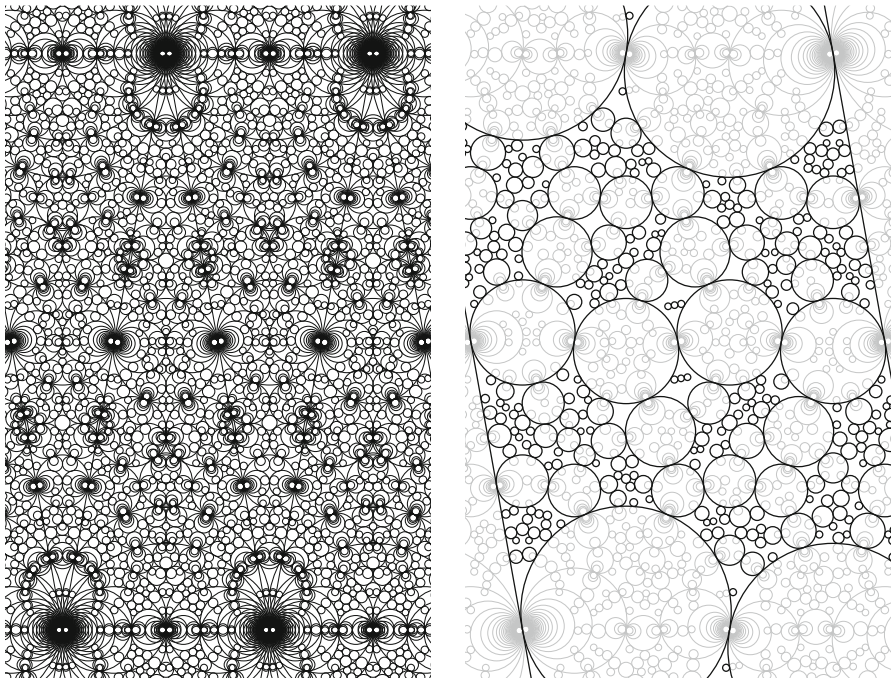


Fig. 4 S_8 for $\mathbb{Q}(i\sqrt{31})$ (left) and an immediate tangency packing (right)

image under a matrix in $\text{PSL}_2(\mathcal{O})$ are either disjoint or equal as Theorem 4.11 requires. (The argument is treated formally in the next corollary.)

This construction does not produce circle packings in every S_D . The first image of Fig. 4 shows S_8 for $\mathbb{Q}(i\sqrt{31})$. Here every circle has two points of tangency where its largest exterior neighbors have overlapping interiors. To find packings in such cases, we search among proper subarrangements. The background circles in the second image are those from S_8 with curvature-center and cocurvature satisfying $i\sqrt{8}\zeta \in (2, (1-i\sqrt{31})/2)$ and $i\sqrt{8}\hat{r} \in (2)$. Then the strategy outlined in the previous paragraph creates the bolded circle packing.

Definition 4.12 Suppose $H(D, \Delta) = 1$. Let \mathcal{A} consist of those $C \in S_D$ for which $i\sqrt{D}\mathbf{v}_C$ is in some fixed cosets of a full-rank sublattice of $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$. An *immediate tangency packing*, $\mathcal{P} \subset \mathcal{A}$, is a circle packing such that each $C \in \mathcal{A}$ is contained in the closure of the interior of a single $C' \in \mathcal{P}$.

While $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$ has rank six over \mathbb{R} , the first two entries of any \mathbf{v}_C are complex conjugates. So only a rank four sublattice of $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$ is actually relevant to determining \mathcal{A} in Definition 4.12.

Note that if $\mathcal{P} \subset \mathcal{A}$ is an immediate tangency packing and $C, C' \in \mathcal{P}$ are tangent, then they must be “immediately tangent” as defined by Stange [28] and as described above regarding Fig. 2. That is, no circle of \mathcal{A} can be caught in between C and C' , tangent and exterior to both. Such a circle could not possibly be in the closure of

the interior of a single element of \mathcal{P} . The immediate tangency property is not stated explicitly in Definition 4.12 due to ambiguity that arises when \mathcal{A} is not connected. Our definition uniquely defines a packing from an initial oriented circle whether \mathcal{A} is connected or not (as seen in the next proof).

Returning to Fig. 4, the sublattice defined by $i\sqrt{8}\zeta \in (2, (1-i\sqrt{31})/2)$ and $i\sqrt{8}\hat{r} \in (2)$ has four cosets because $(2, (1-i\sqrt{31})/2)$ has index two in \mathcal{O} and (2) has index two in \mathbb{Z} . Its trivial coset produces the background circles in the second image, which is Definition 4.12’s \mathcal{A} . Any of the three nontrivial cosets could also be used to create immediate tangency packings because \mathcal{A} in each case has only tangential intersections. It would be interesting to have a general method or criterion for selecting sublattices like this given some S_D . The one used for Fig. 4 was found by experimentation.

Corollary 4.13 *All immediate tangency packings satisfy the asymptotic local–global principle.*

Proof Let $\mathcal{P} \subset \mathcal{A} \subseteq S_D$ as in Definition 4.12. Since the sublattice defining \mathcal{A} is assumed to have full rank, it has finite index in $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$. Call the index L , and let $\Gamma(L) \leq \text{PSL}_2(\mathcal{O})$ be the congruence subgroup of level L . If $M \in \Gamma(L)$ then N from (2) is congruent to the identity mod L , and thus fixes each sublattice coset. In particular, $\Gamma(L)$ fixes \mathcal{A} .

Use C_{in} and C_{ex} to denote the interior and exterior of some $C \in \mathcal{A}$, and let $\overline{C_{\text{in}}} = C_{\text{in}} \cup C$ and $\overline{C_{\text{ex}}} = C_{\text{ex}} \cup C$ be their closures. Let $M \in \Gamma(L)$. We claim that $\mathcal{P} \cap M\mathcal{P}$ is either empty or \mathcal{P} as Theorem 4.11 requires. Let us suppose $C_0 \in M\mathcal{P}$ but $C_0 \notin \mathcal{P}$ and aim to show that $\mathcal{P} \cap M\mathcal{P} = \emptyset$. By assumption, there is $C \in \mathcal{P}$ with $C_0 \subset \overline{C_{\text{in}}}$, as well as $C' \in \mathcal{P}$ with $M^{-1}C \subset \overline{C'_{\text{in}}}$. Then MC'_{in} contains C and thus contains either C_{in} or C_{ex} . It must be the latter: Since MC' and C_0 are both in the circle packing $M\mathcal{P}$, $\overline{MC'_{\text{in}}} \supseteq \overline{C_{\text{in}}} \supset C_0$ forces MC' and C_0 to be the same oriented circle. But $M'C = C = C_0$ contradicts $C \in \mathcal{P}$ and $C_0 \notin \mathcal{P}$. So as claimed, $\overline{MC'_{\text{in}}}$ contains $\overline{C_{\text{ex}}}$ and therefore all of \mathcal{P} . The only element of $M\mathcal{P}$ in $\overline{MC'_{\text{in}}}$ is MC' itself, which is not in \mathcal{P} by the previous sentence. Thus $\mathcal{P} \cap M\mathcal{P} = \emptyset$.

To apply Theorem 4.11, it remains to check that every circle in \mathcal{P} is tangent to at least one other. Fix an element of \mathcal{P} and write it as $M\widehat{\mathbb{R}}$ for some $M \in \text{PGL}_2(K)$ with $|\det M|/\|(M)\| = \sqrt{D}$. Then

$$M\widehat{\mathbb{R}} \quad \text{and} \quad M \begin{bmatrix} 1 & DL\sqrt{\Delta} \\ 0 & -1 \end{bmatrix} \widehat{\mathbb{R}}$$

are tangent with disjoint interiors since the same is true of $\widehat{\mathbb{R}}$ oriented positively and $\widehat{\mathbb{R}} - DL\sqrt{\Delta}$ oriented negatively. Call the second circle above C . Using the same argument from (7), we have

$$C = \left(M \begin{bmatrix} 1 & DL\sqrt{\Delta} \\ 0 & 1 \end{bmatrix} M^{-1} \right) M\widehat{\mathbb{R}} \in \Gamma(L)M\widehat{\mathbb{R}} \subseteq \Gamma(L)\mathcal{A} = \mathcal{A}.$$

Thus $C \subset \overline{C'_{\text{in}}}$ for some $C' \in \mathcal{P}$ by assumption. But then C' and $M\widehat{\mathbb{R}}$ must be tangent at $C \cap M\widehat{\mathbb{R}}$ for their interiors to be disjoint. □

Corollary 4.14 *All Baragar-Lautzenheiser circle packings satisfy the asymptotic local–global principle.*

Proof The rank four sublattice of $\mathcal{O}^2 \times (\sqrt{\Delta}\mathbb{Z})^2$ from Definition 4.12 is generated by \mathbf{v}_{C_1} , \mathbf{v}_{C_2} , \mathbf{v}_{C_3} , and \mathbf{v}_{C_4} from (5) and (6). Their circle packing is constructed to be an immediate tangency packing in the resulting arrangement \mathcal{A} . \square

With respect to circle packings, the author suspects that Theorem 4.9’s hypothesis is not significantly weaker than Corollary 4.13’s. It seems at least slightly weaker, since Fuchs, Stange, and Zhang do not require $M\Gamma N\widehat{\mathbb{R}}$ to have tangential intersections, while immediate tangency packings always have tangential intersections. This difference aside, the author suspects that Definition 4.12 is essentially a geometric formulation of the algebraically-phrased hypothesis of Theorem 4.9 in the special case of circle packings.

5 Relation to the Class Group

5.1 Geometry at a Point

We first study the relationship among oriented circles in \mathcal{S}_D , if any, that contain a fixed point in $\widehat{K} = K \cup \{\infty\}$.

Definition 5.1 A family is a maximal subset of \mathcal{S}_D in which any two circles C and C' satisfy $\langle \mathbf{v}_C, \mathbf{v}_{C'} \rangle_{\mathcal{O}} = 1$. An extended family is a maximal subset in which any two circles intersect at a fixed point with angle θ satisfying $e^{i\theta} \in \mathcal{O}$.

All circles in a family intersect tangentially at a fixed point in \widehat{K} by Proposition 3.5. Also remark that if $\Delta \neq -3, -4$ the only units in \mathcal{O} are 1 and -1 . So an extended family contains two families consisting of the same set of circles with opposite orientation.

Lemma 5.2 *If $C \in \mathcal{S}_D$ and $\alpha \in C \cap K$, then $C = M\widehat{\mathbb{R}}$ for some $M \in \text{PGL}_2(K)$ with left column entries α (top) and 1, $(M) = (\alpha, 1)$, and $|\det M|/\|(M)\| = \sqrt{D}$.*

Proof That $C \cap K$ is nonempty implies $H(D, \Delta) = 1$ by Lemma 3.7, giving $M' \in \text{PGL}_2(K)$ with $M'\widehat{\mathbb{R}} = C$ and $|\det M'|/\|(M')\| = \sqrt{D}$. We have $M'^{-1}(\alpha) \in \widehat{\mathbb{R}} \cap \widehat{K} = \widehat{\mathbb{Q}}$. So there is some $N \in \text{PSL}_2(\mathbb{Z})$ such that $N^{-1}M'^{-1}(\alpha)$ is the point at infinity, $1/0$. In particular, if α', β', γ' and δ' are the entries of $M'N$ (arranged as in (2)), then $M'N(1/0) = \alpha$ means $\alpha'/\beta' = \alpha$. Consider the matrix

$$M = M'N \begin{bmatrix} 1/\beta' & 0 \\ 0 & \|(\alpha, 1)/(M')\|\overline{\beta'} \end{bmatrix} = \begin{bmatrix} \alpha \|(\alpha, 1)/(M')\|\overline{\beta'}\gamma' \\ 1 \|(\alpha, 1)/(M')\|\overline{\beta'}\delta' \end{bmatrix} \in \text{PGL}_2(K).$$

Recall that $(M'N) = (M')$ by Lemma 3.10. The ideal generated by the top-right entry of M is therefore

$$\frac{\|(\alpha, 1)\|\overline{\beta'}\gamma'}{\|(M')\|} = (\alpha, 1) \cdot \frac{(\gamma')}{(M')} \cdot \frac{\overline{(\alpha\beta', \beta')}}{(M')} = (\alpha, 1) \cdot \frac{(\gamma')}{(M'N)} \cdot \frac{\overline{(\alpha', \beta')}}{(M'N)},$$

which is contained in $(\alpha, 1)$. The same is true of the bottom-right entry of M . This shows that $(M) = (\alpha, 1)$, and $|\det M|/\|(M)\| = \|(\alpha, 1)/(M')\| |\det M'|/\|(\alpha, 1)\| = |\det M'|/\|(M')\| = \sqrt{D}$. \square

We use $[\alpha]$ to denote the ideal class of an ideal $\alpha \subset K$.

Theorem 5.3 *At $\alpha \in K$, there is one extended family per integral ideal of norm D in $[(\alpha, 1)]^2$. An intersection angle between elements in extended families corresponding to the ideals \mathfrak{D} and \mathfrak{D}' is the argument of a generator for $\mathfrak{D}/\mathfrak{D}'$. When scaled by $\sqrt{D}/|\Delta|$, curvatures from a single family form a congruence class mod $1/\|(\alpha, 1)\|$.*

Proof If $\mathfrak{D}_0 \in [(\alpha, 1)]^2$ is integral with norm D , then we can find $\gamma, \delta \in (\alpha, 1)\mathfrak{D}_0$ with $(\alpha\delta - 1\gamma) = (\alpha, 1)^2\mathfrak{D}_0$. So let M_0 have entries as in (2) but with $\beta = 1$. Then $(M_0) = (\alpha, 1)$ and $|\det M_0|/\|(M_0)\| = \sqrt{\|\mathfrak{D}_0\|} = \sqrt{D}$.

For an integral ideal $\mathfrak{D} \in [(\alpha, 1)]^2$ of norm D , we will show that

$$\left\{ M_0 \begin{bmatrix} 1 & \lambda \\ 0 & \mu \end{bmatrix} \widehat{\mathbb{R}} \mid \lambda \in \mathcal{O}, (\mu) = \mathfrak{D}/\mathfrak{D}_0 \right\} \tag{8}$$

is an extended family, and that a family is defined by fixing the generator μ .

But first, the angle between two circles from (8), one with matrix entries λ and μ as above and the other with entries λ' and μ' , is the angle between $\widehat{\mathbb{R}}$ and

$$\left(M_0 \begin{bmatrix} 1 & \lambda \\ 0 & \mu \end{bmatrix} \right)^{-1} M_0 \begin{bmatrix} 1 & \lambda' \\ 0 & \mu' \end{bmatrix} \widehat{\mathbb{R}} = \begin{bmatrix} 1 & (\lambda'\mu - \lambda\mu')/\mu \\ 0 & \mu'/\mu \end{bmatrix} \widehat{\mathbb{R}}. \tag{9}$$

This angle is evidently the argument of μ'/μ , which generates $\mathfrak{D}'/\mathfrak{D}$ assuming $(\mu) = \mathfrak{D}/\mathfrak{D}_0$ and $(\mu') = \mathfrak{D}'/\mathfrak{D}_0$. In particular, if $\mathfrak{D} = \mathfrak{D}'$ then $\mu'/\mu \in \mathcal{O}$, showing that each set of the form (8) is contained in a single extended family. Moreover, if $\mu = \mu'$ then the oriented circle in (9), call it C , has curvature-center i by Lemma 3.8. Since $\mathbf{v}_{\widehat{\mathbb{R}}} = [i \ -i \ 0 \ 0]$, we get $\langle \mathbf{v}_{\widehat{\mathbb{R}}}, \mathbf{v}_C \rangle_{\mathcal{O}} = 1$. So for fixed μ , (8) is contained in a single family.

Next, using the formula from Lemma 3.8, scaling a curvature from (8) by $\sqrt{D}/|\Delta| = |\mu \det M_0|/\sqrt{|\Delta|}\|(\alpha, 1)\|$ gives

$$\frac{2\Im(\mu\delta)}{\sqrt{|\Delta|}\|(\alpha, 1)\|} + \frac{2\Im(\lambda)}{\sqrt{|\Delta|}\|(\alpha, 1)\|}.$$

By fixing μ and varying λ we obtain curvatures from the same family that produce a full congruence class mod $1/\|(\alpha, 1)\|$.

It remains only to check that every circle in \mathcal{S}_D containing α can be expressed as in (8) for some λ and μ . By Lemma 5.2, every such circle is of the form $M\widehat{\mathbb{R}}$ for some $M \in \text{PGL}_2(K)$ with left column entries α and 1, $(M) = (\alpha, 1)$, and $(\det M)/(M)^2 = \mathfrak{D}$ for some $\mathfrak{D} \subseteq \mathcal{O}$ of norm D . Let γ' and δ' denote the right column entries of M . We have

$$M_0^{-1}M = \begin{bmatrix} 1 & (\gamma'\delta - \gamma\delta')/\det M_0 \\ 0 & \det M/\det M_0 \end{bmatrix}.$$

First observe that $(\det M / \det M_0) = \mathfrak{D}(M)^2 / \mathfrak{D}_0(M_0)^2 = \mathfrak{D}(\alpha, 1)^2 / \mathfrak{D}_0(\alpha, 1)^2 = \mathfrak{D} / \mathfrak{D}_0$. Next observe that $(\gamma'\delta - \gamma\delta') \in (\gamma', \delta')(\gamma, \delta) \subseteq (M)(\gamma, \delta) = (\alpha, 1)(\gamma, \delta) = (\alpha, 1)^2 \mathfrak{D}_0 = (\det M_0)$. Therefore the upper-right entry above is some integer λ . \square

As an example, recall from Fig. 3 the claim that certain ideal classes of $\mathbb{Q}(i\sqrt{39})$ can be distinguished in \mathcal{S}_4 . This field has a cyclic class group of order four, generated by $[\mathfrak{p}_2]$ for either prime \mathfrak{p}_2 over 2. The only integral ideal of norm 4 in the principal ideal class is (2) . So according to the theorem, if $[(\alpha, 1)]^2 = [\mathcal{O}]$ (meaning $(\alpha, 1) \in [\mathcal{O}]$ or $(\alpha, 1) \in [\mathfrak{p}_2]^2$) then \mathcal{S}_4 has only one extended family at α . These are points in Fig. 3 with only tangential intersection, as $\mathbb{Z}[(1 + i\sqrt{39})/2]$ has trivial unit group. On the other hand, there are two integral ideals of norm 4 in $[\mathfrak{p}_2]^2$: \mathfrak{p}_2^2 and $\overline{\mathfrak{p}_2}^2$. Therefore when $(\alpha, 1) \in [\mathfrak{p}_2]$ or $(\alpha, 1) \in [\mathfrak{p}_2]^3$ there are two extended families at α . The angle between them is the argument of a generator for $\mathfrak{p}_2^2 / \overline{\mathfrak{p}_2}^2$, which is $(5 \pm i\sqrt{39})/8$ depending on the choice of \mathfrak{p}_2 .

Definition 5.4 An arrangement \mathcal{A} covers α if $\alpha \in C$ for some $C \in \mathcal{A}$. In this case, if $\alpha \in K$ we say \mathcal{A} covers the ideal class corresponding to α , which is $[(\alpha, 1)]$.

Corollary 5.5 A point $\alpha \in K$ is covered by \mathcal{S}_D if and only if $[(\alpha, 1)]^2$ contains an integral ideal of norm D . In particular, whether or not α is covered depends only the coset of the 2-torsion subgroup of the class group to which $[(\alpha, 1)]$ belongs.

Proof A point $\alpha \in \widehat{K}$ is covered if and only if \mathcal{S}_D has at least one extended family at α . By Theorem 5.3, this is equivalent to $[(\alpha, 1)]^2$ containing at least one integral ideal of norm D . \square

So in \mathcal{S}_D , covering an ideal class implies every point corresponding to that ideal class is covered, and indeed every point corresponding to an ideal class in the same 2-torsion coset.

We can also say something about the geometry of \mathcal{S}_D around $\alpha \in K$ that are not covered. These points exhibit the typical repulsion property of rational numbers in Diophantine approximation. Figure 5 shows \mathcal{S}_{39} for $\mathbb{Q}(i\sqrt{39})$, which only covers the two 2-torsion ideal classes in the class group. A red forbidden zone has been drawn around uncovered points in K like $(1 + i\sqrt{39})/4$, the focal point of the image. The radius of a red dot is computed from the next proposition with $r = 1200$, because Fig. 5 shows curvatures up to 1200.

Proposition 5.6 If $\alpha \in K$ is not on some $C \in \mathcal{S}_D$ of curvature r , the distance between C and α is at least $(\sqrt{2} - 1) \min(d, \sqrt{d/|r|})$, where $d = \sqrt{|\Delta|} \|(\alpha, 1)\| / \sqrt{D}$.

Proof Suppose $\alpha \notin C \in \mathcal{S}_D$. Let $\mathbf{v}_C = [\zeta \ \bar{\zeta} \ \hat{r} \ r]$. Recall that when $r = 0$, ζ is a unit vector orthogonal to C . So the distance between α and C is the scalar projection of α onto ζ , which is $|\Re(\zeta \bar{\alpha})| = |2\Im(i\sqrt{D}\zeta \bar{\alpha})| / 2\sqrt{D}$. As $2\Im(i\sqrt{D}\zeta \bar{\alpha})$ is an integer multiple of $\sqrt{|\Delta|} \|(\alpha, 1)\|$, it follows that $|\Re(\zeta \bar{\alpha})| \geq d/2 > (\sqrt{2} - 1)d$.

Now suppose $r \neq 0$. From $|\zeta|^2 = 1 + \hat{r}r$ we have the second equality below:

$$D|\zeta - r\alpha|^2 = D(|\zeta|^2 - 2r\Re(\zeta \bar{\alpha}) + r^2|\alpha|^2) =$$

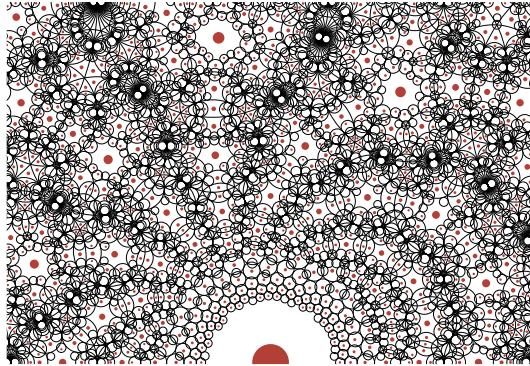


Fig. 5 \mathcal{S}_{39} for $\mathbb{Q}(i\sqrt{39})$ keeping away from uncovered rational points

$$D - i\sqrt{D}r(i\sqrt{D}\hat{r} - 2i\mathfrak{S}(i\sqrt{D}\zeta\bar{\alpha}) + i\sqrt{D}r|\alpha|^2).$$

In parentheses on the bottom line is a nonzero (since $\alpha \notin C$) integer multiple of $\sqrt{D}\|(\alpha, 1)\|$. In particular, $D|\zeta - r\alpha|^2$ is either at least $D + \sqrt{D}|\Delta|\|(\alpha, 1)\||r| = D(1 + d|r|)$ or at most $D - \sqrt{D}|\Delta|\|(\alpha, 1)\||r| = D(1 - d|r|)$. Thus if $|r| < 1/d$, the distance between C and α is at least

$$\begin{aligned} & \left| \frac{\zeta}{r} - \alpha \right| - \frac{1}{|r|} \\ & \geq \frac{1}{|r|} \min \left(\sqrt{1 + d|r|} - 1, -\sqrt{1 - d|r|} + 1 \right) \\ & > \frac{\sqrt{2} - 1}{|r|} > (\sqrt{2} - 1)d. \end{aligned}$$

On the other hand, if $|r| \geq 1/d$ then it is not possible for $D|\zeta - r\alpha|^2$, which is positive, to be at most $D(1 - d|r|)$. So there is no need for the min function above. The distance between C and α is at least

$$\frac{1}{|r|} \left(\sqrt{1 + d|r|} - 1 \right) = \sqrt{\frac{d}{|r|}} \left(\sqrt{\frac{1}{d|r|} + 1} - \sqrt{\frac{1}{d|r|}} \right) \geq (\sqrt{2} - 1)\sqrt{\frac{d}{|r|}},$$

where the last inequality uses $|r| \geq 1/d$ again. □

5.2 Connectivity and the Class Group

Stange proved that \mathcal{O} is Euclidean if and only if the Schmidt arrangement (or \mathcal{S}_1) is topologically connected [27]. The relationship with Euclideanity is extended to

arbitrary \mathcal{S}_D in the author’s dissertation, where it is shown how each step in a pseudo-Euclidean algorithm corresponds to one “step” along a chain of circles [21]. This observation inspired a continued fraction algorithm in which the resulting approximations are chain intersection points from the “walk” through \mathcal{S}_D . The forthcoming example surrounding Fig. 6 hints at the algorithm, but it is not stated here (see [22]). The focus of this section is consequences of such pseudo-Euclideanity for the class group. This is to say that Theorem 5.12 and Corollary 5.13 should be thought of as generalizations of the classical statement “Euclidean implies principal ideal domain.”

Lemma 5.7 *Let $M \in \text{PGL}_2(K)$. If $\alpha \in M\widehat{\mathbb{R}} \cap K$ then $[(\alpha, 1)] = [(M)\mathfrak{d}]$ for some integral ideal \mathfrak{d} dividing $(\det M)/(M)^2$.*

Proof Let M have entries as in (2). Any point in $C \cap K$ is of the form $M(p/q) = (\alpha p + \gamma q)/(\beta p + \delta q)$ for some $p/q \in \widehat{\mathbb{Q}}$. Assuming $p, q \in \mathbb{Z}$ are coprime, we have $(\det M)/(M) \subseteq (\alpha p + \gamma q, \beta p + \delta q) \subseteq (M)$ by Lemma 3.10. Then setting $\mathfrak{d} = (\alpha p + \gamma q, \beta p + \delta q)/(M)$ gives $(\det M)/(M)^2 \subseteq \mathfrak{d} \subseteq \mathcal{O}$ as well as $[(M(p/q), 1)] = [(\alpha p + \gamma q, \beta p + \delta q)] = [(M)\mathfrak{d}]$. \square

Note that the quotient of ideal classes corresponding to two points on the same circle in \mathcal{S}_D is $[\mathfrak{d}/\mathfrak{d}']$ for some $\mathfrak{d}, \mathfrak{d}' \subseteq \mathcal{O}$ with norms dividing D . This prompts the next definition.

Definition 5.8 An arrangement \mathcal{A} is *rationally connected* if for any $C, C' \in \mathcal{A}$ there is a chain $C = C_0, C_1, \dots, C_n = C' \in \mathcal{A}$ such that $C_{k-1} \cap C_k \cap \widehat{K} \neq \emptyset$ for all k .

Proposition 5.9 *The ideal classes covered by a rationally connected subset of \mathcal{S}_D are contained in a single coset of the subgroup generated by ideal classes of primes containing D .*

Proof If $H(D, \Delta) = -1$ then \mathcal{S}_D covers no points of \widehat{K} by Lemma 3.7. The claim holds vacuously in this case.

If $H(D, \Delta) = 1$, then Theorem 3.11 says every circle in \mathcal{S}_D is of the form $M\widehat{\mathbb{R}}$, where $M \in \text{PGL}_2(K)$ and $(\det M)/(M)^2$ has norm D . By Lemma 5.7, the ideal classes covered by such a circle are contained in a single coset of the subgroup generated by ideal classes of primes containing D . So our claim follows by induction on n from Definition 5.8. \square

Proposition 5.10 *The ideal classes covered by \mathcal{S}_D are contained in a single coset of the subgroup generated by ideal classes of primes containing $D\Delta$.*

Proof As in the previous proof, we are done unless $H(D, \Delta) = 1$. So fix $C_0 \in \mathcal{S}_D$ and use Theorem 3.11 to write $C_0 = M_0\widehat{\mathbb{R}}$ as usual. Let Γ_Δ and $\Gamma_{D\Delta}$ denote the subgroups generated by ideal classes of primes containing Δ and $D\Delta$. We claim every ideal class covered by \mathcal{S}_D is in $[(M_0)]\Gamma_{D\Delta}$.

Let $C \in \mathcal{S}_D$ be arbitrary and write $C = M\widehat{\mathbb{R}}$. All ideal classes covered by C are in $[(M)]\Gamma_D$ by Lemma 5.7, so we will be done if we can show $[(M)] \in [(M_0)]\Gamma_{D\Delta}$. Let $\mathfrak{D}_0 = (\det M_0)/(M_0)^2$ and $\mathfrak{D} = (\det M)/(M)^2$. There are at most two ideals in \mathcal{O} above each prime in \mathbb{Z} —a prime ideal and its conjugate—so $\|\mathfrak{D}_0\| = \|\mathfrak{D}\|$ implies

\mathfrak{D}_0 and \mathfrak{D} have the same set of prime ideal divisors up to conjugation. In particular, there is some integral ideal $\mathfrak{a} \mid \mathfrak{D}_0$ satisfying $\bar{\mathfrak{a}}\mathfrak{D}_0 = \mathfrak{a}\mathfrak{D}$. Now divide both sides of $[(\det M)] = [(\det M_0)]$ by $[\mathfrak{D}]$ to get $[(M)]^2 = [(M_0)^2\mathfrak{D}_0/\mathfrak{D}] = [(M_0)^2\mathfrak{a}/\bar{\mathfrak{a}}] = [(M_0\mathfrak{a})^2]$. Thus $[(M)]$ and $[(M_0)\mathfrak{a}]$ are in the same coset of the 2-torsion subgroup, which is exactly Γ_Δ . Finally, $\|\mathfrak{a}\| \mid D$ gives $[(M)] \in [(M_0\mathfrak{a})]\Gamma_\Delta \subseteq [(M_0)]\Gamma_{D\Delta}$. \square

We want to say something substantive about the size of the cosets in the two propositions above, but rational connectivity is not enough by itself. Let us show why with an example.

Recall from Fig. 5 that \mathcal{S}_{39} for $\mathbb{Q}(i\sqrt{39})$ only covers one coset (out of two in this case) of the 2-torsion subgroup of the class group—the bare minimum for any \mathcal{S}_D when $H(D, \Delta) = 1$ by Corollary 5.5. Nevertheless, it appears that every circle is tangent to a larger one, and such intersections are in \hat{K} by Proposition 3.5.

To prove that \mathcal{S}_{39} is indeed rationally connected, we will show that each C_0 is tangent to some C_1 such that $\max\|(\alpha, 1)\|$ among $\alpha \in C_1 \cap K$ is strictly larger than among $\alpha \in C_0 \cap K$ (provided the latter is less than 1). This creates a chain C_0, C_1, \dots, C_n , where C_n contains some α with $\|(\alpha, 1)\| = 1$ (meaning $\alpha \in \mathcal{O}$). By Theorem 5.3, a family at such an α contains a circle of curvature 0. All circles of curvature 0 meet at the point at infinity, thereby proving rational connectivity.

Given some $C_0 \in \mathcal{S}_{39}$ with nonzero curvature, let $\alpha \in C_0 \cap K$ be such that $\|(\alpha, 1)\|$ is maximal among such points. Using Lemma 5.2 we may write $C_0 = M\hat{\mathbb{R}}$ for some $M \in \text{PGL}_2(\mathbb{Q}(i\sqrt{39}))$ with left-column entries α and 1, $(M) = (\alpha, 1)$, and $|\det M|/\|(M)\| = \sqrt{39}$. By right multiplying M by the appropriate upper-triangular matrix in $\text{PSL}_2(\mathbb{Z})$, which does not affect $(M) = (\alpha, 1)$ by Lemma 3.10, we may further assume that its right-column entries, say γ and δ , are in $(M)(i\sqrt{39})$.

For each $\lambda \in \mathcal{O}$ there is a circle tangent to C_0 :

$$C_\lambda = M \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \hat{\mathbb{R}}.$$

Note that $M(\lambda/1) = (\alpha\lambda + \gamma 1)/(1\lambda + \delta 1) \in C_\lambda$. Also, by Lemma 3.10, $(\lambda, 1) \in \mathcal{O}$ implies $(\alpha\lambda + \gamma, \lambda + \delta) \supseteq (\det M)/(M) = (\alpha, 1)(i\sqrt{39})$. But $\gamma, \delta \in (\alpha, 1)(i\sqrt{39})$, which justifies the second equality below:

$$\left\| \left(\frac{\alpha\lambda + \gamma}{\lambda + \delta}, 1 \right) \right\| = \frac{\|(\alpha\lambda + \gamma, \lambda + \delta)\|}{|\lambda + \delta|^2} = \frac{\|(\alpha, 1)(\lambda, i\sqrt{39})\|}{|\lambda + \delta|^2}. \tag{10}$$

The norm above exceeds $\|(\alpha, 1)\|$ if and only if $|\lambda + \delta|^2 < \|(\lambda, i\sqrt{39})\|$. That is, we win with $C_1 = C_\lambda$ if $-\delta$ is contained in the open disc centered on $\lambda \in \mathcal{O}$ with radius squared $\|(\lambda, i\sqrt{39})\|$. Such discs cover the plane!..almost.

Our open discs leave holes (akin to “singular points” on the Ford domain for $\text{PSL}_2(\mathcal{O})$ [29]), three of which can be seen along the vertical centerline of Fig. 6. This image shows a fundamental region for $(i\sqrt{39})$. The holes occur at the two cosets of $(i\sqrt{39})$ represented by $(39 \pm i\sqrt{39})/4$. We claim $-\delta$ cannot be such a point, which would complete the argument that \mathcal{S}_{39} is rationally connected. Note that $((39 \pm i\sqrt{39})/2, 2)$ is a prime over 2, which is not in a 2-torsion ideal class. On the

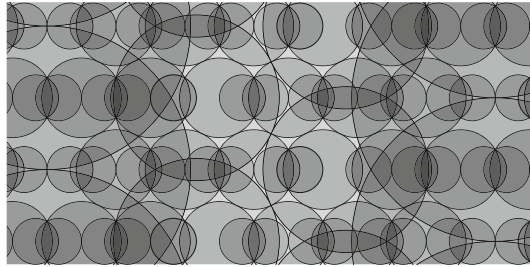


Fig. 6 Discs centered on $\lambda \in \mathcal{O}$ of radius squared $\|(\lambda, i\sqrt{39})\|$

other hand, δ lies on the image of $\widehat{\mathbb{R}}$ under the transpose of M . This is a circle in \mathcal{S}_D , implying $(\delta, 1) = (-\delta, 1)$ is in a 2-torsion ideal class.

Although $-\delta$ cannot actually land on a hole, it can come arbitrarily close. Thus the ratio of $\|(\alpha, 1)\|$ to the norm in (10) can be arbitrarily close to 1, making for a small gain with our step from C_0 to $C_1 = C_\lambda$. The problem is compounded by the fact that C_1 will experience the same issue. The value of “ $-\delta$ ” will slowly inch away from the nearby hole as we create the chain C_0, C_1, \dots . So while \mathcal{S}_39 may be rationally connected, there is no bound on the chain length required to make significant progress toward reaching a circle of curvature 0. Such is the insufficiency of rational connectivity, remedied below.

Definition 5.11 An arrangement \mathcal{A} is *finitely connected* if there is $n \in \mathbb{N}$ such that any $C \in \mathcal{A}$ with nonzero curvature r has a chain $C = C_0, C_1, \dots, C_n \in \mathcal{A}$ for which $C_{k-1} \cap C_k \neq \emptyset$ for all k and C_n has curvature magnitude at most $|r|/2$.

The C_k ’s need not be distinct, meaning the chain can have length less than n . Also remark that we do not require $C_{k-1} \cap C_k \subset \widehat{K}$.

The proof that \mathcal{S}_4 for $\mathbb{Q}(i\sqrt{39})$ in Fig. 3 is finitely (and rationally) connected proceeds by finding disc covers of \mathbb{C} as in Fig. 6. In this case, however, three disc covers must be computed since there are three integral ideals of norm 4. (For the previous example, there is only one integral ideal of norm 39.) A nearly identical scenario is worked out in detail (\mathcal{S}_4 for $\mathbb{Q}(i\sqrt{31})$) in the author’s dissertation [21].

Theorem 5.12 *If \mathcal{S}_D is finitely connected, then it covers all of $\widehat{K} = K \cup \{\infty\}$. In particular, the class group is generated by ideal classes of primes with norm dividing $D\Delta$, and every ideal class in the principal genus contains an integral ideal of norm D .*

Proof Suppose \mathcal{S}_D is finitely connected with chain lengths n as in Definition 5.11. We will show that an arbitrary $\alpha \in K$ is covered by \mathcal{S}_D . Fix any $C \in \mathcal{S}_D$ and $\tau \in C$ (in K or not) which is not the point at infinity. Let $\mathbf{v}_C = [\zeta \ \bar{\zeta} \ r \ \hat{r}]$.

Since $1 \in (\alpha, 1)$, there are infinitely many choices of coprime $\alpha_0, \beta_0 \in \mathcal{O}$ satisfying $\beta_0\alpha - \alpha_01 = 1$. Given such a pair, let $M_0 \in \text{PSL}_2(\mathcal{O})$ be a matrix with left column entries α_0 and β_0 . Call its right column entries γ_0 and δ_0 , and assume δ_0 has minimal magnitude in its coset mod β_0 (achieved via right multiplication by the appropriate upper triangular matrix). By Proposition 2.1, the curvature of M_0C is $2\Re(\zeta\beta_0\delta_0) + |\beta_0|^2\hat{r} + |\delta_0|^2r$. Our choice to make $|\delta_0|$ minimal bounds this curvature in magnitude

from above by $c|\beta_0|^2$ for some constant c independent of β_0 . Also, the distance between M_0C and α is at most $|M_0(\tau) - \alpha| =$

$$\left| \frac{\alpha_0\tau + \gamma_0}{\beta_0\tau + \delta_0} - \alpha \right| \leq \left| \frac{\alpha_0\tau + \gamma_0}{\beta_0\tau + \delta_0} - \frac{\alpha_0}{\beta_0} \right| + \left| \frac{\alpha_0}{\beta_0} - \alpha \right| = \frac{1}{|\beta_0(\beta_0\tau + \delta_0)|} + \frac{1}{|\beta_0|},$$

which is bounded above by $c'/|\beta_0|$ for some constant c' independent of β_0 .

Set $d = \sqrt{|\Delta|} \|(\alpha, 1)\| / \sqrt{D}$, just as in Proposition 5.6. First fix $m \in \mathbb{N}$ large enough so that

$$c' < \frac{(\sqrt{2} - 1)\sqrt{2^m d}}{2\sqrt{c}}. \tag{11}$$

Then fix M_0 as above with β_0 large enough so that

$$\frac{2^{m+1}mn}{c|\beta_0|} < \frac{(\sqrt{2} - 1)\sqrt{2^m d}}{2\sqrt{c}} \quad \text{and} \quad \frac{2^m}{c|\beta_0|^2} < d. \tag{12}$$

Set $C_0 = M_0C$ and let r_0 be its curvature. Take a minimal-length chain of distinct circles $C_0, C_1, \dots, C_j \in \mathcal{S}_D$ such that $|r_j|$, the curvature magnitude of C_j , is at most $c|\beta_0|^2/2^m$. Since c is such that $|r_0| \leq c|\beta_0|^2$, we have $j \leq mn$ by finite connectivity. The diameter of C_k if $0 \leq k < j$ (if there is such an index) is less than $2(2^m/c|\beta_0|^2)$ by minimality of j . So the distance between C_j and α is at most

$$\begin{aligned} |M_0(\tau) - \alpha| + \frac{2^{m+1}j}{c|\beta_0|^2} &\leq \frac{c'}{|\beta_0|} + \frac{2^{m+1}mn}{c|\beta_0|^2} < \\ (\sqrt{2} - 1)\sqrt{\frac{2^m d}{c|\beta_0|^2}} &\leq (\sqrt{2} - 1) \min\left(d, \sqrt{\frac{d}{|r_j|}}\right). \end{aligned}$$

The middle inequality above uses (11) to bound its first summand and the first half of (12) to bound its second summand. The last inequality above uses $|r_j| \leq c|\beta_0|^2/2^m$ and the second half of (12). Thus $\alpha \in C_j$ by Proposition 5.6.

The final two claims of the theorem are Proposition 5.10 and Corollary 5.5. □

Besides Theorem 5.12, our number theory results had to assume that \mathcal{S}_D has intersections in \widehat{K} . It would be interesting to find a criterion that guarantees finite connectivity and takes advantage of Definition 5.11 not requiring $C_{i-1} \cap C_i \subset \widehat{K}$. The author’s only known method for proving finite connectivity is the covering property that \mathcal{S}_{39} for $\mathbb{Q}(i\sqrt{39})$ narrowly failed in Fig. 6. But this property may be stronger than necessary as it also guarantees rational connectivity, which strengthens the conclusion of Theorem 5.12 as follows.

Corollary 5.13 *If \mathcal{S}_D is finitely and rationally connected, then the class group is generated by ideal classes of primes with norm dividing D .*

Proof We are combining Proposition 5.9 with Theorem 5.12. □

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References

1. Baragar, A., Lautzenheiser, D.: A game of packings. [arXiv:2006.00619](https://arxiv.org/abs/2006.00619) (2020)
2. Bogachev, N., Kolpakov, A., Kontorovich, A.: Kleinian sphere packings, reflection groups, and arithmeticity. [arXiv:2203.01973](https://arxiv.org/abs/2203.01973) (2021)
3. Bourgain, J., Kontorovich, A.: On the local-global conjecture for integral Apollonian gaskets. *Invent. Math.* **196**(3), 589–650 (2014)
4. Boyd, D.: A new class of infinite sphere packings. *Pac. J. Math.* **50**(2), 383–398 (1974)
5. Carter, R.W., Segal, G.B., MacDonald, I.G.: *Lectures on Lie Groups and Lie Algebras*, vol. 32. Cambridge University Press, Cambridge (1995)
6. Chait-Roth, D., Cui, A., Stier, Z.: A taxonomy of crystallographic sphere packings. *J. Number Theory* **207**, 196–246 (2020)
7. Elstrodt, J., Grunewald, F., Mennicke, J.: *Groups Acting on Hyperbolic Space: Harmonic Analysis and Number Theory*. Springer, Berlin (2013)
8. Fuchs, E.: Strong approximation in the Apollonian group. *J. Number Theory* **131**(12), 2282–2302 (2011)
9. Fuchs, E., Stange, K.E., Zhang, X.: Local-global principles in circle packings. *Compos. Math.* **155**(6), 1118–1170 (2019)
10. Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., Yan, C.H.: Apollonian circle packings: number theory. *J. Number Theory* **100**(1), 1–45 (2003)
11. Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., Yan, C.H.: Apollonian circle packings: geometry and group theory I: the Apollonian group. *Discrete Comput. Geom.* **34**(4), 547–585 (2005)
12. Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., Yan, C.H.: Apollonian circle packings: geometry and group theory II. Super-Apollonian group and integral packings. *Discrete & Computational Geometry* **35**(1), 1–36 (2006)
13. Graham, R.L., Lagarias, J.C., Mallows, C.L., Wilks, A.R., Yan, C.H.: Apollonian circle packings: geometry and group theory III. Higher dimensions. *Discrete Comput. Geom.* **35**(1), 37–72 (2006)
14. Guettler, G., Mallows, C.: A generalization of Apollonian packing of circles. *J. Comb.* **1**(1), 1–27 (2010)
15. Haag, S., Kertzer, C., Rickards, J., Stange, K.E.: The Local-Global Conjecture for Apollonian circle packings is false. [arXiv:2307.02749](https://arxiv.org/abs/2307.02749) (2023)
16. Kapovich, M., Kontorovich, A.: On Superintegral Kleinian Sphere Packings, Bugs, and Arithmetic Groups. [arXiv:2104.13838](https://arxiv.org/abs/2104.13838) (2021)
17. Kontorovich, A., Nakamura, K.: Geometry and arithmetic of crystallographic sphere packings. *Proc. Natl. Acad. Sci.* **116**(2), 436–441 (2019)
18. Mackenzie, D.: A tisket, a tasket, an Apollonian gasket. *Am. Sci.* **98**(1), 10–14 (2010)
19. Maclachlan, C., Reid, A.W.: *The Arithmetic of Hyperbolic 3-Manifolds*, vol. 219. Springer, New York (2003)
20. Manna, S.S., Herrmann, H.J.: Precise determination of the fractal dimensions of Apollonian packing and space-filling bearings. *J. Phys. A: Math. Gen.* **24**(9), 481–490 (1991)

21. Martin, D.E.: The geometry of imaginary quadratic fields. PhD thesis, University of Colorado at Boulder (2020)
22. Martin, D.E.: Continued fractions over non-Euclidean imaginary quadratic rings. *J. Number Theory* **243**, 688–714 (2022)
23. Niven, I.: *Irrational Numbers*. Mathematical Association of America, Washington, DC (1956)
24. Rapinchuk, A.S.: Strong approximation for algebraic groups. *Math. Sci. Res. Inst. Publ.* **61**, 269–298 (2014)
25. Sarnak, P.: Integral Apollonian packings. *Am. Math. Mon.* **118**(4), 291–306 (2011)
26. Soddy, F.: The bowl of integers and the hexlet. *Nature* **139**(3506), 77–79 (1937)
27. Stange, K.E.: Visualizing the arithmetic of imaginary quadratic fields. *Int. Math. Res. Not.* **2018**(12), 3908–3938 (2017)
28. Stange, K.E.: The Apollonian structure of Bianchi groups. *Trans. Am. Math. Soc.* **370**, 6169–6219 (2018)
29. Swan, R.G.: Generators and relations for certain special linear groups. *Adv. Math.* **6**(1), 1–77 (1971)

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