A Spectral Approach to Polytope Diameter

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Abstract

We prove upper bounds on the graph diameters of polytopes in two settings. The first is a worst-case bound for polytopes defined by integer constraints in terms of the height of the integers and certain subdeterminants of the constraint matrix, which in some cases improves previously known results. The second is a smoothed analysis bound: given an appropriately normalized polytope, we add small Gaussian noise to each constraint. We consider a natural geometric measure on the vertices of the perturbed polytope (corresponding to the mean curvature measure of its polar) and show that with high probability there exists a "giant component" of vertices, with measure $1 - o(1)$ and polynomial diameter. Both bounds rely on spectral gaps—of a certain Schrödinger operator in the first case, and a certain continuous time Markov chain in the second—which arise from the log-concavity of the volume of a simple polytope in terms of its slack variables.

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Springer

1 Introduction

The polynomial Hirsch conjecture asks whether the diameter of an arbitrary bounded polytope $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ is at most a fixed polynomial in m and d (which, since $m > d$ is atmost a fixed polynomial in m). This conjecture is widely open, with the best known upper bounds being $(m - d)^{\log_2 d - \log_2 \log d + O(1)}$ ([31], see also [18, 33]) and $O(m)$ for fixed d ([4, 19]); the best known lower bound is $(1 + \epsilon)m$ for some $\epsilon > 0$ when d is sufficiently large [25]. Given this situation, there has been interest in the following potentially easier questions:

Q1. Assuming A, b have integer entries, bound the diameter of P in terms of their size.
Q2. Assuming A, b are sampled randomly from some distribution, bound the diameter of P with high probability.

Progress on these questions [5, 6, 8, 9, 11, 12, 15, 29, 35] has relied mostly on techniques from polyhedral combinatorics, integral geometry, probability, and operations research (e.g., analysis of the simplex algorithm and its cousins).

On the other hand, the theory of mixed volumes of convex sets has developed largely separately over the past century, with several celebrated achievements including the Alexandrov–Fenchel inequality [1, 2] and more generally the Hodge-Riemann relations for certain algebras associated with simple polytopes [20, 21, 32]. When restricted to strongly isomorphic convex polytopes, the Alexandrov Fenchel inequality is equivalent to a statement about the signature of a certain quadratic form. McMullen, in his two articles [20, 21] greatly generalized this statement, leading to a geometric interpretation of Stanley's proof [30] of the g-conjecture, which concerns the face numbers of simple polytopes.

One consequence of this theory is that a certain Schrödinger operator (weighted adjacency matrix plus diagonal) associated with the graph of *every* bounded polytope has a spectral gap [17] (see Definition 2.1 and Theorem 2.2). We use this fact to make progress on Q1 and Q2. In the first setting, we show the following theorem, where $\| \cdot \|_\infty$ denotes the maximum magnitude entry of a matrix.

Theorem 1.1 *Suppose $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ is a bounded polytope with integer coefficients $A \in \mathbb{Z}^{m \times d}, b \in \mathbb{Z}^m$ such that every $d \times d$ minor of $[A|b]$ has determinant bounded by Δ. Then P has diameter*

$$O(d^2 \Delta \|A\|_\infty \cdot \log(m\|A\|_\infty \|b\|_\infty \Delta)) \tag{1}$$

Theorem 1.1 follows from a more geometric result (Theorem 3.2) stated in terms of the angles between the $d - 2$-faces of the polar of P, which is proven in Sect. 3. It may be contrasted with the best previously known result of this kind due to [11], who achieved a bound of $O(d^3 \Delta_{d-1}^2)$, where Δ_{d-1} is the largest $(d - 1) \times (d - 1)$ subdeterminant of A, in particular independent of b. The two bounds are incomparable in general, but as $\Delta \leq d\|b\|_\infty \Delta_{d-1}$ it is seen that (1) is nearly linear in Δ_{d-1} whereas the result of [11] is quadratic, which yields an improvement for large Δ_{d-1} (compared to $\|A\|_\infty, \|b\|_\infty, \log m$). However, our diameter bound is nonconstructive whereas [11]

show how to efficiently find a path between any two vertices of P; we refer the reader to the introduction of that paper for a more thorough discussion of previous work in this vein (originally initiated by [5, 14]). At a high level, the reason we are able to save a factor of d in comparison with previous works is that they rely on combinatorial expansion arguments, whereas we use spectral expansion, which is amenable to a "square root" improvement using Chebyshev polynomials, first introduced in [28, Thm. 3.1].

Regarding Q2, the study of diameters of random polytopes began with the influential work of Borgwardt [7, 8], who considered A with i.i.d. standard Gaussian entries and $b = 1$. Borgwardt showed the following "for each" guarantee: for any fixed objective functions $c, c' \in \mathbb{R}^d$, the combinatorial distance between the vertices x, x' of P maximizing $\langle c, x, \rangle, \langle c', x' \rangle$ is at most $O(d^{3/2}\sqrt{\log m})$ in expectation, provided $m \to \infty$ sufficiently rapidly. This type of result was extended to the "smoothed unit LP" model by Spielman and Teng in the seminal work [29]; in this model one takes

$$P = \{x \in \mathbb{R}^d : \langle x, v_j \rangle \leq 1\} \tag{2}$$

where $v_j \sim N(a_j, \sigma^2)$ for some fixed vectors $a_1, \ldots, a_m$ normalized to have $\|a_j\| \leq 1$. The original $\mathrm{poly}(m, d, \sigma^{-1})$ path length bound of [29] was improved and simplified in [12, 13, 35]; a key ingredient in each of these results was a "shadow vertex bound" analyzing the expected number of vertices of a two-dimensional projection of P. Note that all of these results provide "for each" guarantees: at best they bound the distance between a single pair of vertices, not between all pairs.

Our second contribution is to prove that for the smoothed unit LP model, *most* pairs of vertices in P are polynomially (in m, d, σ^{-1}) close with high probability, where most is defined with respect to a certain locally defined measure on the vertices known as the *mean curvature measure* χ_2 in convex geometry (see [26, 27]; we recall the definition in Sect. 4). In the language of random graph theory, this means that the graph of P likely contains a "giant component" with respect to χ_2 which is of small diameter. Let χ_2 denote the mean curvature measure on the facets of P°, which corresponds naturally to a measure on the set of vertices of P, denoted Ω, when P is bounded and contains the origin.

Theorem 1.2 *Assume P is a random polytope sampled from the smoothed LP model. Then with probability at least $1 - 1/\mathrm{poly}(m)$, for every $\psi > 0$ there is a subset $G := G(\psi) \subset \Omega$ with $\chi_2(G) \geq (1 - \psi)\chi_2(\Omega)$ such that the vertex diameter of G is at most*

$$O\left(\frac{\mathrm{poly}(m, d)}{\sigma^4 \psi}\right). \tag{3}$$

We prove Theorem 1.2 in Sect. 4.5, where we deduce it from a more refined theorem (Theorem 4.1, which includes explicit powers of m, d) for a certain class of well-rounded polytopes. The idea of the proof is to consider a certain continuous time Markov chain whose states are the vertices of P. This chain automatically has a large spectral gap by Theorem 2.2 and the main challenge is to bound its average transition

rate. This is carried out in Sects. 4.2–4.4 and involves further use of the Alexandrov–Fenchel inequalities, tools from integral geometry, Gaussian anticoncentration, and an application of the shadow vertex bound of [12].

Remark 1.3 It was pointed out to us by an anonymous referee and by D. Dadush that there is a "folklore" result that the average distance between a random pair of vertices (chosen by optimizing two uniformly random objective functions) of P as above is polynomial in m, d, σ^{-1}; this is seen by a Fubini type argument and the shadow vertex bounds of [12, 13, 29, 35]. Our paper considers a different measure on the vertices, and we are not aware of any relation between the two.

Remark 1.4 (*Expansion of Polytopes*) There has been a sustained interest in studying the expansion of graphs of combinatorial polytopes beginning with [22] which conjectured that all 0/1 polytopes have expanding graphs. The recent breakthrough [3] resolved this conjecture for the special case of matroid polytopes using techniques related to high dimensional expanders and the geometry of polynomials, which may be described as capturing "discrete log-concavity". The present work, in contrast, uses "continuous log-concavity" (stemming from the Brunn-Minkowski inequality) to control the spectral gaps of certain matrices associated with the graphs of polytopes with favorable geometric properties.

We note that the Hirsch conjecture is already known to hold for 0/1 polytopes [23].

1.1 Preliminaries and Notation

We recall some basic terminology and facts regarding polytopes; the reader may consult [27, Chap. 4] for a more thorough introduction.

We denote the convex hull of a set of points by $\text{conv}(\cdot)$ and its affine hull by $\text{aff}(\cdot)$. Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m_{>0}$ be a bounded polytope containing the origin in its interior. Its polar is the polytope

$$P^\circ = \text{conv}\{b_j^{-1} a_j\}_{j=1}^m =: K,$$

where $a_1^T, \ldots, a_m^T$ are the rows of A.

A polytope in $\mathbb{R}^d$ is called *simple* if each of its vertices is contained in exactly d (codimension-1) facets, and *simplicial* if each codimension-1 facet contains exactly d vertices. Unless otherwise noted, "facet" refers to a codimension-1 facet. The polar of a simple polytope is simplicial and vice versa.

The 1-dimensional faces of a polytope are called *edges*, and are all line segments when it is bounded. The *vertex diameter* of a bounded polytope P is the diameter of the graph of its vertices and edges. Two facets of a polytope are *adjacent* if their intersection is a $(d-2)$-face of the polytope. The *facet diameter* of a polytope K is the diameter of the graph with vertices given by its facets and edges given by the adjacency relation on facets. By duality, the vertex diameter of a simple polytope P is equal to the facet diameter of P°.

We use dist$(\cdot, \cdot)$ to denote the Euclidean distance between two subsets of $\mathbb{R}^d$, and

$$\text{hdist}(L, K) := \max \left\{ \sup_{x \in L} \text{dist}(x, K), \sup_{y \in K} \text{dist}(y, L) \right\}$$

to denote the *Hausdorff distance* between two sets.

We use $V(K[j], L[d-j])$ to denote the mixed volume of j copies of K and $d - j$ copies of L for convex bodies $K, L \subset \mathbb{R}^d$. The *Alexandrov–Fenchel* inequalities imply that these are log-concave, in the sense that for $j_1, j_2, j = \beta j_1 + (1 - \beta) j_2$ integers in $\{0, \ldots, d\}$ with $\beta \in [0, 1]$ then

$$V(K[j], L[d-j]) \geq V(K[j_1], L[d-j_1])^\beta \cdot V(K[j_2], L[d-j_2])^{1-\beta}. \quad (4)$$

The above inequality follows from the usual form of the Alexandrov–Fenchel inequalities in which j_1, j and j_2 are consecutive positive integers and $\beta = \frac{1}{2}$. We use C to denote absolute constants whose value may change from line to line, unless specified otherwise.

2 Eigenvalues of the Hessian and Spectral Gaps

In this section, we recall that a certain matrix associated with every bounded polytope has exactly one positive eigenvalue.

Definition 2.1 (*Formal Hessian*) For K a bounded polytope containing the origin in its interior with N facets labeled $\{1, \ldots, N\}$, let $H(K)$ denote the $N \times N$ matrix with entries

$$(H(K))_{ij} = \begin{cases} F_{ij} \csc(\theta_{ij}) & i \neq j \\ -\sum_k F_{ik} \cot(\theta_{ik}) & i = j \end{cases} \quad (5)$$

where F_{ij} is the intersection of facets i and j, (and by abuse of notation is also used to denote their $d - 2$-dimensional volume) and θ_{ij} is the angle between the vectors normal to those faces, facing away from the origin.

When K is simple, $H(K)$ is the Hessian of the volume of $K(c) = \{x \mid Mx \leq c\}$ with respect to the slack vector $c > 0$. (see [27, Chap. 4]). Log-concavity of the volume implies that this Hessian has exactly one positive eigenvalue. Izmestiev [17] has shown via an approximation argument that this remains true for the formal Hessian of any polytope.

Theorem 2.2 (Theorem 2.4 of [17]) $H(K)$ *has exactly one positive eigenvalue for any bounded polytope* K.

We include a self-contained proof of Theorem 2.2 in the Appendix of the arXiv version of this paper [24] for completeness.[1]

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[1] Our proof yields a slightly stronger conclusion regarding continuity of the formal Hessian than [17].

We will apply Theorem 2.2 to certain matrices derived from the formal Hessian and the following diagonal scaling, which plays an important role in the remainder of the paper.

Definition 2.3 Let K, F_{ij}, θ_{ij} be as in Definition 2.1. Then let $D(K)$ denote the $N \times N$ positive diagonal matrix with entries $(D(K))_{ii} = \sum_k F_{ik} \tan(\theta_{ik}/2)$. Note that $\theta_{ik} \neq \pi$ whenever $F_{ik} = 0$ since parallel facets of a convex polytope cannot intersect.

Lemma 2.4 (Spectral Gaps from Log-Concavity) *Let K be a polytope and take $H := H(K)$, $D := D(K)$. Let L be the graph Laplacian with entries:*

$$L_{ij} = \begin{cases} -F_{ij} \csc(\theta_{ij}) & i \neq j \\ \sum_k F_{ik} \csc(\theta_{ij}) & i = j \end{cases}. \tag{6}$$

Then

1. $D^{-1/2} H D^{-1/2}$ *has exactly one eigenvalue at* 1 *with the rest of the eigenvalues in* $(-\infty, 0]$. *The eigenvector corresponding to this eigenvalue is* $D^{1/2}\mathbf{1}$.
2. $-D^{-1}L$ *has exactly one eigenvalue at zero, with the rest of the eigenvalues in* $(-\infty, -1]$. *The left corresponding to this eigenvalue is* $D\mathbf{1}$.

Proof Observe that H is "nearly" a graph Laplacian in the sense that:

$$H = -L + D \tag{7}$$

where we have used the identity $\frac{1-\cos\theta}{\sin\theta} = \tan(\theta/2)$. By Sylvester's inertia law, the signature of H matches that of

$$D^{-1/2} H D^{-1/2} = -D^{-1/2} L D^{-1/2} + I, \tag{8}$$

which must therefore have exactly one positive eigenvalue by Theorem 2.2. However, $L \succeq 0$ and $L\mathbf{1} = 0$, so by Sylvester's law $-D^{-1/2}LD^{-1/2} \preceq 0$ with at least one eigenvalue equal to zero. Thus, $D^{-1/2}HD^{-1/2}$ has exactly one eigenvalue equal to one, with eigenvector $D^{1/2}\mathbf{1}$ and the rest of the eigenvalues nonpositive, establishing the first claim. The second claim follows from (8) and the similarity of $D^{-1}L$ and $D^{-1/2}LD^{-1/2}$. □

3 Diameter in Terms of Angles and Bit Length

In this section we use the spectral gap bound of Lemma 2.4(1) to give a bound on the diameter of a polytope specified by integer constraints. We begin by slightly generalizing the argument of [10, 28, 34], who used Chebyshev polynomials to control the diameter of regular (nonnegative weighted) graphs in terms of their spectra, to handle the matrix $D^{-1/2}HD^{-1/2}$ by appropriately controlling its negative entries and top eigenvector.

Lemma 3.1 (Diameter in terms of Spectrum) *Let A be a weighted real symmetric adjacency matrix (possibly with self-loops and negative weights) for a graph G on N vertices. Suppose for some $g > 0$ there is exactly one eigenvalue of A at $1 + g$ with corresponding unit eigenvector v, the smallest absolute entry of which is $v_{\min}$. Further suppose that the rest of the eigenvalues of A are in the interval $[-1, 1]$. Then the diameter of G is at most*

$$\frac{2\log(2N/v_{\min}^2)}{\sqrt{g}}.$$

Proof Note that if $M \in \text{span}(I, A, \ldots, A^k)$ then $e_i^T M e_j \neq 0$ implies that there is a path in G from i to j of length at most k. To this end, consider $T_k(A)$ where T_k is the degree k Chebyshev Polynomial of the first kind. If we find that $T_k(A) \neq 0$ entry-wise, then we can conclude the diameter of G is at most k. Let

$$A = vv^T(1 + g) + \sum_{i=2}^{N} u_i u_i^T \lambda_i$$

be the spectral decomposition of A. Let $|\cdot|$ denote the entry-wise absolute value. Then

$$|T_k(A) - vv^T T_k(1 + g)| = \left|\sum_{i=2}^{N} u_i u_i^T T_k(\lambda_i)\right| \leq \sum_{i=2}^{N} |u_i u_i^T| \, |T_k(\lambda_i)| \leq \sum_{i=2}^{N} |u_i u_i^T| \leq N,$$

since $|T_k(x)| \leq 1$ on $[-1, 1]$. We would therefore have $T_k(A) \neq 0$ entry-wise if N is smaller then the smallest absolute entry of $vv^T T_k(1 + g)$, which is lower bounded by

$$v_{\min}^2 T_k(1 + g) \geq \frac{v_{\min}^2}{2}\left(1 + g + \sqrt{(1 + g)^2 - 1}\right)^k \geq \frac{v_{\min}^2}{2}\left(1 + \sqrt{2g}\right)^k.$$

It suffices to pick

$$k \geq \frac{\log(2N/v_{\min}^2)}{\log(1 + \sqrt{2g})},$$

which is implied by taking

$$k = \frac{2\log(2N/v_{\min}^2)}{\sqrt{g}}$$

and using $\log(1 + \sqrt{2g}) \geq \sqrt{2g} - g + 2\sqrt{2}g^3/3 \geq \sqrt{g}/2$ for $g(0, 1/2]$; if $g \geq 1/2$ we may replace A by $A/2g$ (which does not violate the hypotheses of the Lemma) and reach the desired conclusion. $\qquad\square$

Theorem 3.2 *Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a bounded polytope containing the origin with $m \geq d$, $A \in \mathbb{Z}^{m \times d}$, $b \in \mathbb{Z}^m$. Assume all angles between pairs of adjacent facets of P^o are contained in $[\theta_0, \pi - \theta_0]$ and let the magnitude of the largest $d \times d$ subdeterminant of $[A|b]$ be Δ. Then the vertex diameter of P is at most*

$$\frac{3d \log m + O\left(d \log(d\|A\|_\infty \|b\|_\infty \Delta)\right)}{\sin(\theta_0/2)}.$$

Proof Put $D := D(P^o)$ and $H := H(P^o)$. By Lemma 2.4, $D^{-1/2}HD^{-1/2}$ is real symmetric with one eigenvalue at 1 and the rest at most 0. We can bound its smallest eigenvalue by using Lemma 3.3 and considering the similar matrix $D^{-1}H$. We upper bound the absolute row sum of the ith row of $D^{-1}H$ by

$$\sum_j |(D^{-1}H)_{ij}| \leq \frac{\sum_{i \sim j} F_{ij} \csc(\theta_{ij})}{\sum_{i \sim k} F_{ik} \tan(\theta_{ik}/2)} + \frac{\sum_{i \sim j} F_{ij} |\cot(\theta_{ij})|}{\sum_{i \sim k} F_{ik} \tan(\theta_{ik}/2)}$$

$$\leq \frac{2\sum_{i \sim j} F_{ij} \csc(\theta_{ij})}{\sum_{i \sim k} F_{ik} \tan(\theta_{ik}/2)}.$$

Taking the supremum of the above expression gives

$$\sup_i \frac{2\sum_k F_{ik} \csc(\theta_{ik})}{\sum_k F_{ik} \tan(\theta_{ik}/2)} \leq \sup_{i \sim j} \frac{2 \csc(\theta_{ij})}{\tan(\theta_{ij}/2)} = \sup_{i \sim j} \csc^2(\theta_{ij}/2).$$

Therefore by Lemma 3.3, the smallest eigenvalue of $D^{-1}H$, and consequently of $D^{-1/2}HD^{-1/2}$ is at least $-\csc^2(\theta_0/2)$. Then

$$M = \frac{D^{-1/2}HD^{-1/2} + \csc^2(\theta_0/2)I}{\csc^2(\theta_0/2)}$$

has exactly one eigenvalue at $\frac{1+\csc^2(\theta_0/2)}{\csc^2(\theta_0/2)} = 1 + \csc^{-2}(\theta_0/2)$ with the rest contained in the interval $[0, 1]$. We can apply Lemma 3.1 with $g = \csc^{-2}(\theta_0/2)$ to obtain a diameter of at most:

$$\frac{d \log(m) + 2\log(v_{\min}^{-1})}{\sin(\theta_0/2)}.$$

The eigenvector v corresponding to eigenvalue $1 + \csc^2(\theta_0/2)$ is simply $\mathbf{1}^T D^{1/2}$ normalized, so

$$v_{\min} = \frac{\min_i(\mathbf{1}^T\sqrt{D})_i}{\|\mathbf{1}^T\sqrt{D}\|_2} \geq \frac{1}{\sqrt{N}}\sqrt{\frac{\min_i D_{ii}}{\max_i D_{ii}}} \geq \frac{1}{\sqrt{N}}\frac{\min_i \sum_k F_{ik}\tan(\theta_{ik}/2)}{\max_i \sum_k F_{ik}\tan(\theta_{ik}/2)}$$

$$\geq \frac{\sin^2(\theta_0)}{4N^{3/2}}\frac{\min_{i,j} F_{ij}}{\max_{i,j} F_{ij}},$$

where we used that $\theta_{ik} \in [\theta_0, \pi - \theta_0]$ implies $\sin(\theta_0/2) \leq \tan(\theta_{ik}/2) \leq \frac{1}{\sin(\theta_0/2)}$ and $\sin(x/2) \geq \sin(x)/2$. Finally use $N \leq \binom{m}{d} \leq m^d/2$ as well as the estimates from Lemma 3.4 to see that

$$\log(v_{\min}^{-1}) \leq \log(4N^{3/2}) + 2\log(2d\|A\|_\infty \Delta)$$
$$+ d\log(\sqrt{d}\|A\|_\infty) + d\log d + 2d\log(\|b\|_\infty)$$
$$\leq 2d\log(m) + O(d\log(d\|A\|_\infty\|b\|_\infty) + \log(\Delta)),$$

yielding the advertised bound. $\qquad\square$

Lemma 3.3 (Gershgorin's circle theorem) *The smallest (real) eigenvalue of M is at least $-\sup_i \sum_j |M_{ij}|$.*

Lemma 3.4 (Worst Case Volumes and Angles) *Let $P^o = \mathrm{conv}(a_1/b_1, \ldots, a_m/b_m)$ be a polytope where each $a_i/b_i \in \mathbb{R}^d$ is a vertex and $a_i \in \mathbb{Z}^d, b_i \in \mathbb{Z}$. Then:*

1. *The smallest co-dimension 2 face of P^o has volume at least $1/(d!\|b\|_\infty^{2d})$, and the largest co-dimension 2 face has volume at most $(\sqrt{d}\|A\|_\infty)^d$.*
2. *If the largest $d \times d$ minor of $[A|b]$ is bounded in magnitude by Δ, then the angle between any two adjacent facets of P^o satisfies $\csc(\theta) \leq 2d\Delta\|A\|_\infty$.*

Proof Every co-dimension 2 face can be written as the convex hull of some subset of size at least $d - 1$ of the vertices $a_1/b_1, \ldots, a_m/b_m$. Without loss of generality, say that $F = \mathrm{conv}(a_1/b_1, \ldots, a_{d-1}/b_{d-1})$ is the smallest co-dimension 2 face. Then its volume is:

$$\mathrm{Vol}(\mathrm{conv}(a_1/b_1, \ldots, a_{d-1}/b_{d-1})) = \frac{1}{(d-2)!}\sqrt{|\det(M^T M)|}$$
$$\geq \frac{1}{d!(b_1 \ldots b_{d-2})b_{d-1}^{d-2}} \geq \frac{1}{d!\|b\|_\infty^{2d}},$$

where M is the $d \times (d-2)$ matrix whose ith column is $a_i/b_i - a_{d-1}/b_{d-1}$, and we have used that the determinant of a nonsingular integer matrix is at least one. On the other hand, P^o is contained inside the ℓ_2 ball of radius $d^{1/2}\|A\|_\infty$, and so each co-dimension 2 face of P^o is contained in a cross section of that ball, and consequently has volume at most $(\sqrt{d}\|A\|_\infty)^d$, establishing (1).

Regarding the angles, consider without loss of generality two adjacent facets of P^o, with vertices numbered so that $F = \mathrm{conv}(a_1/b_1, \ldots, a_d/b_d)$ and $F' = \mathrm{conv}(a_2/b_2, \ldots, a_d/b_d, a_j/b_j)$, $j > d$, and $|b_j| \leq |b_1|$. Observe that the angle θ between the normals to these adjacent facets satisfies:

$$\csc(\theta) = \frac{\mathrm{dist}(a_j/b_j, F)}{\mathrm{dist}(a_j/b_j, \mathrm{aff}(F))}.$$

The numerator is at most the distance between a_j/b_j and any vertex of F, which is at most

$$\|a_1/b_1 - a_j/b_j\|_2 \leq \sqrt{d}\frac{\|a_1\|_\infty + \|a_j\|_\infty}{|b_j|} \leq \frac{2\sqrt{d}\|A\|_\infty}{|b_j|},$$

by our choice of j. The denominator is given by

$$\text{dist}(a_j/b_j, \text{aff}(F)) \geq \text{dist}((a_j/b_j, 1)^T, \text{span}(\hat{a}_1, \ldots \hat{a}_d)) \quad \text{where } \hat{a}_i := (a_i, b_i)^T \in \mathbb{R}^{d+1}$$

$$= \frac{1}{|b_j|\|e_j^T M^{-1}\|}$$

where M is the $(d+1) \times (d+1)$ matrix with columns $(\hat{a}_1, \ldots, \hat{a}_d, \hat{a}_j)$, which must be invertible since $\text{conv}(a_1/b_1, \ldots, a_d/b_d, a_j/b_j)$ is a full dimensional simplex. By the adjugate formula, the entries of M^{-1} are of magnitude at most Δ, so we have

$$\text{dist}(a_j, \text{aff}(F)) \geq \frac{1}{\sqrt{d}|b_j|\Delta}.$$

Combining these bounds and cancelling the $|b_j|$ yields

$$\csc(\theta) \leq 2d\|A\|_\infty\Delta,$$

establishing (3). □

Finally, we can prove the bound advertised in the introduction.

Proof of Theorem 1.1 Applying Theorem 3.2, Lemma 3.4(2), and $\sin(x/2) \geq \sin(x)/2$, we find that the diameter of P is at most

$$O(d\log(m\|A\|_\infty\|b\|_\infty\Delta)) \times d\|A\|_\infty\Delta,$$

as advertised. □

4 Smoothed Analysis

In this section we consider the "smoothed unit LP" model defined in (2). Suppose P_0 is a fixed polytope specified as

$$P_0 = \{x \in \mathbb{R}^d : \langle a_j, x \rangle \leq 1, j \in [m]\},$$

for some vectors $\|a_j\| \leq 1$, and consider the random polytope

$$P = \{x \in \mathbb{R}^d : \langle v_j, x \rangle \leq 1\},$$

where $v_j = a_j + g_j$ for $g_j \sim N(0, \sigma^2 I_d)$ i.i.d spherical Gaussians. Denote the polars of P_0 and P by

$$K_0 := P_0^\circ = \mathrm{conv}(a_1, \ldots, a_m) \subset B_2^d,$$
$$K := P^\circ = \mathrm{conv}(v_1, \ldots, v_m).$$

Note that K is simplicial with probability one, so each of its k-dimensional faces has exactly $k+1$ vertices. We will use the notation $F_S := \mathrm{conv}\{v_j : j \in S\}$ to denote faces of K and $\mathcal{F}_k(K) := \left\{ S \in \binom{[m]}{k+1} : F_S \text{ is a } k - \text{ dimensional face of } K \right\}$ to denote the set of all faces of K. The k-dimensional volume of a face F_S, $S \in \mathcal{F}_k(K)$ will be denoted by $|F_S|$ or $\mathrm{Vol}_k(F_S)$. We will often abbreviate $F_{S \cap T}$ as F_{ST} for adjacent S, T. For two $S, T \in \binom{[m]}{d}$, let $\theta_{ST} \in (0, \pi)$ denote the angle between the unit normals u_S, u_T to F_S, F_T, respectively; note that almost surely $\theta_{ST} \neq 0, \pi$ for every $S, T \in \binom{[m]}{d}$.

We will pay special attention to the set of $(d-1)$-faces of K, which we denote as

$$\Omega := \mathcal{F}_{d-1}(K) \subset \binom{[m]}{d}.$$

Define the measures $\chi_2, \pi, \delta : \Omega \to \mathbb{R}_{\geq 0}$ by

$$\chi_2(S) := \sum_{T \in \Omega} |F_{ST}| \theta_{ST}, \tag{9}$$

$$\pi(S) := \sum_{T \in \Omega} |F_{ST}| \tan(\theta_{ST}/2), \tag{10}$$

$$\delta(S) := \sum_{T \in \Omega} |F_{ST}| \csc(\theta_{ST}). \tag{11}$$

It will be convenient to make two further **technical assumptions** on K_0 and σ for the proofs of our results; in Sect. 4.5 we will show that any instance of the smoothed unit LP model may be reduced to one satisfying both assumptions with parameter

$$r = \Omega(\sigma m^3), \tag{12}$$

incurring only a poly(m) loss in the diameter. Let $K_0^{(j)} = \mathrm{conv}(a_i : i \neq j)$ be the polytope obtained from K_0 by deleting vertex a_j.

(R) Roundedness of Subpolytopes: There is an $r \in (0, 1)$ such that for every $j \leq m$ there is a vector $v_j \in \mathbb{R}^d$ such that:

$$r B_2^d + v_j \subset K_0^{(j)}.$$

(S) Smallness of σ:

$$\alpha := 6\sigma \sqrt{d \log m} < r/d^2. \tag{13}$$

The main result of this section is the following "almost-diameter" bound with respect to the measure π.

Theorem 4.1 *Assume* (**S**), (**R**). *Then with probability at least* $1 - C/m^2$, *for every* $\phi > 0$ *there is a subset* $G := G(\phi) \subset \Omega$ *with* $\pi(G) \geq (1 - \phi)\pi(\Omega)$ *such that the facet diameter of* G *is at most* $\tilde{O}(m^3 d^8/\sigma^2 r \phi)$.

Remark 4.2 The probability in Theorem 4.1 may be upgraded to $1 - m^{-c}$ for any c at the cost of an additional m^c factor in the diameter bound, by applying Markov's inequality in the proof of Lemma 4.5 with a different threshold.

The proof of Theorem 4.1 relies on three properties of the (random) continuous time Markov chain with state space Ω and infinitesimal generator[2]

$$Q := -D^{-1}L, \tag{14}$$

where L is as in (6). The corresponding Markov semigroup

$$P(t) := \exp(-tD^{-1}L), \quad t \geq 0,$$

has stationary distribution proportional to $D\mathbf{1} = \pi(\cdot)$ by Lemma 2.4(2); call the normalized stationary distribution $\overline{\pi}(\cdot) := \pi(\cdot)/\pi(\Omega)$

The first property is that the stationary distribution $\overline{\pi}$ is (in a quite mild sense) non-degenerate, with high probability. Apart from being essential in our proofs, this relates the measure π to well-studied measures in convex geometry such as the surface measure and mean curvature measure $\chi_2(\cdot)$, clarifying the meaning of Theorem 4.1. The proof of Lemma 4.3 appears in Sect. 4.1.

Lemma 4.3 (Non-degeneracy of $\overline{\pi}$) *Assume* (**S**), (**R**). *With probability at least* $1 - 1/m^2$:

1. $\min_{S \in \Omega} \overline{\pi}(S) \geq \overline{\pi}_{min} := C \frac{m^{-2d^2} r^2}{d^3}$.
2. $c\mathrm{Vol}_{d-1}(\partial K) \leq \pi(\Omega) \leq O(d^3 r^{-2})\mathrm{Vol}_{d-1}(\partial K)$.
3. *For every* $S \in \Omega$, $\chi_2(S)/2 \leq \pi(S) \leq O(r^{-1})\chi_2(S)$.

The second property is that Q (almost surely) has a spectral gap of at least one, by Lemma 2.4(2). This implies that the chain (14) mixes rapidly to $\overline{\pi}$ (in the sense of continuous time) from any well-behaved starting distribution. In particular let us say that a probability measure p on Ω is an *M-warm start* if

$$\sup_{S \in \Omega} \frac{p(S)}{\overline{\pi}(S)} \leq M.$$

Let $\ell_2(\overline{\pi})$ denote the inner product space on defined on $\mathbb{R}^\Omega$, where the inner product is given by $\langle f, g \rangle_{\ell_2(\overline{\pi})} := \sum_{S \in \Omega} \overline{\pi}(S) f(S) g(S)$, and let $\ell_1(\overline{\pi})$ be the corresponding ℓ_1 space. Let Π be the $\Omega \times \Omega$ diagonal matrix whose S^{th} diagonal entry is $\overline{\pi}(S)$. We define the density of p with respect to $\overline{\pi}$ to be the the vector with entries $\frac{p(S)}{\overline{\pi}(S)}$. We omit the proof of the following standard fact.

---

[2] The reader may consult e.g. [16, Chap. 6] for an introduction to continuous time Markov processes.

Lemma 4.4 (Warm Start Mixing) *If p is M-warm, then for $\tau > 0$, $t = \Omega(\log(M/\tau))$ time, one has*

$$||\overline{\pi} - pP(t)||_{TV} \leq \tau.$$

The third and final property is a bound on the rate at which the continuous chain makes discrete transitions between states. Let J_{avg} denote the average number of state transitions made by the continuous time chain in unit time, from stationarity, and note that

$$J_{\mathsf{avg}} = \frac{\sum_{S \in \Omega} \pi(S)|Q(S, S)|}{\sum_{S \in \Omega} \pi(S)} = \frac{\sum_{S \in \Omega} \delta(S)}{\sum_{S \in \Omega} \pi(S)}$$

as the diagonal entries of the generator Q are equal to $-\delta(S)/\pi(S)$. The most technical part of the proof is the following probabilistic bound.

Lemma 4.5 (Polynomial Jump Rate) *Assume* **(S)**, **(R)**. *With probability at least* $1 - 1/m^2$, *the continuous time Markov chain defined by* (14) *satisfies:*

$$J_{\mathsf{avg}} \leq \tilde{O}(m^3 d^6/\sigma^2 r).$$

The proof of this lemma involves showing that the facets of K are well-shaped and have non-degenerate angles between them in a certain average sense, and is carried out in Sects. 4.2, 4.3, and 4.4.

Combining these ingredients, we can prove Theorem 4.1

Proof of Theorem 4.1 Let T be a fixed positive time to be chosen later. Consider the continuous time chain (14), and for $F \in \Omega$ let the random variable J_F^T denote the number of transitions in $[0, T]$ when the chain is started at F. With probability $1 - 1/m^2$ we have

$$\sum_{F \in \Omega} \overline{\pi}(F)\mathbb{E}J_F^T = T J_{\mathsf{avg}} \leq \tilde{O}(m^3 d^6/\sigma^2 r) \cdot T$$

by Lemma 4.5 so there is a facet $F_0 \in \Omega$ satisfying

$$\mathbb{E}J_{F_0}^T \leq \tilde{O}(m^3 d^6/\sigma^2 r) \cdot T. \tag{15}$$

By Lemma 4.3(1), the distribution δ_{F_0} concentrated on F_0 is $\overline{\pi}_{min}^{-1}$-warm with probability $1 - 1/m^2$. Invoking Lemma 4.4 with starting distribution δ_{F_0} and parameters

$$T = O(\log(1/\overline{\pi}_{min})) = \tilde{O}(d^2 \log(1/r)), \quad M = \overline{\pi}_{min}^{-1}, \quad \tau = \overline{\pi}_{min}/2$$

we have

$$||\overline{\pi} - \delta_{F_0} P(T)||_{TV} \leq \overline{\pi}_{min}/2.$$

Combining this with (15), we obtain a distribution on discrete paths γ in Ω (with respect to the adjacency relation $\sim$) such that each path has source F_0,

$$\mathbb{E}\mathrm{length}(\gamma) \leq \tilde{O}(m^3 d^6/\sigma^2 r) \cdot T,$$

and the distribution of target(γ) is within total variation distance $\overline{\pi}_{min}/2$ of $\overline{\pi}$. Letting

$$G = \{\mathrm{target}(\gamma) : \mathrm{length}(\gamma) \leq 2\mathbb{E}\mathrm{length}(\gamma)/\phi\}$$

we immediately have that the diameter of G is at most

$$\tilde{O}(m^3 d^6/\sigma^2 r) \cdot 2T/\phi = \tilde{O}(m^3 d^8/\sigma^2 r\phi)$$

and by Markov's inequality $\overline{\pi}(G) \geq 1 - \phi$, as desired. $\qquad\qquad\square$

Before proceeding with the proofs of Lemmas 4.3 and 4.5, we collect the probabilistic notation used throughout the sequel. We will often truncate on the following two high probability events. Fix

$$\epsilon := m^{-5d} \tag{16}$$

and define:

$$\mathcal{B} := \left\{ \min_{S \in \binom{[m]}{d}, j \in [m] \setminus S} \mathrm{dist}(v_j, \mathrm{aff}(F_S)) \geq \epsilon \right\},$$

$$\mathcal{C} := \left\{ \max_{j \in [m]} \|g_j\| \leq \alpha \right\}.$$

Note that whenever $\sigma > m^{-d}$ (which we may assume without loss of generality, as otherwise the diameter is trivially at most $1/\sigma$):

$$\mathbb{P}[\mathcal{B}] \geq 1 - O(m^{-4d}/\sigma) \geq 1 - 1/m^3, \tag{17}$$

since the density of the component of v_j orthogonal to $\mathrm{aff}(F_S)$ is bounded by $1/\sigma$ and there are at most m^d facets. We also have

$$\mathbb{P}[\mathcal{C}] \geq 1 - 1/m^3, \tag{18}$$

by standard Gaussian concentration and a union bound.

We will repeatedly use that on $\mathcal{C}$, we have the Hausdorff distance bounds

$$\mathrm{hdist}(K, K_0) \leq \alpha, \qquad \mathrm{hdist}(K^{(j)}, K_0^{(j)}) \leq \alpha \quad \forall j \leq m, \tag{19}$$

for α as in (13), since if $x = \sum_{j \leq m} c_j(a_j + g_j) \in K$ for some convex coefficients c_j then $x_0 = \sum_{j \leq m} c_j a_j \in K_0$ and $\|x - x_0\| \leq \alpha$.

For an index $j \in [m]$ let $\hat{\mathbf{g}}_j := (g_1, \ldots, g_{j-1}, g_{j+1}, \ldots g_m)$ and let $K^{(j)} = \operatorname{conv}(v_i : i \neq j)$. Note that $K^{(j)}$ is a deterministic function of $\hat{\mathbf{g}}_j$. Define the indicator random variables

$$K_S := \{F_S \in \mathcal{F}_{d-1}(K)\}, \qquad K_S^{(j)} := \{F_S \in \mathcal{F}_{d-1}(K^{(j)})\}$$

for subsets $S \in \binom{[m]}{d}$. It will be convenient to fix in advance a total order $<$ on $\binom{[m]}{d}$.

We will occasionally refer to

$$\sum_{F \in \mathcal{F}_k(K)} \operatorname{Vol}_k(F)$$

as the *k-perimeter* of K.

4.1 Nondegeneracy of $\overline{\pi}$

We will repeatedly use the following fact relating Hausdorff distance and containment of convex bodies.

Lemma 4.6 (Containment of Small Perturbations) *If* $\operatorname{hdist}(K, K_0) \leq \alpha < \frac{r}{2}$ *for any two convex bodies and* $r B_2^d \subset K_0$, *then*

$$(1 + 2\alpha/r)^{-1} K_0 \subset K \subset (1 + \alpha/r) K_0.$$

Proof The second containment is immediate from

$$K \subset K_0 + \alpha B_2^d \subset K_0 + (\alpha/r) K_0.$$

The condition $\operatorname{hdist}(K, K_0) \leq \alpha$ also implies $K_0 \subset K + \alpha B_2^d$. To turn this into a multiplicative containment, we claim that $(r/2) B_2^d \subset K$. If not, there is a point $z \in \partial(r/2) B_2^d \setminus K$. Choose a halfspace H supported at z containing K. Let y be a point in $\partial(r B_2^d)$ at distance at least $r/2$ from H and note that $y \in K_0$. But now $\operatorname{dist}(y, K) \geq \operatorname{dist}(y, H) \geq r/2 > \alpha$, violating that $K_0 \subset K + \alpha B_2^d$. Thus, we conclude that $K_0 \subset (1 + 2\alpha/r) K$, establishing the first containment. $\square$

Proof of Lemma 4.3 Condition on $\mathcal{C}$. By (**S**), (**R**), (19), and Lemma 4.6, we have

$$K \supset (1 + 2/d^2)^{-1} K_0 \supset (r/2) B_2^d, \tag{20}$$

and also $K \subset (1 + \alpha) B_2^d$. Consequently, the angle between any two adjacent facets F_S, F_T of K must satisfy

$$|\theta_{ST} - \pi| = \Omega(1/r),$$

which implies

$$\theta_{ST}/2 \leq \tan(\theta_{ST}/2) \leq O(1/r)\theta_{ST} \tag{21}$$

for all θ_{ST}. Thus, for each facet $S \in \Omega$:

$$\chi_2(S)/2 \leq \pi(S) \leq O(r^{-1})\chi_2(S), \tag{22}$$

establishing Lemma 4.3(3).

Equation (20) further implies:

$$\frac{|\partial K|}{|K|} \leq \frac{2d}{r}.$$

By e.g. [27, Sect. 4.2], we have the quermassintegral formulas:

$$d \cdot V(K[d-1], B_2^d[1]) = \sum_{S \in \Omega} |F_S| = |\partial K|, \tag{23}$$

$$\binom{d}{2} V(K[d-2], B_2^d[2]) = \sum_{S < T \in \Omega} |F_{ST}|\theta_{ST}. \tag{24}$$

By the Alexandrov–Fenchel inequality with $\beta = 1/2$:

$$\begin{aligned}
\chi_2(\Omega) = \binom{d}{2} V(K[d-2], B_2^d[2]) &\leq \binom{d}{2} \frac{V(K[d-1], B_2^d[1])^2}{V(K[d])} \\
&\leq \binom{d}{2} \frac{|\partial K|^2}{|K|} \\
&\leq O(d^3) \frac{|\partial K|}{r} \\
&\leq O(d^3/r)|\partial K|.
\end{aligned}$$

By Alexandrov–Fenchel with $\beta = 1/(d-1)$, we also have

$$\begin{aligned}
\chi_2(\Omega) = \binom{d}{2} V(K[d-2], B_2^d[2]) &\geq \binom{d}{2} V(K[d-1], B_2^d[1])^{\frac{d-2}{d-1}} V(B_2^d[d])^{\frac{1}{d-1}} \\
&= \binom{d}{2} \left(d^{-1}|\partial K|\right)^{\frac{d-2}{d-1}} |B_2^d|^{\frac{1}{d-1}} \\
&\geq \binom{d}{2} \sqrt{\frac{2\pi e}{d^3}} |\partial K|^{\frac{d-2}{d-1}} \\
&\geq C|\partial K|.
\end{aligned}$$

The last step follows from the fact that $K \subset (1+\alpha)B_2^d$ so $|\partial K|^{\frac{1}{d-1}} = O(1)$. Combining these inequalities with (21),(22), we conclude that:

$$C|\partial K| \leq \pi(\Omega) \leq O(d^3 r^{-2})|\partial K|, \tag{25}$$

establishing Lemma 4.3(2).

The event $\mathcal{B}$ implies that for every $S \in \Omega$:

$$
\begin{aligned}
\pi(S) &= \sum_{T \sim S} |F_{ST}| \tan(\theta_{ST}/2) \\
&\geq C\epsilon \sum_{T \sim S} |F_{ST}| \quad \text{since } \tan(\theta_{ST}/2) \geq C\epsilon \\
&\geq C\epsilon(d-1)|F_S|^{\frac{d-2}{d-1}}|B_2^d|^{\frac{1}{d-1}} \quad \text{by the isoperimetric inequality} \\
&\geq C\epsilon d|F_S|,
\end{aligned}
$$

where in the last step we used $|F_S| = O(1)$. Conditional on $\mathcal{B}$ Lemma 4.7 implies that $|F_S| \geq \frac{\epsilon^{d-1}}{d}$ for every $S \in \Omega$, so we conclude that

$$
\pi(S) \geq C\epsilon^d \quad \forall S \in \Omega,
$$

and consequently by (25)

$$
\overline{\pi}(S) \geq \frac{C\epsilon^d r^2}{d^3},
$$

yielding Lemma 4.3(1), as desired. □

Lemma 4.7 (Inradius of a Simplex) *If $L = \operatorname{conv}(v_1, \ldots, v_{t+1})$ is a t-dimensional simplex such that each vertex of L is at distance s from the affine span of the remaining vertices, then L contains a ball of radius $s/(t+1)$.*

Proof We see that the distance of the centroid $\frac{\sum_{i=1}^{t+1} v_i}{t+1}$ of the simplex from any face is at least $\frac{s}{t+1}$, proving the lemma. □

4.2 Average Jump Rate Bound

In this section we establish the following Lemma, which immediately implies Lemma 4.5 by $\mathbb{P}[\mathcal{BC}] \geq 1 - 2/m^3$ and Markov's inequality applied to the expectation below (absorbing the $\log(m)$ factor into the $\tilde{O}$).

Lemma 4.8 (Main Estimate) *Assume* (**S**), (**R**) *in the above setting. Then*

$$
\mathbb{E} \frac{\sum_{S \in \Omega} \delta(S)}{\sum_{S \in \Omega} \pi(S)} \cdot \mathcal{BC} \leq \tilde{O}(m^3 d^6/\sigma^2 r). \tag{26}
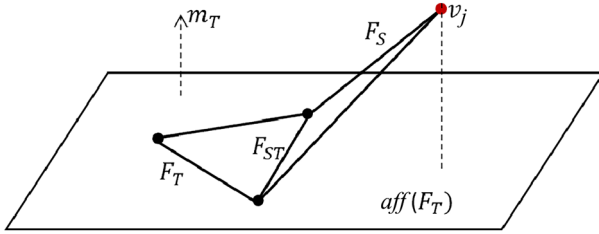$$

Fig. 1 Proof of Lemma 4.9

Proof

$$
\mathbb{E}\frac{\sum_{S\in\Omega}\delta(S)}{\sum_{S\in\Omega}\pi(S)}\cdot\mathcal{BC} \le O(1)\cdot\mathbb{E}\frac{\sum_{S\in\Omega}\delta(S)}{\mathrm{Vol}_{d-1}(\partial K_0)}\cdot\mathcal{BC}
$$

$$
\text{by (25) and (20)}
$$

$$
= \frac{O(1)}{\mathrm{Vol}_{d-1}(\partial K_0)}
$$

$$
\times \mathbb{E}\left[\mathcal{C}\cdot\sum_{S<T\in\binom{[m]}{d}}\mathcal{B}|F_{ST}|\csc\theta_{ST}K_S K_T K_{ST}\right]
$$

$$
\le \frac{O(1)\cdot O(d\log m/\sigma)}{\mathrm{Vol}_{d-1}(\partial K_0)}\mathbb{E}\left[\mathcal{C}\cdot\sum_{S<T\in\binom{[m]}{d}}|F_{ST}|K_{ST}^{(S\setminus T)}\right]
$$

$$
\text{by Lemma 4.9}
$$

$$
\le \frac{O(1)\cdot O(d\log m/\sigma)}{\mathrm{Vol}_{d-1}(\partial K_0)}\mathbb{E}\left[\mathcal{C}\cdot\sum_{j\le m}\sum_{S<T\in\binom{[m]}{d}}|F_{ST}|K_{ST}^{(j)}\right]
$$

$$
\le \frac{O(1)\cdot O(d\log m/\sigma)}{\mathrm{Vol}_{d-1}(\partial K_0)}
$$

$$
\times O(m^2 d^{9/2}\log^{5/2}(m)/\sigma r)\sum_{j\le m}\mathrm{Vol}_{d-1}(\partial K_0^{(j)})
$$

$$
\text{by Lemma 4.10}
$$

$$
\le \frac{O(1)\cdot O(d\log m/\sigma)}{\mathrm{Vol}_{d-1}(\partial K_0)}
$$

$$
\times O(m^2 d^{9/2}\log^{5/2}(m)/\sigma r)\cdot m\,\mathrm{Vol}_{d-1}(\partial K_0) \quad\text{since } K_0^{(j)}\subset K_0
$$

$$
\le m\cdot O(1)\cdot O(d\log m/\sigma)\cdot O(m^2 d^{9/2}\log^{5/2}(m)/\sigma r),
$$

implying the desired conclusion. $\qquad\square$

Lemma 4.9 (Angles Large On Average) *For every $S, T \in \binom{[m]}{d}$ with $S\setminus T = \{j\}$:*

$$
\mathbb{E}\left[\mathcal{B}|F_{ST}|\csc\theta_{ST}K_{ST}\,\Big|\,\hat{\mathbf{g}}_j, \mathcal{C}\right] \le O(d\log m/\sigma)\cdot\mathbb{E}\left[|F_{ST}|K_{ST}^{(j)}\,\Big|\,\hat{\mathbf{g}}_j, \mathcal{C}\right]. \quad (27)
$$

Proof By trigonometry,

$$\csc \theta_{ST} = \frac{\text{dist}(v_j, \text{aff}(F_{ST}))}{\text{dist}(v_j, \text{aff}(F_T))} \leq \frac{3}{\text{dist}(v_j, \text{aff}(F_T))},$$

conditional on $\mathcal{C}$, since K has diameter at most $2 + 2\alpha \leq 3$. The distance in the denominator can be rewritten as

$$\text{dist}(v_j, \text{aff}(F_T)) = \text{dist}(g_j + a_j, \text{aff}(F_T)) = \text{dist}(g_j, \text{aff}(F_T) - a_j) = |h_j - x_T|$$

where $h_j = \langle g_j, m_T \rangle$ and $x_T = \text{dist}(0, \text{aff}(F_T) - a_j) \leq 4$ for m_T the unit normal to $\text{aff}(F_T)$. Moreover,

$$|F_{ST}|K_{ST} \leq |F_{ST}|K_{ST}^{(j)}$$

with probability one conditional on $\hat{g}_j$ since $S \cap T \in \mathcal{F}_{d-2}(K)$ implies $S \cap T \in \mathcal{F}_{d-2}(K^{(j)})$ as $j \notin S \cap T$. Combining these facts, the left hand side of (27) is at most

$$\mathbb{E}\left[\mathcal{B}|F_{ST}|K_{ST}^{(j)} \frac{3}{\text{dist}(v_j, \text{aff}(F_T))} \middle| \hat{\mathbf{g}}_j, \mathcal{C}\right] = 3|F_{ST}|K_{ST}^{(j)}\mathbb{E}\left[\mathcal{B}\frac{1}{|h_j - x_T|} \middle| \hat{\mathbf{g}}_j, \mathcal{C}\right],$$

Notice that h_j has density on $\mathbb{R}$ bounded by

$$t \mapsto \frac{1}{\sqrt{2\pi}\sigma}e^{-t^2/2\sigma^2}\frac{1}{\mathbb{P}[\|g_j\| \leq \alpha]} \leq \frac{1}{\sigma},$$

and $\epsilon \leq |h_j - x_T| \leq |h_j| + x_T \leq 4 + \alpha < 5$ conditioned on $\mathcal{B}, \mathcal{C}$, so the last conditional expectation is at most

$$3\int_\epsilon^5 \frac{1}{\sigma t}dt = 2(\log(1/\epsilon) + \log 5)/\sigma \leq O(d\log m/\sigma),$$

completing the proof. □

The most technical part of the proof is the following $(d - 2)$-perimeter estimate, whose proof is deferred to Sect. 4.3. The conceptual meaning of this estimate is that on average, the $(d - 2)$-dimensional surface area of a random facet of $K^{(j)}$ is well-bounded by its $(d - 1)$-dimensional volume.

Lemma 4.10 (Codimension 2 Perimeter versus Surface Area) *Assume* (**R**), (**S**). *For every* $j \in [m]$*:*

$$\mathbb{E}\mathcal{C} \sum_{S<T\in\binom{[m]}{d}} |F_{ST}|K_{ST}^{(j)} \leq O(m^2 d^{9/2} \log^{5/2}(m)/\sigma r) \cdot \text{Vol}_{d-1}(\partial K_0^{(j)}).$$

4.3 Proof of Lemma 4.10

The key step in the proof is to show that for any well-rounded polytope L_0, there a distribution on two-dimensional planes W such that the $(d-2)$-perimeter of every nearby polytope L is accurately reflected in the average number of vertices of $W \cap L$. Since this number of vertices is small in expectation by [12], we can then conclude that the codimension 2 perimeter is small.

In this section and the next only, the variable ϵ will refer to a quantity tending to zero (as opposed to the definition (16)).

Lemma 4.11 (Quadrature by Planes) *Let $r_1 B_2^d \subset L_0 \subset r_2 B_2^d$. There there is a probability distribution on two dimensional planes W in $\mathbb{R}^d$ such that for sufficiently small $\epsilon > 0$ the following holds uniformly over every polytope L with at most m^d facets satisfying*

$$\mathrm{hdist}(L, L_0) \leq \eta < \frac{r_1}{2d} :$$ (28)

every $(d-2)$-dimensional disk S_ϵ of radius ϵ contained in the interior of a $(d-2)$-dimensional face of L satisfies

$$\mathrm{Vol}_{d-1}(\partial L_0) \cdot \mathbb{P}[W \cap S_\epsilon \neq \emptyset] \geq \Omega\left(\frac{r_1}{d^{3/2}(r_2 + \eta)}\right) \cdot \mathrm{Vol}_{d-2}(S_\epsilon).$$

Moreover, for every $(d-2)$-dimensional affine subspace $H \subset \mathbb{R}^d$, $\mathbb{P}[W \cap H > 1] = 0$.

The proof of Lemma 4.11 is deferred to Sect. 4.4.

We rely on the following result of Dadush and Huiberts [12, Thm. 1.13] (they prove something a little stronger, but we use a simplified bound).

Theorem 4.12 (Shadow Vertex Bound) *Suppose W is a fixed two dimensional plane and $Q = \mathrm{conv}\{v_1, \ldots, v_m\}$ where $v_i \sim N(a_i, \sigma^2 I)$ with $\|a_i\| \leq 1$. Then*

$$\mathbb{E}[|\mathcal{F}_0(W \cap Q)|] = O(d^{2.5} \log^2(m)/\sigma^2).$$ (29)

Combining these two ingredients, we can prove Lemma 4.10.

Proof of Lemma 4.10 Fix $j \leq m$ and recall that $r B_2^d \subset K_0^{(j)} \subset B_2^d$ by (**R**). Conditioning on $\mathcal{C}$, we also have $\mathrm{hdist}(K^{(j)}, K_0^{(j)}) \leq \alpha$. Thus we may invoke Lemma 4.11 with $L_0 = K_0^{(j)}$, $L = K^{(j)}$, $r_1 = r$, $r_2 = 1$, and $\eta = \alpha = \Omega(\sigma\sqrt{d \log m})$ to obtain a probability measure ν on two dimensional planes $W \subset \mathbb{R}^d$ with the advertised properties; note that crucially W depends only on K_0 and is independent of K. Let I_ϵ be a maximal collection of disjoint $(d-2)$-dimensional disks S_ϵ of radius ϵ, with each

S_ϵ contained in some $(d-2)$-face of L. Notice that

$$\int d\nu(W) \mathbb{E} \sum_{S<T\in\binom{[m]}{d}} [\mathcal{C}\{|W \cap F_{ST}| \neq 0\} K_{ST}^{(j)}]$$

$$= \mathbb{E} \sum_{S<T\in\binom{[m]}{d}} \int d\nu(W) \mathcal{C}\{|W \cap F_{ST}| \neq 0\} K_{ST}^{(j)}$$

$$\geq \mathbb{E} \sum_{S<T\in\binom{[m]}{d}} \int d\nu(W) \mathcal{C} \sum_{S_\epsilon \in I_\epsilon} \{|W \cap S_\epsilon| \neq 0\} K_{ST}^{(j)}$$

by the "Moreover" part of Lemma 4.11

$$\geq \mathbb{E} \mathrm{Vol}_{d-1}(\partial L_0)^{-1} \Omega\left(\frac{r_1}{d^{3/2}(r_2+\eta)}\right) \cdot \left[\mathcal{C} \sum_{S<T\in\binom{[m]}{d}} |F_{ST}| K_{ST}^{(j)}\right]$$

by Lemma 4.11, choosing $\epsilon \ll 1$

The integrand in the first expression above above is at most

$$m^2 \cdot \mathbb{E}[\mathcal{F}_0(W \cap K)\mathcal{C}]$$

since each set in $\binom{[m]}{d-2}$ appears as the intersection of at most m^2 adjacent pairs S, T. Therefore by Theorem 4.12 the first expression above is bounded above by $O(m^2 d^{5/2} \log^2(m)/\sigma^2)$. Rearranging yields

$$\mathbb{E}\left[\mathcal{C} \sum_{S<T\in\binom{[m]}{d}} |F_{ST}| K_{ST}^{(j)}\right] \leq O(m^2 d^{2.5} \log^2(m)/\sigma^2)$$

$$\times O(d^{3/2}\sigma\sqrt{d\log m}/r)\mathrm{Vol}_{d-1}(\partial K_0^{(j)}),$$

implying the desired conclusion.

4.4 Proof of Lemma 4.11

We provide an explicit construction for the distribution of W. Let $\tilde{L} = L_0 + 2\eta B_2^d$ and note that its boundary $\partial \tilde{L}$ is smooth; let ψ be the $d-1$-dimensional surface measure on $\partial \tilde{L}$. This equals both the $d-1$ dimensional Hausdorff measure and the Minkowski content of $\partial \tilde{L}$. Then let $W = V + a$ where a is a point sampled according to ψ, and V is sampled by taking the span of two Gaussian vectors (or any radially symmetric random vectors). In order to compute $\mathbb{P}(V + a \cap S_\epsilon \neq \emptyset)$, it will help to first reduce it to the related probability $\mathbb{P}(W' \cap S_\epsilon \neq \emptyset)$ for $W = V + a'$ where a' is sampled uniformly from the unit ball which shares a center with S_ϵ. In particular, let x be the center of S_ϵ and denote $B_x = B_2^d + x$. Let ψ' be the $d-1$-dimensional Hausdorff

measure on ∂B_x. Then we will reduce to the case of $\mathbb{P}(V + a' \cap S_\epsilon \neq \emptyset)$ for a' sampled according to ψ'. For any z, define the radial projection Π_z by

$$\Pi_z(y) = \frac{y - z}{\|y - z\|} + z.$$

Note that Π_x is a bijection between $\partial \tilde{L}$ and ∂B_x since every ray originating from x intersects $\partial \tilde{L}$ in exactly one point because x is in the interior of $\tilde{L}$, which is convex.
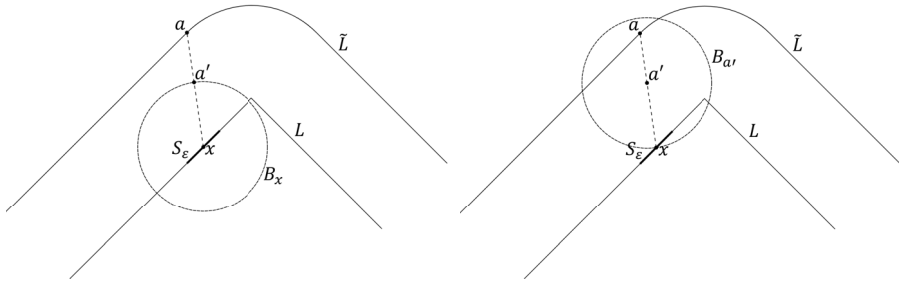
Claim 4.13 The push-forward of ψ by Π_x is absolutely continuous with respect to ψ' with Radon-Nikodym derivative

$$\frac{d(\psi \circ \Pi_x^{-1})(a')}{d\psi'(a')} = \frac{\sin \phi}{\|x - a\|^{d-1}}, \qquad a' = \Pi_x(a) \in \partial B_x,$$

where ϕ is the angle in $[0, \pi]$ between the tangent plane to $\partial \tilde{L}$ at a and the line segment $\overline{xa}$.

Proof An explicit Jacobian calculation given the definition of Π_x and smoothness of $\partial \tilde{L}$ gives the result. $\qquad \square$

Lemma 4.14 Let $z \notin \mathrm{aff}(S_\epsilon)$ be a point such that Π_z is injective on S_ϵ. Let V be a random two-dimensional subspace. Then

$$\mathbb{P}(V + z \cap S \neq 0) = \mu(\Pi_z(S))/A_{d-2}$$

where μ is the Hausdorff measure of $\Pi_z(\mathrm{aff}(S))$ and $A_{d-2} = \mu(\Pi_z(\mathrm{aff}(S)))$ (half the surface area of S^{d-2}).

Proof Since $\mathrm{aff}(S_\epsilon)$ misses z, we have that $\mathrm{aff}(\{z\} \cup S_\epsilon)$ is $d - 1$ dimensional. On the other hand, Π_z is smooth and injective on $\mathrm{aff}(S_\epsilon)$ so $\Pi_z(S_\epsilon)$ itself is $d - 2$ dimensional. Condition on $(V + z) \not\subset \mathrm{aff}(\{z\} \cup S)$, which occurs with probability 1. Then $(V + z) \cap \mathrm{aff}(\{z\} \cup S)$ is a line through z. By symmetry, the intersection of that line with $B_2^d + z$ will be a uniformly random antipodal pair. Exactly one point from each pair will fall in $\Pi_z(\mathrm{aff}(S))$. Thus, the event we care about is the event that $y \in \Pi_z(S)$ where y is sampled uniformly from μ. $\qquad \square$

The following Lemma takes a and a' to be fixed, and depends only on the randomness of V.

Lemma 4.15 *Let a be a point not in* $\mathrm{aff}(S_\epsilon)$ *and* $a' = \Pi_x(a)$. *Let θ be the angle between S_ϵ and the ray emanating from a through x. Then, for a uniformly random 2-plane V,*

$$\mathbb{P}(V + a \cap S_\epsilon \neq \emptyset) = \frac{\mathrm{Vol}_{d-2}(S_\epsilon)}{A_{d-2}} \frac{\cos\theta}{\|x - a\|^{d-2}}(1 + O(d\epsilon))$$

and

$$\mathbb{P}(V + a' \cap S_\epsilon \neq \emptyset) = \frac{\mathrm{Vol}_{d-2}(S_\epsilon)}{A_{d-2}}(\cos\theta)(1 + O(d\epsilon)).$$

where the convergence is uniform in a, a'. In particular, the ratio of the above two quantities is $\|x - a\|^{d-2}$.

Proof We apply Lemma 4.14 twice, both times with S_ϵ playing the role of S. The first time we take a to play the role of z, and the second time a'. This gives

$$\mathbb{P}(V + a \cap S_\epsilon \neq \emptyset) = \mu_a(\Pi_a(S_\epsilon))/A_{d-2} \quad \text{and}$$
$$\mathbb{P}(V + a' \cap S_\epsilon \neq \emptyset) = \mu_{a'}(\Pi_{a'}(S_\epsilon))/A_{d-2}$$

where $\mu_a, \mu_{a'}$ are the Hausdorff measures on $\Pi_a(\mathrm{aff}(S_\epsilon))$, $\Pi_{a'}(\mathrm{aff}(S_\epsilon))$ respectively. Let μ' be the surface measure on $\mathrm{aff}(S_\epsilon)$. Then the Radon-Nikodym derivatives of μ' and the pull-backs of μ_a and $\mu_{a'}$ are

$$\frac{d(\mu_a \circ \Pi_a)(y)}{d\mu'(y)} = \frac{\cos\theta_y^a}{\|y - a\|^{d-2}} \quad \text{and} \quad \frac{d(\mu_{a'} \circ \Pi_{a'})(y)}{d\mu'(y)}$$
$$= \frac{\cos\theta_y^{a'}}{\|y - a'\|^{d-2}}$$

where $\theta_y^a, \theta_y^{a'}$ are the angles between S_ϵ and the rays from a, a' to y respectively. This allows us to compute

$$\mu_a(\Pi_a(S_\epsilon)) = (\mu_a \circ \Pi_a)(S_\epsilon) = \int_{S_\epsilon} \frac{\cos\theta_y^a}{\|y - a\|^{d-2}} d\mu(y)$$
$$= \mathrm{Vol}_{d-2}(S_\epsilon)\frac{\cos\theta_x^a}{\|x - a\|^{d-2}}(1 + O(d\epsilon)).$$

The same is true for a' in place of a. Note that $\theta_x^{a'} = \theta_x^a = \theta$, and that $\|x - a'\| = 1$. That gives the desired result. □

Lemma 4.16 (Reduction to ∂B_x) *Let W be as above and let W' be a uniformly random two dimensional plane through a uniformly random point a' chosen from ∂B_x. Then for sufficiently small $\epsilon > 0$ (depending only on L_0):*

$$\text{Vol}_{d-1}(\partial \tilde{L})\mathbb{P}[W \cap S_\epsilon \neq \emptyset] \geq \frac{r_1}{8(r_2 + \eta)}\text{Vol}_{d-1}(\partial B_x)\mathbb{P}[W' \cap S_\epsilon \neq \emptyset]$$

Proof Note that a, a' miss $\text{aff}(S_\epsilon)$ with probability 1, so we implicitly condition on that event in the following.

$$\text{Vol}_{d-1}(\partial \tilde{L})\mathbb{P}[W \cap S_\epsilon \neq \emptyset]$$

$$= \int \mathbb{P}[W \cap S_\epsilon \neq \emptyset | a]d\psi(a)$$

$$= \int \mathbb{P}[W \cap S_\epsilon \neq \emptyset | a' = T(a)]d(\psi \circ \Pi_x^{-1})(a') \quad \text{by invertibility of } \Pi_x (4.13)$$

$$= \int \mathbb{P}[W \cap S_\epsilon \neq \emptyset | a']\frac{\sin \phi}{\|x - a\|^{d-1}}d\psi'(a') \quad \text{by Claim 4.13}$$

$$\geq \int \left(\mathbb{P}[W' \cap S_\epsilon \neq \emptyset | a'](1/2)\right)\frac{\sin \phi}{\|x - a\|}d\psi'(a') \quad \text{by Claim 4.15,}$$

for sufficiently small ϵ

$$\geq \frac{r_1}{8(r_2 + \eta)}\int \mathbb{P}[W' \cap S_\epsilon \neq \emptyset | a']d\psi'(a') = \frac{r_1}{8(r_2 + \eta)}\text{Vol}_{d-1}(\partial B_x)\mathbb{P}[W' \cap S_\epsilon \neq \emptyset],$$

where in the final inequality we have used $\|x - a\| \leq 2(r_2 + \eta)$ and $\sin \phi \geq \frac{r_1}{4r_2}$ because $\tilde{L} \supset L \supset (r_1 - \eta)B_2^d \supset (r_1/2)B_2^d$ and $\tilde{L} \subset (r_2 + \eta)B_2^d \subset 2r_2 B_2^d$. □

Lemma 4.17 (Intersection Probability for ∂B_x)

$$\mathbb{P}\left(W' \cap S_\epsilon \neq \emptyset\right) = \frac{\text{Vol}_{d-2}(S_\epsilon)}{A_{d-2}}\frac{C_d}{\sqrt{d}}(1 + O(d\epsilon))$$

for some constant $C_d = \Theta(1)$ depending on d.

Proof Using iterated expectation, we can write

$$\mathbb{P}\left(W' \cap S_\epsilon \neq \emptyset\right) = \mathbb{E}\left(\mathbb{P}\left(W' \cap S_\epsilon \neq \emptyset : a'\right)\right)$$

where the outer expectation is over the randomness of a' and inner probability over V. The inner probability is given by 4.15 as

$$\frac{\text{Vol}_{d-2}(S_\epsilon)}{A_{d-2}}\cos(\theta_x^{a'})(1 + O(d\epsilon)).$$

The only dependence on a' is in $\cos(\theta_x^{a'})$. However, by symmetry of the distribution of a', $\theta_x^{a'}$ might as well measure the angle between a uniform random vector selected

from ∂B_x and any fixed line. Thus

$$\mathbb{E}[\cos \theta_x^{a'}] = C_d/\sqrt{d},$$

for some constant $C_d = \Theta(1)$ depending on d. □

We can now complete the proof of Lemma 4.11. Combining Lemmas 4.16 and 4.17, we have for sufficiently small $\epsilon > 0$:

$$\mathrm{Vol}_{d-1}(\partial \tilde{L})\mathbb{P}[W \cap S_\epsilon] \geq \frac{r_2}{2r_1\eta}\mathrm{Vol}_{d-1}(\partial B_x) \cdot \frac{\Omega(1)}{\sqrt{d}A_{d-2}}\mathrm{Vol}_{d-2}(S_\epsilon)$$

$$= \Omega\left(\frac{r_1}{d^{3/2}(r_2 + \eta)}\right)\mathrm{Vol}_{d-2}(S_\epsilon)$$

since $\mathrm{Vol}_{d-2}(\partial B_x)/A_{d-2} = 2\pi/d$, as desired.

4.5 Removing Assumptions (S), (R)

In this section we explain how any instance of the smoothed unit LP model may be reduced to one for which (S), (R) hold with parameter (12), incurring only a polynomial loss in m.

Proof of Theorem 1.2 The idea is to add the noise vector g_j as the sum of two independent Gaussians $g_{j,1} \sim N(0, \sigma_1^2)$ and $g_{j,2} \sim N(0, \sigma_2^2)$ with σ_1 guaranteeing roundedness and σ_2 supplying the necessary anticoncentration and concentration for the main part of the proof. Given $\sigma < 1/d$, set

$$\sigma_1 = m^8\sigma_2$$

and $\sigma_1^2 + \sigma_2^2 = \sigma^2$ and let K_1 be equal to K_0 perturbed by g_1 only. Applying Lemma 4.18 to each $K_0^{(j)}$ and taking a union bound, we have

$$K_1^{(j)} \supset rB_2^d \quad \forall j \leq m, \qquad r = \Omega(\sigma m^{-5}) = \Omega(\sigma_2 m^3),$$

with probability $1 - O(m^{-2})$. Since $\sigma < 1/d$, another union bound reveals that

$$K_1 \subset 2B_2^d$$

with probability $1 - O(m^{-2})$; let $K_2 = K_1/2$. Now K_2 is an instance of the smoothed unit LP model, (K_2, σ_2) satisfy (R) with $r = \Omega(\sigma_2 m^3) = \Omega(\sigma/m^5)$, and

$$6\sqrt{d \log m}\sigma_2 = o(r/m^2),$$

so (K_2, σ_2) also satisfy (S), establishing (12) with the role of (K_0, σ) now played by (K_2, σ_2).

Invoking Theorem 4.1, we conclude that with probability $1 - 1/m^2$, for every $\phi \in (0, 1)$ there is a subset $G \subset \Omega$ with $\pi(G) \geq (1 - \phi)\pi(\Omega)$ and facet diameter

$$\tilde{O}(m^3 d^8/(\sigma/m^8)^2(\sigma/m^5)\phi) = \text{poly}(m, d)/\sigma^3\phi.$$

Moreover, by Lemma 4.3(3), we have

$$\chi_2(G) \leq 2\pi(G) \leq 2\phi \cdot \pi(\Omega) \leq \phi \cdot O(m^5/\sigma)\chi_2(\Omega),$$

so we conclude that $\chi_2(G) \geq (1 - \psi)\chi_2(\Omega)$ for $\psi = O(m^5\phi/\sigma)$. Rewriting the diameter bound in terms of ψ yields the desired conclusion. The probability may be upgraded to $1 - 1/\text{poly}(m)$ by Remark 4.2.

Lemma 4.18 (Roundedness of Smoothed Polytopes) *Suppose we have $m \geq d + 1$ points $a_1, \ldots, a_m \in \mathbb{R}^d$, and these are perturbed to $v_1, \ldots, v_m$ by adding independent $g_j \sim N(0, \sigma_1^2 I_d)$ to each respective a_j. Then, with probability at least $1 - O(m^{-3})$, the convex hull K of $v_1, \ldots, v_m$ contains a ball of radius $r_{in} \geq \Omega(\sigma_1 m^{-5})$.*

Proof Without loss of generality, taking the first $d + 1$ points a_i, we may assume that $m = d + 1$. Then K is the convex hull of $d + 1$ points $v_1, \ldots, v_{d+1}$. The probability that the affine span of these points equals $\mathbb{R}^d$ is 1. Let r_{in} be the inradius of K; by Lemma 4.7, we have

$$r_{in} \geq \frac{\min_i \text{dist}(v_i, \text{aff}(F_i))}{d + 1}.$$

Let us now fix an i and obtain and obtain a probabilistic lower bound on $\frac{\text{dist}(v_i, \text{aff}(F_i))}{d+1}$. Reorder the points (if necessary) so that $i = d + 1$. It now follows that given the the affine span A of the points $v_1, \ldots, v_d$ and given a_{d+1}, the distribution of $\text{dist}(v_{d+1}, A)$ is the same as the distribution of $|\tilde{g} + \text{dist}(a_{d+1}, A)|$, where $\tilde{g} \sim N(0, \sigma_1^2)$ has the distribution of a one dimensional Gaussian with variance σ_1^2. However, the probability that $|\tilde{g} + \text{dist}(a_{d+1}, A)|$ is less than $\sigma_1 m^{-4}$ is at most $O(m^{-4})$. Therefore, by the union bound,

$$\mathbb{P}\left[\min_i \text{dist}(v_i, \text{aff}(F_i)) > \sigma d^{-4}\right] > 1 - O(m^{-3}).$$

It follows that

$$\mathbb{P}\left[r_{in} > \frac{\sigma m^{-4}}{d + 1}\right] > 1 - O(m^{-3}),$$

as desired. □

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