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Three-Dimensional Graph Products with Unbounded Stack-Number

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Abstract
We prove that the stack-number of the strong product of three n-vertex paths is $\Theta(n^{1/3})$. The best previously known upper bound was $O(n)$. No non-trivial lower bound was known. This is the first explicit example of a graph family with bounded maximum degree and unbounded stack-number. The main tool used in our proof of the lower bound is the topological overlap theorem of Gromov. We actually prove a stronger result in terms of so-called triangulations of Cartesian products. We conclude that triangulations of three-dimensional Cartesian products of any sufficiently large connected graphs have large stack-number. The upper bound is a special case of a more general construction based on families of permutations derived from Hadamard matrices. The strong product of three paths is also the first example of a bounded degree graph with bounded queue-number and unbounded stack-number. A natural question that follows from our result is to determine the smallest Δ_0 such that there exists a graph family with unbounded stack-number, bounded queue-number and maximum degree Δ_0. We show that $\Delta_0 \in \{6, 7\}$.

Keywords Stack layout · Stack-number · Strong product · Topological overlap theorem

Mathematics Subject Classification 05C10 · 05C62

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1 Introduction

Stack layouts are ubiquitous objects at the intersection of combinatorics, geometry and topology with applications in computational complexity [14, 15, 26, 34], RNA folding [38], graph drawing in two [5, 56] and three dimensions [59], traffic light scheduling [45], and fault-tolerant multiprocessing [19, 55].

For a graph[1] G and ordering $(v_1, \dots, v_n)$ of $V(G)$, two edges $v_i v_j, v_k v_\ell \in E(G)$ *cross* with respect to $(v_1, \dots, v_n)$ if $i < k < j < \ell$. For $s \in \mathbb{N}_0$, an *s-stack layout* of G consists of an ordering $(v_1, \dots, v_n)$ of $V(G)$ together with a function $\phi\colon E(G) \to \{1, \dots, s\}$ such that for each $a \in \{1, \dots, s\}$ no two edges in $\phi^{-1}(a)$ cross with respect to $(v_1, \dots, v_n)$; see Fig. 1 for an example. Each set $\phi^{-1}(a)$ is called a *stack*. Edges in a stack behave in a last-in-first-out manner (hence the name). Stack layouts can also be viewed topologically via embeddings into books (first defined by Ollmann [50]). The *stack-number* $\operatorname{sn}(G)$ of a graph G is the minimum $s \in \mathbb{N}_0$ for which there exists an s-stack layout of G (also known as *page-number*, *book-thickness*, or *fixed outer-thickness*).

Stack layouts have been studied for planar graphs [9, 17, 41, 62, 63], graphs of given genus [31, 42, 49], treewidth [29, 30, 35, 58], minor-closed graph classes [12], 1-planar graphs [2, 7, 8, 16], and graphs with a given number of edges [48], amongst other examples.

This paper studies stack layouts of 3-dimensional products. As illustrated in Fig. 2, for graphs G_1 and G_2, the *Cartesian product* $G_1 \square G_2$ is the graph with vertex-set $V(G_1) \times V(G_2)$ with an edge between two vertices (x, y) and (x', y') if $x = x'$ and $yy' \in E(G_2)$, or $y = y'$ and $xx' \in E(G_1)$. The *strong product* $G_1 \boxtimes G_2$ is the graph obtained from $G_1 \square G_2$ by adding edges $(x, y)(x', y')$ and $(x, y')(x', y)$ for all edges $xx' \in E(G_1)$ and $yy' \in E(G_2)$. Since the Cartesian and strong products are associative, we may write $G_1 \square G_2 \square G_3$ and $G_1 \boxtimes G_2 \boxtimes G_3$ (identifying pairs of the forms $((v_1, v_2), v_3)$ and $(v_1, (v_2, v_3))$ with the triple (v_1, v_2, v_3)).

Let P_n denote the n-vertex path. Our first main result is the following tight bound on the stack-number of the strong product of three paths (the 3-dimensional grid plus crosses).

Theorem 1.1 $\operatorname{sn}(P_n \boxtimes P_n \boxtimes P_n) \in \Theta(n^{1/3})$.

Note that $(P_n \boxtimes P_n) \square P_n$ and $(P_n \square P_n) \boxtimes P_n$ both have bounded stack-number, as we now explain. Bernhart and Kainen [10] showed that if G_1 and G_2 are graphs with bounded stack-number and G_1 is bipartite with bounded maximum degree, then the stack-number of $G_1 \square G_2$ is bounded. Pupyrev [54] showed that if additionally G_2 has bounded pathwidth, then the stack-number of $G_1 \boxtimes G_2$ is also bounded. Since the Cartesian product is commutative, these results imply that $(P_n \boxtimes P_n) \square P_n$ and $(P_n \square P_n) \boxtimes P_n$ indeed have bounded stack-number. This shows that in Theorem 1.1, we cannot replace even one 'strong product' by a 'Cartesian product'.

No non-trivial lower bound on $\operatorname{sn}(P_n \boxtimes P_n \boxtimes P_n)$ was previously known. Indeed, Theorem 1.1 provides the first explicit example of a graph family with bounded maximum degree and unbounded stack-number. Malitz [48] first proved that graphs of

[1] All graphs in this paper are simple and, unless explicitly stated otherwise, undirected and finite. Let $V(G)$ and $E(G)$ denote the vertex-set and edge-set of a graph G. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

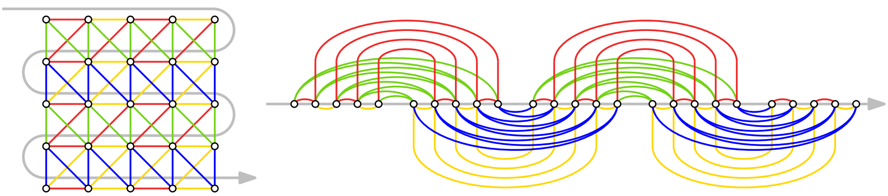

Fig. 1 A 4-stack layout of the strong product $P_5 \boxtimes P_5$ of two paths

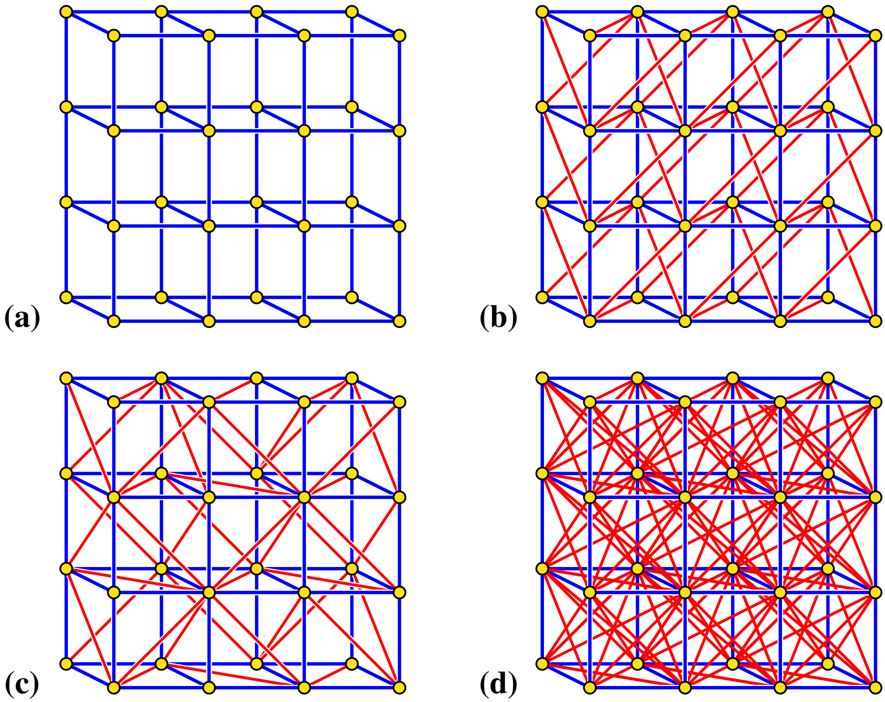

Fig. 2 (a) $P_4 \square P_4 \square P_2$, (b) $P_4 \boxslash P_4 \boxslash P_2$, (c) a triangulation of $P_4 \square P_4 \square P_2$, (d) $P_4 \boxtimes P_4 \boxtimes P_2$

maximum degree 3 have unbounded stack-number (using a probabilistic argument). Further motivation for Theorem 1.1 is provided in Sect. 2 where we present various connections to related graph parameters, shallow/small minors and growth.

We now discuss the lower bound in Theorem 1.1. We actually prove a stronger result that depends on the following definitions. For graphs G_1 and G_2, a *triangulation* of $G_1 \square G_2$ is any graph obtained from $G_1 \square G_2$ by adding the edge $(x, y)(x', y')$ or $(x, y')(x', y)$ for each $xx' \in E(G_1)$ and $yy' \in E(G_2)$. A *triangulation* of $G_1 \square G_2 \square G_3$ is any graph obtained by triangulating all subgraphs induced by sets of the form $\{v_1\} \times V(G_2) \times V(G_3)$, $V(G_1) \times \{v_2\} \times V(G_3)$, and $V(G_1) \times V(G_2) \times \{v_3\}$ with $v_i \in V(G_i)$; see Fig. 2(c) for an example.

For directed graphs G_1 and G_2, if U_i is the undirected graph underlying G_i, then let $G_1 \boxslash G_2$ be the triangulation of $U_1 \square U_2$ containing the edge $(x, y)(x', y')$ for all directed edges $(x, x') \in E(G_1)$ and $(y, y') \in E(G_2)$. Similarly, for directed graphs G_1, G_2, and G_3, let $G_1 \boxslash G_2 \boxslash G_3$ be the appropriate triangulation of $U_1 \square U_2 \square U_3$. When using the notation $G_1 \boxslash G_2$ or $G_1 \boxslash G_2 \boxslash G_3$, if some G_i is a path P_n, then P_n is a directed path.

Every triangulation of $G_1 \square G_2$ is a subgraph of $G_1 \boxtimes G_2$ and every triangulation of $G_1 \square G_2 \square G_3$ is a subgraph of $G_1 \boxtimes G_2 \boxtimes G_3$. So the next result immediately implies the lower bound in Theorem 1.1.

Theorem 1.2 *Let T_1, T_2, and T_3 be n-vertex trees with maximum degree Δ_1, Δ_2, and Δ_3 respectively, where $\Delta_1 \geq \Delta_2 \geq \Delta_3$. Then for every triangulation G of $T_1 \square T_2 \square T_3$,*

$$\operatorname{sn}(G) \in \Omega\left(\left(\frac{n}{\Delta_1 \Delta_2}\right)^{1/3}\right).$$

Theorem 1.2 is similar in spirit to a recent result of Dujmović et al. [22], who showed that if S_n is the n-leaf star, then

$$\operatorname{sn}(S_n \square (P_n \boxslash P_n)) \in \Omega(\sqrt{\log \log n}). \tag{1.1}$$

Their proof actually establishes the following stronger result[2]:

Theorem 1.3 [22] *For every triangulation G of $P_n \square P_n$,*

$$\operatorname{sn}(S_n \square G) \in \Omega(\sqrt{\log \log n}).$$

Theorem 1.2 has the advantage over Theorem 1.3 in that it applies to bounded degree graphs (for example when each T_i is a path). Moreover, the lower bound in Theorem 1.1 (as a function of the number of vertices) is much stronger than the lower bound of Theorem 1.3.

We prove Theorem 1.2 by relating the stack-number of a graph to the topological properties of its triangle complex. The *triangle complex* of a graph G, denoted by $\operatorname{Tr}(G)$, is the 2-dimensional simplicial complex with 0-faces corresponding to vertices of G, 1-faces corresponding to edges of G, and 2-faces corresponding to triangles of G. For topological spaces X and Y, define

$$\operatorname{overlap}(X, Y) = \min_{f \in C(X,Y)} \max_{p \in Y} |f^{-1}(p)|,$$

where $C(X, Y)$ denotes the space of all continuous functions $f : X \to Y$. In Sects. 3 and 4, respectively, we prove the following two lemmas.

Lemma 1.4 *For all n-vertex trees T_1, T_2, and T_3, and for every triangulation G of $T_1 \square T_2 \square T_3$,*

$$\operatorname{overlap}(\operatorname{Tr}(G), \mathbb{R}^2) \in \Omega(n).$$

[2] A key step in the proof of (1.1) is the application of the Hex Lemma, which holds for any triangulation of $P_n \square P_n$ (see [61] for example). Theorem 1.3 then follows by the method in [22].

Lemma 1.5 *For every graph G such that every vertex is in at most c triangles,*

$$\operatorname{sn}(G) \geq \left(\frac{\operatorname{overlap}(\operatorname{Tr}(G), \mathbb{R}^2)}{3c}\right)^{1/3}.$$

Lemma 1.4 has a natural geometric consequence: A *straight-line drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by the line segment between its endpoints, such that the only vertices an edge intersects are its own endpoints. Let T_1, T_2, and T_3 be n-vertex trees and let G be a triangulation of $T_1 \square T_2 \square T_3$. Lemma 1.4 implies that for every straight-line drawing of G, there exists a point in the plane within $\Omega(n)$ triangles of G.

Theorem 1.2 (and thus the lower bound in Theorem 1.1) follows from Lemmas 1.4 and 1.5 since (using the notation from Theorem 1.2) each vertex of G is in at most $2(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3) \leq 6\Delta_1\Delta_2$ triangles. So Lemma 1.5 is applicable with $c = 6\Delta_1\Delta_2$.

The lower bound on $\operatorname{sn}(G)$ in Theorem 1.2 is non-trivial only if $\Delta_1\Delta_2 \in o(n)$. Nevertheless, we have the following result with no assumption on the maximum degree. Theorems 1.2 and 1.3 imply that the stack-numbers of triangulations of $P_n \square P_n \square P_n$ and $S_n \square P_n \square P_n$ grow with n. Moreover, $S_n \square S_n$ contains a subgraph isomorphic to a 1-subdivision of $K_{n,n}$, so its stack-number grows with n as well (see [13, 32]). Since every sufficiently large connected graph contains a copy of P_n or S_n, we deduce the following.

Corollary 1.6 *For every $s \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all n-vertex connected graphs G_1, G_2, and G_3, if G is any triangulation of $G_1 \square G_2 \square G_3$, then $\operatorname{sn}(G) > s$.*

The best previously known upper bound on $\operatorname{sn}(P_n \boxtimes P_n \boxtimes P_n)$ was $O(n)$, which follows from Theorem 1 of Dujmović et al. [24] or from Corollary 1 of Pupyrev [54]. The upper bound in Theorem 1.1 follows from a more general result in Sect. 5, which says that for every graph G with bounded stack-number and bounded maximum degree, $\operatorname{sn}(G \boxtimes P_n) \in O(n^{1/2-\varepsilon})$ for some $\varepsilon > 0$ that depends on the chromatic number of G. The proof is based on families of permutations derived from Hadamard matrices.

Our final results concern maximum degree. Theorem 1.2 implies that $(P_n \boxtimes P_n \boxtimes P_n)_{n\in\mathbb{N}}$ is a family of graphs with maximum degree 12, unbounded stack-number and bounded queue-number (defined in Sect. 2). It is natural to ask what is the smallest bound on the maximum degree in such a family. We prove the answer is 6 or 7.

Theorem 1.7 *The least integer Δ_0 such that there exists a graph family with maximum degree Δ_0, unbounded stack-number and bounded queue-number satisfies $\Delta_0 \in \{6, 7\}$.*

The proof of the upper bound in Theorem 1.7 uses the same topological machinery used to prove Theorem 1.2, and is based on the tessellation of $\mathbb{R}^3$ by truncated octahedra. The proof of the lower bound exploits a connection with clustered colouring.

2 Connections

This section provides further motivation for our results by discussing connections with related graph parameters, minors and growth.

Consider a graph G. The *geometric thickness* of G is the minimum $k \in \mathbb{N}_0$ for which there is a straight-line drawing of G and a partition of $E(G)$ into k plane subgraphs; see [4, 20, 29]. A k-stack layout of G defines such a drawing and edge-partition, where the vertices are drawn on a circle in the order given by the stack layout. Thus the geometric thickness of G is at most its stack-number. The *slope-number* of G is the minimum $k \in \mathbb{N}_0$ for which there is a straight-line drawing of G with k distinct edge slopes; see [3, 4, 23, 27, 46, 51]. Since edges of the same slope do not cross, the geometric thickness of G is at most its slope-number.

Note that $P_n \boxtimes P_n \boxtimes P_n$ has slope-number and geometric thickness at most 6 (simply project the natural 3-dimensional representation to the plane). Hence Theorem 1.1 provides a family of graphs with bounded slope-number, bounded geometric thickness, and unbounded stack-number. Eppstein [32] previously constructed a graph family with bounded geometric thickness and unbounded stack-number, but not with bounded slope-number (since the graphs in question have unbounded maximum degree).

For a graph G and ordering $(v_1, \dots, v_n)$ of $V(G)$, two edges $v_i v_j, v_k v_\ell \in E(G)$ *nest* with respect to $(v_1, \dots, v_n)$ if $i < k < \ell < j$. For $q \in \mathbb{N}_0$, a *q-queue layout* of G consists of an ordering $(v_1, \dots, v_n)$ of $V(G)$ together with a function $\phi\colon E(G) \to \{1, \dots, q\}$ such that for each $a \in \{1, \dots, q\}$ no two edges in $\phi^{-1}(a)$ nest with respect to $(v_1, \dots, v_n)$. Each set $\phi^{-1}(a)$ is called a *queue*. Edges in a queue behave in a first-in-first-out manner (hence the name). The *queue-number* $\operatorname{qn}(G)$ of a graph G is the minimum $q \in \mathbb{N}_0$ for which there exists a q-queue layout of G.

Stack and queue layouts are considered to be dual to each other [43]. However, for many years it was open whether there is a graph family with bounded queue-number and unbounded stack-number, or bounded stack-number and unbounded queue-number. Theorem 1.3 by Dujmović et al. [22] resolved the first question, since they also showed that $\operatorname{qn}(S_n \square (P_n \boxtimes P_n)) \leq 4$. Observe that a 4-queue layout of $P_n \boxtimes P_n \boxtimes P_n$ can be obtained by taking the lexicographical ordering $(v_1, \dots, v_{n^3})$ of the vertices and letting $\phi(v_i v_j)$ be determined by which of the sets $\{1\}$, $\{n, n+1\}$, $\{n^2, n^2+1\}$, $\{n^2+n\}$ contains $|i-j|$. By Theorem 1.1, $(P_n \boxtimes P_n \boxtimes P_n)_{n \in \mathbb{N}}$ is therefore another graph family with queue-number at most 4 and unbounded stack-number. It remains open whether there is a graph family with bounded stack-number and unbounded queue-number.

While stack-number is not bounded by a function of queue number in general, it is of interest to find graph classes for which this property holds. In Sect. 6, we show that the class of graphs with maximum degree 5 has this property, but the class of graphs with maximum degree 7 does not. Bipartite graphs are another class where stack-number is bounded by a function of queue-number [25, 53]. On the other hand, 3-colourable graphs do not have this property, since $P_n \boxtimes P_n \boxtimes P_n$ and $S_n \square (P_n \boxtimes P_n)$ are 3-colourable with bounded queue-number and unbounded stack-number by Theorems 1.2 and 1.3.

We now discuss the behaviour of stack- and queue-number with respect to taking minors. Let G, H, and J be graphs. Let $r, s \in \mathbb{N}_0$ and let $k \geq 0$ be a half-integer (that is, $2k \in \mathbb{N}_0$). H is a *minor* of G if a graph isomorphic to H can be obtained from G by vertex deletions, edge deletions, and edge contractions. A *model* of H in G is a function μ with domain $V(H)$ such that: $\mu(v)$ is a connected subgraph of G; $\mu(v) \cap \mu(w) = \emptyset$ for all distinct $v, w \in V(G)$; and $\mu(v)$ and $\mu(w)$ are adjacent for every edge $vw \in E(H)$. Observe that H is a minor of G if and only if G contains a model of H. For $r \in \mathbb{N}_0$, if there exists a model μ of H in G such that $\mu(v)$ has radius at most r for all $v \in V(H)$, then H is an *r-shallow minor* of G. For $s \in \mathbb{N}_0$, if there exists a model μ of H in G such that $|V(\mu(v))| \leq s$ for all $v \in V(H)$, then H is an *s-small minor* of G. We say that J is an *$(\leq s)$-subdivision* of H if J can be obtained from H by replacing each edge by a path with at most s internal vertices. If every path that replaces an edge has exactly s vertices, then J is the *s-subdivision* of H. We say that H is a *k-shallow topological minor* of G if a subgraph of G is isomorphic to a $(\leq 2k)$-subdivision of H. Note that if a graph H is an s-small minor or an s-shallow topological minor of a graph G, then H is an r-shallow minor of G whenever $s \leq r$.

Blankenship and Oporowski [13] conjectured that stack-number is 'well-behaved' under shallow topological minors in the following sense:

Conjecture 2.1 [13] *There exists a function f such that for every graph G and half-integer $k \geq 0$, if H is any k-shallow topological-minor of G, then* $\operatorname{sn}(H) \leq f(\operatorname{sn}(G), k)$.

Dujmović et al. [22] disproved Conjecture 2.1. Their proof used the following lemma by Dujmović and Wood [28].

Lemma 2.2 [28] *For every graph G, if $s = 1 + 2\lceil \log_2 \operatorname{qn}(G) \rceil$ then the s-subdivision of G has a 3-stack layout.*

Lemma 2.2 implies that the 5-subdivision of $S_n \square (P_n \boxtimes P_n)$ admits a 3-stack layout. Using Theorem 1.3, Dujmović et al. [22] concluded there exists a graph class $\mathcal{G}$ with bounded stack-number for which the class of $(5/2)$-shallow topological minors of graphs in $\mathcal{G}$ has unbounded stack-number. Thus stack-number is not well-behaved under shallow topological minors. We now prove an analogous result for small minors.

Theorem 2.3 *There exists a graph class $\mathcal{G}$ with bounded stack-number for which the class of 2-small minors of graphs in $\mathcal{G}$ has unbounded stack-number.*

Proof Lemma 2.2 implies that the 5-subdivision of $P_n \boxtimes P_n \boxtimes P_n$ admits a 3-stack layout. Since $P_n \boxtimes P_n \boxtimes P_n$ has maximum degree at most 12 and $12\lceil 5/2 \rceil + 1 = 37$, it follows that $P_n \boxtimes P_n \boxtimes P_n$ is a 37-small minor of its 5-subdivision. Let $\mathcal{G}_0$ be the class of graphs with stack-number at most 3, and for each $i \in \mathbb{N}_0$, let $\mathcal{G}_{i+1}$ be the class of 2-small minors of graphs in $\mathcal{G}_i$. Thus, $\mathcal{G}_{37}$ contains all 37-small minors of graphs in $\mathcal{G}_0$, including all graphs of the form $P_n \boxtimes P_n \boxtimes P_n$. Hence, there exists $i \in \{0, \ldots, 37\}$ such that stack-number is bounded for $\mathcal{G}_i$ and unbounded for $\mathcal{G}_{i+1}$. □

In contrast to Theorem 2.3, queue-number is well-behaved under small minors. In fact, the following lemma shows it is even well-behaved under shallow minors, which is a key distinction between these parameters.

Lemma 2.4 [44] *For every graph G and for every r-shallow minor H of G,*

$$\operatorname{qn}(H) \leq 2r(2\operatorname{qn}(G))^{2r}.$$

We now compare stack and queue layouts with respect to growth. The *growth* of a graph G is the function $f_G\colon \mathbb{N} \to \mathbb{N}$ such that $f_G(r)$ is the maximum number of vertices in a subgraph of G with radius at most r. Similarly, the *growth* of a graph class $\mathcal{G}$ is the function $f_{\mathcal{G}}\colon \mathbb{N} \to \mathbb{N}\cup\{\infty\}$ where $f_{\mathcal{G}}(r) = \sup\{f_G(r) : G \in \mathcal{G}\}$. A graph class $\mathcal{G}$ has *linear/quadratic/cubic/polynomial growth* if $f_{\mathcal{G}}(r) \in O(r)/O(r^2)/O(r^3)/O(r^d)$ for some $d \in \mathbb{N}$. Let $\mathbb{Z}_n^d$ be the d-fold strong product $P_n \boxtimes \cdots \boxtimes P_n$. Every subgraph of $\mathbb{Z}_n^d$ with radius at most r has at most $(2r+1)^d$ vertices. Thus $(P_n \boxtimes P_n \boxtimes P_n)_{n\in\mathbb{N}}$ has cubic growth, so Theorem 1.1 implies that graph classes with cubic growth can have unbounded stack-number. In contrast, we now show that graphs with polynomial growth have bounded queue-number. Krauthgamer and Lee [47] established the following characterisation of graphs with polynomial growth.

Theorem 2.5 [47] *Let G be a graph with growth $f_G(r) \leq cr^d$ for some $c, d > 0$. Then G is isomorphic to a subgraph of $\mathbb{Z}_n^{\lfloor c' \cdot d \log d \rfloor}$ for some n, and for some c' depending only on c.*

It follows from the upper bound on the queue-number of products by Wood [60] that $\operatorname{qn}(\mathbb{Z}_n^d) \leq (3^d - 1)/2$. Theorem 2.5 therefore implies the following.

Corollary 2.6 *If a graph G has growth $f_G(r) \leq cr^d$ for some $c, d > 0$, then $\operatorname{qn}(G) \leq 2^{c' d \log d}$, for some c' depending only on c.*

3 Proof of Lemma 1.4

The topological arguments in this paper exclusively involve finite 2-dimensional topological simplicial complexes[3], and so for brevity we refer to such an object as simply a *complex*. The proof of Lemma 1.4 relies on a Topological Overlap Theorem of Gromov [36]. To use it we need a technical variation of the overlap parameter. For a complex X and topological space Y, define

$$\operatorname{overlap}_\triangle(X, Y) = \min_{f \in C(X,Y)} \max_{p \in Y} |\{F \in X^{=2} \mid p \in f(F)\}|,$$

where $X^{=2}$ denotes the set of 2-dimensional cells of X. We now state the 2-dimensional case of Gromov's theorem.

Theorem 3.1 [36, p. 419, topological Δ-inequality] *There exists $\alpha > 0$ such that*

$$\operatorname{overlap}_\triangle(\operatorname{Tr}(K_n), \mathbb{R}^2) \geq \alpha n^3$$

for every integer $n \geq 3$.

We use Theorem 3.1 in combination with the following straightforward lemma to lower bound $\operatorname{overlap}(X, \mathbb{R}^2)$ for a complex X in terms of $\operatorname{overlap}_\triangle(\operatorname{Tr}(K_n), X)$.

3 See [11, Sect. 12] for background on simplicial complexes.

Lemma 3.2 *For every complex X_0 and for all topological spaces X and Y,*

$$\operatorname{overlap}(X, Y) \geq \frac{\operatorname{overlap}_{\triangle}(X_0, Y)}{\operatorname{overlap}_{\triangle}(X_0, X)}.$$

Proof Let $f_0 \colon X_0 \to X$ be a continuous function such that

$$\operatorname{overlap}_{\triangle}(X_0, X) = \max_{p \in X} |\{F \in X_0^{=2} \mid p \in f_0(F)\}|.$$

Let $f \colon X \to Y$ be an arbitrary continuous function. Then $f_0 \circ f \colon X_0 \to Y$ is continuous and so there exists $p_0 \in Y$ such that

$$|\{F \in X_0^{=2} \mid p_0 \in (f_0 \circ f)(F)\}| \geq \operatorname{overlap}_{\triangle}(X_0, Y).$$

But

$$\begin{aligned}|\{F \in X_0^{=2} \mid p_0 \in (f_0 \circ f)(F)\}| &\leq \sum_{p \in f^{-1}(p_0)} |\{F \in X_0^{=2} \mid p \in f_0(F)\}| \\ &\leq |f^{-1}(p_0)| \cdot \operatorname{overlap}_{\triangle}(X_0, X).\end{aligned}$$

It follows that

$$|f^{-1}(p_0)| \geq \frac{\operatorname{overlap}_{\triangle}(X_0, Y)}{\operatorname{overlap}_{\triangle}(X_0, X)},$$

as desired. □

A family $\mathcal{B}$ of subcomplexes of a complex X is a *bramble over X*[4] if:

- every $B \in \mathcal{B}$ is non-empty,
- $B_1 \cup B_2$ is path-connected[5] for every pair of distinct $B_1, B_2 \in \mathcal{B}$,
- $B_1 \cup B_2 \cup B_3$ is simply connected[6] for every triple of distinct $B_1, B_2, B_3 \in \mathcal{B}$.

The *congestion* $\operatorname{cong}(\mathcal{B})$ of a bramble $\mathcal{B}$ is the maximum size of a collection of elements in $\mathcal{B}$ that all share a point in common; that is,

$$\operatorname{cong}(\mathcal{B}) = \max_{p \in X} |\{B \in \mathcal{B} \mid p \in B\}|.$$

[4] This definition is inspired by the definition of a *bramble* in a graph, which is used in graph minor theory. A bramble of a graph is only required to satisfy the first and second among the conditions we impose.

[5] A topological space X is *path-connected* if there is a path joining any two points of X; that is, for all $x, y \in X$ there exists a continuous map $f \colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

[6] A topological space is *simply connected* if it is path-connected and any two paths with the same endpoints can be continuously transformed into each other.

Lemma 3.3 *There exists $\beta > 0$ such that for every complex X and bramble $\mathcal{B}$ over X,*

$$\operatorname{overlap}(X, \mathbb{R}^2) \geq \frac{\beta |\mathcal{B}|}{\operatorname{cong}(\mathcal{B})}.$$

Proof Let $\mathcal{B} = \{B_1, \dots, B_n\}$. Let $X_0 = \operatorname{Tr}(K_n)$ be the triangle complex of the complete graph with vertex set $\{v_1, \dots, v_n\}$. We construct a continuous map $f: X_0 \to X$ as follows. Let $p_i \in B_i$ be chosen arbitrarily, and set $f(v_i) = p_i$ for every $i \in \{1, \dots, n\}$. Extend f to the 1-skeleton of X_0 by mapping each edge $v_i v_j$ to a path $\pi_{ij} \subseteq B_i \cup B_j$ from p_i to p_j.

Finally, we need to continuously extend f to the interior of every 2-simplex F_{ijk} of X_0 with vertices v_i, v_j, and v_k. Since $B_i \cup B_j \cup B_k$ is simply connected, and the boundary of F_{ijk} is mapped to a closed curve in $B_i \cup B_j \cup B_k$, there is such an extension such that $f(F_{ijk}) \subseteq B_i \cup B_j \cup B_k$.

Consider arbitrary $p \in X$ and let $I = \{i \in \{1, \dots, n\} \mid p \in B_i\}$. Then $|I| \leq \operatorname{cong}(\mathcal{B})$ and

$$|\{F \in X^{=2} \mid p \in f(F)\}| \leq |\{\{i, j, k\} \subseteq \{1, \dots, n\} \mid \{i, j, k\} \cap I \neq \emptyset\}| \leq 3|I|\binom{n}{2}.$$

It follows that $\operatorname{overlap}_\triangle(X_0, X) \leq 3n^2 \operatorname{cong}(\mathcal{B})/2$. Thus by Theorem 3.1 and Lemma 3.2,

$$\operatorname{overlap}(X, \mathbb{R}^2) \geq \frac{\operatorname{overlap}_\triangle(X_0, \mathbb{R}^2)}{\operatorname{overlap}_\triangle(X_0, X)} \geq \frac{\alpha n^3}{3n^2 \operatorname{cong}(\mathcal{B})/2},$$

where α is from Theorem 3.1. It follows that $\beta = 2\alpha/3$ satisfies the lemma. □

Lemma 3.3 allows one to lower bound $\operatorname{overlap}(X, \mathbb{R}^2)$ by exhibiting a bramble with size large in comparison to its congestion. To simplify the verification of the conditions in the definition of a bramble we use a simple consequence of van Kampen's theorem.[7]

Lemma 3.4 *Let B_1, B_2 be subcomplexes of a complex X. If B_1 and B_2 are simply connected, and $B_1 \cap B_2$ is non-empty and path-connected, then $B_1 \cup B_2$ is simply connected.*

Corollary 3.5 *Let X be a complex. Let $\mathcal{B}$ be a family of subcomplexes of X such that:*

- *every $B \in \mathcal{B}$ is simply connected,*
- *$B_1 \cap B_2$ is path-connected for every pair of distinct $B_1, B_2 \in \mathcal{B}$,*
- *$B_1 \cap B_2 \cap B_3$ is non-empty for every triple of distinct $B_1, B_2, B_3 \in \mathcal{B}$.*

Then $\mathcal{B}$ is a bramble over X.

[7] See [39, Thm. 1.20] for the general statement of van Kampen's theorem, or [18, Sect. 2.1.3] for an example of the application of the theorem in the setting generalizing Lemma 3.4.

Proof The first condition in the definition of a bramble trivially holds. The second condition holds since $B_1 \cup B_2$ is simply connected for all distinct $B_1, B_2 \in \mathcal{B}$ by Lemma 3.4.

It remains to show $B_1 \cup B_2 \cup B_3$ is simply connected for all distinct $B_1, B_2, B_3 \in \mathcal{B}$. Since $B_1 \cup B_2$ and B_3 are simply connected, this follows from Lemma 3.4 as long as $B_3 \cap (B_1 \cup B_2) = (B_3 \cap B_1) \cup (B_3 \cap B_2)$ is non-empty and path-connected. By the assumptions of the corollary, each of $B_3 \cap B_1$ and $B_3 \cap B_2$ is path-connected and they share a point in common. Thus their union is path-connected as desired. □

We are now ready to prove Lemma 1.4.

Proof of Lemma 1.4 Let G be any triangulation of $T_1 \square T_2 \square T_3$, where T_1, T_2, and T_3 are n-vertex trees. To simplify our notation we assume that T_1, T_2, and T_3 are vertex-disjoint. We denote by $\widehat{T_i}$ the 1-dimensional complex corresponding to T_i, to avoid confusion between discrete and topological objects.

For $u \in V(T_1)$, let $G(u)$ be the subgraph of G induced by all the vertices of G with the first coordinate u. That is, $V(G(u)) = \{(u, u_2, u_3) \mid u_2 \in V(T_2),\ u_3 \in V(T_3)\}$. So $G(u)$ is isomorphic to a triangulation of $T_2 \square T_3$. Let $X(u) = \mathrm{Tr}(G(u))$. As a topological space, $X(u)$ is homeomorphic to $\widehat{T_2} \times \widehat{T_3}$. Define $G(u)$ and $X(u)$ for $u \in V(T_2) \cup V(T_3)$ analogously.

For $(u_1, u_2, u_3) \in V(G)$, let $B(u_1, u_2, u_3) = X(u_1) \cup X(u_2) \cup X(u_3)$, and let $\mathcal{B} = \{B(u_1, u_2, u_3) \mid (u_1, u_2, u_3) \in V(G)\}$. We claim that $\mathcal{B}$ is a bramble over Tr(G). It suffices to check that it satisfies the conditions of Corollary 3.5.

To verify the first condition for $\mathcal{B}$, we use Corollary 3.5 to show that $\{X(u_1), X(u_2), X(u_3)\}$ is a bramble over Tr(G) for every $(u_1, u_2, u_3) \in V(G)$. Note that each $\widehat{T_i}$ is simply connected. As the product of simply connected spaces is simply connected, it follows that $X(u)$ is simply connected for every $u \in V(T_1) \cup V(T_2) \cup V(T_3)$. Consider now $(u_1, u_2, u_3) \in V(G)$. Then $X(u_1) \cap X(u_2)$ is homeomorphic to $\widehat{T_3}$ and is path-connected. Similarly, $X(u_1) \cap X(u_3)$ and $X(u_2) \cap X(u_3)$ are path-connected. Finally, $X(u_1) \cap X(u_2) \cap X(u_3)$ consists of a single point (and is path-connected). By Corollary 3.5 the set $\{X(u_1), X(u_2), X(u_3)\}$ is a bramble. In particular, $B(u_1, u_2, u_3) = X(u_1) \cup X(u_2) \cup X(u_3)$ is simply connected. Thus the first condition in Corollary 3.5 for $\mathcal{B}$ holds.

For the second condition, consider distinct $B(u_1, u_2, u_3), B(v_1, v_2, v_3) \in \mathcal{B}$. Let $R = B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3)$ for brevity. Assume first, for simplicity, that $u_i \neq v_i$ for $i \in \{1, 2, 3\}$. Then

$$R = \bigcup_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} X(u_i) \cap X(v_j).$$

Each set $X(u_i) \cap X(v_j)$ in this decomposition is path-connected, since it is homeomorphic to $\widehat{T_k}$ for $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$. Moreover, when ordering these sets,

$$\begin{array}{lll} X(u_1) \cap X(v_2), & X(u_3) \cap X(v_2), & X(u_3) \cap X(v_1), \\ X(u_2) \cap X(v_1), & X(u_2) \cap X(v_3), & X(u_1) \cap X(v_3), \end{array}$$

each pair of consecutive sets share a point, for example, $(u_1, v_2, u_3) \in (X(u_1) \cap X(v_2)) \cap (X(u_3) \cap X(v_2))$. It follows that their union is path-connected.

It remains to consider the case $u_i = v_i$ for some $i \in \{1, 2, 3\}$. Assume $i = 1$ without loss of generality. If $u_2 \neq v_2$ and $u_3 \neq v_3$ then $R = X(u_1) \cup (X(u_2) \cap X(v_3)) \cup (X(v_2) \cap X(u_3))$. Each of the sets in this decomposition is again path-connected and the second and third sets intersect the first, implying that their union is path-connected. Finally, if say $u_2 = v_2$ then $R = X(u_1) \cup X(u_2)$ and is again a union of path-connected intersecting sets. This verifies the second condition of Corollary 3.5 for $\mathcal{B}$.

For the last condition, for any $B(u_1, u_2, u_3), B(v_1, v_2, v_3), B(w_1, w_2, w_3) \in \mathcal{B}$,

$$(u_1, v_2, w_3) \in B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3) \cap B(w_1, w_2, w_3),$$

and so $B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3) \cap B(w_1, w_2, w_3) \neq \emptyset$. By Corollary 3.5, $\mathcal{B}$ is a bramble.

As noted above, $|\mathcal{B}| = n^3$. Moreover, for every $p \in \mathrm{Tr}(G)$ and every $i \in \{1, 2, 3\}$ there exists at most one $v_i \in V(T_i)$ such that $p \in X(v_i)$. Thus $\mathrm{cong}(\mathcal{B}) \leq 3n^2$, and by Lemma 3.3,

$$\mathrm{overlap}(\mathrm{Tr}(G), \mathbb{R}^2) \geq \frac{\beta|\mathcal{B}|}{\mathrm{cong}(\mathcal{B})} \geq \frac{\beta n}{3},$$

as desired. □

Note that $\mathrm{Tr}(P_n \boxtimes P_n \boxtimes P_n)$ has a natural affine embedding into $\mathbb{R}^3$ with vertices mapped to points in $\{1, \ldots, n\}^3$. Since any line in a direction sufficiently close to an axis direction intersects this complex in $n + O(1)$ points, projecting along such a direction we obtain an affine map $\mathrm{Tr}(G) \to \mathbb{R}^2$ that covers every point at most $n + O(1)$ times. Thus $\mathrm{overlap}(\mathrm{Tr}(P_n \boxtimes P_n \boxtimes P_n), \mathbb{R}^2) \leq n + O(1)$ and so the bound in Lemma 1.4 cannot be substantially improved.

4 Proof of Lemma 1.5

The proof of Lemma 1.5 depends on the following result.

Lemma 4.1 *Let $T_1, \ldots, T_m$ be pairwise vertex-disjoint pairwise intersecting triangles in $\mathbb{R}^2$ with all the vertices on a circle S. Assume that the edges of $T_1, \ldots, T_m$ can be partitioned into k non-crossing sets. Then $m \leq k^3$.*

Proof Number the vertices of $T_1, \ldots, T_m$ by $1, \ldots, 3m$ in clockwise order starting at an arbitrary point on S. By assumption there is a function $\phi\colon \bigcup_i E(T_i) \to \{1, \ldots, k\}$ such that $\phi(e_1) \neq \phi(e_2)$ for all crossing edges $e_1, e_2 \in \bigcup_i E(T_i)$. Say the vertices of T_i are (a_i, b_i, c_i) with $a_i < b_i < c_i$. Define the function $f\colon \{1, \ldots, m\} \to \{1, \ldots, k\}^3$ by $f(i) = (\phi(a_i b_i), \phi(a_i c_i), \phi(b_i c_i))$. Suppose for the sake of contradiction that $f(i) = f(j)$ for distinct $i, j \in \{1, \ldots, m\}$. Thus $a_i b_i$ does not cross $a_j b_j$, $a_i c_i$ does not cross $a_j c_j$, and $b_i c_i$ does not cross $b_j c_j$. Without loss of generality, $a_i < a_j$. Then $a_j < c_i$ since T_i and T_j intersect. If $c_i < c_j$ then $a_i c_i$ crosses $a_j c_j$, so $a_i < a_j <$

$b_j < c_j < c_i$. Now consider b_i. If $a_i < b_i < a_j$ or $c_j < b_i < c_i$, then T_i and T_j do not intersect. So $a_j < b_i < c_j$. If $a_j < b_i < b_j$, then $a_i b_i$ crosses $a_j b_j$. Otherwise, $b_j < b_i < c_j$ implying $b_j c_j$ crosses $b_i c_i$. This contradiction shows that $f(i) \neq f(j)$ for distinct $i, j \in \{1, \dots, m\}$. Hence $m \leq k^3$. □

Proof of Lemma 1.5 Let $k = \text{sn}(G)$, and let $(v_1, \dots, v_n)$ together with a function $\phi\colon E(G) \to \{1, \dots, k\}$ be a k-stack layout of G. Let $p_1, p_2, \dots, p_n$ be pairwise distinct points chosen on a circle S in $\mathbb{R}^2$, numbered in cyclic order around S. Let $X = \text{Tr}(G)$. Define a continuous function $f\colon X \to \mathbb{R}^2$ by setting $f(v_i) = p_i$ for all $i \in \{1, \dots, n\}$, extending f affinely to 2-simplices of X (triangles of G), and, for simplicity, mapping every edge of G that does not belong to a triangle continuously to a curve internally disjoint from S and its interior, so that every point of $\mathbb{R}^2$ belongs to at most two such curves.

Let $m' = \text{overlap}(X, \mathbb{R}^2)$. Then there exists $p \in \mathbb{R}^2$ such that $|f^{-1}(p)| \geq m'$. If $m' \leq 2$ the lemma trivially holds, and so we assume $m' > 2$. Since the restriction of f to each simplex of X is injective, there exist triangles $T_1, T_2, \dots, T_{m'} \subseteq \mathbb{R}^2$ corresponding to images of distinct 2-simplices of X so that $p \in \bigcap_{i=1}^{m'} T_i$. Let $m = \lceil m'/(3c-2) \rceil > m'/(3c)$. Since every triangle shares a vertex with at most $3c - 3$ others, we may assume that $T_1, T_2, \dots, T_m$ are pairwise vertex-disjoint by greedily selecting vertex-disjoint triangles among $T_1, T_2, \dots, T_{m'}$. By Lemma 4.1, $m \leq k^3$. Thus

$$\text{sn}(G) = k \geq m^{1/3} \geq \left(\frac{m'}{3c}\right)^{1/3} = \left(\frac{\text{overlap}(X, \mathbb{R}^2)}{3c}\right)^{1/3}. \qquad \square$$

This lemma completes the proof of the lower bound in Theorem 1.1 as well as Theorem 1.2.

We finish this section by showing that Lemma 4.1 is best possible up to a constant factor.

Proposition 4.2 *For infinitely many $k \in \mathbb{N}$, there is a set of $(k/3)^3$ pairwise intersecting and pairwise vertex-disjoint triangles with vertices on a circle in $\mathbb{R}^2$, such that the edges of the triangles can be k-coloured with crossing edges assigned distinct colours.*

This result is implied by the following (with $k = 3 \cdot 2^\ell$).

Lemma 4.3 *Let S be a circle in $\mathbb{R}^2$ partitioned into three pairwise disjoint arcs A, B, C. For every $\ell \in \mathbb{N}_0$ there is a set $\mathcal{T}$ of 8^ℓ triangles, each with one vertex in each of A, B, C, such that the AB-edges can be 2^ℓ-coloured with crossing edges assigned distinct colours, the BC-edges can be 2^ℓ-coloured with crossing edges assigned distinct colours, and the CA-edges can be 2^ℓ-coloured with crossing edges assigned distinct colours.*

Proof We proceed by induction on ℓ. The claim is trivial with $\ell = 0$. Assume that for some integer $\ell \geq 0$, there is a set $\mathcal{T}$ of 8^ℓ triangles satisfying the claim. Let X be the set of 2^ℓ colours used for the AB-edges. Let Y be the set of 2^ℓ colours used for the BC-edges. Let Z be the set of 2^ℓ colours used for the CA-edges.

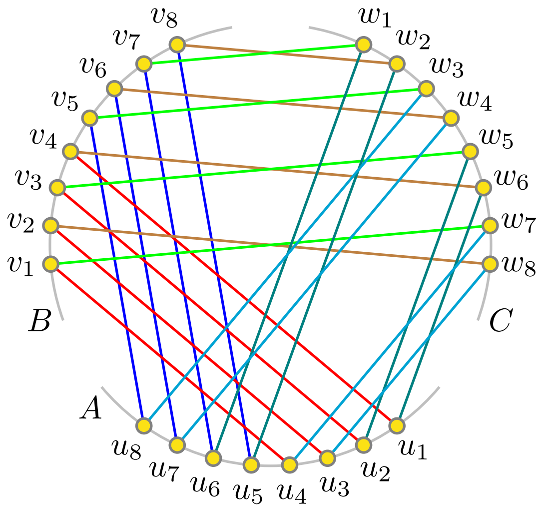

Fig. 3 Construction in the proof of Lemma 4.3

Let $X' = \{x', x' \mid x \in X\}$ be a set of $2^{\ell+1}$ colours. Let $Y' = \{y', y' \mid y \in Y\}$ be a set of $2^{\ell+1}$ colours. Let $Z' = \{z', z'' \mid z \in Z\}$ be a set of $2^{\ell+1}$ colours. Let $\mathcal{T}'$ be the set of triangles obtained by replacing each $T \in \mathcal{T}$ by eight triangles as follows. Say the vertices of T are u, v, w where $u \in A$, $v \in B$, and $w \in C$. Replace u by $u_1, \ldots, u_8$ in clockwise order in A, replace v by $v_1, \ldots, v_8$ in clockwise order in B, and replace w by $w_1, \ldots, w_8$ in clockwise order in C. By this we mean that if p, q are consecutive vertices in the original ordering, then $p_1, \ldots, p_8, q_1, \ldots, q_8$ are consecutive vertices in the enlarged ordering. Add the triangles $u_1v_4w_6, u_2v_3w_5, u_3v_2w_8, u_4v_1w_7, u_5v_8w_2, u_6v_7w_1, u_7v_6w_4, u_8v_5w_3$ to $\mathcal{T}'$. Say uv is coloured $x \in X$, vw is coloured $y \in Y$, and wv is coloured $z \in Z$. Colour each of $u_1v_4, u_2v_3, u_3v_2, u_4v_1$ by $x' \in X'$, and colour each of $u_5v_8, u_6v_7, u_7v_6, u_8v_5$ by $x'' \in X'$. Colour each of $v_1w_7, v_3w_5, v_5w_3, v_7w_1$ by $y' \in Y'$, and colour each of $v_2w_8, v_4w_6, v_6w_4, v_8w_2$ by $y'' \in Y'$. Colour each of $w_1u_6, w_2u_5, w_5u_2, w_6u_1$ by $z' \in Z'$, and colour each of $w_3u_8, w_4u_7, w_7u_4, w_8u_3$ by $z'' \in Z'$. As illustrated in Fig. 3, crossing edges are assigned distinct colours. Thus $\mathcal{T}'$ is the desired set of $8^{\ell+1}$ triangles. □

5 Upper Bound

This section proves the upper bound in Theorem 1.1 showing that $P_n \boxtimes P_n \boxtimes P_n$ has an $O(n^{1/3})$-stack layout. Assume $V(P_n) = \{1, \ldots, n\}$ and $E(P_n) = \{i(i+1) \mid i \in \{1, \ldots, n-1\}\}$. We start with a sketch of the construction. Take a particular $O(1)$-stack layout and a proper 4-colouring of $P_n \boxtimes P_n$. In the corresponding ordering of $V(P_n \boxtimes P_n)$ replace each vertex (x, y) by vertices $((x, y, z_1), \ldots, (x, y, z_n))$ where $(z_1, \ldots, z_n)$ is a permutation of $\{1, \ldots, n\}$ determined by the colour of (x, y). An

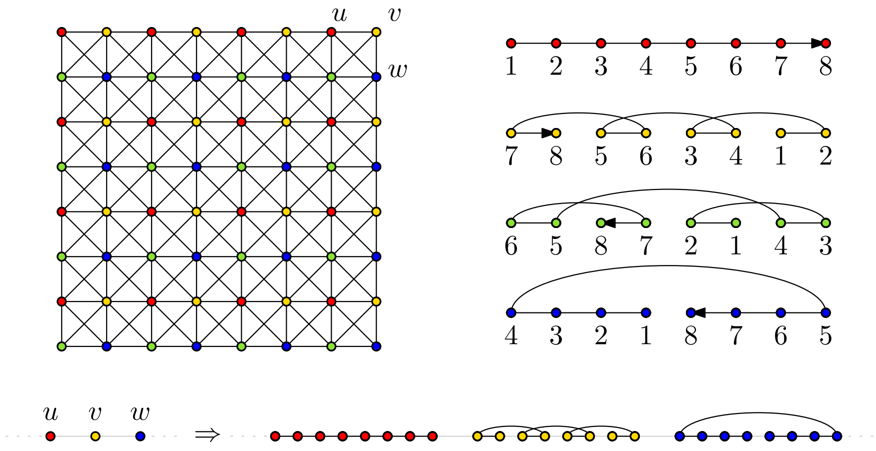

Fig. 4 A proper 4-colouring of $P_8 \boxtimes P_8$ and four permutations of $\{1, \ldots, 8\}$. For $n = 8$, the stack layout of the graph $P_n \boxtimes P_n \boxtimes P_n$ is obtained from a stack layout of $P_n \boxtimes P_n$ by replacing each vertex by a permutation of vertices on the corresponding copy of P_n. The permutation is determined by the colour of the vertex

appropriate choice of the permutations ensures that the edges of $P_n \boxtimes P_n \boxtimes P_n$ can be partitioned into $O(n^{1/3})$ stacks. See Fig. 4 for an illustration of the case $n = 8$.

We actually prove a more general result, Theorem 5.1 below, which relies on the following definition from the literature. An s-stack layout $((v_1, \ldots, v_n), \psi)$ of a graph G is *dispersable* [1, 10, 54] (also called *pushdown* [26, 34]) if $\psi^{-1}(a)$ is a matching in G for each $a \in \{1, \ldots, s\}$. The *dispersable stack-number* $\operatorname{dsn}(G)$ is the minimum $s \in \mathbb{N}_0$ for which there is a dispersable s-stack layout of G. For example, $\operatorname{dsn}(P_n \boxtimes P_n) = 8$ since in the 4-stack layout of $P_n \boxtimes P_n$ illustrated in Fig. 1, each stack is a linear forest; putting alternative edges from each path in distinct stacks produces a dispersable 8-stack layout. In general, $\max\{\Delta(G), \operatorname{sn}(G)\} \leq \operatorname{dsn}(G) \leq (\Delta(G) + 1)\operatorname{sn}(G)$ by Vizing's Theorem. So a graph family has bounded dispersable stack-number if and only if it has bounded stack-number and bounded maximum degree.

Theorem 5.1 *Let G be a graph with chromatic number χ and dispersable stack-number d. Let $n \in \mathbb{N}$ and $p = \max\{2^{\lceil \log_2 \chi \rceil}, 2\}$. Then*

$$\operatorname{sn}(G \boxtimes P_n) \leq 2^{p/2-1} d(2p-1) \cdot n^{1/2-1/(2p-2)} + 2p - 3.$$

Since $\chi(P_n \boxtimes P_n) \leq 4$ and $\operatorname{dsn}(P_n \boxtimes P_n) = 8$, Theorem 5.1 implies that

$$\operatorname{sn}(P_n \boxtimes P_n \boxtimes P_n) \leq 2 \cdot 8 \cdot 7n^{1/3} + 5 = 112n^{1/3} + 5.$$

Furthermore, $\chi(G) \leq \Delta(G) + 1 \leq \operatorname{dsn}(G) + 1$. So Theorem 5.1 shows that graphs G with bounded dispersable stack-number satisfy $\operatorname{sn}(G \boxtimes P_n) \in O(n^{1/2-\varepsilon})$ for some $\varepsilon > 0$.

For an even positive integer p, an *Hadamard matrix* of order p is a $p \times p$ matrix H with all entries in $\{+1, -1\}$ such that every pair of distinct rows differs in exactly

$p/2$ entries. An example of an Hadamard matrix of order 4 is

$$\begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.$$

Sylvester [57] proved that the order of an Hadamard matrix is 2 or is divisible by 4. He also constructed an Hadamard matrix whose order is any power of 2. Paley [52] constructed an Hadamard matrix of order $q + 1$ for any prime power $q \equiv 3 \pmod 4$, and an Hadamard matrix of order $2(q + 1)$ for any prime power $q \equiv 1 \pmod 4$. The Hadamard Conjecture proposes that there exists an Hadamard matrix of order p whenever p is divisible by 4. This conjecture has been verified for numerous values of p; see [21] for example.

To prove Theorem 5.1, we construct a stack layout of $G \boxtimes P_n$ from a d-dispersable stack layout of G by replacing each vertex by a copy of the corresponding path P_n. To determine the orderings of these paths, we use a family of permutations without long common subsequences by Beame et al. [6]. To avoid many pairwise crossing edges between two paths, it is necessary to choose permutations without long common subsequences. However, this is not sufficient because of the edges in $G \boxtimes P_n$ between the i-th vertex and the $(i+1)$-th vertex in distinct paths. To deal with this, the following lemma provides additional properties of the construction by Beame et al. [6].

Lemma 5.2 *Assume there exists an Hadamard matrix of order p. Let $m \in \mathbb{N}$ and $n = m^{p-1}$. Then there exist permutations $\pi_1, \dots, \pi_p : \{1, \dots, n\} \to \{1, \dots, n\}$ such that for all distinct $k, \ell \in \{1, \dots, p\}$,*

$$|\{\pi_k(i) + \pi_\ell(j) \mid i, j \in \{1, \dots, n\}, |i - j| \leq 1\}| \leq (2p - 1)n^{1/2 - 1/(2p-2)},$$

and for each $k \in \{1, \dots, p\}$ there exists a $(2p-3)$-stack layout of P_n using the vertex ordering $(\pi_k^{-1}(1), \dots, \pi_k^{-1}(n))$.

Proof Since there exists an Hadamard matrix of order p, there exist ± 1-vectors $h_1, \dots, h_p$ of length p each, such that any two of them differ in exactly $p/2$ entries. Denote $h_k = (h_k(0), \dots, h_k(p - 1))$ for each $k \in \{1, \dots, p\}$. By possibly negating all entries in some of these vectors, we may assume $h_k(p - 1) = +1$ for all $k \in \{1, \dots, p\}$.

For each $i \in \{1, \dots, n\}$, let $d(i, 0), \dots, d(i, p - 2) \in \{0, \dots, m - 1\}$ denote the digits of $i - 1$ in base m, so that

$$i = 1 + \sum_{a=0}^{p-2} d(i, a)m^a.$$

Note that the mapping $\{1, \dots, n\} \ni i \mapsto (d(i, 0), \dots, d(i, p-2)) \in \{0, \dots, m-1\}^{p-1}$ is bijective. For all $k \in \{1, \dots, p\}$, $i \in \{1, \dots, n\}$, and $a \in \{0, \dots, p-2\}$, let

$$d_k(i, a) = \begin{cases} d(i, a) & \text{if } h_k(a) = +1, \\ m - 1 - d(i, a) & \text{if } h_k(a) = -1. \end{cases}$$

Observe that for any $k, \ell \in \{1, \dots, p\}$ and $a \in \{0, \dots, p-2\}$, if $h_k(a) = -h_\ell(a)$ then $d_k(i, a) + d_\ell(i, a) = m - 1$ for all $i \in \{1, \dots, n\}$. For every $k \in \{1, \dots, p\}$, define π_k by

$$\pi_k(i) = 1 + \sum_{a=0}^{p-2} d_k(i, a) m^a.$$

Note that each π_k is an involution; that is, $\pi_k = \pi_k^{-1}$.

We claim that $\pi_1, \dots, \pi_p$ satisfy the lemma. Fix distinct $k, \ell \in \{1, \dots, p\}$. For $i, j \in \{1, \dots, n\}$ and $a \in \{0, \dots, p-2\}$, define $\tau_a(i, j) = d_k(i, a) + d_\ell(j, a)$, and let

$$\tau(i, j) = (\tau_0(i, j), \dots, \tau_{p-2}(i, j)).$$

Since $\pi_k(i)+\pi_\ell(j) = 2+\sum_{a=0}^{p-2} \tau_a(i, j)m^a$, the value of $\tau(i, j)$ determines the value of $\pi_k(i) + \pi_\ell(j)$. As such, to prove the first part of the lemma it suffices to show that

$$|\{\tau(i, j) \mid i, j \in \{1, \dots, n\},\ |i - j| \le 1\}| \le (2p - 1)n^{1/2-1/(2p-2)}.$$

Let $A^+ = \{a \in \{0, \dots, p-2\} \mid h_k(a) = h_\ell(a)\}$ and $A^- = \{a \in \{0, \dots, p-2\} \mid h_k(a) = -h_\ell(a)\}$. Since h_k and h_ℓ differ in exactly $p/2$ entries and $h_k(p-1) = +1 = h_\ell(p-1)$, we have $|A^+| = p/2 - 1$ and $|A^-| = p/2$.

Let $i \in \{1, \dots, n\}$. For every $a \in A^-$, we have $\tau_a(i, i) = m - 1$ since $h_k(a) = -h_\ell(a)$. For every $a \in A^+$ we have $d_k(i, a) = d_\ell(i, a) \in \{0, \dots, m-1\}$, and thus $\tau_a(i, i)$ is an even integer between 0 and $2m - 2$. Hence

$$\begin{aligned} |\{\tau(i, i) \mid i \in \{1, \dots, n\}\}| &\le 1^{|A^-|}m^{|A^+|} = m^{p/2-1} \\ &= n^{(p/2-1)/(p-1)} = n^{1/2-1/(2p-2)}. \end{aligned}$$

Now, let $i, j \in \{1, \dots, n\}$ be such that $j - i = 1$. In particular, we have $j > 1$, so there exists $c \in \{0, \dots, p-2\}$ such that $d(j, c) \neq 0$. Let $c_{i,j}$ denote the least such c. As $j - i = 1$ and $d(j, 0), \dots, d(j, p-2)$ are the digits of $j - 1$ in base m, the value of $c_{i,j}$ determines the value of $d(j, a) - d(i, a)$ for each $a \in \{0, \dots, p-2\}$:

$$d(j, a) - d(i, a) = \begin{cases} -(m - 1) & \text{if } a < c_{i,j}, \\ 1 & \text{if } a = c_{i,j}, \\ 0 & \text{if } a > c_{i,j}. \end{cases}$$

Hence for any $c \in \{0, \dots, p-2\}$ and $a \in A^-$, for all pairs (i, j) with $j - i = 1$ and $c_{i,j} = c$, the value of $\tau_a(i, j) = d_k(i, a) + d_\ell(j, a)$ is determined and is independent of i and j: if $h_k(a) = +1$, $h_\ell(a) = -1$, then $\tau_a(i, j) = m - 1 - (d(j, a) - d(i, a))$, and if $h_k(a) = -1$ and $h_\ell(a) = +1$, then $\tau_a(i, j) = m - 1 + (d(j, a) - d(i, a))$.

Let $a \in A^+$. If $a < c_{i,j}$, then $d(i, a) = m - 1$ and $d(j, a) = 0$ whence $\{d_k(i, a), d_\ell(j, a)\} = \{0, m - 1\}$ and thus $\tau_a(i, j) = m - 1$. If $a = c_{i,j}$, then $|d_k(i, a) - d_\ell(j, a)| = 1$ since $d(j, c_{i,j}) - d(i, c_{i,j}) = 1$, and thus $\tau_a(i, j)$ is an odd integer between 1 and $2m - 3$. If $a > c_{i,j}$, then $d_k(i, a) = d_\ell(j, a)$ since $d(i, a) = d(j, a)$, and thus $\tau_a(i, j)$ is an even integer between 0 and $2m - 2$. In each of these three cases we can see that, for any $c \in \{0, \dots, p-2\}$ and $a \in A^+$, there are at most m possible values of $\tau_a(i, j)$ for pairs (i, j) with $j - i = 1$ and $c_{i,j} = c$.

Summarizing, for any $a, c \in \{0, \dots, p-2\}$,

$$|\{\tau_a(i, j) \mid i, j \in \{1, \dots, n\},\ j - i = 1,\ c_{i,j} = c\}| \le \begin{cases} 1 & \text{if } a \in A^-, \\ m & \text{if } a \in A^+. \end{cases}$$

Hence

$$\begin{aligned} &|\{\tau(i, j) \mid i, j \in \{1, \dots, n\},\ j - i = 1\}| \\ &\quad \le \sum_{c=0}^{p-2} |\{\tau(i, j) \mid i, j \in \{1, \dots, n\},\ j - i = 1,\ c_{i,j} = c\}| \\ &\quad \le (p-1)1^{|A^-|} m^{|A^+|} = (p-1)m^{p/2-1} = (p-1)n^{1/2-1/(2p-2)}. \end{aligned}$$

By a symmetric argument,

$$|\{\tau(i, j) \mid i, j \in \{1, \dots, n\},\ j - i = -1\}| \le (p-1)n^{1/2-1/(2p-2)}.$$

Therefore

$$\begin{aligned} |\{\tau(i, j) \mid i, j \in \{1, \dots, n\},\ |j - i| \le 1\}| &\le (1 + 2(p-1))n^{1/2-1/(2p-2)} \\ &= (2p-1)n^{1/2-1/(2p-2)}, \end{aligned}$$

which completes the proof of the first part of the lemma.

It remains to show that for each $k \in \{1, \dots, p\}$, the set $E(P_n)$ can be partitioned into $2p - 3$ stacks with respect to $(\pi_k^{-1}(1), \dots, \pi_k^{-1}(m^{p-1}))$. Partition $E(P_n)$ into sets $E_0, \dots, E_{p-2}$ where E_a consists of all edges $i(i + 1)$ such that i is divisible by m^a but not by m^{a+1}. By our construction of π_k, any edge in E_0 is between two consecutive vertices in $(\pi_k^{-1}(1), \dots, \pi_k^{-1}(m^{p-1}))$, so the set E_0 is a valid stack. Now it suffices to show that for every $a \in \{1, \dots, p-2\}$, the set E_a can be partitioned into two stacks. Fix $a \in \{1, \dots, p-2\}$, and partition $(\pi_k^{-1}(1), \dots, \pi_k^{-1}(m^{p-1}))$ into blocks $B_{a,1}, \dots, B_{a,m^{p-1-a}}$ each consisting of m^a consecutive elements, so that $B_{a,j} = \{\pi_k^{-1}((j-1)m^a + 1), \dots, \pi_k^{-1}(jm^a)\}$. By our construction of π_k, each block

$B_{a,j}$ consists of m^a consecutive integers, where the largest integer in the block is divisible by m^a. Furthermore, every edge $i(i+1) \in E_a$ is between two consecutive blocks $B_{a,j}$ and $B_{a,j+1}$. Assign the edge $i(i+1)$ to a set E_a^0 when j is even and to a set E_a^1 when j is odd. The resulting sets E_a^0 and E_a^1 are indeed stacks; in fact, no two edges cross or nest in these sets. Thus $E(P_n)$ can be partitioned into $2p-3$ stacks. □

Theorem 5.1 is a consequence of the following technical variant.

Lemma 5.3 *Assume there exists an Hadamard matrix of order p. Let $m \in \mathbb{N}$ and $n = m^{p-1}$. Let G be a p-colourable graph with a dispersable d-stack layout. Then*

$$\mathrm{sn}(G \boxtimes P_n) \le d(2p-1)n^{1/2-1/(2p-2)} + 2p - 3.$$

Before proving Lemma 5.3, we show that it implies Theorem 5.1.

Proof of Theorem 5.1 Let $m = \lceil n^{1/(p-1)} \rceil$. Since p is a power of 2, there exists an Hadamard matrix of order p. Since $P_n \subseteq P_{m^{p-1}}$, Lemma 5.3 implies

$$\begin{aligned}
\mathrm{sn}(G \boxtimes P_n) \le \mathrm{sn}(G \boxtimes P_{m^{p-1}}) &\le d(2p-1)m^{(p-1)(1/2-1/(2p-2))} + 2p - 3 \\
&= d(2p-1)m^{p/2-1} + 2p - 3 \\
&< d(2p-1)(n^{1/(p-1)} + 1)^{p/2-1} + 2p - 3 \\
&\le d(2p-1)(2n^{1/(p-1)})^{p/2-1} + 2p - 3 \\
&= 2^{p/2-1} d(2p-1)n^{1/2-1/(2p-2)} + 2p - 3.
\end{aligned}$$
□

Proof of Lemma 5.3 Let $\rho\colon V(G) \to \{1, \dots, p\}$ be a proper colouring of G. Let $\pi_1, \dots, \pi_p$ be the permutations of $\{1, \dots, n\}$ given by Lemma 5.2. For each $v \in V(G)$, let P^v denote the path in $G \boxtimes P_n$ induced by $\{v\} \times \{1, \dots, n\}$, and let $\overrightarrow{P^v} = \big((v, \pi_{\rho(v)}^{-1}(1)), \dots (v, \pi_{\rho(v)}^{-1}(n))\big)$. Let $N = |V(G)|$, and let $((v_1, \dots, v_N), \psi)$ be a dispersable d-stack layout of G. Let $\overrightarrow{V} = (\overrightarrow{P^{v_1}}; \overrightarrow{P^{v_2}}; \dots; \overrightarrow{P^{v_N}})$ be an ordering of $V(G \boxtimes P_n)$. By our choice of the permutations $\pi_1, \dots, \pi_p$, for each $v \in V(G)$, the set $E(P^v)$ can be partitioned into $2p-3$ stacks. Since the paths P^v occupy disjoint parts of $\overrightarrow{V}$, it follows that $\bigcup_{v \in V(G)} E(P^v)$ admits a partition into $2p-3$ stacks with respect to $\overrightarrow{V}$.

We partition the set $E(G \boxtimes P_n) \setminus \bigcup_{v \in V(G)} E(P^v)$ into sets E_{uv} indexed by the edges of G. For an edge $uv \in E(G)$, let $E_{uv} = \{xy \in E(G \boxtimes P_n) \mid x \in V(P^u),\ y \in V(P^v)\}$. For each $xy \in E_{uv}$, let $\gamma(xy) = \psi(uv)$. We claim that for all $k \in \{1, \dots, d\}$, $\gamma^{-1}(k)$ can be partitioned into at most $(2p-1)n^{1/2-1/(2p-2)}$ stacks with respect to $\overrightarrow{V}$. Since ψ is a dispersable stack-layout, it suffices to show that for every $uv \in E(G)$, E_{uv} can be partitioned into at most $(2p-1)n^{1/2-1/(2p-2)}$ stacks.

Fix an edge $uv \in E(G)$. For each edge $e = (u, i)(v, j) \in E_{uv}$, define

$$\phi(e) = \pi_{\rho(u)}(i) + \pi_{\rho(v)}(j).$$

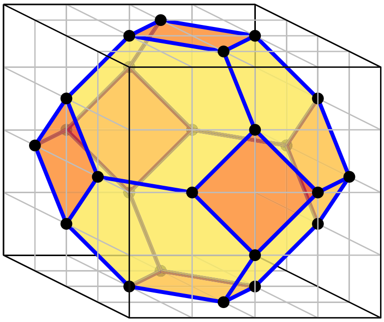

Fig. 5 The truncated octahedron Q_0 inscribed in a cube

By our choice of the permutations $\pi_1, \dots, \pi_p$, the size of the image of ϕ is at most $(2p-1)n^{1/2-1/(2p-2)}$. It remains to show that ϕ partitions E_{uv} into stacks with respect to $\overrightarrow{V}$. Let $e = (u,i)(v,j)$ and $e' = (u,i')(v,j')$ be two edges from E_{uv} which cross. Without loss of generality, assume that the vertices are in the order $(u,i), (u',i'), (v,j), (v',j')$ in $\overrightarrow{V}$. This means that $\pi_{\rho(u)}(i) < \pi_{\rho(u)}(i')$ and $\pi_{\rho(v)}(j) < \pi_{\rho(v)}(j')$, so $\phi(e) = \pi_{\rho(u)}(i)+\pi_{\rho(v)}(j) < \pi_{\rho(u)}(i')+\pi_{\rho(v)}(j') = \phi(e')$, so $\phi(e) \neq \phi(e')$. Hence ϕ partitions E_{uv} into $(2p-1)n^{1/2-1/(2p-2)}$ stacks, as required. □

Note that when p is not a power of 2 but there exists an Hadamard matrix of order p, Lemma 5.3 gives a stronger bound than Theorem 5.1.

6 Smaller Maximum Degree

This section proves Theorem 1.7, which says that if Δ_0 is the minimum integer for which there exists a graph family with maximum degree Δ_0, unbounded stack-number and bounded queue-number, then $\Delta_0 \in \{6, 7\}$. These upper and lower bounds are respectively proved in Theorems 6.1 and 6.4 below.

Theorem 6.1 *There exists a graph family with maximum degree* 7*, unbounded stack-number and bounded queue-number.*

The construction for Theorem 6.1 is based on a tessellation of $\mathbb{R}^3$ with truncated octahedra, first studied by Fedorov [33]. Let Q_0 denote the convex hull of all points $(x,y,z) \in \mathbb{R}^3$ such that $\{|x|,|y|,|z|\} = \{0,1,2\}$; see Fig. 5. A *truncated octahedron* is any polyhedron embedded in $\mathbb{R}^3$ that is geometrically similar to Q_0. At each corner of a truncated octahedron three faces meet: one square and two regular hexagons. Let $Q_0 + (x,y,z)$ be the translation of Q_0 by a vector (x,y,z).

Let $\mathcal{T}_\infty$ be the family of translations of Q_0 defined as

$$\mathcal{T}_\infty = \{Q_0 + (4x,4y,4z) \mid x,y,z \in \mathbb{Z}\} \\ \cup \{Q_0 + (4x+2,4y+2,4z+2) \mid x,y,z \in \mathbb{Z}\}.$$

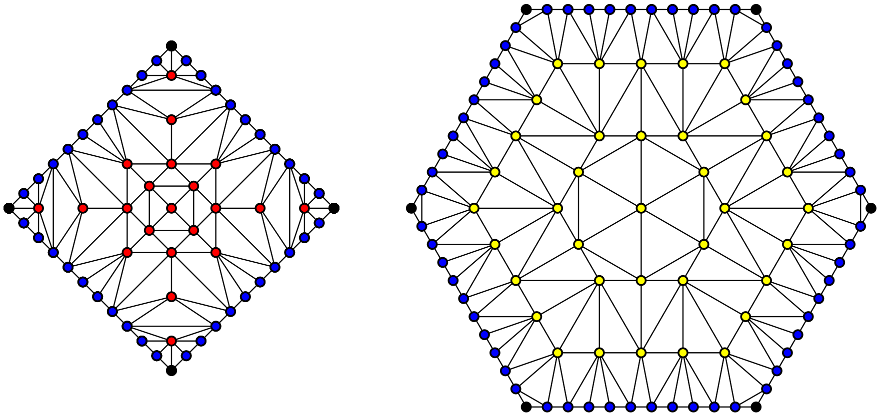

Fig. 6 The graphs $J_\diamond$ and $J_{\hexagon}$

$\mathcal{T}_\infty$ is a 3-*dimensional tessellation*; that is, a family of interior-disjoint polyhedra whose union is $\mathbb{R}^3$. A *corner* of $\mathcal{T}_\infty$ is any corner of a truncated octahedron from $\mathcal{T}_\infty$; *edges* and *faces* of $\mathcal{T}_\infty$ are defined similarly. Every (hexagonal or square) face of $\mathcal{T}_\infty$ is shared by two truncated octahedra in $\mathcal{T}_\infty$. At each edge of $\mathcal{T}_\infty$ a square face and two hexagonal faces meet, and at each corner of $\mathcal{T}_\infty$ two square and four hexagonal faces meet.

We construct an infinite graph G_∞ whose vertices are points in $\mathbb{R}^3$ and edges are line segments between their endpoints. G_∞ is the union of copies of the plane graphs $J_\diamond$ and $J_{\hexagon}$ (depicted in Fig. 6), where the copies of $J_\diamond$ are placed at the square faces of $\mathcal{T}_\infty$ and the copies of $J_{\hexagon}$ are placed at the hexagonal faces of $\mathcal{T}_\infty$. Each copy of $J_\diamond$ or $J_{\hexagon}$ is contained within its corresponding face so that the exterior cycle coincides with the union of the edges contained in that face, black vertices are at the corners of the face, and each edge of the face is split into equal segments by 10 vertices from the exterior face. A vertex of G_∞ is called a *corner vertex* if it coincides with a corner of $\mathcal{T}_\infty$, an *edge vertex* if it lies on an edge of $\mathcal{T}_\infty$ and is not a corner vertex, or a *face vertex* if it is neither a corner vertex nor an edge vertex.

Lemma 6.2 *The maximum degree of* G_∞ *is* 7.

Proof Let $v \in V(G_\infty)$. We proceed by case analysis. If v is a face vertex, then its degree is at most 7 because $J_\diamond$ and $J_{\hexagon}$ have maximum degree 7.

Now suppose v is a corner vertex. Since v belongs to four edges of $\mathcal{T}_\infty$, it is adjacent to four edge vertices in G_∞. Furthermore, v is adjacent to a face vertex in each of the two copies of $J_\diamond$ containing v and is not adjacent to any face vertex in a copy of $J_{\hexagon}$. Therefore, v has degree 6.

It remains to consider the case when v is an edge vertex. Let ε be the edge of the tessellation $\mathcal{T}_\infty$ that contains v. Let $v_0 \dots v_{11}$ be the path induced by the vertices of G_∞ which lie on ε, so that v_0 and v_{11} are corner vertices and $v_1, \dots, v_{10}$ are edge vertices (and one of them is v). The vertices $v_1, \dots, v_{10}$ and all their neighbours lie in one copy of $J_\diamond$ and two copies of $J_{\hexagon}$. The degrees of the vertices $v_1, \dots, v_{10}$ in the copy of $J_\diamond$ containing them are 3, 3, 5, 5, 3, 3, 5, 5, 3, 3, respectively, and their

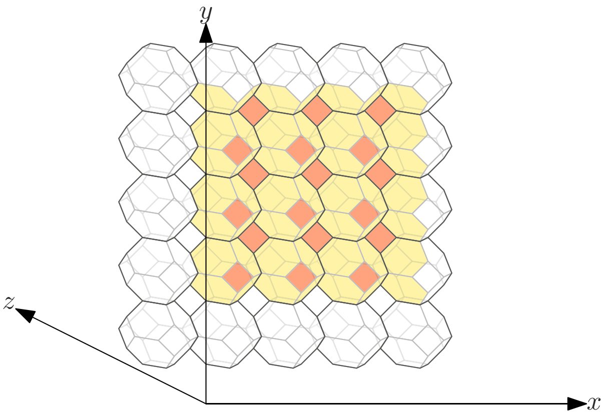

Fig. 7 The set X_i^a for $n = 4$, $a = 3$, $i = 2$. The truncated octahedra are of the form $Q_0 + (4i + 2, 4j + 2, 4k - 2)$ for $(j, k) \in \{0, \dots, n\}^2$

degrees in each copy of $J_{\hexagon}$ containing them are 4, 4, 3, 3, 4, 4, 3, 3, 4, 4, respectively. Since $3 + 2 \cdot 4 = 5 + 2 \cdot 3 = 11$, for each $i \in \{1, \dots, 10\}$, the total sum of degrees of v_i in the copy of $J_{\diamond}$ and the two copies of $J_{\hexagon}$ is 11. However, we counted the vertices v_{i-1} and v_{i+1} thrice, so the degree of v_i in G_∞ is $11 - 4 = 7$, as required. □

Let $[a, b]$ denote the closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. Observe that for every $(i, j, k) \in \mathbb{Z}^3$, the cube $[4i - 2, 4i + 2] \times [4j - 2, 4j + 2] \times [4k - 2, 4k + 2]$ contains the same finite number of vertices of G_∞. For every $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the set of all faces of $\mathcal{T}_\infty$ contained in $[4, 4n + 2]^3$ and let G_n be the subgraph of G_∞ induced by the vertices lying on the faces in $\mathcal{F}_n$. Then $|V(G_n)| \in \Theta(n^3)$. Furthermore, $(G_n)_{n \in \mathbb{N}}$ has cubic growth as the distance between any pair of adjacent vertices in G_n is $O(1)$. By Corollary 2.6 and Lemma 6.2, $(G_n)_{n \in \mathbb{N}}$ has bounded queue-number and maximum degree 7. Thus Theorem 6.1 follows from the next lemma.

Lemma 6.3 $\operatorname{sn}(G_n) \in \Omega(n^{1/3})$.

Proof $J_{\diamond}$ and $J_{\hexagon}$ are plane graphs in which every internal face is a triangle. Therefore, every subgraph induced by a face of $\mathcal{T}_\infty$ has a triangle complex homeomorphic to that face (and to a closed disk). Furthermore, since faces intersect only at their edges and corners, every subgraph of G_n induced by a union of faces from $\mathcal{F}_n$ has a triangle complex homeomorphic to that union.

For $a \in \{1, 2, 3\}$ and $i \in \{1, \dots, n\}$, let X_i^a be the union of all faces $F \in \mathcal{F}_n$ such that $4i \leq x_a \leq 4i + 2$ for all $(x_1, x_2, x_3) \in F$. Each set X_i^a is homeomorphic to a closed disk (see Fig. 7). Let L_i^a denote the subgraph of G_n induced by $X_i^a \cap V(G_\infty)$. As observed earlier, X_i^a as a union of faces from $\mathcal{F}_n$ is homeomorphic to $\operatorname{Tr}(L_i^a)$. Hence we identify X_i^a with $\operatorname{Tr}(L_i^a)$.

Observe that there are $(2n - 1)^3$ hexagonal faces in $\mathcal{F}_n$, each of which is contained in a different cube of the form $[2x + 2, 2x + 4] \times [2y + 2, 2y + 4] \times [2z + 2, 2z + 4]$ with $(x, y, z) \in \{1, \dots, 2n - 1\}^3$. Furthermore, each square face has its centre in a corner of one of these cubes (but not every corner is a centre of a square face). Hence, for $(i, j) \in \{1, \ldots, n\}^2$, the intersection $X_i^1 \cap X_j^2$ is the union of the $2n-1$ hexagonal faces from $\mathcal{F}_n$ contained in $[4i, 4i+2] \times [4j, 4j+2] \times \mathbb{R}$. These hexagons form a sequence such that any pair of consecutive hexagons share an edge. The intersection $X_i^1 \cap X_j^2$ is thus path-connected. Symmetric arguments show that

$$X_i^a \cap X_j^b \text{ is path-connected for } (i, j) \in \{1, \ldots, n\}^2 \text{ and } (a, b) \in \{1, 2, 3\}^2,\ a \neq b. \tag{6.1}$$

Furthermore, for $(i, j, k) \in \{1, \ldots, n\}^3$, the intersection $X_i^1 \cap X_j^2 \cap X_k^3$ is the hexagonal face contained in $[4i, 4i+2] \times [4j, 4j+2] \times [4k, 4k+2]$, so in particular

$$X_i^1 \cap X_j^2 \cap X_k^3 \text{ is path-connected (and non-empty) for } (i, j, k) \in \{1, \ldots, n\}^3. \tag{6.2}$$

Let $\mathcal{B} = \{B(i, j, k)\}_{(i,j,k) \in \{1,\ldots,n\}^3}$ where $B(i, j, k) = X_i^1 \cup X_j^2 \cup X_k^3$. We claim that $\mathcal{B}$ is a bramble over $\mathrm{Tr}(G_n)$. We proceed by verifying the preconditions of Corollary 3.5.

Let $(i, j, k) \in \{1, \ldots, n\}^3$. We first show that $B(i, j, k)$ is simply connected. We have $B(i, j, k) = X_i^1 \cup X_j^2 \cup X_k^3$, and each of X_i^1, X_j^2 and X_k^3 is homeomorphic to a closed disk and is thus simply connected. By (6.1) and (6.2), the sets $X_i^1 \cap X_j^2$ and $X_i^1 \cap X_j^2 \cap X_k^3$ are path-connected. Hence, by Lemma 3.4, $B(i, j, k)$ is simply connected.

Now, let $(i_1, i_2, i_3), (j_1, j_2, j_3) \in \{1, \ldots, n\}^3$, and let $B_1 = B(i_1, i_2, i_3)$ and $B_2 = B(j_1, j_2, j_3)$. We now show that $B_1 \cap B_2$ is path-connected. We have

$$B_1 \cap B_2 = (X_{i_1}^1 \cup X_{i_2}^2 \cup X_{i_3}^3) \cap (X_{j_1}^1 \cup X_{j_2}^2 \cup X_{j_3}^3) = \bigcup_{a=1}^{3} \bigcup_{b=1}^{3} \left(X_{i_a}^a \cap X_{j_b}^b\right).$$

By (6.1), $X_{i_a}^a \cap X_{j_b}^b$ is path-connected when $a \neq b$. Furthermore, if $\{a, b, c\} = \{1, 2, 3\}$, then, by (6.2), $X_{i_a}^a \cap X_{j_b}^b$ intersects $X_{i_a}^a \cap X_{j_c}^c$ and $X_{i_c}^c \cap X_{j_b}^b$. Hence, the union of all $X_{i_a}^a \cap X_{j_b}^b$ with $a \neq b$ is path-connected. Furthermore, for each $a \in \{1, 2, 3\}$, the intersection $X_{i_a}^a \cap X_{j_a}^a$ is $X_{i_a}^a$ if $i_a = j_a$, or an empty set if $i_a \neq j_a$. In the former case, $X_{i_a}^a \cap X_{j_a}^a$ intersects $X_{i_a}^a \cap X_{j_b}^b$ for $b \in \{1, 2, 3\} \setminus \{a\}$. Hence, $B_1 \cap B_2$ is indeed path-connected.

Finally, for any $(i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3) \in \{1, \ldots, n\}^3$, the intersection $B(i_1, j_1, k_1) \cap B(i_2, j_2, k_2) \cap B(i_3, j_3, k_3)$ contains $X_{i_1}^1 \cap X_{j_2}^2 \cap X_{k_3}^3$ which is non-empty by (6.2). Hence, by Corollary 3.5, $\mathcal{B}$ is a bramble.

By definition, $|\mathcal{B}| = n^3$. Moreover, for every $p \in \mathrm{Tr}(G_n)$ and every $a \in \{1, 2, 3\}$ there exists at most one $i \in \{1, \ldots, n\}$ such that $p \in \mathrm{Tr}(L_i^a)$. Thus $\mathrm{cong}(\mathcal{B}) \leq 3n^2$. By Lemma 3.3,

$$\mathrm{overlap}(\mathrm{Tr}(G_n), \mathbb{R}^2) \geq \frac{\beta |\mathcal{B}|}{\mathrm{cong}(\mathcal{B})} = \frac{\beta n^3}{3n^2} = \frac{\beta n}{3}.$$

Since the maximum degree of G_n is 7, each vertex is contained in at most $\binom{7}{2} = 21$ triangles. Therefore, by Lemma 1.5,

$$\operatorname{sn}(G_n) \geq \left(\frac{\operatorname{overlap}(\operatorname{Tr}(G_n), \mathbb{R}^2)}{63}\right)^{1/3} \geq \left(\frac{\beta n}{189}\right)^{1/3}. \qquad \square$$

We now prove the lower bound for Δ_0 (defined at the start of the section).

Theorem 6.4 *Every graph class with maximum degree* 5 *and bounded queue-number has bounded stack-number.*

The proof of Theorem 6.4 depends on the following definitions. For $k, c \in \mathbb{N}$, a graph G is *k-colourable with clustering c* if each vertex of G can be assigned one of k colours such that each monochromatic component has at most c vertices. Here a *monochromatic component* is a maximal monochromatic connected subgraph. The *clustered chromatic number* of a graph class $\mathcal{G}$ is the minimum $k \in \mathbb{N}$ such that for some $c \in \mathbb{N}$ every graph in $\mathcal{G}$ is k-colourable with clustering c. See [61] for a survey on clustered graph colouring. Haxell et al. [40] proved that the class of graphs with maximum degree at most 5 has clustered chromatic number 2 (which is best possible, since $(P_n \boxslash P_n)_{n \in \mathbb{N}}$ has maximum degree 6 and clustered chromatic number 3 by the Hex Lemma). Thus, Theorem 6.4 is an immediate consequence of the following result.

Theorem 6.5 *Every graph class $\mathcal{G}$ with bounded queue-number and clustered chromatic number at most* 2 *has bounded stack-number.*

The proof of Theorem 6.5 depends on the following lemmas.

Lemma 6.6 [25, 53] *For every bipartite graph G,*

$$\operatorname{sn}(G) \leq 2 \operatorname{qn}(G).$$

Lemma 6.7 *For every graph G and $t \in \mathbb{N}$,*

$$\operatorname{sn}(G \boxtimes K_t) \leq 3t \ \operatorname{sn}(G) + \left\lceil \frac{t}{2} \right\rceil.$$

Proof We may assume that G is connected and $V(K_t) = \{1, \dots, t\}$. If $|V(G)| = 1$ then the claim holds since $\operatorname{sn}(G) = 0$ and $\operatorname{sn}(G \boxtimes K_t) = \operatorname{sn}(K_t) \leq \lceil t/2 \rceil$; see [10]. Now assume that $|V(G)| \geq 2$ and thus $E(G) \neq \emptyset$. Let $s = \operatorname{sn}(G) \geq 1$. Let $(v_1, \dots, v_n)$ together with $\psi : E(G) \to \{1, \dots, s\}$ be an s-stack layout of G. For each $k \in \{1, \dots, s\}$, let G_k be the spanning subgraph of G with $E(G_k) = \psi^{-1}(k)$. Thus G_k admits a 1-stack layout and is therefore outerplanar. Hakimi et al. [37] proved that every outerplanar graph has an edge-partition into three star forests. For each $k \in \{1, \dots, s\}$, let $G_{k,1}, G_{k,2}, G_{k,3}$ be an edge-partition of G_k into three star forests.

Consider the vertex-ordering

$$\big((v_1, 1), \dots, (v_1, t); (v_2, 1), \dots, (v_2, t); \dots; (v_n, 1), \dots, (v_n, t)\big).$$

We now define the stack assignment. An edge $(u, i)(v, j) \in E(G \boxtimes K_t)$ is an *intra-K_t edge* if $u = v$, otherwise $uv \in E(G)$ and it is a *cross-K_t edge*. We use separate stacks for the intra-K_t edges and the cross-K_t edges. Since the intra-K_t edges induce copies of K_t that are disjoint with respect to the vertex-ordering, we may partition them into $\mathrm{sn}(K_t) = \lceil t/2 \rceil$ stacks. Now for some $k \in \{1, \dots, s\}$ and $j \in \{1, 2, 3\}$, consider the spanning subgraph $\tilde{G}_{k,j}$ of $G_{k,j} \boxtimes K_t$ that contains only the cross-K_t edges. By the construction of $G_{k,t}$, edges from different components in $\tilde{G}_{k,j}$ do not cross with respect to the above vertex-ordering. Moreover, as each component of $\tilde{G}_{k,t}$ that contains an edge is obtained from a star by blowing-up each vertex with t non-adjacent vertices and replacing each edge with a copy of $K_{t,t}$, it is thus isomorphic to a bipartite graph of the form $K_{t,r}$ for some multiple r of t. As such, we may partition the edges of $\tilde{G}_{k,j}$ into t stacks. Therefore, we may partition the cross-K_t edges into $3st$ stacks, as required. □

Proof of Theorem 6.5 By assumption, there exist $c, \ell \in \mathbb{N}$ such that for every graph $G \in \mathcal{G}$, we have $\mathrm{qn}(G) \leq c$ and G is 2-colourable with each monochromatic component having at most ℓ vertices. Contracting each monochromatic component to a single vertex gives a bipartite ℓ-small minor H of G. Lemma 2.4 implies $\mathrm{qn}(H) \leq 2\ell(2c)^{2\ell}$. Lemma 6.6 implies $\mathrm{sn}(H) \leq 4\ell(2c)^{2\ell}$. By construction, G is isomorphic to a subgraph of $H \boxtimes K_\ell$. Thus $\mathrm{sn}(G) \leq \mathrm{sn}(H \boxtimes K_\ell)$, which is at most $3\ell\,\mathrm{sn}(H) + \lceil \ell/2 \rceil \leq 12\ell^2(2c)^{2\ell} + \lceil \ell/2 \rceil$ by Lemma 6.7. Hence $\mathcal{G}$ has bounded stack-number. □

7 Open Problems

We finish with some open problems:

- Does there exist a graph class with bounded stack-number and unbounded queue-number? This is equivalent to the question of whether graphs with stack-number 3 have bounded queue-number [28].
- Do graphs with queue-number 2 (or 3) have bounded stack-number [22]?
- Do graph classes with quadratic (or linear) growth have bounded stack-number?
- Does there exist a graph family with unbounded stack-number, bounded queue-number and maximum degree 6?
- The best known lower bound on the maximum stack-number of n-vertex graphs with fixed maximum degree Δ is $\Omega(n^{1/2-1/\Delta})$, proved by Malitz [48] using a probabilistic argument. Is there a constructive proof of this bound?
- The best upper bound on the stack-number of n-vertex graphs with fixed maximum degree Δ is $O(n^{1/2})$, also due to Malitz [48]. Closing the gap between the lower and upper bounds is an interesting open problem. For example, the best bounds for graphs of maximum degree 3 are $\Omega(n^{1/6})$ and $O(n^{1/2})$.

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