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A Family of Convex Sets in the Plane Satisfying the (4, 3)-Property can be Pierced by Nine Points

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Abstract
We prove that every finite family of convex sets in the plane satisfying the (4, 3)-property can be pierced by nine points. This improves the bound of 13 proved by Kleitman et al. (Combinatorica **21**(2), 221–232 (2001)).

Keywords Piercing · Helly · Hadwiger–Debrunner

Mathematics Subject Classification 52A35

1 Introduction

For positive integers $p \geq q$, a family of sets $\mathscr{C}$ is said to satisfy the (p, q)-*property* if for every p sets, some q have a point in common. We say that $\mathscr{C}$ *can be pierced by* m *points* if there exists a set of size at most m intersecting every element in $\mathscr{C}$. The piercing number $\tau(\mathscr{C})$ of $\mathscr{C}$ is the minimum m so that $\mathscr{C}$ can be pierced by m points.

In 1957 Hadwiger and Debrunner [2] conjectured that for every given positive integers $p \geq q > d$, there exists a (smallest) constant $\mathrm{HD}_d(p, q)$ such that every finite family $\mathscr{C}$ of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property has $\tau(\mathscr{C}) \leq \mathrm{HD}_d(p, q)$. This conjecture was proved by Alon and Kleitman in 1992 [1].

In general, the bounds on $\mathrm{HD}_d(p, q)$ given by Alon and Kleitman's proof are far from optimal. The first case where $\mathrm{HD}_d(p, q)$ is not known is when $d = 2$, $p = 4$, and $q = 3$. In this case, the bound in $\mathrm{HD}_d(p, q)$ given by the Alon–Kleitman proof is 343, while there is no known example of a family of convex sets in the plane that satisfy

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the (4, 3)-property and cannot be pierced by three points. We note that improvements on general upper bounds for $\mathrm{HD}_d(p,q)$ were made in [4].

In 2001, Gyárfás et al. [5] proved that $\mathrm{HD}_2(4,3) \le 13$, and since then this bound has seen no improvement. In this paper we prove that $\mathrm{HD}_2(4,3) \le 9$:

Theorem 1.1 *If $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^2$ such that for any four sets, three have a point in common, then $\tau(\mathscr{C}) \le 9$.*

The main tools in the proof are the following two theorems, and a geometrical analysis. Let $\Delta^{n-1} \subset \mathbb{R}^n$ denote the $n-1$-dimensional simplex with vertex set $e_1, \dots, e_n$ (the standard basis vectors in $\mathbb{R}^n$). The following version of the KKM Theorem was proven in [7].

Theorem 1.2 *Let $A_1, \dots, A_n$ be open sets such that for every $I \subseteq \{1, \dots, n\}$, $\bigcup_I A_i \supseteq \operatorname{conv}\{e_i \mid i \in I\}$. Then $\bigcap_{i=1}^n A_i \neq \emptyset$.*

We note that Theorem 1.2 stated for closed sets $A_1, \dots, A_n$ is the original KKM Theorem, which was proven in [6].

A matching in a family of sets $\mathscr{F}$ is a subset of pairwise disjoint sets in $\mathscr{F}$. The matching number $\nu(\mathscr{F})$ is the maximum size of a matching in $\mathscr{F}$. Let L_1, L_2 be two homeomorphic copies of the real line. A 2-*interval* is a union $I_1 \cup I_2$, where I_i is an interval on L_i.

Theorem 1.3 (Tardos [8]) *If $\mathscr{F}$ is a family of 2-intervals then $\tau(\mathscr{F}) \le 2\nu(\mathscr{F})$.*

2 Using the KKM Theorem

Let $\mathscr{C}$ be a finite family of convex sets satisfying the (4, 3)-property. We may assume that the sets are compact by considering a set S containing a point in each intersection of sets in $\mathscr{C}$, and replacing every set $C \in \mathscr{C}$ by $\operatorname{conv}\{s \in S \mid s \in C\}$. Furthermore, we may assume that each set in $\mathscr{C}$ has a non-empty interior. To see this, let B_ϵ be the closed ball of radius ϵ with the center at the origin, and let $\mathscr{C}_\epsilon = \{C + B_\epsilon \mid C \in \mathscr{C}\}$. Then $\mathscr{C}_\epsilon$ also satisfies the (4, 3)-property. It follows from the compactness of the sets in $\mathscr{C}$ that if $\mathscr{C}_\epsilon$ can be pierced by nine points for all $\epsilon > 0$, then $\mathscr{C}$ can be pierced by nine points. Therefore, we may assume each set in $\mathscr{C}$ has a non-empty interior. In particular, $\mathscr{C}$ contains neither points nor line segments.

We may clearly assume $|\mathscr{C}| \ge 4$. We scale the plane so that all the sets in $\mathscr{C}$ are contained in the open unit disk, which we denote by D. Let f be a parameterization of the unit circle defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

for $t \in [0,1]$. For two points a, b in the plane, let $\overline{ab}$ be the line through a and b and let $[a,b]$ be the line segment with a and b as endpoints.

Let $\Delta = \Delta^3 = \operatorname{conv}\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^4$ be the standard 3-dimensional simplex, and let $x = (x_1, x_2, x_3, x_4) \in \Delta$. Note that $x_i \in [0,1]$ and $\sum_{i=1}^4 x_i = 1$. For $1 \le i \le 4$, define R_x^i to be the interior of the region bounded by the arc along the

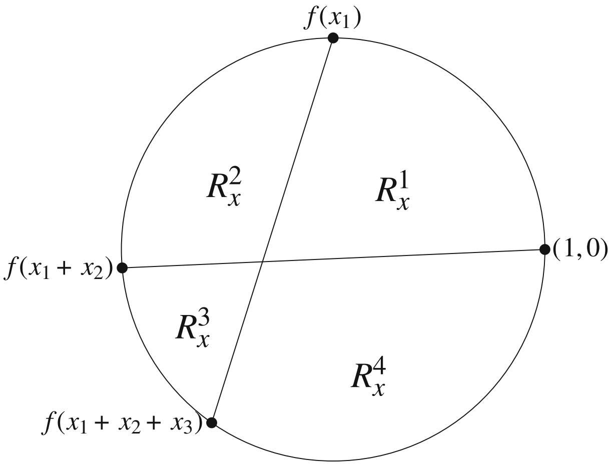

Fig. 1 A point $x \in \Delta^3$ corresponds to four regions R_x^i

circle from $f\bigl(\sum_{j=1}^{i-1} x_j\bigr)$ to $f\bigl(\sum_{j=1}^{i} x_j\bigr)$ (an empty sum is understood to be 0) and by the line segments $[(1,0), f(x_1+x_2)]$ and $[f(x_1), f(x_1+x_2+x_3)]$ (see Fig. 1). Notice that if $x_i = 0$, then $R_x^i = \emptyset$.

For every $1 \le i \le 4$ define a subset A_i of Δ as follows: $x \in \Delta^3$ is in A_i if and only if there exist three sets $C_1, C_2, C_3 \in \mathscr{C}$ such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R_x^i$ for all $1 \le j < k \le 3$ (see Fig. 2). Observe that A_i is open. For every $x \in \Delta$ and $C \in \mathscr{C}$ let I_C be the (possibly empty) 2-interval

$$(C \cap [(1,0), f(x_1+x_2)]) \cup (C \cap [f(x_1), f(x_1+x_2+x_3)]).$$

Lemma 2.1 *Suppose there exists $x \in \Delta \setminus \bigcup_{i=1}^4 A_i$. Then there exist two points a, b such that if $a, b \notin C$ then $I_C \neq \emptyset$.*

Proof Assume that $x \in \Delta \setminus \bigcup_{i=1}^4 A_i$. Note that since $\mathscr{C}$ does not contain three pairwise non-intersecting sets, at most two of the regions R_x^i can contain a set in $\mathscr{C}$.

We claim for every $i \le 4$, the region R_x^i contains at most two sets in $\mathscr{C}$. Indeed, assume to the contrary that R_x^i contains three sets $C_1, C_2, C_3 \in \mathscr{C}$. Then $C_1 \cap C_2 \cap C_3 = \emptyset$ since $x \notin A_i$. Applying the $(4,3)$ property to C_1, C_2, C_3 and some additional set $F \in \mathscr{C}$, we obtain that $C_j \cap C_k \cap F \neq \emptyset$ for some $1 \le j < k \le 3$, and all pairwise intersections of C_j, C_k, F are contained in R_x^i, contradicting $x \notin A_i$.

If there is only one region R_x^i containing sets in $\mathscr{C}$, then since there are at most two such sets, there are two points that pierce them. If there are two regions R_x^i and R_x^j containing sets in $\mathscr{C}$, then if there are two sets contained in R_x^i (or R_x^j), they must intersect. Otherwise these two sets together with a set in R_x^j (or R_x^i, respectively) will be three pairwise non-intersecting sets, a contradiction since $\mathscr{C}$ has the $(4,3)$-property. Therefore, there is a point piercing the sets contained in R_x^i and a point piercing the sets in R_x^j and we are done. □

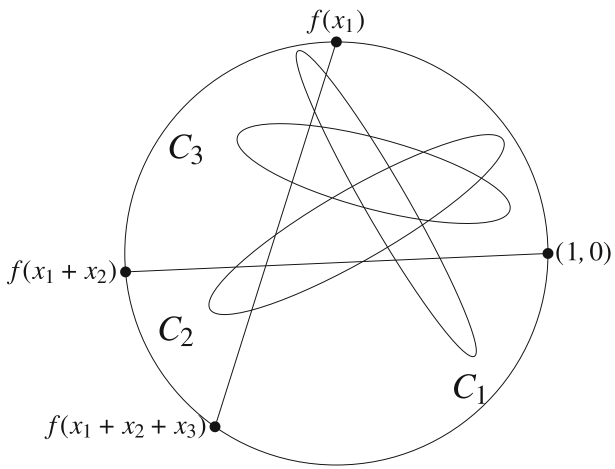

Fig. 2 Three sets $C_1, C_2, C_3 \in C$ with $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R_x^1$ for all $1 \leq j < k \leq 3$, implying $x = (x_1, x_2, x_3, x_4) \in A_1$

Theorem 2.2 *If there exists $x \in \Delta \setminus \bigcup_{i=1}^4 A_i$, then $\tau(\mathscr{C}) \leq 8$.*

Proof Let $\mathscr{D} = \{C \in \mathscr{C} \mid I_C \neq \emptyset\}$. We will show that $\tau(\mathscr{D}) \leq 6$. Together with Lemma 2.1 this will imply the theorem.

Let $\mathscr{I} = \{I_C \mid C \in \mathscr{D}\}$. Let $C_1, C_2, C_3, C_4 \in \mathscr{D}$ be four sets. Some three, say C_1, C_2, C_3, intersect by the (4, 3)-property. Since $x \notin \bigcup_{i=1}^4 A_i$, the intersection of two of these three sets, say $C_1 \cap C_2$, must intersect either $[(1, 0), f(x_1 + x_2)]$ or $[f(x_1), f(x_1 + x_2 + x_3)]$. In other words, $I_{C_1} \cap I_{C_2} \neq \emptyset$. This shows that $\mathscr{I}$ has no four pairwise disjoint elements, implying $\nu(\mathscr{I}) \leq 3$. Thus, by Theorem 1.3, $\tau(\mathscr{D}) \leq \tau(\mathscr{I}) \leq 6$. □

By Theorem 2.2 we may assume that $\Delta \subset \bigcup_{i=1}^4 A_i$. We claim that in this case the sets $A_1, \ldots, A_4$ satisfy the conditions of Theorem 1.2. Indeed, let $I \subset [4]$, and let $y \in \operatorname{conv}\{e_i \mid i \in I\}$. Then for all $j \in [4] \setminus I$, we have $R_y^j = \emptyset$, implying $y \notin A_j$. Since $y \in \bigcup_{i=1}^4 A_i$, we have that $y \in \bigcup_{i \in I} A_i$. Thus, by Theorem 1.2 we have:

Theorem 2.3 *If $\Delta \subset \bigcup_{i=1}^4 A_i$, then there exists $x \in \bigcap_{i=1}^4 A_i$.*

For the rest of the paper we fix $x \in \bigcap_{i=1}^4 A_i$. Let $R_x^i = R^i$, and let $f_1 = (1, 0)$, $f_2 = f(x_1)$, $f_3 = f(x_1 + x_2)$, and $f_4 = f(x_1 + x_2 + x_3)$. Let c be the intersection point of $[f_1, f_2]$ and $[f_2, f_4]$, and let $\mathscr{C}^* = \{C \in \mathscr{C} \mid c \notin C\}$. Note that $\bigcap_{i=1}^4 A_i$ is an open set, so we may shift x slightly to ensure that c does not lie on the boundary of any set in $\mathscr{C}$ and neither of the segments $[f_1, f_3]$ or $[f_2, f_4]$ meets the boundary of any set in $\mathscr{C}$ and contains the set in one of its closed halfspaces. We use $\overline{R^i}$ to denote the topological closure of R^i.

Proposition 2.4 *If $C \in \mathscr{C}^*$, then there exists some i for which $C \cap R^i = \emptyset$.*

Proof Assume C has a point p_i in each R^i. Then since C is convex, it contains the points $q_1 = [p_1, p_2] \cap [f_2, f_4]$ and $q_2 = [p_3, p_4] \cap [f_2, f_4]$. Since q_1 and q_2 lie in two different hyperplanes defined by the line $\overline{f_1 f_3}$, C must contain c, a contradiction. □

Let $\mathscr{C}_i$ denote the family of sets in $\mathscr{C}^*$ that are disjoint from R^i. By Proposition 2.4, we have $\mathscr{C}^* = \bigcup_{i=1}^{4} \mathscr{C}_i$. In the remainder of the paper we prove the following:

Theorem 2.5 *For every $i \leq 4$, $\tau(\mathscr{C}_i) \leq 2$.*

This will imply that $\mathscr{C}$ can be pierced by nine points: two points for each $\mathscr{C}_i$ and the point c.

3 Piercing $\mathscr{C}_i$ by Two Points

In this section we prove Theorem 2.5. Without loss of generality we prove the theorem for $\mathscr{C}_1$.

3.1 Preliminary Definitions and Observations

Let $C_1, C_2, C_3 \in \mathscr{C}$ be the three sets witnessing the fact that $x \in A_1$; so $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and $C_j \cap C_k \subset R^1$ for all $1 \leq j < k \leq 3$.

If there are two sets $F_1, F_2 \in \mathscr{C}_1$ that do not intersect, then F_1, F_2, C_1, C_2 do not satisfy the (4, 3)-property. Thus every two sets in $\mathscr{C}_1$ intersect. Also, if for some $1 \leq i \leq 3$ we have $C_i \subset R^1$, then again by the (4, 3)-property every three sets in $\mathscr{C}_1$ have a common point. This implies by Helly's theorem [3] that $\tau(\mathscr{C}_1) = 1$. So we may assume that no C_i is contained in R^1.

Let L_1 be the line $\overline{f_1 f_3}$ and let L_2 be the line $\overline{f_2 f_4}$ (see Fig. 3).

By our assumption C_i is not contained in R^1 for $1 \leq i \leq 3$, and thus $C_i \setminus R^1$ has at least one non-empty connected component. The next proposition shows that $C_i \setminus R^1$ has at most two connected components.

Proposition 3.1 *For every $1 \leq i \leq 3$, the set $C_i \setminus R^1$ has at most two connected components. Moreover, if $C_i \setminus R^1$ has two components, then the components are $C_i \cap \overline{R^2}$ and $C_i \cap \overline{R^4}$ and hence are convex.*

Proof If C_i contains c, then $C_i \setminus R^1$ has one component because the line segment from any point in $\mathbb{R}^2 \setminus R^1$ to c is contained in $\mathbb{R}^2 \setminus R^1$. So assume C_i does not contain c. Then it must have a point in either $\overline{R^2}$ or $\overline{R^4}$, without loss of generality, in $\overline{R^2}$.

Suppose C_i contains a point in $\overline{R^3}$. Since C_i does not contain c but contains points in the three regions $\overline{R^1}$, $\overline{R^2}$, $\overline{R^3}$, then by Proposition 2.4 it cannot contain a point in $\overline{R^4}$. Thus $C_i \setminus R^1 = C_i \cap (\overline{R^2} \cup \overline{R^3})$. This means that $C_i \setminus R^1$ is an intersection of two convex sets, hence it is convex and has only one component.

Thus, if $C_i \setminus R^1$ has more than one component, then C_i does not have a point in $\overline{R^3}$. In this case the components of $C_i \setminus R^1$ are $C_i \cap \overline{R^2}$ and $C_i \cap \overline{R^4}$ both of which are convex. □

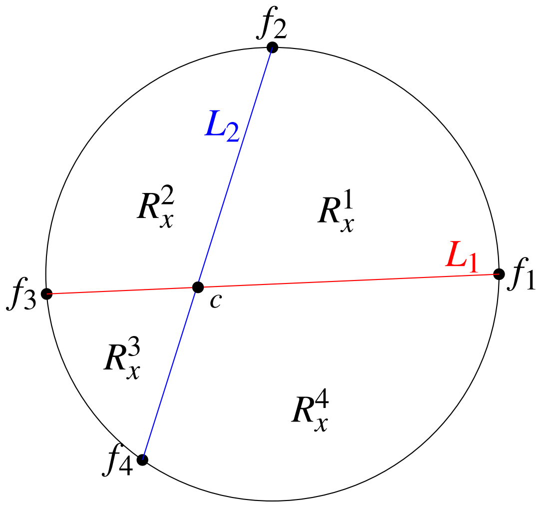

Fig. 3 The lines L_1 (in red) and L_2 (in blue)

Let $Z = [f_1, c] \cup [c, f_2]$. We think of Z as starting at f_1 and ending at f_2. Thus a point $a \in Z$ comes before a point $b \in Z$ if the distance along Z from a to f_1 on Z is smaller than the distance from b to f_1 on Z.

Let $I_i^1 = C_i \cap [f_1, c]$, $I_i^2 = C_i \cap [c, f_2]$, and $I_i = C_i \cap Z$. Because each C_i has a non-empty interior and our choice of c, none of I_i^1, I_i^2, or I_i consists of a single point, or has c as one of its endpoints. It is possible, however, that one of I_i^1 or I_i^2 are empty.

For any interval (i.e., connected set) I on Z, let $r(I)$ be the endpoint of I that comes first on Z, and let $\ell(I)$ be the other endpoint. Given a convex set C and a point p on the boundary of C, a *supporting line for* C at p is a line L passing through p that contains C in one of the closed halfspaces defined by L. For $1 \le i \le 3$, let $C_i' = C_i \setminus R^1$.

Definition 3.2 Let X be a connected component of C_i', and let $I = X \cap Z$ (so I is an interval on Z). Define $S_i^r(I)$ and $S_i^\ell(I)$ to be some supporting line for C_i at the point $r(I)$ and $\ell(I)$, respectively (see Fig. 4).

Because we chose $x \in \bigcap_i A_i$ so that neither L_1 nor L_2 meet the boundary of any set in $\mathscr{C}$ and contains the set in one its halfspaces, $S_i^r(I)$ and $S_i^\ell(I)$ are not equal to L_1 or L_2 for all i.

Definition 3.3 Assume C_i' has two components $X_1 = C_i' \cap \overline{R^4}$ and $X_2 = C_i' \cap \overline{R^2}$. We define S_i' to be a piece-wise linear curve as follows (see Fig. 5):

$$S_i' = (S_i^\ell(I_i^1) \cap \overline{R^4}) \cup [\ell(I_i^1), r(I_i^2)] \cup (S_i^r(I_i^2) \cap \overline{R^2}).$$

Note that S_i' lies in the closed halfspace defined by the line between the points $r(I_i^1)$ and $\ell(I_i^2)$ containing f_1 and f_2.

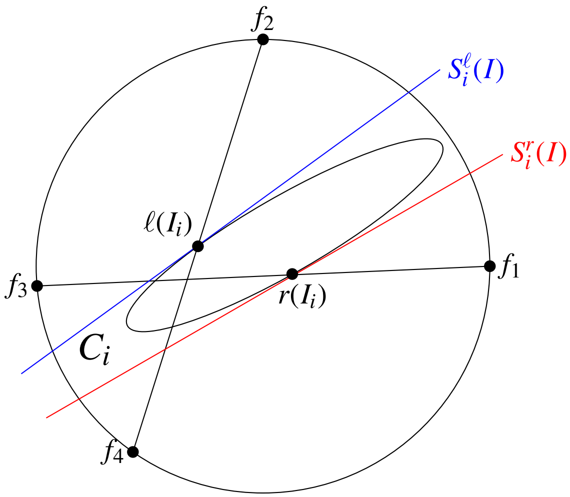

Fig. 4 Definition 3.2: the lines $S_i^\ell(I_i)$ (in blue) and $S_i^r(I_i)$ (in red)

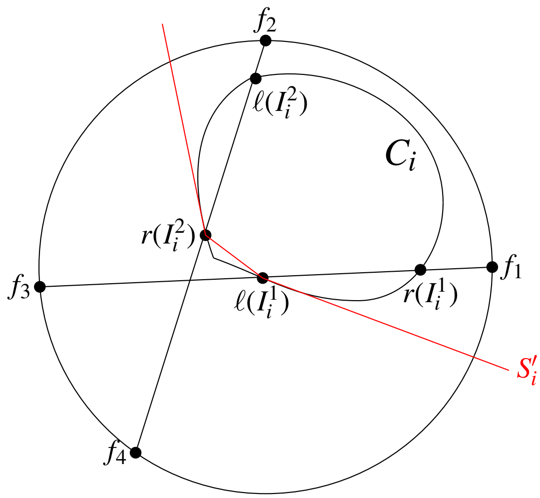

Fig. 5 Definition 3.3: The piece-wise linear curve S_i' (in red)

3.2 Five Lemmas

For two intervals I and J of Z, we say that I *comes before* J *on* Z if the point $\ell(I)$ comes before $r(J)$ on Z. Recall that D is the open unit disc.

Lemma 3.4 *Let* $1 \leq i \neq j \leq 3$. *Let* X *be a component of* C_i' *and* $I = X \cap Z$. *Let* Y *be a component of* C_j' *and* $J = Y \cap Z$. *Suppose that* $F \subset D$ *such that* $F \cap R^1 = \emptyset$, $F \cap X \neq \emptyset$ *and* $F \cap Y \neq \emptyset$. *If* I *comes before* J *on* Z, *then* $F \cap S_i^\ell(I) \neq \emptyset$ *and* $F \cap S_j^r(J) \neq \emptyset$ *(see Fig. 6).*

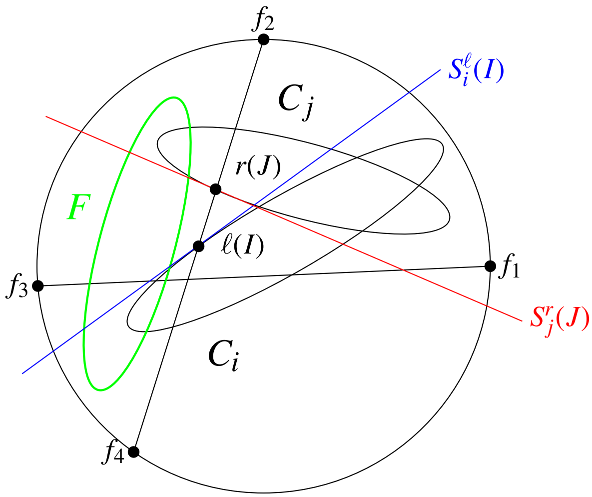

Fig. 6 Lemma 3.4

Proof We will show that $F \cap S_i^\ell(I) \neq \emptyset$. The fact that $F \cap S_j^r(J) \neq \emptyset$ will follow similarly.

Let $p \in C_i \cap C_j$. Let $a \in F \cap X$ and $b \in F \cap Y$. Note that the segment $[a, p]$ lies in C_i and hence the point of intersection, i_a, of $[a, p]$ with Z does not come after $\ell(I)$ on Z. Similarly, the point of intersection, i_b, of $[b, p]$ with Z does not come before $\ell(J)$ on Z. Since $[a, b] \subset F \subset D \setminus R^1$, the triangle with the vertices a, b, and p (a, b, and p cannot lie on a line because $C_i \cap C_j \subset R^1$) contains the interval $[i_a, i_b]$ on Z. Since $S_i^\ell(I)$ passes through $\ell(I)$, either $S_i^\ell(I_i)$ intersects the interior of the triangle or it contains the segment $[a, p]$. If $S_i^\ell(I)$ intersects the interior of the triangle, then it cannot intersect $[a, p]$ since $S_i^\ell(I)$ is a supporting line for C_i. Therefore, $S_i^\ell(I)$ must intersect $[a, b]$ which is contained in F. If $S_i^\ell(I)$ contains the segment $[a, p]$, then $S_i^\ell(I)$ contains a, which is in F. This concludes the proof. □

Similar arguments can be applied to prove the following lemma.

Lemma 3.5 *Let $1 \leq i \neq j \leq 3$. Assume that C_i' has two components $X_1 = C_i' \cap \overline{R^4}$ and $X_2 = C_i' \cap \overline{R^2}$. Let Y be a component of C_j' and $J = Y \cap Z$. Suppose $F \subset D$ is a convex set such that $F \cap R^1 = \emptyset$ and $F \cap Y \neq \emptyset$. If $F \cap X_1 \neq \emptyset$ and J comes after I_i^1 on Z, or if $F \cap X_2 \neq \emptyset$ and I_i^2 comes after J on Z, then $F \cap S_i' \neq \emptyset$ (see Fig. 7).*

Proof Assume that F intersects X_1 and J comes after I_i^1 on Z. By Lemma 3.4, F intersects $S_i^\ell(I_i^1)$. However, if F intersects $S_i^\ell(I_i^1) \cap \overline{R^2}$, then by convexity, F intersects R^1, a contradiction. Therefore, F intersects $S_i^\ell(I_i^1) \cap \overline{R^4} \subset S_i'$. Similarly, if F intersects X_2 and I_i^2 comes after J on Z, then F intersects $S_i^r(I_i^2) \cap \overline{R^2} \subset S_i'$. □

Let C_i' have two components. We say that a set $F \subset D$ *lies above* $S_i^\ell(I_i^1)$ in $\overline{R^2}$ if F is contained in the open halfspace defined by $S_i^\ell(I_i^1)$ containing C_i and $F \subset \overline{R^2}$ (see Fig. 8).

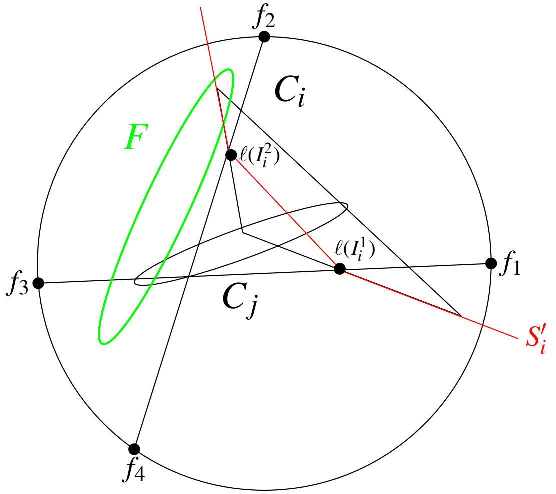

Fig. 7 Lemma 3.5

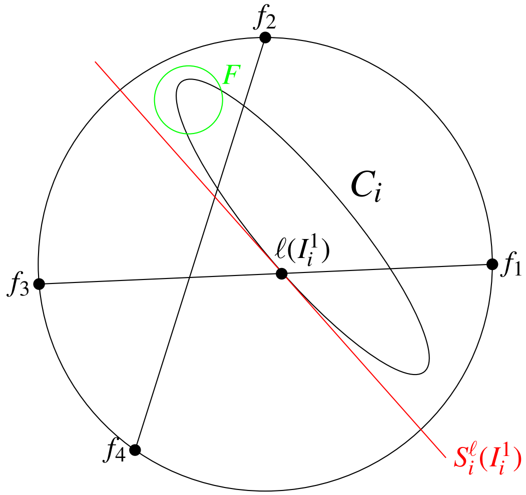

Fig. 8 The set F (in green) lies above $S_i^\ell(I_i^1)$ (in red) in $\overline{R^2}$

Similarly, we say that a set $F \subset D$ *lies above* $S_i^r(I_i^2)$ in $\overline{R^4}$ if F is contained in the open halfspace defined by $S_i^r(I_i^2)$ containing C_j and $F \subset \overline{R^4}$ (see Fig. 10).

Lemma 3.6 *Assume that for $1 \leq i \neq j \leq 3$, C_i' and C_j' both have two components: $X_1 = C_i' \cap \overline{R^4}$, $X_2 = C_i' \cap \overline{R^2}$, $Y_1 = C_j' \cap \overline{R^4}$, and $Y_2 = C_j' \cap \overline{R^2}$. Assume that I_i^1 comes before I_j^1 on Z and I_i^2 comes before I_j^2 on Z. Then one of the following statements hold (see Fig. 9):*

– *For every two distinct sets $F_1, F_2 \in \mathscr{C}_1$, if $F_1 \cap F_2$ intersects C_i' and C_j' or $F_1 \cap F_2$ intersects Y_2, then $(F_1 \cap F_2) \cap S_i^\ell(I_i^1) \neq \emptyset$.*

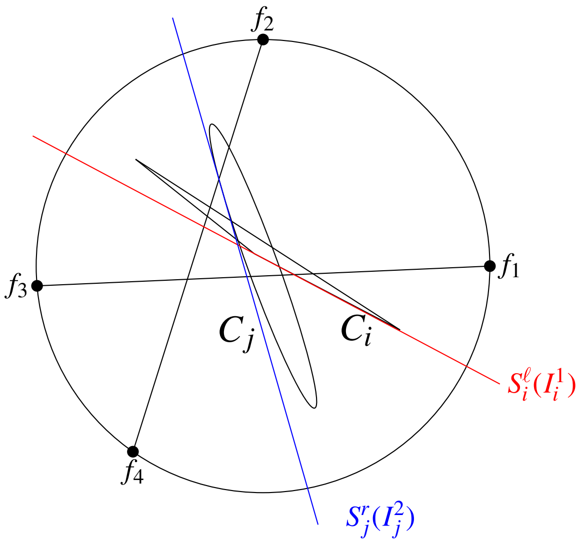

Fig. 9 Lemma 3.6. The relevant lines for the first and second statement in Lemma 3.6 (in red and blue respectively)

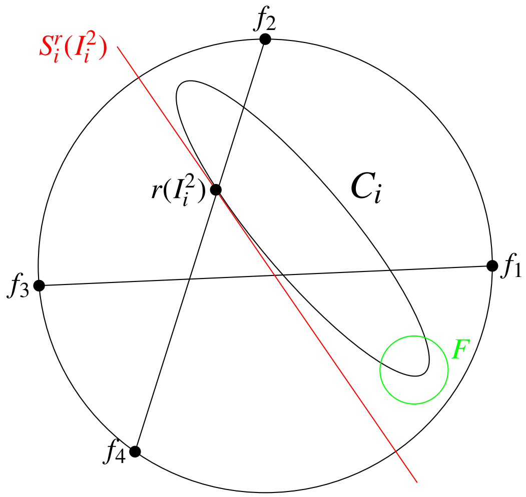

Fig. 10 The set F (in green) lies above $S_i^r(I_i^2)$ (in red) in $\overline{R^4}$

– *For every two distinct sets* $F_1, F_2 \in \mathscr{C}_1$*, if* $F_1 \cap F_2$ *intersects* C_i' *and* C_j' *or* $F_1 \cap F_2$ *intersects* X_1*, then* $(F_1 \cap F_2) \cap S_j^r(I_j^2) \neq \emptyset$*.*

Proof First note that X_1 lies above $S_j^r(I_j^2)$ in $\overline{R^4}$ and Y_2 lies above $S_i^\ell(I_i^1)$ in $\overline{R^2}$. For instance, if X_1 contains a point p in the closed halfspace defined by $S_j^r(I_j^2)$ not containing C_j, then the segment $[p, r(I_i^2)]$ intersects $[c, f_1]$ at a point that comes after

$r(I_j^1)$ on Z, contradicting the fact that I_i^1 comes before I_j^1 on Z. A similar argument applies to the corresponding statement for Y_2.

We will show that either there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects Y_2 and lies above $S_i^\ell(I_i^1)$ in $\overline{R^2}$, or there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects X_1 and lies above $S_j^r(I_j^2)$ in $\overline{R^4}$. This will imply the lemma as shown in the last two paragraphs of this proof.

Assume for contradiction that there exist two distinct sets $F_1, F_2 \in \mathscr{C}_1$ such that $F_1 \cap F_2$ lies above $S_j^r(I_j^2)$ in $\overline{R^4}$ and $X_1 \cap (F_1 \cap F_2) \neq \emptyset$, and two distinct sets $F_3, F_4 \in \mathscr{C}_1$ such that $F_3 \cap F_4$ lies above $S_i^\ell(I_i^1)$ in $\overline{R^2}$ and $Y_2 \cap (F_3 \cap F_4) \neq \emptyset$.

Note that neither F_3 nor F_4 can be equal to F_1 or F_2. For instance, if $F_3 = F_1$, then F_1 contains a point p above $S_j^r(I_j^2)$ in $\overline{R^4}$, and a point q in Y_2. Since $q \in R^2$ and lies in the same halfspace defined by $S_j^r(I_j^2)$ as p, the segment $[p, q]$ intersects R^1. However, this is a contradiction to the fact that $F_1 \in \mathscr{C}_1$. Therefore, the sets F_1, F_2, F_3, F_4 are pairwise distinct.

By the $(4, 3)$-property, three sets out of F_1, F_2, F_3, F_4 have a common point. If F_1, F_2, F_3 intersect, then F_3 intersects Y_2 and has a point above $S_j^r(I_j^2)$ in $\overline{R^4}$, which implies that F_3 has a point in R^1, a contradiction. Similarly, F_1, F_2, F_4 cannot intersect. An analogous argument show that neither F_1, F_3, F_4 nor F_2, F_3, F_4 can intersect. Therefore, there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects Y_2 and lies above $S_i^\ell(I_i^1)$ in $\overline{R^2}$, or there is no pair of sets in $\mathscr{C}_1$ whose intersection intersects X_1 and lies above $S_j^r(I_j^2)$ in $\overline{R^4}$.

Assume that there is no pair of sets in $\mathscr{C}_1$ whose intersection lies above $S_i^\ell(I_i^1)$ in $\overline{R^2}$ and intersects Y_2, and take $L = S_i^\ell(I_i^1)$. Let F be the intersection of any pair of sets in $\mathscr{C}_1$. If F intersects Y_2, then by the above F does not lie above L in $\overline{R^2}$. This implies that F intersects L since Y_2 lies above L in $\overline{R^2}$. If F intersects Y_1 and X_1, then F intersects L by Lemma 3.4. If F intersects Y_1 and X_2, then F intersects L since Y_1 lies in the halfspace defined by L that does not contain C_i.

If there is no pair of sets in $\mathscr{C}_1$ whose intersection lies above $S_j^r(I_j^2)$ in $\overline{R^4}$ and intersects X_1, then a similar argument shows that the corresponding statements follow for $S_j^r(I_j^2)$. □

Lemma 3.7 *If $F \in \mathscr{C}_1$ and C_i' has two components, then $F \cap S_i'$ is an interval.*

Proof Clearly, $F \cap (S_i' \cap \overline{R^2})$, $F \cap (S_i' \cap \overline{R^4})$ are intervals, and $F \cap S_i' = (S_i' \cap \overline{R^2}) \cup (S_i' \cap \overline{R^4})$, so it suffices to show that F cannot intersect both $S_i' \cap \overline{R^2}$ and $S_i' \cap \overline{R^4}$. Suppose it does. Let T be the line passing through $\ell(I_i^1)$ and $r(I_i^2)$. By the definition of S_i', both $S_i' \cap \overline{R^2}$ and $S_i' \cap \overline{R^4}$ lie on the closed halfspace defined by T containing f_1 and f_2. Since F is convex, this implies that F has a point in R^1, a contradiction. □

Lemma 3.8 *Let $F_1, F_2 \in \mathscr{C}_1$, then $F_1 \cap F_2$ intersects at least two of C_1, C_2, C_3.*

Proof Suppose $F_1 \cap F_2$ does not intersect C_1. Since $C_1 \cap C_2 \subset R^1$ and $F \cap R^1 = \emptyset$, by the $(4, 3)$-property for the sets C_1, C_2, F_1, F_2, we have that C_2 must intersect $F_1 \cap F_2$. Similarly, C_3 intersects $F_1 \cap F_2$. □

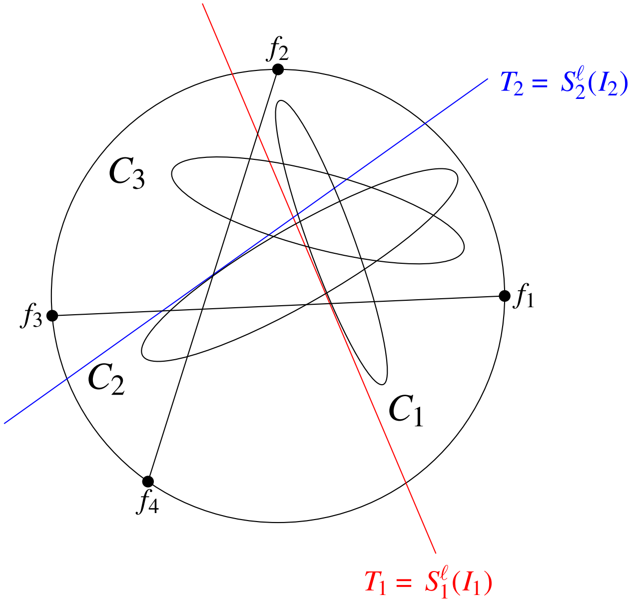

Fig. 11 Case 1: each C_i' has one component

3.3 Proof of Theorem 2.5

We wish to show that $\tau(\mathscr{C}_1) \le 2$. We split into four cases. In each case and subcase, we find two homeomorphic copies of the real line T_1 and T_2, and show that the family of 2-intervals $\mathscr{I} = \{(F \cap T_1) \cup (F \cap T_2) \mid F \in \mathscr{C}_1\}$ satisfies $\nu(\mathscr{I}) = 1$. By Theorem 1.3, this implies $\tau(\mathscr{I}) \le 2$. The curves T_1, T_2 will be of the form $S_i^\ell(I)$ or $S_i^r(I)$ for some interval I on Z, or S_i', and Lemma 3.7 ensures that $\mathscr{I}$ is indeed a family of 2-intervals. Recall that $I_i^1 = C_i \cap [f_1, c]$, $I_i^2 = C_i \cap [c, f_2]$, and $I_i = C_i \cap Z$.

Also, if C_i' and C_j' have two components, $i \ne j$, I_i^1 comes before I_j^1 on Z, and I_i^2 comes before I_j^2 on Z, then one of the two statements in Lemma 3.6 holds. If the first statement holds, *the curve obtained by applying Lemma* 3.6 *to* C_i *and* C_j will be understood to be $S_i^\ell(I_i^1)$. Otherwise, the curve obtained by applying Lemma 3.6 to C_i and C_j will be understood to be $S_j^r(I_j^2)$.

Throughout, $F_1, F_2 \in \mathscr{C}_1$ are two arbitrary, distinct sets, and $F = F_1 \cap F_2$. Because of Lemma 3.8, we assume that F intersects two of the C_i's throughout. Recall in order to show that $\nu(\mathscr{I}) = 1$, we must show that F intersects $T_1 \cup T_2$ or, in other words, that F intersects T_1 or T_2.

Case 1: C_i' has one component for each i (see Fig. 11). Notice in this case each I_i is an interval on Z. Assume without loss of generality that I_1 comes before I_2 and I_2 comes before I_3 on Z. Set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_2^\ell(I_2)$. By Lemma 3.4, F intersects T_1 or T_2 (recall we assume that F intersects two of the C_i's). It follows that our collection of 2-intervals, $\mathscr{I}$, has matching number 1.

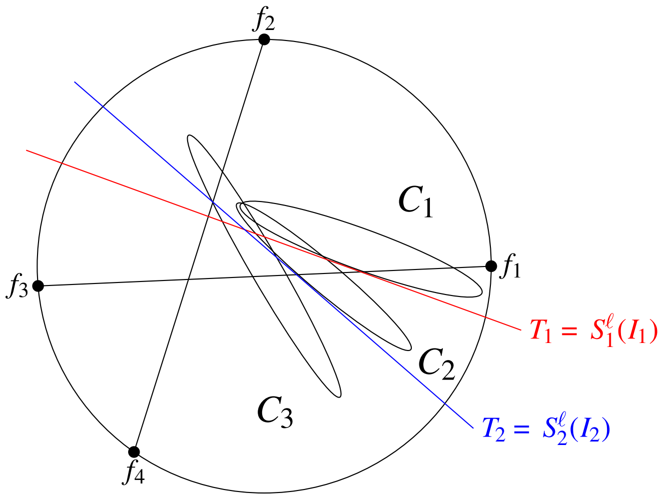

Fig. 12 Subcase 2.1

Case 2: One of the C_i''s has two components. We can assume without loss of generality that C_3' has two components, and that I_1 comes before I_2 on Z.

Subcase 2.1. If the order of the intervals on Z is I_1, I_2, I_3^1, I_3^2, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_2^\ell(I_2)$ (see Fig. 12). Similarly to Case 1, it follows from Lemma 3.4 that $\mathscr{I}$ has matching number 1.

Subcase 2.2. If the order of the intervals is I_1, I_3^1, I_2, I_3^2, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_3'$ (see Fig. 13). If F intersects C_1 and C_2, then F intersects T_1 by Lemma 3.4. If F intersects C_1 and C_3, then F intersects T_1 by Lemma 3.4. If F intersect C_2 and C_3, then F intersects T_2 by Lemma 3.5.

Subcase 2.3. If the order of the intervals is I_1, I_3^1, I_3^2, I_2, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_2^r(I_2)$ (see Fig. 14). Similarly to Case 1, it follows from Lemma 3.4 that $\mathscr{I}$ has matching number 1.

Subcase 2.4. If the order of the intervals is I_3^1, I_1, I_2, I_3^2, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_3'$ (see Fig. 15). If F intersects C_1 and C_2, then F intersects T_1 by Lemma 3.4. If F intersect C_3 and C_1 or C_2, then F intersects T_2 by Lemma 3.5. Therefore, $\mathscr{I}$ has matching number 1.

The remaining subcases of Case 2 are symmetrical. For instance, the case where the order of the intervals is I_3^1, I_3^2, I_1, I_2 follows similarly to the case where the order of the intervals is I_1, I_2, I_3^1, I_3^2.

Case 3: Two of the C_i''s have two components. Without loss of generality, assume C_2' and C_3' have two components.

Subcase 3.1. Assume the order of the intervals is $I_1, I_2^1, I_3^1, I_3^2, I_2^2$, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_2'$ (see Fig. 16). If F intersects C_1 and one of C_2 or C_3,

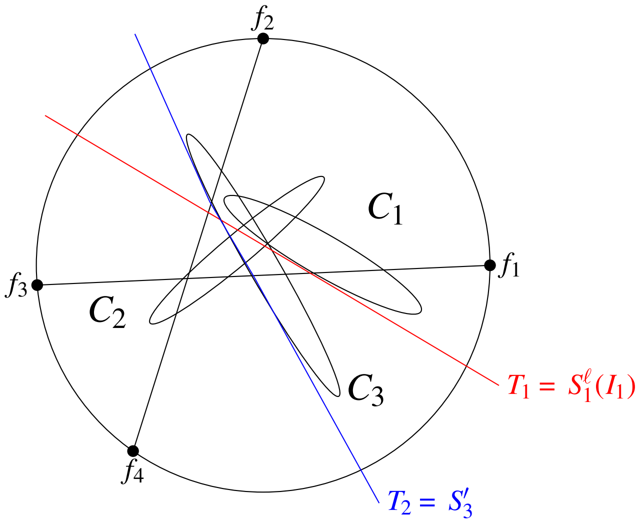

Fig. 13 Subcase 2.2

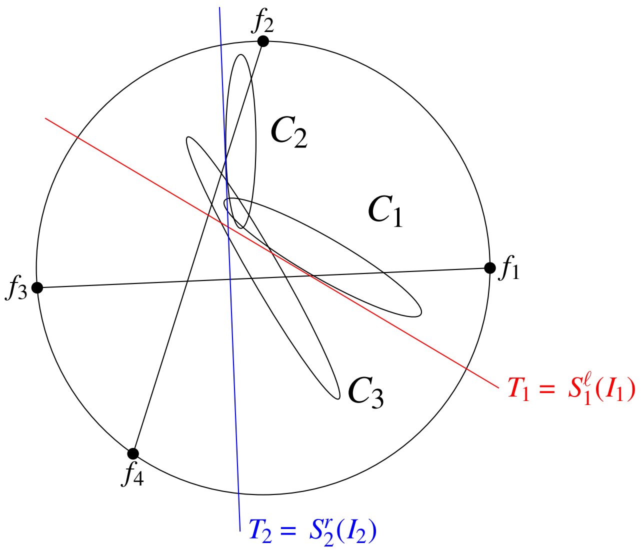

Fig. 14 Subcase 2.3

then F intersects T_1 by Lemma 3.4. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.5. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 3.2. If the order of the intervals is $I_1, I_2^1, I_3^1, I_2^2, I_3^2$, then set $T_1 = S_1^\ell(I_1)$ and T_2 to be the line obtained by applying Lemma 3.6 to C_2 and C_3 (see Fig. 17). If F intersects C_1 and one of C_2 or C_3, then F intersects T_1 by Lemma 3.4. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.6. Therefore, $\mathscr{I}$ has matching number 1.

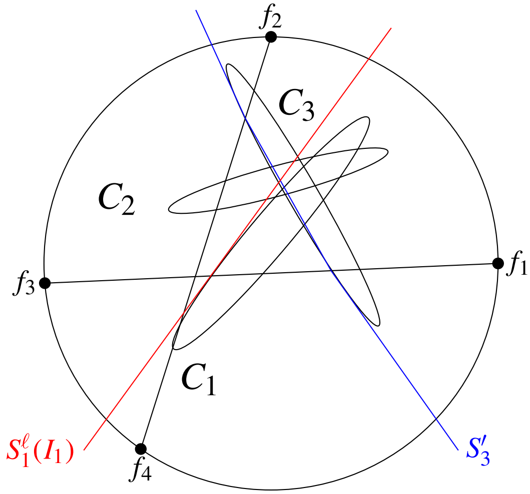

Fig. 15 Subcase 2.4

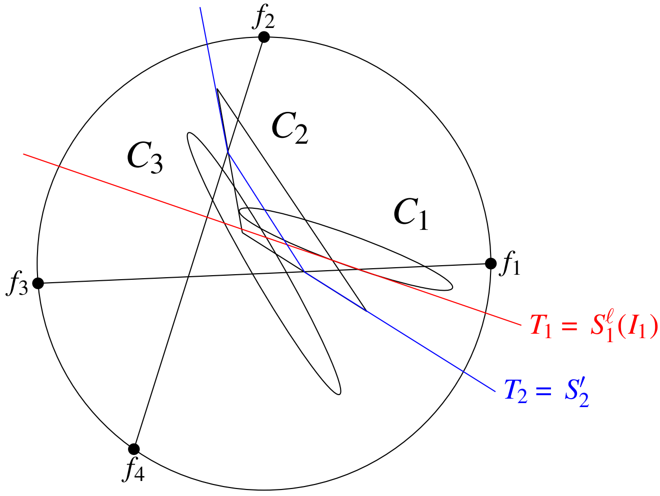

Fig. 16 Subcase 3.1

Subcase 3.3. If the order of the intervals is $I_2^1, I_1, I_3^1, I_3^2, I_2^2$, then set $T_1 = S_1^\ell(I_1)$ and $T_2 = S_2'$ (see Fig. 18). If F intersects C_1 and C_3, then F intersects T_1 by Lemma 3.4. If F intersect C_2 and one of C_1 or C_3, then F intersects T_2 by Lemma 3.5. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 3.4. If the order of the intervals is $I_2^1, I_1, I_3^1, I_2^2, I_3^2$, then set $T_1 = S_1^\ell(I_1)$ and T_2 to be the line obtained by applying Lemma 3.6 to C_2 and C_3 (see Fig. 19). If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.6. If F intersects C_1 and C_3 or C_1 and $C_2' \cap \overline{R^2}$, then F intersects T_1 by Lemma 3.4. If $T_2 = S_2^\ell(I_2^1)$ and F intersects C_1 and $C_2' \cap \overline{R^4}$, then F

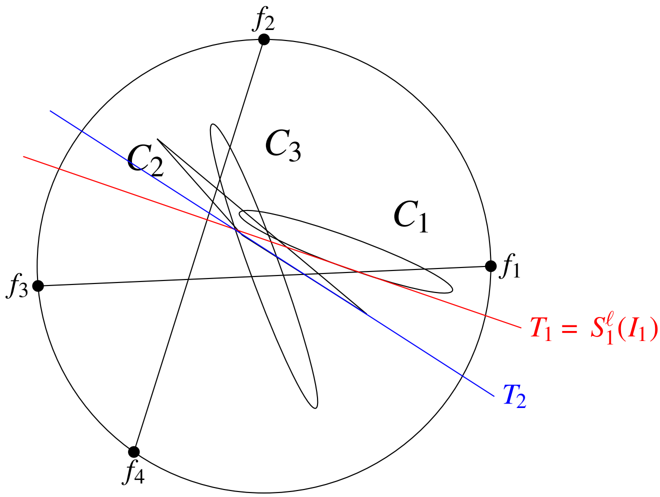

Fig. 17 Subcase 3.2. Here, the curve in blue is one of the two possibilities for T_2

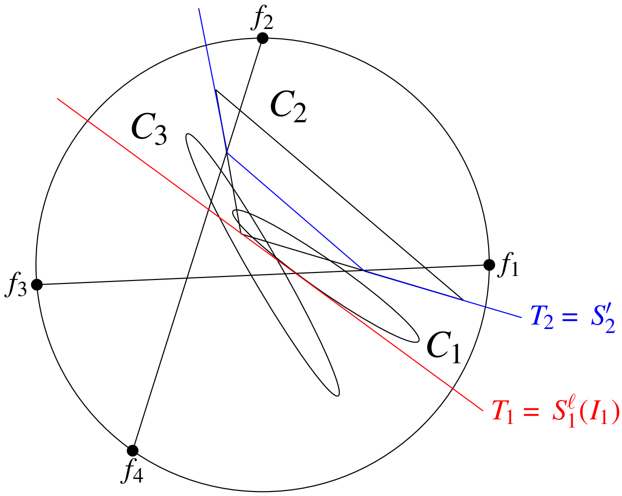

Fig. 18 Subcase 3.3

intersects T_2 by Lemma 3.4. If $T_2 = S_3^r(I_3^2)$ and F intersects C_1 and $C_2' \cap \overline{R^4}$, then F intersects T_2 by Lemma 3.6. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 3.5. If the order of the intervals is $I_2^1, I_3^1, I_1, I_3^2, I_2^2$, then set $T_1 = S_2^r$ and $T_2 = S_3^r$ (see Fig. 20). If F intersects C_2 and one of C_1 or C_3, then F intersects T_1 by Lemma 3.5. If F intersects C_1 and C_3, then F intersects T_2 by Lemma 3.5. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 3.6. If the order of the intervals is $I_2^1, I_3^1, I_1, I_2^2, I_3^2$, then set $T_1 = S_2^r$ and $T_2 = S_3^r$ (see Fig. 21). If F intersects C_1 and one of C_2 or C_3, then F

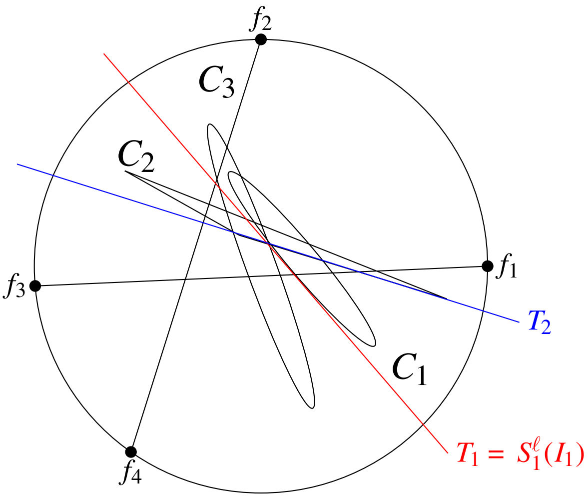

Fig. 19 Subcase 3.4. Here, the blue curve is one of two possibilities for T_2

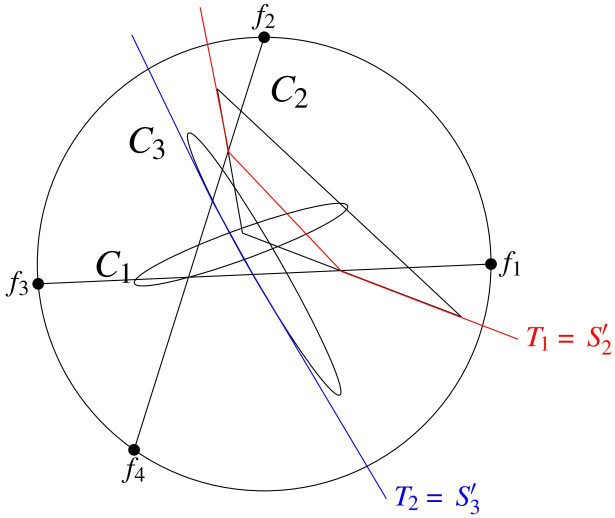

Fig. 20 Subcase 3.5

intersects T_1 or T_2, respectively, by Lemma 3.5. If F intersects $C_2' \cap \overline{R^2}$ and C_3, then F intersects T_2 by Lemma 3.5. If F intersects C_2 and $C_3' \cap \overline{R^4}$, then F intersects T_1 by Lemma 3.5. Finally, F cannot intersect $C_2' \cap \overline{R^4}$ and $C_3' \cap \overline{R^2}$, otherwise, since $C_3' \cap \overline{R^2}$ lies above $S_2^\ell(I_2^1)$ in $\overline{R^2}$ (this fact was mentioned in the proof of Lemma 3.5), F has a point in R^1 by convexity. Therefore, $\mathscr{I}$ has matching number 1.

The remaining subcases are symmetrical to one of the above cases.

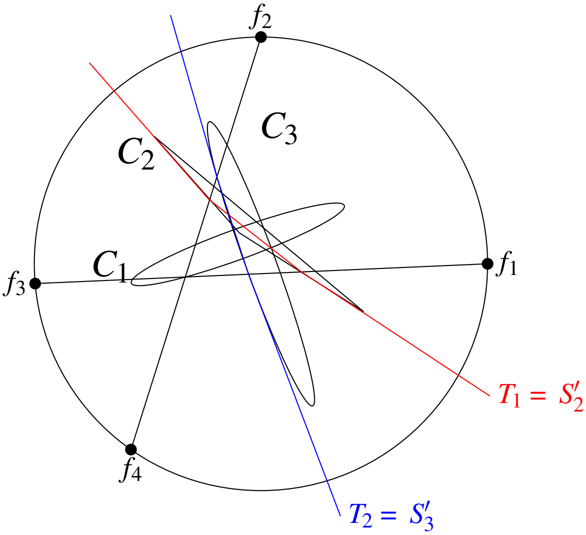

Fig. 21 Subcase 3.6

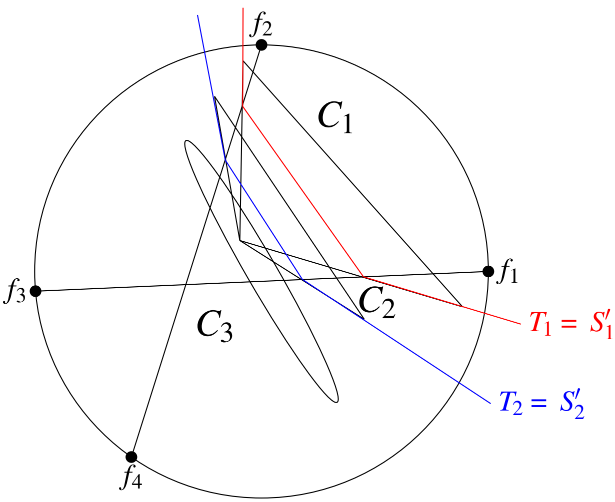

Fig. 22 Subcase 4.1

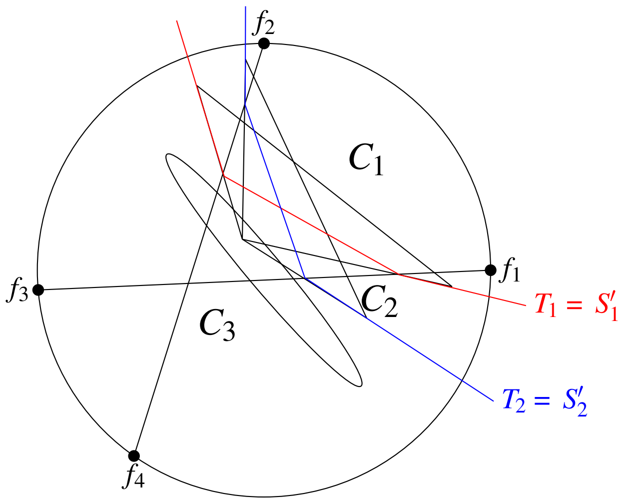

Fig. 23 Subcase 4.2

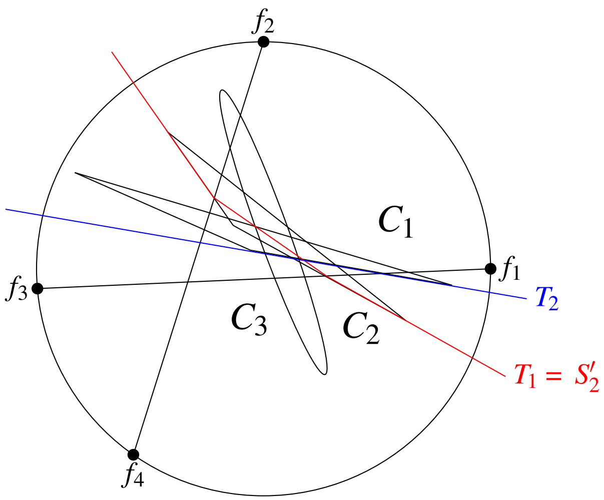

Fig. 24 Subcase 4.3. Here, the blue curve is one of two possibilities for T_2

Case 4: Each C'_i has two components.

Subcase 4.1. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_3^2, I_2^2, I_1^2$, then set $T_1 = S'_1$ and $T_2 = S'_2$ (see Fig. 22). If F intersects C_1 and one of C_2 or C_3, then F intersects T_1 by Lemma 3.5. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.5. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 4.2. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_3^2, I_1^2, I_2^2$, then set $T_1 = S'_1$ and $T_2 = S'_2$ (see Fig. 23). Similarly to Subcase 3.6 (where C_1, C_2, C_3 here

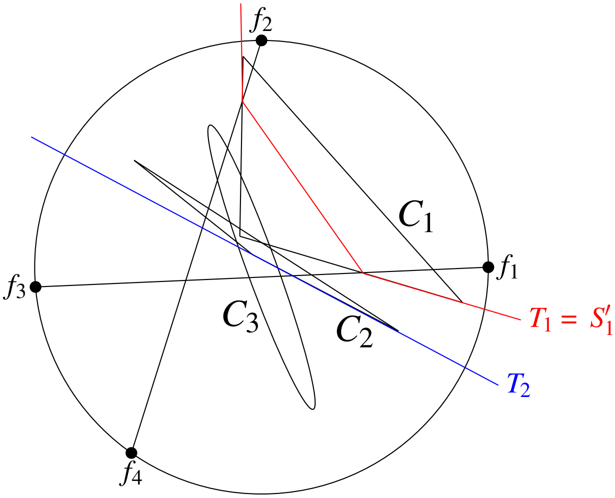

Fig. 25 Subcase 4.4. Here, the blue curve is one of two possibilities for T_2

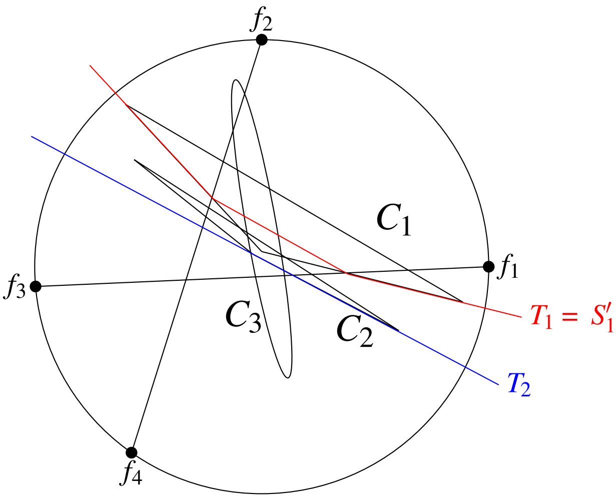

Fig. 26 Subcase 4.5. The curve in blue is one of two possibilities for T_2

are analogous to C_2, C_3, C_1 from Subcase 3.6, respectively), it follows from Lemma 3.5 that $\mathscr{I}$ has matching number 1.

Subcase 4.3. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_1^2, I_2^2, I_3^2$, then set $T_1 = S_2'$ and T_2 to be the line obtained by applying Lemma 3.6 to C_1 and C_3 (see Fig. 24). If F intersects C_1 and C_3, then F intersects T_2 by Lemma 3.6. If F intersects C_2 and one of $C_1' \cap \overline{R^2}$ or $C_3' \cap \overline{R^4}$, then F intersects T_1 by Lemma 3.5. If $T_2 = S_1^\ell(I_1^1)$ and F intersects $C_3' \cap \overline{R^2}$, then F intersects T_2 by Lemma 3.6. If F intersects $C_2' \cap \overline{R^4}$ and $C_1' \cap \overline{R^4}$, then F intersects T_2 by Lemma 3.4. If $T_2 = S_3^r(I_3^2)$ and F intersects $C_1' \cap \overline{R^2}$, then F

intersects T_2 by Lemma 3.6. If F intersects $C_2' \cap \overline{R^2}$ and $C_3' \cap \overline{R^2}$, then F intersects T_2 by Lemma 3.4. Similary to the reasoning in Subcase 3.6, F does not intersect $C_2' \cap \overline{R^2}$ and $C_1' \cap \overline{R^4}$, and F does not intersect $C_2' \cap \overline{R^4}$ and $C_3' \cap \overline{R^2}$. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 4.4. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_2^2, I_3^2, I_1^2$, then set $T_1 = S_1'$ and T_2 to be the line obtained by applying Lemma 3.6 to C_2 and C_3 (see Fig. 25). If F intersects C_1 and one of C_2 or C_3, then F intersects T_1 by Lemma 3.5. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.6. Therefore, $\mathscr{I}$ has matching number 1.

Subcase 4.5. If the order of the intervals is $I_1^1, I_2^1, I_3^1, I_2^2, I_1^2, I_3^2$, then set $T_1 = S_1'$ and T_2 to be the line obtained by applying Lemma 3.6 to C_2 and C_3 (see Fig. 26). If F intersects C_1 and C_2, then F intersects T_1 by Lemma 3.5. If F intersects C_2 and C_3, then F intersects T_2 by Lemma 3.6. If F intersects C_1 and $C_3' \cap \overline{R^4}$, then F intersects T_1 by Lemma 3.5. If $T_2 = S_2^{\ell}(I_2^1)$, then if F intersects $C_3' \cap \overline{R^2}$, F intersects T_2 by Lemma 3.6. If $T_2 = S_3^r(I_3^2)$, then if F intersects $C_1' \cap \overline{R^2}$ and $C_3' \cap \overline{R^2}$, F intersects T_2 by Lemma 3.4. Similarly to the reasoning in Subcase 3.6, F cannot intersect $C_3' \cap \overline{R^2}$ and $C_1' \cap \overline{R^4}$. Therefore, $\mathscr{I}$ has matching number 1.

Again, the remaining possible subcases are symmetrical to one of the above subcases.

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