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New Bounds for Range Closest-Pair Problems

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Abstract

Given a dataset S of points in $\mathbb{R}^2$, the range closest-pair (RCP) problem aims to preprocess S into a data structure such that when a query range X is specified, the closest-pair in $S \cap X$ can be reported efficiently. The RCP problem can be viewed as a range-search version of the classical closest-pair problem, and finds applications in many areas. Due to its non-decomposability, the RCP problem is much more challenging than many traditional range-search problems. This paper revisits the RCP problem, and proposes new data structures for various query types including quadrants, strips, rectangles, and halfplanes. Both worst-case and average-case analyses (in the sense that the data points are drawn uniformly and independently from the unit square) are applied to these new data structures, which result in new bounds for the RCP problem. Some of the new bounds significantly improve the previous results, while the others are entirely new.

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1 Introduction

The closest-pair problem is one of the most fundamental problems in computational geometry and finds many applications, e.g., collision detection, similarity search, traffic control, etc. In this paper, we study a range-search version of the closest-pair problem called the *range closest-pair* (RCP) problem. Let $\mathcal{X}$ be a certain collection of ranges called *query space*. The RCP problem with query space $\mathcal{X}$ (or the $\mathcal{X}$-RCP problem for short) aims to preprocess a given dataset S of points into a low-space data structure such that when a query range $X \in \mathcal{X}$ is specified, the closest-pair in $S \cap X$ can be reported efficiently. The motivation for the RCP problem is clear and similar to that of range search: in many situations, one is interested in local information (i.e., local closest-pairs) inside specified ranges rather than global information (i.e., global closest-pair) of the dataset.

The RCP problem is quite challenging due to a couple of reasons. First, in the RCP problem, the objects of interest are in fact point-pairs instead of single points, and in a dataset there is a quadratic number of point-pairs to be dealt with. Moreover, the RCP problem is non-decomposable in the sense that even if the query range $X \in \mathcal{X}$ can be written as $X = X_1 \cup X_2$, the closest-pair in $S \cap X$ cannot be computed efficiently from the closest-pairs in $S \cap X_1$ and $S \cap X_2$. The non-decomposability makes many traditional range-search techniques inapplicable to the RCP problem, and thus makes the problem much more challenging.

The RCP problem in $\mathbb{R}^2$ has been studied in prior work over the last fifteen years, e.g., [1, 5, 6, 8, 9, 12, 13]. In this paper, we revisit this problem and make significant improvements to the existing solutions. Following the existing work, the query types considered in this paper are orthogonal ranges (specifically, quadrants, strips, rectangles) and halfplanes.

1.1 Our Contributions, Techniques, and Related Work

The closest-pair problem and range search are both classical topics in computational geometry; see [2, 10] for references. The RCP problem is relatively new. The best existing bounds in $\mathbb{R}^2$ and our new results are summarized in Table 1 (Space refers to space cost and Qtime refers to query time), and we give a brief explanation below.

- **Related Work** The RCP problem for orthogonal queries was studied in [1, 5, 6, 8, 9, 12, 13], where [13] is a preliminary version of this paper. The best known solution for quadrant queries was given by [6], while [9] gave the best known solution for strip queries. For (orthogonal) rectangle queries, there are two best known solutions (in terms of worst-case bounds) given by [6] and [9] respectively. The above results only considered worst-case performance of the data structures. The authors of [6] for the first time applied average-case analysis to RCP data structures

Table 1 Summary of the best existing bounds and our new results for the RCP problem in $\mathbb{R}^2$ (each row corresponds to an RCP data structure for the corresponding query space)

Query	Source	Worst-case		Average-case	
		Space	Qtime	Space	Qtime
Quadrant	[6]	$O(n \log n)$	$O(\log n)$	–	–
	Theorem 2.2	$\boldsymbol{O(n)}$	$\boldsymbol{O(\log n)}$	$\boldsymbol{O(\log^2 n)}$	$\boldsymbol{O(\log \log n)}$
Strip	[9]	$O(n \log^2 n)$	$O(\log n)$	–	–
	Theorem 3.3	$\boldsymbol{O(n \log n)}$	$\boldsymbol{O(\log n)}$	$\boldsymbol{O(n)}$	$\boldsymbol{O(\log n)}$
Rectangle	[6]	$O(n \log^5 n)$	$O(\log^2 n)$	–	–
	[9]	$O(n \log^3 n)$	$O(\log^3 n)$	–	–
	[6]	–	–	$O(n \log^4 n)$	$O(\log^4 n)$
	Theorem 4.10	$\boldsymbol{O(n \log^2 n)}$	$\boldsymbol{O(\log^2 n)}$	$\boldsymbol{O(n \log n)}$	$\boldsymbol{O(\log n)}$
Halfplane	[1]	$O(n \log n)$	$O(n^{0.5+\varepsilon})$	–	–
	[1]	$O(n \log^2 n)$	$O(n^{0.75+\varepsilon})$	–	–
	Theorem 5.3	$\boldsymbol{O(n)}$	$\boldsymbol{O(\log n)}$	$\boldsymbol{O(\log^2 n)}$	$\boldsymbol{O(\log \log n)}$

in the model where the data points are drawn independently and uniformly from the unit square. Unfortunately, they only gave a rectangle RCP data structure with low average-case *preprocessing* time, while its average-case space cost and query time are even higher than the worst-case counterparts of the data structure given by [9] (in fact, its worst-case space cost is super-quadratic). In fact, in terms of space cost and query time, no nontrivial average-case bounds were known for any kind of query before this paper. The RCP problem for halfplane query was studied in [1]. Two data structures were proposed. The second one, while having higher space cost and query time than the first one, can be built more efficiently than the first one (in $O(n \log^2 n)$ time versus $O(n^2 \log^2 n)$ time). Both data structures require (worst-case) super-linear space cost and polynomial query time. The paper [12] studies an approximate version of the RCP problem in which the answer pair returned is allowed to be slightly outside the query range. Under this approximation model, efficient data structures for disk and ball queries were proposed.

- **Our Contributions** In this paper, we improve all the above results by giving new RCP data structures for various query types. The improvements can be seen in Table 1. In terms of worst-case bounds, the highlights are our rectangle RCP data structure which simultaneously improves the two best known results (given by [6] and [9]) and our halfplane RCP data structure which is *optimal* and significantly improves the bounds in [1]. Furthermore, by applying average-case analysis to our new data structures, we establish the first nontrivial average-case bounds for all the query types studied. Our average-case analysis applies to datasets generated not only in the unit square but also in an arbitrary axes-parallel rectangle. These average-case bounds demonstrate that our new data structures might have much better performance in practice than one can expect from the worst-case bounds. In

addition, all of our new RCP data structures presented in Table 1 can be constructed in near-linear worst-case time. Specifically, the quadrant, strip, and halfplane RCP data structures can be built in $O(n \log^2 n)$ time, and the rectangle RCP data structure can be built in $O(n \log^7 n)$ time.

- **Our Techniques** An important notion in our techniques is that of a *candidate pair*, i.e., a pair of data points that is the answer to some RCP query. Our solutions for the quadrant and strip RCP problems use the candidate pairs to construct a planar subdivision and take advantage of point-location techniques to answer queries. The data structures themselves are simple, and our main technical contribution here occurs in the average-case analysis of the data structures. The analysis requires a nontrivial study of the expected number of candidate pairs in a random dataset, which is of both geometric and combinatorial interest.
  Our data structure for the rectangle RCP problem is more complicated; it is constructed by combining two simpler data structures, each of which partially achieves the desired bounds. The high-level framework of the two simpler data structures is identical: it first "decomposes" a rectangle query into four quadrant queries and then simplifies the problem via some geometric observations similar to those in the standard divide-and-conquer algorithm for the classical closest-pair problem. Also, the analysis of the data structures is technically interesting.
  Our solution for the halfplane RCP problem applies the duality technique to map the candidate pairs to wedges in the dual space and form a planar subdivision, which allows us to solve the problem by using point-location techniques on the subdivision, similarly to the approach for the quadrant and strip RCP problems. However, unlike the quadrant and strip cases, to bound the complexity of the subdivision here is much more challenging, which requires non-obvious observations made by properly using the properties of duality and the problem itself. The average-case bounds of the data structure follow from a technical result bounding the expected number of candidate pairs.
- **Organization** Section 1.2 presents the notations and preliminaries that are used throughout the paper. Our solutions for quadrant, strip, rectangle, and halfplane queries are presented in Sects. 2, 3, 4, and 5, respectively. In Sect. 6, we conclude our results and give some open questions for future work. To make the paper more readable and preserve a high-level view of the main results, some technical proofs are deferred to Appendix A.

1.2 Notations and Preliminaries

We introduce the notations and preliminaries that are used throughout the paper.

- **Query Spaces** The following notations denote various query spaces (i.e., collections of ranges in $\mathbb{R}^2$): $\mathcal{Q}$ quadrants, $\mathcal{P}$ strips, $\mathcal{U}$ 3-sided rectangles, $\mathcal{R}$ rectangles, $\mathcal{H}$ halfplanes (quadrants, strips, 3-sided rectangles, rectangles under consideration are all axes-parallel). Define $\mathcal{Q}^{\nearrow} = \{[x, \infty) \times [y, \infty) : x, y \in \mathbb{R}\} \subseteq \mathcal{Q}$ as the sub-collection of all northeast quadrants, and define $\mathcal{Q}^{\nwarrow}, \mathcal{Q}^{\searrow}, \mathcal{Q}^{\swarrow}$ similarly. Define $\mathcal{P}^{\mathrm{v}} = \{[x_1, x_2] \times \mathbb{R} : x_1, x_2 \in \mathbb{R}\} \subseteq \mathcal{P}$ as the sub-collection of all vertical strips, and similarly $\mathcal{P}^{\mathrm{h}}$ horizontal strips. If l is a vertical (resp.,

horizontal) line, an *l-anchored* strip is a vertical (resp., horizontal) strip containing l; define $\mathcal{P}_l \subseteq \mathcal{P}$ as the sub-collection of all l-anchored strips. Define $\mathcal{U}^{\downarrow} = \{[x_1, x_2] \times (-\infty, y] : x_1, x_2, y \in \mathbb{R}\} \subseteq \mathcal{U}$ as the sub-collection of all bottom-unbounded rectangles, and define $\mathcal{U}^{\uparrow}, \mathcal{U}^{\leftarrow}, \mathcal{U}^{\rightarrow}$ similarly. If l is a non-vertical line, denote by $l^{\uparrow}$ (resp., $l^{\downarrow}$) the halfplane above (resp., below) l; define $\mathcal{H}^{\uparrow} \subseteq \mathcal{H}$ (resp., $\mathcal{H}^{\downarrow} \subseteq \mathcal{H}$) as the sub-collection of all such halfplanes.

- **Candidate Pairs** For a dataset S and query space $\mathcal{X}$, a *candidate pair* of S with respect to $\mathcal{X}$ refers to a pair of points in S which is the closest-pair in $S \cap X$ for some $X \in \mathcal{X}$. We denote by $\Phi(S, \mathcal{X})$ the set of the candidate pairs of S with respect to $\mathcal{X}$. If l is a line, we define $\Phi_l(S, \mathcal{X}) \subseteq \Phi(S, \mathcal{X})$ as the subset consisting of the candidate pairs that cross l (i.e., whose two points are on opposite sides of l).
- **Data Structures** For a data structure $\mathcal{D}$, we denote by $\mathcal{D}(S)$ the data structure instance of $\mathcal{D}$ built on the dataset S. The notations $\mathsf{Space}(\mathcal{D}(S))$ and $\mathsf{Qtime}(\mathcal{D}(S))$ denote the space cost and query time (i.e., the maximum time for answering a query) of $\mathcal{D}(S)$, respectively.
- **Random Datasets** If X is a region in $\mathbb{R}^2$ (or more generally in $\mathbb{R}^d$), we write $S \propto X^n$ to mean that S is a dataset of n random points drawn independently from the uniform distribution $\mathsf{Uni}(X)$ on X. More generally, if $X_1, \ldots, X_n$ are regions in $\mathbb{R}^2$ (or more generally in $\mathbb{R}^d$), we write $S \propto \prod_{i=1}^{n} X_i$ to mean that S is a dataset of n random points drawn independently from $\mathsf{Uni}(X_1), \ldots, \mathsf{Uni}(X_n)$ respectively.
- **Other Notions** For a point $a \in \mathbb{R}^2$, we denote by $a.x$ and $a.y$ the x-coordinate and y-coordinate of a, respectively. For two points $a, b \in \mathbb{R}^d$, we use $\mathrm{dist}(a, b)$ to denote the Euclidean distance between a and b, and use $[a, b]$ to denote the segments connecting a and b (in $\mathbb{R}^1$ this coincides with the notation for a closed interval). We say $I_1, \ldots, I_n$ are vertical (resp., horizontal) *aligned* segments in $\mathbb{R}^2$ if there exist $r_1, \ldots, r_n, \alpha, \beta \in \mathbb{R}$ such that $I_i = \{r_i\} \times [\alpha, \beta]$ (resp., $I_i = [\alpha, \beta] \times \{r_i\}$). The *length* of a pair $\phi = (a, b)$ of points is the length of the segment $[a, b]$. For $S \subseteq \mathbb{R}^2$ of size at least 2, the notation $\kappa(S)$ denotes the *closest-pair distance* of S, i.e., the length of the closest-pair in S.

The following result regarding the closest-pair distance of a random dataset will be used to bound the expected number of candidate pairs with respect to various query spaces. The proof of this result (and also of several other results in the remaining sections) can be found in Appendix A.

Lemma 1.1 *Let R be a rectangle of size $\Delta \times \Delta'$ where $\Delta \leq \Delta'$, and $A \propto R^m$. Then*

$$\mathbb{E}[\kappa^p(A)] = \Theta\left(\max\left\{(\Delta'/m^2)^p, (\sqrt{\Delta\Delta'}/m)^p\right\}\right) \quad \textit{for any constant } p > 1.$$

In particular, if R is a segment of length ℓ, then $\mathbb{E}[\kappa^p(A)] = \Theta((\ell/m^2)^p)$.

2 Quadrant Query

We consider the RCP problem for quadrant queries, i.e., the $\mathcal{Q}$-RCP problem. In order to solve the $\mathcal{Q}$-RCP problem, it suffices to consider the $\mathcal{Q}^{\swarrow}$-RCP problem. Let $S \subseteq \mathbb{R}^2$

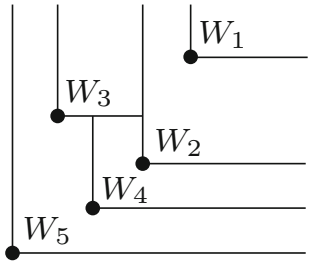


Fig. 1 A subdivision induced by successively overlaying the quadrants

be a dataset of size n. Suppose $\Phi(S, \mathcal{Q}^{\swarrow}) = \{\phi_1, \ldots, \phi_m\}$ where $\phi_i = (a_i, b_i)$, and assume $\phi_1, \ldots, \phi_m$ are sorted in increasing order of their lengths. It was shown in [6] that $m = O(n)$. We construct a mapping $\Phi(S, \mathcal{Q}^{\swarrow}) \to \mathbb{R}^2$ as $\phi_i \mapsto w_i$ where $w_i = (\max\{a_i.x, b_i.x\}, \max\{a_i.y, b_i.y\})$, and observe that for a query range $Q \in \mathcal{Q}^{\swarrow}$, ϕ_i is contained in Q iff $w_i \in Q$. Let W_i be the northeast quadrant with vertex w_i. Then we further have $w_i \in Q$ iff $q \in W_i$ where q is the vertex of Q. As such, the closest-pair in $S \cap Q$ to be reported is ϕ_η for $\eta = \min\{i : q \in W_i\}$.

We create a planar subdivision Γ, by successively overlaying $W_1, \ldots, W_m$ (see Fig. 1). Note that the complexity of Γ is $O(m)$, since overlaying each quadrant creates at most two vertices of Γ. By the above observation, the answer for Q is ϕ_i iff q is in the cell $W_i \setminus \bigcup_{j=1}^{i-1} W_j$. Thus, we can use the optimal planar point-location data structures (e.g., [4, 7]) to solve the problem in $O(m)$ space with $O(\log m)$ query time. Since $m = O(n)$, we obtain a $\mathcal{Q}$-RCP data structure using $O(n)$ space with $O(\log n)$ query time in worst-case.

Next, we analyze the average-case performance of the above data structure. In fact, it suffices to bound the expected number of the candidate pairs. Surprisingly, we have the following poly-logarithmic bound.

Lemma 2.1 *For a random dataset $S \propto R^n$ where R is an axes-parallel rectangle, $\mathbb{E}[|\Phi(S, \mathcal{Q})|] = O(\log^2 n)$.*

Using the above lemma, we can immediately conclude that our data structure uses $O(\log^2 n)$ space in average-case. The average-case query time is in fact $O(\mathbb{E}[\log |\Phi(S, \mathcal{Q})|])$. Note that $\mathbb{E}[\log x] \leq \log \mathbb{E}[x]$ for a positive random variable x, thus $\mathbb{E}[\log |\Phi(S, \mathcal{Q})|] = O(\log \log n)$.

Finally, we consider how to build the above data structure efficiently. To this end, we need an efficient algorithm to overlay quadrants. Let $W_1, \ldots, W_m$ be m northeast quadrants, and our goal to overlay them to obtain the subdivision Γ described above. (We can store Γ as a DCEL structure [3], for example.) We process $W_1, \ldots, W_m$ in this order. When processing W_i, we want to compute the cell $W_i \setminus \bigcup_{j=1}^{i-1} W_j$. To this end, we maintain the union U of the quadrants that have been processed. Note that U is always a staircase shape and its boundary ∂U is an orthogonal chain from top-left to bottom-right. Therefore, we can use a (balanced) binary search tree $\mathcal{T}$ to maintain ∂U, where the nodes store the segments of ∂U (which are vertical or horizontal) in the order they appear on ∂U.

To compute the cell $W_i \setminus \bigcup_{j=1}^{i-1} W_j$, we first find all the segments of ∂U that intersect W_i, denoted by $\sigma_1, \ldots, \sigma_k$. Note that $\sigma_1, \ldots, \sigma_k$ must be consecutive on ∂U,

and we can find them by simply searching in the BST $\mathcal{T}$ in $O(\log m + k)$ time. Once $\sigma_1, \ldots, \sigma_k$ are found, the cell $W_i \setminus \bigcup_{j=1}^{i-1} W_j$ can be directly computed and stored in the DCEL. After W_i is processed, we have to update ∂U in $\mathcal{T}$. We delete from $\mathcal{T}$ the segments among $\sigma_1, \ldots, \sigma_k$ that are completely contained in W_i, because they are no longer edges of ∂U. Besides, among $\sigma_1, \ldots, \sigma_k$, there are (at most) two segments partially contained in W_i; we need to modify them in $\mathcal{T}$ as they get changed when we insert W_i. Furthermore, there are (at most) two new segments appearing on ∂U due to the insertion of W_i (which correspond to a portion of ∂W_i), and we need to insert them into $\mathcal{T}$. All of the above work can be done in $O(k \log m)$ time. Since k is at most the number of the edges of the cell $W_i \setminus \bigcup_{j=1}^{i-1} W_j$ and the complexity of the final subdivision is linear in m, the overall time cost for processing $W_1, \ldots, W_m$ is $O(m \log m)$.

Using the above algorithm, if we already have the set $\Phi(S, \mathcal{Q}^{\swarrow})$ of candidate pairs in hand, then our RCP data structure can be built in $O(n \log n)$ time. Unfortunately, it is currently not known how to compute the candidate pairs (with respect to quadrants) efficiently. To handle this issue, we use a result by Abam et al. [1]. Before describing this result, we need to introduce the notion of *spanners* and *local spanners*. Let $S \subseteq \mathbb{R}^2$ be a dataset and $t \geq 1$ be a number. A *geometric t-spanner* (or *t-spanner* for short) of S is a set Ψ of pairs of points in S such that if we regard the points in S as vertices and the pairs in Ψ as edges (where the weight of each edge is equal to the Euclidean distance between the pair of points), the resulting graph G_Ψ satisfies $d_{G_\Psi}(a, b) \leq t \operatorname{dist}(a, b)$ for all $a, b \in S$, where $d_{G_\Psi}(a, b)$ denotes the shortest-path distance between a and b in G_Ψ. Let $\mathcal{X}$ be a query space consisting of ranges in $\mathbb{R}^2$. An *$\mathcal{X}$-local t-spanner* of S is a set Ψ of pairs of points in S such that for every $X \in \mathcal{X}$, Ψ_X is a t-spanner of $S \cap X$, where $\Psi_X \subseteq \Psi$ consists of the pairs in Ψ whose two points are both inside X. Note that if $t < 2$, then a t-spanner Ψ of S must contain the closest pair of S[1]. Therefore, if $t < 2$, then an $\mathcal{X}$-local t-spanner Ψ of S must contains all candidate pairs of S with respect to $\mathcal{X}$, i.e., $\Phi(S, \mathcal{X}) \subseteq \Psi$.

Exploiting the so-called *semi-separated pair decomposition* (SSPD), Abam et al. [1] showed that, given a dataset $S \subseteq \mathbb{R}^2$ of size n, one can compute in $O(n \log^2 n)$ time a $\mathcal{Q}^{\swarrow}$-local t-spanner Ψ of S for some $t < 2$ such that $|\Psi| = O(n \log n)$. As argued above, we have $\Phi(S, \mathcal{Q}^{\swarrow}) \subseteq \Psi$. To build our RCP data structure, instead of computing the set $\Phi(S, \mathcal{Q}^{\swarrow})$ of candidate pairs, we directly use the pairs in Ψ to create a planar subdivision described above.

Formally, let $\Psi = \{\phi_1, \ldots, \phi_M\}$ where $\phi_1, \ldots, \phi_M$ are sorted in increasing order of their lengths, and let $W_1, \ldots, W_M$ be their corresponding northeast quadrants. We overlay $W_1, \ldots, W_M$ in order, and let Γ' be the resulting subdivision. We claim that Γ' is the same as the subdivision Γ constructed using the candidate pairs in $\Phi(S, \mathcal{Q}^{\swarrow})$. Since $\Phi(S, \mathcal{Q}^{\swarrow}) \subseteq \Psi$, for any southwest quadrant $Q \in \mathcal{Q}^{\swarrow}$, the closest-pair in $S \cap Q$ is contained in Ψ, and in fact is the shortest element in Ψ that is contained in Q. It follows that the corresponding cell of $\phi \in \Psi$ in Γ' is just the subset of $\mathbb{R}^2$ consisting of the vertices of all $Q \in \mathcal{Q}^{\swarrow}$ such that the closest-pair in $S \cap Q$ is ϕ. Therefore, the cell corresponding to any $\phi \in \Psi \setminus \Phi(S, \mathcal{Q}^{\swarrow})$ in Γ' must be empty, because ϕ

[1] Indeed, if (a, b) is the closest pair of S and $(a, b) \notin \Psi$, then $d_{G_\Psi}(a, b) \geq 2 \operatorname{dist}(a, b) > t \operatorname{dist}(a, b)$, because the shortest path between a and b in G_Ψ consists of at least two edges whose weights are at least $\operatorname{dist}(a, b)$.

is not the closest-pair in any southwest quadrant. Also, the cell corresponding to any $\phi \in \Phi(S, \mathcal{Q}^{\swarrow})$ in Γ' must be the same as the one in Γ. So we have $\Gamma' = \Gamma$. Note that $|\Psi| = O(n \log n)$, thus the time for constructing Γ' is $O(n \log^2 n)$, which is also the preprocessing time of our RCP data structure.

Theorem 2.2 *There exists a $\mathcal{Q}$-RCP data structure $\mathcal{A}$ such that:*

– *For any $S \subseteq \mathbb{R}^2$ of size n,*

$$\mathsf{Space}(\mathcal{A}(S)) = O(n) \quad \textit{and} \quad \mathsf{Qtime}(\mathcal{A}(S)) = O(\log n).$$

– *For a random $S \propto R^n$ where R is the unit square or more generally an arbitrary axes-parallel rectangle,*

$$\mathbb{E}[\mathsf{Space}(\mathcal{A}(S))] = O(\log^2 n) \quad \textit{and} \quad \mathbb{E}[\mathsf{Qtime}(\mathcal{A}(S))] = O(\log \log n).$$

Furthermore, the above data structure can be constructed in $O(n \log^2 n)$ worst-case time.

3 Strip Query

We consider the RCP problem for strip queries, i.e., the $\mathcal{P}$-RCP problem. In order to solve the $\mathcal{P}$-RCP problem, it suffices to consider the $\mathcal{P}^{\mathrm{v}}$-RCP problem. Let $S \subseteq \mathbb{R}^2$ be a dataset of size n. Suppose $\Phi(S, \mathcal{P}^{\mathrm{v}}) = \{\phi_1, \ldots, \phi_m\}$ where $\phi_i = (a_i, b_i)$, and assume $\phi_1, \ldots, \phi_m$ are sorted in increasing order of their lengths. It was shown in [9] that $m = O(n \log n)$. We construct a mapping $\Phi(S, \mathcal{P}^{\mathrm{v}}) \to \mathbb{R}^2$ as $\phi_i \mapsto w_i$ where $w_i = (\min\{a_i.x, b_i.x\}, \max\{a_i.x, b_i.x\})$, and observe that for a query range $P = [x_1, x_2] \times \mathbb{R} \in \mathcal{P}^{\mathrm{v}}$, ϕ_i is contained in P iff w_i is in the southeast quadrant $[x_1, \infty) \times (-\infty, x_2]$. Let W_i be the northwest quadrant with vertex w_i. Then we further have $w_i \in [x_1, \infty) \times (-\infty, x_2]$ iff $p \in W_i$ where $p = (x_1, x_2)$. As such, the closest-pair in $S \cap P$ is ϕ_η for $\eta = \min\{i : p \in W_i\}$. Thus, as in Sect. 2, we can successively overlay $W_1, \ldots, W_m$ to create a planar subdivision Γ, and use point-location to solve the problem in $O(m)$ space and $O(\log m)$ query time. Since $m = O(n \log n)$ here, we obtain a $\mathcal{P}$-RCP data structure using $O(n \log n)$ space with $O(\log n)$ query time in worst-case.

Next, we analyze the average-case performance of our data structure. Again, it suffices to bound the expected number of the candidate pairs. For later use, we study here a more general case in which the candidate pairs are considered with respect to 3-sided rectangle queries.

Lemma 3.1 *Let $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are distinct vertical (resp., horizontal) aligned segments sorted from left to right (resp., from bottom to top). Suppose $a_i \in S$ is the point drawn on I_i. Then for $i, j \in \{1, \ldots, n\}$ with $i < j$ and $\mathcal{X} \in \{\mathcal{U}^{\downarrow}, \mathcal{U}^{\uparrow}\}$ (resp., $\mathcal{X} \in \{\mathcal{U}^{\leftarrow}, \mathcal{U}^{\rightarrow}\}$),*

$$\Pr[(a_i, a_j) \in \Phi(S, \mathcal{X})] = O\left(\frac{\log(j-i)}{(j-i)^2}\right).$$

From the above lemma, a direct calculation gives us the following corollary.

Corollary 3.2 *For a random dataset $S \propto R^n$ where R is an axes-parallel rectangle, $\mathbb{E}[|\Phi(S, \mathcal{U})|] = \Theta(n)$ and $\mathbb{E}[|\Phi(S, \mathcal{P})|] = \Theta(n)$.*

Proof Without loss of generality, assume $R = [0, 1] \times [0, \Delta]$. Since $\Phi(S, \mathcal{P}) \subseteq \Phi(S, \mathcal{U})$ for any S, it suffices to show $\mathbb{E}[|\Phi(S, \mathcal{P})|] = \Omega(n)$ and $\mathbb{E}[|\Phi(S, \mathcal{U})|] = O(n)$. The former is clear, since every pair of x-adjacent or y-adjacent points in S is a candidate pair with respect to $\mathcal{P}$. The latter can be shown using Lemma 3.1 as follows. We only need to bound $\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|]$. We first show that if $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are vertical aligned segments sorted from left to right, then $\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|] = O(n)$. In fact, this follows directly from Lemma 3.1. Let a_i be the random point drawn on I_i. Then

$$\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\left[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})\right].$$

We plug in the bound $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})] = O(\log(j - i)/(j - i)^2)$ shown in Lemma 3.1 to the above equation. Noting the fact that $\sum_{t=1}^{\infty} \log t/t^2 = O(1)$, a direct calculation then gives us $\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|] = O(n)$. Now assume $S \propto R^n$. Define a random multi-set $X = \{a.x : a \in S\}$, which consists of the x-coordinates of the n random points in S. We shall show that for all $x_1, \ldots, x_n \in [0, 1]$ such that $x_1 < \ldots < x_n$,

$$\mathbb{E}\left[|\Phi(S, \mathcal{U}^{\downarrow})| \,\middle|\, X = \{x_1, \ldots, x_n\}\right] = O(n), \tag{1}$$

which implies that $\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|] = O(n)$, because the random points in S have distinct x-coordinates with probability 1. Let $I_i = x_i \times [0, \Delta]$ for $i \in \{1, \ldots, n\}$, then $I_1, \ldots, I_n$ are vertical aligned segments sorted from left to right. Note that, under the condition $X = \{x_1, \ldots, x_n\}$, the n random points in S can be viewed as independently drawn from the uniform distributions on $I_1, \ldots, I_n$, respectively. Thus, (1) follows directly from our previous argument for the case $S \propto \prod_{i=1}^{n} I_i$. As a result, $\mathbb{E}[|\Phi(S, \mathcal{U}^{\downarrow})|] = O(n)$. □

Using the above result and our previous data structure, we immediately conclude that the average-case space cost of our $\mathcal{P}$-RCP data structure is $O(n)$. To build this data structure, we use the same method as in Sect. 2. Abam et al. [1] showed that one can compute in $O(n \log^2 n)$ time a $\mathcal{P}^{\mathrm{v}}$-local t-spanner Ψ of S for some $t < 2$ such that $|\Psi| = O(n \log n)$; see Sect. 2 for the definition of local spanners. As argued in Sect. 2, we have $\Phi(S, \mathcal{P}^{\mathrm{v}}) \subseteq \Psi$. We then overlay the corresponding northeast quadrants of the pairs in Ψ to construct a planar subdivision Γ'. This can be done in $O(n \log^2 n)$ using the algorithm in Sect. 2 for overlaying quadrants. Using the same argument as in Sect. 2, we see that Γ' is the same as the subdivision Γ constructed using the candidate pairs in $\Phi(S, \mathcal{P}^{\mathrm{v}})$ and hence can be used to answer RCP queries. Therefore, the overall preprocessing time of our $\mathcal{P}$-RCP data structure is $O(n \log^2 n)$.

Theorem 3.3 *There exists a $\mathcal{P}$-RCP data structure $\mathcal{B}$ such that:*

– *For any $S \subseteq \mathbb{R}^2$ of size n,*

$$\mathsf{Space}(\mathcal{B}(S)) = O(n \log n) \quad \textit{and} \quad \mathsf{Qtime}(\mathcal{B}(S)) = O(\log n).$$

– *For a random $S \propto R^n$ where R is the unit square or more generally an arbitrary axes-parallel rectangle,*

$$\mathbb{E}[\mathsf{Space}(\mathcal{B}(S))] = O(n) \quad \textit{and} \quad \mathbb{E}[\mathsf{Qtime}(\mathcal{B}(S))] = O(\log n).$$

Furthermore, the above data structure can be constructed in $O(n \log^2 n)$ worst-case time.

4 Rectangle Query

We consider the RCP problem for rectangle queries, i.e., the $\mathcal{R}$-RCP problem. Unlike the data structures presented in the previous sections, our $\mathcal{R}$-RCP data structure is somewhat complicated, so we give an brief overview below before discussing the details.

Overview. Interestingly, our final data structure for the $\mathcal{R}$-RCP problem is a combination of two simpler $\mathcal{R}$-RCP data structures, each of which partially achieves the desired bounds. Specifically, the first data structure has the desired worst-case space cost and query time, while the second data structure has the desired average-case space cost and has an even better (worst-case) query time of $O(\log n)$. By properly combining the two data structures (Sect. 4.4), we obtain our final $\mathcal{R}$-RCP data structure, which achieves simultaneously the desired worst-case and average-case bounds. Both of the simpler data structures are based on range trees [3] and our results for quadrant and strip RCP problems presented in the previous sections.

In what follows, we first describe the common part of the two simpler $\mathcal{R}$-RCP data structures. Let $S \subseteq \mathbb{R}^2$ be a dataset of size n. The common component of our two data structures is a standard 2D range tree built on S [3]. The main tree (or primary tree) $\mathcal{T}$ is a range tree storing at its leaves the x-coordinates of the points in S. Each node $\mathbf{u} \in \mathcal{T}$ corresponds to a subset $S(\mathbf{u})$ of x-consecutive points in S, called the *canonical subset* of $\mathbf{u}$. At $\mathbf{u}$, there is an associated secondary tree $\mathcal{T}_{\mathbf{u}}$, which is a range tree whose leaves store the y-coordinates of the points in $S(\mathbf{u})$. With an abuse of notation, for each node $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$, we still use $S(\mathbf{v})$ to denote the canonical subset of $\mathbf{v}$, which is a subset of y-consecutive points in $S(\mathbf{u})$. As in [6], for each (non-leaf) primary node $\mathbf{u} \in \mathcal{T}$, we fix a vertical line $l_{\mathbf{u}}$ such that the points in the canonical subset of the left (resp., right) child of $\mathbf{u}$ are to the left (resp., right) of $l_{\mathbf{u}}$. Similarly, for each (non-leaf) secondary node $\mathbf{v}$, we fix a horizontal line $l_{\mathbf{v}}$ such that the points in the canonical subset of the left (resp., right) child of $\mathbf{v}$ are above (resp., below) $l_{\mathbf{v}}$. Let $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$ be a secondary node. Then at $\mathbf{v}$ we have two lines $l_{\mathbf{v}}$ and $l_{\mathbf{u}}$, which partition $\mathbb{R}^2$ into four quadrants. We denote by $S_1(\mathbf{v}), \ldots, S_4(\mathbf{v})$ the subsets of $S(\mathbf{v})$ contained in these quadrants; see Fig. 2 for the correspondence.

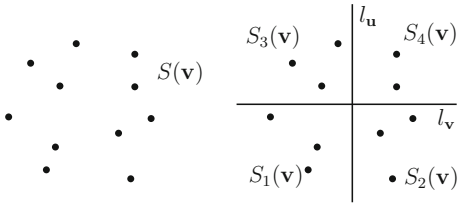


Fig. 2 Illustrating the subsets $S_1(\mathbf{v}), \ldots, S_4(\mathbf{v})$

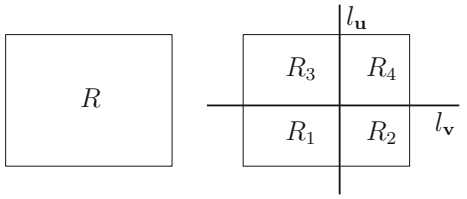


Fig. 3 Illustrating the rectangles $R_1, \ldots, R_4$

In order to solve the problem, we need to store some additional data structures (called *sub-structures*) at the nodes of the tree. At each secondary node $\mathbf{v}$, we store four $\mathcal{Q}$-RCP data structures $\mathcal{A}(S_1(\mathbf{v})), \ldots, \mathcal{A}(S_4(\mathbf{v}))$ (Theorem 2.2).

Now let us explain what we can do by using this 2D range tree (with the sub-structures). Let $R = [x_1, x_2] \times [y_1, y_2] \in \mathcal{R}$ be a query rectangle. We first find in $\mathcal{T}$ the *splitting* node $\mathbf{u} \in \mathcal{T}$ corresponding to the range $[x_1, x_2]$, which is by definition the LCA of all the leaves whose corresponding points are in $[x_1, x_2] \times \mathbb{R}$. Then we find in $\mathcal{T}_{\mathbf{u}}$ the splitting node $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$ corresponding to the range $[y_1, y_2]$. If either of the splitting nodes does not exist or is a leaf node, then $|S \cap R| \leq 1$ and nothing should be reported. So assume $\mathbf{u}$ and $\mathbf{v}$ are non-leaf nodes. By the property of splitting node, we have $S \cap R = S(\mathbf{v}) \cap R$, and the lines $l_{\mathbf{u}}$ and $l_{\mathbf{v}}$ both intersect R. Thus, $l_{\mathbf{u}}$ and $l_{\mathbf{v}}$ decompose R into four smaller rectangles $R_1, \ldots, R_4$; see Fig. 3 for the correspondence. By construction, we have $S(\mathbf{v}) \cap R_i = S_i(\mathbf{v}) \cap R_i$. In order to find the closest-pair in $S \cap R$, we first compute the closest-pair in $S \cap R_i$ for all $i \in \{1, \ldots, 4\}$. This can be done by querying the sub-structures stored at $\mathbf{v}$. Indeed, $S \cap R_i = S(\mathbf{v}) \cap R_i = S_i(\mathbf{v}) \cap R_i = S_i(\mathbf{v}) \cap Q_i$, where Q_i is the quadrant obtained by removing the two sides of R_i that coincide with $l_{\mathbf{u}}$ and $l_{\mathbf{v}}$. Therefore, we can query $\mathcal{A}(S_i(\mathbf{v}))$ with Q_i to find the closest-pair in $S \cap R_i$. Once the four closest-pairs are computed, we take the shortest one (i.e., the one of the smallest length) among them and denote it by ϕ.

Clearly, ϕ is not necessarily the closest-pair in $S \cap R$ as the two points in the closest-pair may belong to different R_i's. However, as we will see, with ϕ in hand, finding the closest-pair in $S \cap R$ becomes easier. Suppose $l_{\mathbf{u}} : x = \alpha$ and $l_{\mathbf{v}} : y = \beta$, where $x_1 \leq \alpha \leq x_2$ and $y_1 \leq \beta \leq y_2$. Let δ be the length of ϕ. We define $P_\alpha = [\alpha - \delta, \alpha + \delta] \times \mathbb{R}$ (resp., $P_\beta = \mathbb{R} \times [\beta - \delta, \beta + \delta]$) and $R_\alpha = R \cap P_\alpha$ (resp., $R_\beta = R \cap P_\beta$); see Fig. 4. We have the following key observation.

Lemma 4.1 *The closest-pair in $S \cap R$ is the shortest one among $\{\phi, \phi_\alpha, \phi_\beta\}$, where ϕ_α (resp., ϕ_β) is the closest-pair in $S \cap R_\alpha$ (resp., $S \cap R_\beta$).*

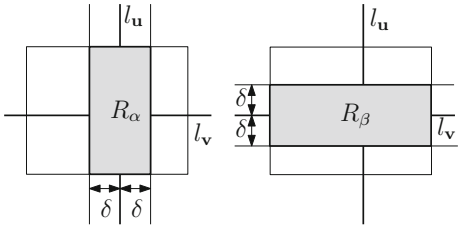


Fig. 4 Illustrating the rectangles R_α and R_β

Proof Let $\phi^* = (a^*, b^*)$ be the closest-pair in $S \cap R$. Since $\phi, \phi_\alpha, \phi_\beta$ are all point-pairs in $S \cap R$, it suffices to show that $\phi^* \in \{\phi, \phi_\alpha, \phi_\beta\}$. If $\phi^* = \phi$, we are done. So assume $\phi^* \neq \phi$. Then a^* and b^* must be contained in different R_i's. It follows that the segment $[a^*, b^*]$ intersects either $l_{\mathbf{u}}$ or $l_{\mathbf{v}}$. Note that the length of ϕ^* is at most δ (recall that δ is the length of ϕ), which implies $|a^*.x - b^*.x| \leq \delta$ and $|a^*.y - b^*.y| \leq \delta$. If $[a^*, b^*]$ intersects $l_{\mathbf{u}}$, then $a^*, b^* \in P_\alpha$ (because $|a^*.x - b^*.x| < \delta$). Thus, $a^*, b^* \in R_\alpha$ and $\phi^* = \phi_\alpha$. Similarly, if $[a^*, b^*]$ intersects $l_{\mathbf{v}}$, we have $\phi^* = \phi_\beta$. As a result, $\phi^* \in \{\phi, \phi_\alpha, \phi_\beta\}$. □

Due to the above lemma, it now suffices to compute ϕ_α and ϕ_β. Note that R_α and R_β are rectangles, so computing ϕ_α and ϕ_β still requires rectangle RCP queries. Fortunately, there are some additional properties which make it easy to search for the closest-pairs in $S \cap R_\alpha$ and $S \cap R_\beta$. For a set A of points in $\mathbb{R}^2$ and $a, b \in A$, we define the *x-gap* (resp., *y-gap*) between a and b in A as the number of the points in $A \setminus \{a, b\}$ whose x-coordinates (resp., y-coordinates) are in between $a.x$ and $b.x$ (resp., $a.y$ and $b.y$).

Lemma 4.2 *There exists a constant integer k such that the y-gap (resp., x-gap) between the two points of ϕ_α (resp., ϕ_β) in $S \cap R_\alpha$ (resp., $S \cap R_\beta$) is at most k.*

Proof We only need to consider ϕ_α. Let $k = 100$. The reason for this choice will become clear shortly. Suppose $\phi_\alpha = (a, b)$. We denote by w the left-right width of R_α, i.e., the distance between the left and right boundaries of R_α. By the construction of R_α, we have $w \leq 2\delta$. We consider two cases: $|a.y - b.y| \geq 2w$ and $|a.y - b.y| < 2w$. Suppose $|a.y - b.y| \geq 2w$. Assume there are more than k points in $(S \cap R_\alpha) \setminus \{a, b\}$ whose y-coordinates are in between $a.y$ and $b.y$. Then by the pigeonhole principle, we can find, among these points, two points a' and b' such that $|a'.y - b'.y| \leq |a.y - b.y|/k$. Since $a', b' \in R_\alpha$, we have $|a'.x - b'.x| \leq w \leq |a.y - b.y|/2$. It follows that

$$\text{dist}(a', b') \leq |a'.x - b'.x| + |a'.y - b'.y| < |a.y - b.y| \leq \text{dist}(a, b),$$

which contradicts the fact that ϕ_α is the closest-pair in $S \cap R_\alpha$. Next, suppose $|a.y - b.y| < 2w$. Then $|a.y - b.y| < 4\delta$. Consider the rectangle $R^* = R_\alpha \cap (\mathbb{R} \times [a.y, b.y])$. Note that $(S \cap R^*) \setminus \{a, b\}$ consists of exactly the points in $(S \cap R_\alpha) \setminus \{a, b\}$ whose y-coordinates are in between $a.y$ and $b.y$. Therefore, it suffices to show $|S \cap R^*| \leq k$. Let $R_i^* = R^* \cap R_i$ for $i \in \{1, \ldots, 4\}$. Since $R_i^* \subseteq R_i$, the pairwise distances of the points in $S \cap R_i^*$ are at least δ. Furthermore, the left-right width of each R_i^* is at most δ and the top-bottom width of each R_i^* is at most $|a.y - b.y|$ (which is smaller

than 4δ). Therefore, a simple argument using the pigeonhole principle shows that $|S \cap R_i^*| \leq 16 < 25 = k/4$. As such, $|S \cap R^*| \leq k$. □

We shall use the above lemma to help compute ϕ_α and ϕ_β. At this point, our two data structures diverge.

4.1 Preliminary: Extreme Point Data Structures

Before presenting our results, we introduce the so-called *top/bottom extreme point* (TBEP) and *left/right extreme point* (LREP) data structures. For a query space $\mathcal{X}$ and a constant integer k, an $(\mathcal{X}, k)$-TBEP (resp. $(\mathcal{X}, k)$-LREP) data structure stores a given set S of points in $\mathbb{R}^2$ and can report the k topmost/bottommost (resp., leftmost/rightmost) points in $S \cap X$ for a query range $X \in \mathcal{X}$.

Lemma 4.3 *Let k be a constant integer. There exists a $(\mathcal{P}^{\mathrm{v}}, k)$-TBEP data structure $\mathcal{K}^{\mathrm{v}}$ such that for any $S \subseteq \mathbb{R}^2$ of size n, $\mathsf{Space}(\mathcal{K}^{\mathrm{v}}(S)) = O(n)$ and $\mathsf{Qtime}(\mathcal{K}^{\mathrm{v}}(S)) = O(\log n)$. Furthermore, the above data structure can be constructed in $O(n \log n)$ worst-case time. Symmetrically, there also exists a $(\mathcal{P}^{\mathrm{h}}, k)$-LREP data structure $\mathcal{K}^{\mathrm{h}}$ satisfying the same bounds.*

Proof Let $S \subseteq \mathbb{R}^2$ be a dataset of size n. The $(\mathcal{P}^{\mathrm{v}}, k)$-TBEP data structure instance $\mathcal{K}^{\mathrm{v}}(S)$ is a standard 1D range tree $\mathcal{T}$ built on the x-coordinates of the points in S. By the construction of a range tree, each node $\mathbf{u} \in \mathcal{T}$ corresponds to a subset $S(\mathbf{u})$ of x-consecutive points in S, called the *canonical subset* of $\mathbf{u}$. The leaves of $\mathcal{T}$ one-to-one correspond to the points in S. At each node $\mathbf{u} \in \mathcal{T}$, we store the k topmost and k bottommost points in $S(\mathbf{u})$; we denote the set of these $2k$ points by $K(\mathbf{u})$. The overall space cost of the range tree (with the stored points) is clearly $O(n)$, as k is a constant.

To answer a query $P = [x_1, x_2] \times \mathbb{R} \in \mathcal{P}^{\mathrm{v}}$, we first find the $t = O(\log n)$ canonical nodes $\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathcal{T}$ corresponding to the range $[x_1, x_2]$. This is a standard range-tree operation, which can be done in $O(\log n)$ time. We compute $K = \bigcup_{i=1}^{t} K(\mathbf{u}_i)$ in $O(\log n)$ time. We then use selection to find the k topmost and k bottommost points in K; this can be done in $O(\log n)$ time since $|K| = 2kt = O(\log n)$. These $2k$ points are just the k topmost and k bottommost points in $S \cap P$.

The above data structure can be easily built in $O(n \log n)$ time, because finding $K(\mathbf{u})$ for each node $\mathbf{u} \in \mathcal{T}$ takes $|S(\mathbf{u})|$ time (via selection). The $(\mathcal{P}^{\mathrm{h}}, k)$-LREP data structure $\mathcal{K}^{\mathrm{h}}$ is constructed in a symmetric way. □

Lemma 4.4 *Let l be a vertical (resp., horizontal) line and k be a constant integer. There exists a $(\mathcal{P}_l, k)$-TBEP (resp., $(\mathcal{P}_l, k)$-LREP) data structure $\mathcal{K}_l$ such that for $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are distinct vertical (resp., horizontal) aligned segments, $\mathbb{E}[\mathsf{Space}(\mathcal{K}_l(S))] = O(\log n)$ and $\mathbb{E}[\mathsf{Qtime}(\mathcal{K}_l(S))] = O(\log \log n)$. Furthermore, the above data structure can be built in $O(n \log n)$ worst-case time.*

4.2 First Data Structure

We now introduce our first $\mathcal{R}$-RCP data structure, which achieves the desired worst-case bounds. Let k be the constant integer in Lemma 4.2. In our first data structure,

besides the 2D range tree presented before, we build additionally two 1D range trees $\mathcal{T}'$ and $\mathcal{T}''$ on S, where $\mathcal{T}'$ (resp., $\mathcal{T}''$) is built on y-coordinates (resp., x-coordinates). For $\mathbf{u}' \in \mathcal{T}'$ (resp., $\mathbf{u}'' \in \mathcal{T}''$), we still use $S(\mathbf{u}')$ (resp., $S(\mathbf{u}'')$) to denote the canonical subset of $\mathbf{u}'$ (resp., $\mathbf{u}'' \in \mathcal{T}''$). At each node $\mathbf{u}' \in \mathcal{T}'$, we store a $\mathcal{P}$-RCP data structure $\mathcal{B}(S(\mathbf{u}'))$ (Theorem 3.3) and a $(\mathcal{P}^{\mathrm{v}}, k)$-TBEP data structure $\mathcal{K}^{\mathrm{v}}(S(\mathbf{u}'))$ (Lemma 4.3). Similarly, at each node $\mathbf{u}'' \in \mathcal{T}''$, we store a $\mathcal{P}$-RCP data structure $\mathcal{B}(S(\mathbf{u}''))$ (Theorem 3.3) and a $(\mathcal{P}^{\mathrm{h}}, k)$-LREP data structure $\mathcal{K}^{\mathrm{h}}(S(\mathbf{u}''))$ (Lemma 4.3).

We now explain how to compute ϕ_α and ϕ_β. Suppose $R_\alpha = [x_\alpha, x'_\alpha] \times [y_\alpha, y'_\alpha]$. Let $P_x = [x_\alpha, x'_\alpha] \times \mathbb{R}$ and $P_y = \mathbb{R} \times [y_\alpha, y'_\alpha]$. To compute ϕ_α, we first find in $\mathcal{T}'$ the $t = O(\log n)$ canonical nodes $\mathbf{u}'_1, \ldots, \mathbf{u}'_t \in \mathcal{T}'$ corresponding to the range $[y_\alpha, y'_\alpha]$. Then $\bigcup_{i=1}^{t} S(\mathbf{u}'_i) = S \cap P_y$, and each $S(\mathbf{u}'_i)$ is a set of y-consecutive points in $S \cap P_y$. Furthermore, $S \cap R_\alpha = \bigcup_{i=1}^{t} S(\mathbf{u}'_i) \cap P_x$. We query the sub-structures $\mathcal{B}(S(\mathbf{u}'_1)), \ldots, \mathcal{B}(S(\mathbf{u}'_t))$ with P_x to find the closest-pairs $\phi_1, \ldots, \phi_t$ in $S(\mathbf{v}_1) \cap P_x, \ldots, S(\mathbf{v}_t) \cap P_x$, respectively. We also query $\mathcal{K}^{\mathrm{v}}(S(\mathbf{u}'_1)), \ldots, \mathcal{K}^{\mathrm{v}}(S(\mathbf{u}'_t))$ with P_x to obtain the k topmost and bottommost points in $S(\mathbf{u}'_1) \cap P, \ldots, S(\mathbf{u}'_t) \cap P$, respectively; we denote by K the set of the $2kt$ reported points. Then we find the closest-pair ϕ_K in K using the standard divide-and-conquer algorithm.

We claim that ϕ_α is the shortest one among $\{\phi_1, \ldots, \phi_t, \phi_K\}$. Suppose $\phi_\alpha = (a, b)$. If the two points of ϕ_α are both contained in some $S(\mathbf{u}'_i)$, then clearly $\phi_\alpha = \phi_i$. Otherwise, by Lemma 4.2 and the choice of k, the two points of ϕ_α must belong to K and hence $\phi_\alpha = \phi_K$. It follows that $\phi_\alpha \in \{\phi_1, \ldots, \phi_t, \phi_K\}$. Furthermore, because the pairs $\phi_1, \ldots, \phi_t, \phi_K$ are all contained in R_α, ϕ_α must be the shortest one among $\{\phi_1, \ldots, \phi_t, \phi_K\}$. Therefore, with $\phi_1, \ldots, \phi_t, \phi_K$ in hand, ϕ_α can be easily computed. The pair ϕ_β is computed symmetrically using $\mathcal{T}''$. Finally, taking the shortest one among $\{\phi, \phi_\alpha, \phi_\beta\}$, the query R can be answered.

The 2D range tree together with the two 1D range trees $\mathcal{T}'$ and $\mathcal{T}''$ forms our first $\mathcal{R}$-RCP data structure. A straightforward analysis gives us the worst-case performance of this data structure.

Theorem 4.5 *There exists an $\mathcal{R}$-RCP data structure $\mathcal{D}_1$ such that for any $S \subseteq \mathbb{R}^2$ of size n,* $\mathsf{Space}(\mathcal{D}_1(S)) = O(n \log^2 n)$ *and* $\mathsf{Qtime}(\mathcal{D}_1(S)) = O(\log^2 n)$. *Furthermore, the above data structure can be constructed in $O(n \log^4 n)$ worst-case time.*

Proof We first analyze the space cost. Let $\mathbf{v}$ be a secondary node of the 2D range tree. By Theorem 2.2, the space cost of the sub-structures stored at $\mathbf{v}$ is $O(|S(\mathbf{v})|)$. Therefore, for a primary node $\mathbf{u} \in \mathcal{T}$ of the 2D range tree, the space cost of $\mathcal{T}_\mathbf{u}$ (with the sub-structures) is $O(|S(\mathbf{u})| \log |S(\mathbf{u})|)$. As a result, the entire space cost of the 2D range tree is $O(n \log^2 n)$. Let $\mathbf{u}' \in \mathcal{T}'$ be a node of the 1D range tree $\mathcal{T}'$. By Theorem 3.3 and Lemma 4.3, the space cost of the sub-structures stored at $\mathbf{u}'$ is $O(|S(\mathbf{u}')| \log |S(\mathbf{u}')|)$. As such, the entire space cost of $\mathcal{T}'$ is $O(n \log^2 n)$. For the same reason, the space cost of $\mathcal{T}''$ is $O(n \log^2 n)$, and hence the entire space cost of $\mathcal{D}_1$ is $O(n \log^2 n)$.

Next, we analyze the query time. When answering a query, we need to compute the pairs $\phi, \phi_\alpha, \phi_\beta$ in Lemma 4.1. To compute ϕ, we first find the splitting nodes $\mathbf{u} \in \mathcal{T}$ and $\mathbf{v} \in \mathcal{T}_\mathbf{u}$. This is done by a top-down walk in $\mathcal{T}$ and $\mathcal{T}_\mathbf{u}$, which takes $O(\log n)$ time. Then we query the sub-structures $\mathcal{A}(S_1(\mathbf{v})), \ldots, \mathcal{A}(S_4(\mathbf{v}))$, which can be done in $O(\log n)$ time by Theorem 2.2. Thus, the time for computing ϕ is $O(\log n)$. To

compute ϕ_α, we first find the $t = O(\log n)$ canonical nodes $\mathbf{u}'_1, \ldots, \mathbf{u}'_t \in \mathcal{T}'$, which can be done in $O(\log n)$ time. Then we query the sub-structures $\mathcal{B}(S(\mathbf{u}'_1)), \ldots, \mathcal{B}(S(\mathbf{u}'_t))$ and $\mathcal{K}^{\mathrm{v}}(S(\mathbf{u}'_1)), \ldots, \mathcal{K}^{\mathrm{v}}(S(\mathbf{u}'_t))$ to obtain the pairs $\phi_1, \ldots, \phi_t$ and the set K of $2kt$ points. By Theorem 3.3 and Lemma 4.3, this step can be done in $O(\log^2 n)$ time. Finally, we compute the closest-pair ϕ_K in K using the standard divide-and-conquer algorithm, which takes $O(\log n \log\log n)$ time since $|K| = O(\log n)$. Thus, the time for computing ϕ_α is $O(\log^2 n)$, so is the time for computing ϕ_β. As a result, the overall query time is $O(\log^2 n)$.

Finally, we analyze the preprocessing time. By Theorem 2.2, to build the sub-structures stored at each secondary node $\mathbf{v}$ of the 2D range tree takes $O(|S(\mathbf{v})| \log^2 |S(\mathbf{v})|)$ time. Therefore, the time for building the 2D range tree is $O(n \log^4 n)$. By Theorem 3.3 and Lemma 4.3, to build the sub-structures at each node $\mathbf{u}$ of $\mathcal{T}'$ or $\mathcal{T}''$ takes $O(|S(\mathbf{u})| \log^2 |S(\mathbf{u})|)$ time. Therefore, the time for building the two 1D range trees $\mathcal{T}'$ and $\mathcal{T}''$ is $O(n \log^3 n)$. As a result, the overall preprocessing time is $O(n \log^4 n)$. □

Our first data structure itself already achieves the desired worst-case bounds, which simultaneously improves the results given in [6, 9].

4.3 Second Data Structure

We now introduce our second $\mathcal{R}$-RCP data structure, which has the desired average-case space cost and an $O(\log n)$ query time (even in worst-case). Furthermore, this data structure can be constructed efficiently in average-case. In our second data structure, we only use the 2D range tree presented before, but we need some additional sub-structures stored at each secondary node. Let k be the constant integer in Lemma 4.2. Define $S_{\blacktriangle}(\mathbf{v}) = S_3(\mathbf{v}) \cup S_4(\mathbf{v})$ (resp., $S_{\blacktriangledown}(\mathbf{v}) = S_1(\mathbf{v}) \cup S_2(\mathbf{v})$) as the subset of $S(\mathbf{v})$ consisting of the points above (resp., below) $l_{\mathbf{v}}$. Similarly, define $S_{\blacktriangleleft}(\mathbf{v})$ and $S_{\blacktriangleright}(\mathbf{v})$ as the subsets to the left and right of $l_{\mathbf{u}}$, respectively. Let $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$ be a secondary node. Besides $\mathcal{A}(S_1(\mathbf{v})), \ldots, \mathcal{A}(S_4(\mathbf{v}))$, we store at $\mathbf{v}$ two $(\mathcal{P}_{l_{\mathbf{u}}}, k)$-TBEP data structures $\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}))$, $\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangledown}(\mathbf{v}))$ (Lemma 4.4) and two $(\mathcal{P}_{l_{\mathbf{v}}}, k)$-LREP data structures $\mathcal{K}_{l_{\mathbf{v}}}(S_{\blacktriangleleft}(\mathbf{v}))$, $\mathcal{K}_{l_{\mathbf{v}}}(S_{\blacktriangleright}(\mathbf{v}))$ (Lemma 4.4).

Furthermore, we need a new kind of sub-structures called *range shortest-segment* (RSS) data structures. For a query space $\mathcal{X}$, an $\mathcal{X}$-RSS data structure stores a given set of segments in $\mathbb{R}^2$ and can report the shortest segment contained in a query range $X \in \mathcal{X}$. For the case $\mathcal{X} = \mathcal{U}$, we have the following RSS data structure.

Lemma 4.6 *There exists a $\mathcal{U}$-RSS data structure $\mathcal{C}$ such that for any set G of m segments,* $\mathsf{Space}(\mathcal{C}(G)) = O(m^2)$ *and* $\mathsf{Qtime}(\mathcal{C}(G)) = O(\log m)$. *Furthermore, the above data structure can be built in $O(m^2 \log m)$ worst-case time.*

Proof It suffices to design the $\mathcal{U}^{\downarrow}$-RSS data structure. We first notice the existence of a $\mathcal{P}^{\mathrm{v}}$-RSS data structure using $O(m)$ space with $O(\log m)$ query time, which can be built in $O(m \log m)$ time. Indeed, by applying the method in Sect. 3, we immediately obtain this data structure (a segment here corresponds to a candidate pair in Sect. 3).

With this $\mathcal{P}^{\mathrm{v}}$-RSS data structure in hand, it is quite straightforward to design the desired $\mathcal{U}^{\downarrow}$-RSS data structure. Let $G = \{\sigma_1, \ldots, \sigma_m\}$ be a set of m segments where

$\sigma_i = [a_i, b_i]$. Define $y_i = \max\{a_i.y, b_i.y\}$ and assume $y_1 \leq \ldots \leq y_m$. Now build a (balanced) binary search tree with keys $y_1, \ldots, y_m$. We denote by $\mathbf{u}_i$ the node corresponding to y_i. At $\mathbf{u}_i$, we build a $\mathcal{P}^{\mathrm{v}}$-RSS data structure described above built on the subset $G_i = \{\sigma_1, \ldots, \sigma_i\} \subseteq G$. The overall space cost is clearly $O(m^2)$, and the time for constructing the data structure is $O(m^2 \log m)$.

To answer a query $U = [x_1, x_2] \times (-\infty, y] \in \mathcal{U}^{\downarrow}$, we first use the binary search tree to find the maximum y_i that is less than or equal to y. This can be done in $O(\log m)$ time. Let $P = [x_1, x_2] \times \mathbb{R} \in \mathcal{P}^{\mathrm{v}}$. Note that a segment $\sigma \in G$ is contained in U iff $\sigma \in G_i$ and σ is contained in P. Thus, we can find the desired segment by querying the $\mathcal{P}^{\mathrm{v}}$-RSS data structure stored at $\mathbf{u}_i$ with P, which takes $O(\log m)$ time. □

We now define $\Phi_{\blacktriangle}(\mathbf{v}) = \Phi_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}), \mathcal{U}^{\downarrow})$, $\Phi_{\blacktriangledown}(\mathbf{v}) = \Phi_{l_{\mathbf{u}}}(S_{\blacktriangledown}(\mathbf{v}), \mathcal{U}^{\uparrow})$, $\Phi_{\blacktriangleleft}(\mathbf{v}) = \Phi_{l_{\mathbf{v}}}(S_{\blacktriangleleft}(\mathbf{v}), \mathcal{U}^{\rightarrow})$, and $\Phi_{\blacktriangleright}(\mathbf{v}) = \Phi_{l_{\mathbf{v}}}(S_{\blacktriangleright}(\mathbf{v}), \mathcal{U}^{\leftarrow})$. We can view $\Phi_{\blacktriangle}(\mathbf{v}), \Phi_{\blacktriangledown}(\mathbf{v}), \Phi_{\blacktriangleleft}(\mathbf{v}), \Phi_{\blacktriangleright}(\mathbf{v})$ as four sets of segments by identifying each point-pair (a, b) as a segment $[a, b]$. Then we apply Lemma 4.6 to build and store at $\mathbf{v}$ four $\mathcal{U}$-RSS data structures $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$, $\mathcal{C}(\Phi_{\blacktriangledown}(\mathbf{v}))$, $\mathcal{C}(\Phi_{\blacktriangleleft}(\mathbf{v}))$, $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$.

We now explain how to compute ϕ_α and ϕ_β. Let us consider ϕ_α. Recall that ϕ_α is the closest-pair in $S \cap R_\alpha$, i.e., in $S(\mathbf{v}) \cap R_\alpha$. Let P be the $l_{\mathbf{u}}$-anchored strip obtained by removing the top/bottom bounding line of R_α. If the two points of ϕ_α are on opposite sides of $l_{\mathbf{v}}$, then by Lemma 4.2 its two points must be among the k bottommost points in $S_{\blacktriangle}(\mathbf{v}) \cap P$ and the k topmost points in $S_{\blacktriangledown}(\mathbf{v}) \cap P$ respectively. Using $\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}))$ and $\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangledown}(\mathbf{v}))$, we report these $2k$ points, and compute the closest-pair among them by brute-force. If the two points of ϕ_α are on the same side of $l_{\mathbf{v}}$, then they are both contained in either $S_{\blacktriangle}(\mathbf{v})$ or $S_{\blacktriangledown}(\mathbf{v})$. So it suffices to compute the closest-pairs in $S_{\blacktriangle}(\mathbf{v}) \cap R_\alpha$ and $S_{\blacktriangledown}(\mathbf{v}) \cap R_\alpha$. Without loss of generality, we only need to consider the closest-pair in $S_{\blacktriangle}(\mathbf{v}) \cap R_\alpha$. We denote by U the 3-sided rectangle obtained by removing the bottom boundary of R_α, and by Q_1 (resp., Q_2) the quadrant obtained by removing the right (resp., left) boundary of U. We query $\mathcal{A}(S_1(\mathbf{v}))$ with Q_1, $\mathcal{A}(S_2(\mathbf{v}))$ with Q_2, and $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$ with U. Clearly, the shortest one among the three answers is the closest-pair in $S_{\blacktriangle}(\mathbf{v}) \cap R_\alpha$. Indeed, the three answers are all point-pairs in $S_{\blacktriangle}(\mathbf{v}) \cap R_\alpha$. If the two points of the closest-pair in $S_{\blacktriangle}(\mathbf{v}) \cap R_\alpha$ are both to the left (resp., right) of $l_{\mathbf{u}}$, $\mathcal{A}(S_1(\mathbf{v}))$ (resp., $\mathcal{A}(S_2(\mathbf{v}))$) reports it; otherwise, the closest-pair crosses $l_{\mathbf{u}}$, and $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$ reports it. Now we see how to compute ϕ_α, and ϕ_β can be computed symmetrically. Finally, taking the shortest one among $\{\phi, \phi_\alpha, \phi_\beta\}$, the query R can be answered.

A straightforward analysis shows that the overall query time is $O(\log n)$ even in worst-case. The worst-case space cost is not near-linear, as the $\mathcal{U}$-RSS data structure $\mathcal{C}$ may occupy quadratic space by Lemma 4.6. Interestingly, we can show that the average-case space cost is in fact $O(n \log n)$. The crucial thing is to bound the average-case space of the sub-structures stored at the secondary nodes. The intuition for bounding the average-case space of the $\mathcal{Q}$-RCP and TBEP/LREP sub-structures comes directly from the average-case performance of our $\mathcal{Q}$-RCP data structure (Theorem 2.2) and TBEP/LREP data structure (Lemma 4.4).

However, to bound the average-case space of the $\mathcal{U}$-RSS sub-structures is much more difficult. By our construction, the segments stored in these sub-structures are 3-sided candidate pairs that cross a line. As such, we have to study the expected num-

ber of such candidate pairs in a random dataset. To this end, we recall Lemma 3.1. Let l be a vertical line, and $S \propto \prod_{i=1}^{n} I_i$ be a random dataset drawn from vertical aligned segments $I_1, \ldots, I_n$ as in Lemma 3.1. Suppose we build a $\mathcal{U}$-RSS data structure $\mathcal{C}(\Phi)$ on $\Phi = \Phi_l(S, \mathcal{U}^{\downarrow})$. Using Lemma 3.1, a direct calculation gives us $\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|] = O(\log^2 n)$. Unfortunately, this is not sufficient for bounding the average-case space of $\mathcal{C}(\Phi)$, because $\mathbb{E}[\mathsf{Space}(\mathcal{C}(\Phi))] = O(\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|^2])$ and in general $\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|^2] \neq \mathbb{E}^2[|\Phi_l(S, \mathcal{U}^{\downarrow})|]$. Therefore, we need a bound for $\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|^2]$, which can also be obtained using Lemma 3.1, but requires nontrivial work.

Lemma 4.7 *Let l be a vertical (resp., horizontal) line and $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are distinct vertical (resp., horizontal) aligned segments. Then for $\mathcal{X} \in \{\mathcal{U}^{\downarrow}, \mathcal{U}^{\uparrow}\}$ (resp., $\mathcal{X} \in \{\mathcal{U}^{\leftarrow}, \mathcal{U}^{\rightarrow}\}$), $\mathbb{E}[|\Phi_l(S, \mathcal{X})|] = O(\log^2 n)$ and $\mathbb{E}[|\Phi_l(S, \mathcal{X})|^2] = O(\log^4 n)$.*

With the above lemma in hand, we are ready to bound the average-case space cost of our second data structure. Before this, let us first consider the preprocessing of this data structure. We are not able to bound the worst-case preprocessing time (as the worst-case space cost of the data structure is not well-bounded). Therefore, what we want here is a preprocessing algorithm with near-linear average-case running time. The 2D range tree itself and its $\mathcal{Q}$-RCP sub-structures can be built in the same way as in our first data structure using $O(n \log^4 n)$ worst time. The TBEP/LREP sub-structures on each secondary node $\mathbf{v}$ can be built in $O(|S(\mathbf{v})| \log |S(\mathbf{v})|)$ time by Lemma 4.4; hence the overall time cost for this part is $O(n \log^3 n)$. It suffices to consider the $\mathcal{U}$-RSS sub-structures. The difficulty here is how to find efficiently the 3-sided candidate pairs that cross a line; as long as we have these candidate pairs in hand, the $\mathcal{U}$-RSS data structure can be built in $O(m^2 \log m)$ time by Lemma 4.6 (where m is the number of the pairs).

Lemma 4.8 *Let l be a vertical (resp., horizontal) line and $\mathcal{X} \in \{\mathcal{U}^{\downarrow}, \mathcal{U}^{\uparrow}\}$ (resp., $\mathcal{X} \in \{\mathcal{U}^{\leftarrow}, \mathcal{U}^{\rightarrow}\}$). Then there exists an algorithm computing $\Phi_l(S, \mathcal{X})$ for an input dataset $S \subseteq \mathbb{R}^2$ which runs in $O(n \log^4 n)$ average-case time for a random $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are distinct vertical (resp., horizontal) aligned segments. Therefore, one can construct the $\mathcal{U}$-RSS data structure $\mathcal{C}(\Phi_l(S, \mathcal{X}))$ described in Lemma 4.6 in $O(n \log^4 n)$ average-case time for a random $S \propto \prod_{i=1}^{n} I_i$.*

Proof It suffices to consider the case in which l is a vertical line. Suppose $\mathcal{X} = \mathcal{U}^{\downarrow}$. Let $S \subseteq \mathbb{R}^2$ be a dataset of size n. We compute $\Phi_l(S, \mathcal{U}^{\downarrow})$ as follows. First, we compute the rectangle candidate pairs $\Phi(S, \mathcal{R})$. As observed in [6], $\Phi(S, \mathcal{R})$ can be computed in $O(n \log n + \Delta)$ time where $\Delta = |\Phi(S, \mathcal{R})|$. Note that $\Phi_l(S, \mathcal{U}^{\downarrow}) \subseteq \Phi(S, \mathcal{U}^{\downarrow}) \subseteq \Phi(S, \mathcal{R})$.

To further compute $\Phi(S, \mathcal{U}^{\downarrow})$ from $\Phi(S, \mathcal{R})$, we build an $\mathcal{R}$-RCP data structure $\mathcal{D}_1(S)$ as described in Theorem 4.5, which takes $O(n \log^4 n)$ time. For each $\phi = (a, b) \in \Phi(S, \mathcal{R})$, let $U_\phi = [\min\{a.x, b.x\}, \max\{a.x, b.x\}] \times (-\infty, \max\{a.y, b.y\}] \in \mathcal{U}^{\downarrow}$ be the smallest bottom-unbounded 3-sided rectangle that contains ϕ. Clearly, $\phi \in \Phi_l(S, \mathcal{U}^{\downarrow})$ iff ϕ crosses l and is the closest-pair in $S \cap U_\phi$. Therefore, for each $\phi \in \Phi(S, \mathcal{R})$ that crosses l, we can query $\mathcal{D}_1(S)$ with U_ϕ (note that an $\mathcal{R}$-RCP data structure can also answer 3-sided rectangle queries) to

check if $\phi \in \Phi_l(S, \mathcal{U}^{\downarrow})$. Since the query time of $\mathcal{D}_1(S)$ is $O(\log^2 n)$, we can compute $\Phi_l(S, \mathcal{U}^{\downarrow})$ in $O(\Delta \log^2 n)$ time after $\mathcal{D}_1(S)$ is built. Therefore, the total time for computing $\Phi_l(S, \mathcal{U}^{\downarrow})$ is $O(n \log^4 n + \Delta \log^2 n)$.

Gupta et al. [6] proved that $\mathbb{E}[\Delta] = O(n \log n)$ when S is generated uniformly and independently from the unit square. Although $S \propto \prod_{i=1}^{n} I_i$ in our setting, the same proof actually results in the same bound, i.e., $\mathbb{E}[\Delta] = O(n \log n)$. As such, our algorithm for computing $\Phi_l(S, \mathcal{U}^{\downarrow})$ takes $O(n \log^4 n)$ average-case time when $S \propto \prod_{i=1}^{n} I_i$. To further build the data structure $\mathcal{C}(\Phi_l(S, \mathcal{X}))$, we apply Lemma 4.6, which takes $O(m^2 \log m)$ time where $m = |\Phi_l(S, \mathcal{X})|$. Lemma 4.7 shows that $\mathbb{E}[m^2] = O(\log^4 n)$, which implies $\mathbb{E}[m^2 \log m] = O(\log^5 n)$ because the maximum possible value of the random variable m is bounded by $O(n^2)$. Therefore, $\mathcal{C}(\Phi_l(S, \mathcal{X}))$ can be constructed in $O(n \log^4 n)$ average-case time when $S \propto \prod_{i=1}^{n} I_i$, including the time for computing $\Phi_l(S, \mathcal{U}^{\downarrow})$. □

Now we are ready to prove the performance of our second data structure.

Theorem 4.9 *There exists an $\mathcal{R}$-RCP data structure $\mathcal{D}_2$ such that:*

- *For any $S \subseteq \mathbb{R}^2$ of size n, $\mathsf{Qtime}(\mathcal{D}_2(S)) = O(\log n)$.*
- *For a random $S \propto R^n$ where R is the unit square or more generally an arbitrary axes-parallel rectangle, $\mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S))] = O(n \log n)$.*

Furthermore, the above data structure can be constructed in $O(n \log^6 n)$ average-case time.

4.4 Combining the Two Data Structures

We now combine the two data structures $\mathcal{D}_1$ (Theorem 4.5) and $\mathcal{D}_2$ (Theorem 4.9) to obtain a *single* data structure $\mathcal{D}$ that achieves the desired worst-case and average-case bounds simultaneously (and can be constructed in near-linear time). For a dataset $S \subseteq \mathbb{R}^2$ of size n, if $\mathsf{Space}(\mathcal{D}_2(S)) \geq n \log^2 n$ or the time for constructing $\mathcal{D}_2(S)$ (using Theorem 4.9) is greater than $n \log^7 n$, we set $\mathcal{D}(S) = \mathcal{D}_1(S)$, otherwise we set $\mathcal{D}(S) = \mathcal{D}_2(S)$. The reason for the choice of the thresholds $n \log^2 n$ and $n \log^7 n$ will be clear shortly (in the proof of Theorem 4.10). The worst-case bounds of $\mathcal{D}$ follow directly, while to see the average-case bounds of $\mathcal{D}$ requires a careful analysis using Markov's inequality.

Theorem 4.10 *There exists an $\mathcal{R}$-RCP data structure $\mathcal{D}$ such that:*

- *For any $S \subseteq \mathbb{R}^2$ of size n,*

$$\mathsf{Space}(\mathcal{D}(S)) = O(n \log^2 n) \quad \textit{and} \quad \mathsf{Qtime}(\mathcal{D}(S)) = O(\log^2 n).$$

- *For a random $S \propto R^n$ where R is the unit square or more generally an arbitrary axes-parallel rectangle,*

$$\mathbb{E}[\mathsf{Space}(\mathcal{D}(S))] = O(n \log n) \quad \textit{and} \quad \mathbb{E}[\mathsf{Qtime}(\mathcal{D}(S))] = O(\log n).$$

Furthermore, the above data structure can be constructed in $O(n \log^7 n)$ worst-case time.

Proof As mentioned above, our data structure $\mathcal{D}$ is obtained by combining $\mathcal{D}_1$ (Theorem 4.5) and $\mathcal{D}_2$ (Theorem 4.9) as follows. For any $S \subseteq \mathbb{R}^2$ of size n, if $\mathsf{Space}(\mathcal{D}_2(S)) \geq n\log^2 n$ or the time for constructing $\mathcal{D}_2(S)$ (using Theorem 4.9) is greater than $n\log^7 n$, we set $\mathcal{D}(S) = \mathcal{D}_1(S)$, otherwise we set $\mathcal{D}(S) = \mathcal{D}_2(S)$. (The reason for choosing $n\log^7 n$ will be clear at the end of the proof.) We claim that $\mathcal{D}$ has the desired perfomance.

We first consider the space cost and query time. Let $S \subseteq \mathbb{R}^2$ be a dataset of size n. It is clear from the construction that $\mathsf{Space}(\mathcal{D}(S)) = O(n\log^2 n)$. Also, $\mathsf{Qtime}(\mathcal{D}(S)) = O(\log^2 n)$, since $\mathsf{Qtime}(\mathcal{D}_1(S)) = O(\log^2 n)$ and $\mathsf{Qtime}(\mathcal{D}_2(S)) = O(\log n)$. To analyze the average-case space and query time of $\mathcal{D}$, let $S \propto R^n$ for an axes-parallel rectangle R. Define E as the event $\mathsf{Space}(\mathcal{D}_2(S)) \geq n\log^2 n$ and E' as the event that the time for constructing $\mathcal{D}_2(S)$ (using Theorem 4.9) is greater than $n\log^7 n$. Let $\neg E$ and $\neg E'$ be the complement events of E and E', respectively. Since $\mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S))] = O(n\log n)$, we have $\Pr[E] = O(1/\log n)$ by Markov's inequality. Similarly, since the average-case preprocessing time of $\mathcal{D}_2$ is $O(n\log^6 n)$, we also have $\Pr[E'] = O(1/\log n)$ by Markov's inequality. Therefore, $\Pr[E''] = O(1/\log n)$ for $E'' = E \vee E'$.

To bound the average-case space cost, we observe

$$\begin{aligned}\mathbb{E}[\mathsf{Space}(\mathcal{D}(S))] = {} & \Pr[E''] \cdot \mathbb{E}[\mathsf{Space}(\mathcal{D}_1(S)) \,|\, E''] \\ & + \Pr[\neg E''] \cdot \mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S)) \,|\, \neg E''].\end{aligned}$$

Note that $\Pr[E''] \cdot \mathbb{E}[\mathsf{Space}(\mathcal{D}_1(S)) \,|\, E''] = O(n\log n)$, since $\mathsf{Space}(\mathcal{D}_1(S)) = O(n\log^2 n)$ and $\Pr[E''] = O(1/\log n)$. Also, $\Pr[\neg E''] \cdot \mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S)) \,|\, \neg E''] \leq \mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S))] = O(n\log n)$. Thus, $\mathbb{E}[\mathsf{Space}(\mathcal{D}(S))] = O(n\log n)$. To bound the average-case query time, let T_i be the worst-case query time of $\mathcal{D}_i$ built on a dataset of size n, for $i \in \{1, 2\}$. Then $\mathbb{E}[\mathsf{Qtime}(\mathcal{D}(S))] \leq \Pr[E''] \cdot T_1 + \Pr[\neg E''] \cdot T_2$. Since $T_1 = O(\log^2 n)$, $T_2 = O(\log n)$, $\Pr[E] = O(1/\log n)$, we have $\mathbb{E}[\mathsf{Qtime}(\mathcal{D}(S))] = O(\log n)$.

Finally, we consider how to construct $\mathcal{D}(S)$ in $O(n\log^7 n)$ time for a given dataset $S \subseteq \mathbb{R}^2$ of size n. We first try to build $\mathcal{D}_2(S)$ using the algorithm in Theorem 4.9. We run the algorithm for $n\log^7 n$ time units. If $\mathcal{D}_2(S)$ is not successfully constructed in such amount of time, we terminate the algorithm and build $\mathcal{D}_1(S)$ in $O(n\log^4 n)$ time using Theorem 4.5, since we have $\mathcal{D}(S) = \mathcal{D}_1(S)$ in this case. Assume $\mathcal{D}_2(S)$ is successfully constructed in $n\log^7 n$ time. We then check if $\mathsf{Space}(\mathcal{D}_2(S)) \leq n\log^2 n$. If so, $\mathcal{D}(S) = \mathcal{D}_2(S)$ and we are done. Otherwise, we have $\mathcal{D}(S) = \mathcal{D}_1(S)$, and thus we build $\mathcal{D}_1(S)$ in $O(n\log^4 n)$ time. By the above argument, we can construct $\mathcal{D}(S)$ in $O(n\log^7 n)$ worst-case time. □

5 Halfplane Query

We consider the RCP problem for halfplane queries, i.e., the $\mathcal{H}$-RCP problem. In order to solve the $\mathcal{H}$-RCP problem, it suffices to consider the $\mathcal{H}^{\uparrow}$-RCP problem. Let $S \subseteq \mathbb{R}^2$ be the dataset of size n. Before discussing the details, we first give an overview of our method.

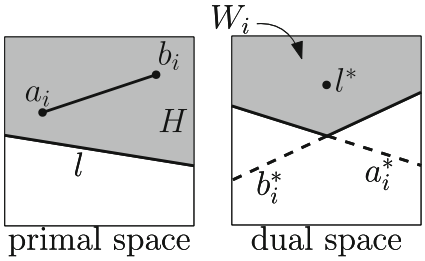


Fig. 5 Illustrating the upward-open wedge W_i

Overview. Similar to what we did for the quadrant and strip RCP problems, we consider candidate pairs. It was known [1] that the candidate pairs with respect to the query space $\mathcal{H}^{\uparrow}$ do not cross each other (when regarded as segments), which implies that the candidate pairs are edges of a planar graph (with vertex set S) and hence $\Phi(S, \mathcal{H}^{\uparrow}) = O(n)$. The main difficulty here is to store these candidate pairs in a linear-space data structure that can answer halfplane queries in logarithmic time. To this end, we use duality [3] to map each candidate pair in the primal space to a wedge in the dual space. We then establish a key combinatorial result which shows that the arrangement obtained by overlaying the wedges in the dual space has a *linear* complexity. This result relies heavily on the fact that the candidate pairs are non-crossing. As a byproduct of this result, we also obtain an optimal halfspace RSS data structure for *non-crossing* segments (see Sect. 5.2). To achieve the average-case bounds, we prove that the number of candidate pairs with respect to halfplane queries is $O(\log^2 n)$ in average-case.

We begin by introducing the standard duality technique [3]. A non-vertical line $l : y = ux + v$ in $\mathbb{R}^2$ is dual to the point $l^* = (u, -v)$ and a point $p = (s, t) \in \mathbb{R}^2$ is dual to the line $p^* : y = sx - t$. A basic property of duality is that $p \in l^{\uparrow}$ (resp., $p \in l^{\downarrow}$) iff $l^* \in (p^*)^{\uparrow}$ (resp., $l^* \in (p^*)^{\downarrow}$). To make the exposition cleaner, we distinguish between *primal space* and *dual space*, which are two copies of $\mathbb{R}^2$. The dataset S and query ranges are assumed to lie in the primal space, while their dual objects are assumed to lie in the dual space.

Duality allows us to transform the $\mathcal{H}^{\uparrow}$-RCP problem into a point location problem as follows. Let $H = l^{\uparrow} \in \mathcal{H}^{\uparrow}$ be a query range. The line l bounding H is dual to the point l^* in the dual space; for convenience, we also call l^* the dual point of H. If we decompose the dual space into "cells" such that the query ranges whose dual points lie in the same cell have the same answer, then point location techniques can be applied to solve the problem directly. Note that this decomposition must be a polygonal subdivision Γ of $\mathbb{R}^2$, which consists of vertices, straight-line edges, and polygonal faces (i.e., cells). This is because the cell-boundaries must be defined by the dual lines of the points in S.

In order to analyze the space cost and query time, we need to study the complexity $|\Gamma|$ of Γ. An $O(n^2)$ trivial upper bound for $|\Gamma|$ follows from the fact that the subdivision formed by the n dual lines of the points in S has an $O(n^2)$ complexity. Surprisingly, using additional properties of the problem, we can show that $|\Gamma| = O(n)$, which is a key ingredient of our result in this section.

Suppose $\Phi(S, \mathcal{H}^{\uparrow}) = \{\phi_1, \ldots, \phi_m\}$ where $\phi_i = (a_i, b_i)$ and $\phi_1, \ldots, \phi_m$ are sorted in increasing order of their lengths. It was shown in [1] that $m = O(n)$, and the candidate pairs do not cross each other (when identified as segments), i.e., the segments $[a_i, b_i]$ and $[a_j, b_j]$ do not cross for any $i \neq j$. The non-crossing property of the candidate pairs is important and will be used later for proving Lemma 5.1. With this in hand, we now consider the subdivision Γ.

Let $H = l^{\uparrow} \in \mathcal{H}^{\uparrow}$ be a query range. By the property of duality, ϕ_i is contained in H iff $l^* \in (a_i^*)^{\uparrow}$ and $l^* \in (b_i^*)^{\uparrow}$, i.e., l^* is in the upward-open *wedge* W_i generated by the lines a_i^* and b_i^* (in the dual space); see Fig. 5. As such, the closest-pair in $S \cap H$ to be reported is ϕ_η for $\eta = \min\{i : l^* \in W_i\}$. Therefore, Γ can be constructed by successively overlaying the wedges $W_1, \ldots, W_m$ (similarly to what we see in Sect. 2).

Formally, we begin with a trivial subdivision Γ_0 of $\mathbb{R}^2$, which consists of only one face, the entire plane. Suppose Γ_{i-1} is constructed, which has an *outer face* F_{i-1} equal to the complement of $\bigcup_{j=1}^{i-1} W_j$ in $\mathbb{R}^2$. Now we construct a new subdivision Γ_i by "inserting" W_i to Γ_{i-1}. Specifically, Γ_i is obtained from Γ_{i-1} by decomposing the outer face F_{i-1} via the wedge W_i; that is, we decompose F_{i-1} into several smaller faces: one is $F_{i-1} \setminus W_i$ and the others are the connected components of $F_{i-1} \cap W_i$. Note that $F_{i-1} \setminus W_i$ is the complement of $\bigcup_{j=1}^{i} W_j$, which is connected (as one can easily verify) and becomes the outer face F_i of Γ_i. In this way, we construct $\Gamma_1, \ldots, \Gamma_m$ in order, and it is clear that $\Gamma_m = \Gamma$. The linear upper bound for $|\Gamma|$ follows from the following technical result.

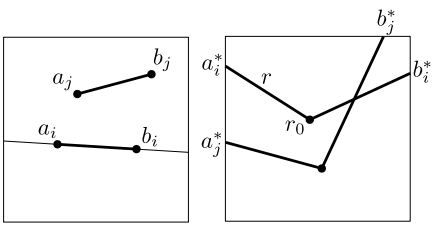
Lemma 5.1 $|\Gamma_i| - |\Gamma_{i-1}| = O(1)$ *for* $i \in \{1, \ldots, m\}$. *In particular,* $|\Gamma| = O(m)$.

Proof Let F_i be the outer face of Γ_i, and ∂W_i be the boundary of the wedge W_i (which consists of two rays emanating from the intersection point of a_i^* and b_i^*). We first note that, to deduce that $|\Gamma_i| - |\Gamma_{i-1}| = O(1)$, it suffices to show that the number of the connected components of $\partial W_i \cap F_{i-1}$ is constant. This is because every connected component of $\partial W_i \cap F_{i-1}$ contributes to Γ_i exactly one new face, a constant number of new vertices, and a constant number of new edges. Indeed, we only need to check one branch of ∂W_i (i.e., one of the two rays of ∂W_i), say the ray contained in a_i^* (we denote it by r). We will show that $r \cap F_{i-1}$ has $O(1)$ connected components.

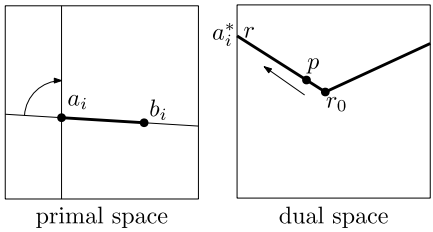
Without loss of generality, we may assume that a_i is to the left of b_i. Then each point on r is dual to a line in the primal space, which goes through the point a_i with the segment $[a_i, b_i]$ above it. Note that $r \cap F_{i-1} = r \setminus \bigcup_{j=1}^{i-1} W_j = r \setminus \bigcup_{j=1}^{i-1} (r \cap W_j)$, and each $r \cap W_j$ is a connected portion of r. We consider each $j \in \{1, \ldots, i-1\}$ and analyze the intersection $r \cap W_j$. Let l_i be the line through a_i and b_i. There are three cases to be considered separately: (1) $a_j, b_j \in l_i^{\uparrow}$, (2) $a_j, b_j \in l_i^{\downarrow}$, or (3) one of a_j, b_j is in $l_i^{\uparrow} \setminus l_i$ (i.e., strictly above l_i) while the other is in $l_i^{\downarrow} \setminus l_i$ (i.e., strictly below l_i).

[Case 1] In this case, $a_j, b_j \in l_i^{\uparrow}$. The wedge W_j must contain the initial point r_0 of r (i.e., the intersection point of a_i^* and b_i^*, which is the dual of the line l_i), because $r_0 \in (a_j^*)^{\uparrow}$ and $r_0 \in (b_j^*)^{\uparrow}$. (See Fig. 6a.)

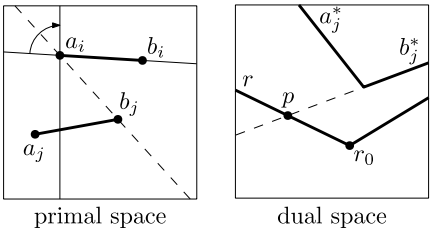
[Case 2] In this case, $a_j, b_j \in l_i^{\downarrow}$. We claim that either $r \cap W_j$ is empty or it contains the infinite end of r (i.e., the point at infinity along r). Imagine that we have a point p moving along r from r_0 to the infinite end of r. Then p is dual to a line in the primal



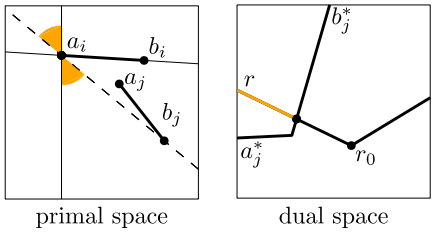
(a) An illustration of Case 1



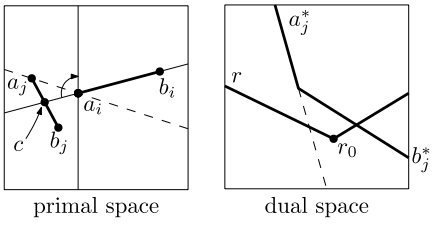
(b) Moving p and rotating line $a_i b_i$ clockwise around a_i



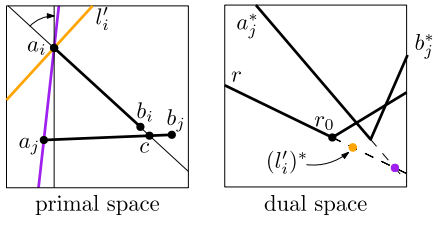
(c) An example for Case 2 where $r \cap W_j = \emptyset$



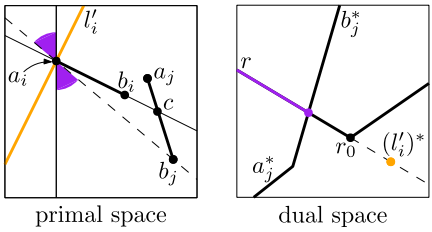
(d) An example for Case 2 where $r \cap W_j \neq \emptyset$



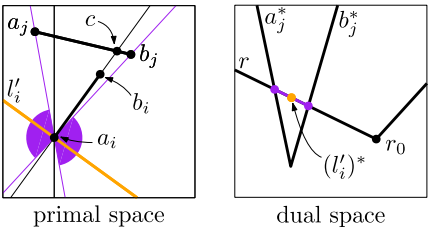
(e) An example for Case 3 where $a_j b_j$ is to the left and $r \cap W_j = \emptyset$



(f) An example for Case 3 where $(l_i')^* \notin r$ and $r \cap W_j = \emptyset$



(g) An example for Case 3 where $(l_i')^* \notin r$ and $r \cap W_j \neq \emptyset$



(h) An example for Case 3 where $(l_i')^* \in r$ and $r \cap W_j$ contains $(l_i')^*$ but does not contain r_0 or the infinite end of r

Fig. 6 Illustrating the various cases in Lemma 5.1

space rotating clockwise around a_i from the line l_i to the vertical line through a_i; see Fig. 6b. Note that $p \in r \cap W_j$ (in the dual space) only when $a_j, b_j \in (p^*)^{\uparrow}$ (in the primal space). But $a_j, b_j \in l_i^{\downarrow}$ in this case. When p is moving, the region $l_i^{\downarrow} \cap (p^*)^{\uparrow}$

expands. As such, one can easily see that $r \cap W_j$ must contain the infinite end of r if it is nonempty. (See Fig. 6, c and d.)

[Case 3] In this case, one point of a_j, b_j is in $l_i^{\uparrow} \setminus l_i$ while the other is in $l_i^{\downarrow} \setminus l_i$. Thus, the segment $[a_j, b_j]$ must intersect the line l_i. However, as argued before, the segments $[a_j, b_j]$ and $[a_i, b_i]$ do not cross. So the intersection point c of $[a_j, b_j]$ and l_i is either to the left of a_i or to the right of b_i (recall that a_i is assumed to be to the left of b_i).

If c is to the left of a_i, we claim that $r \cap W_j$ is empty. Observe that the dual line of any point on r is through a_i and below b_i, meaning that it must be above c (as c is to the left of a_i). In other words, the dual line of any point on r is above at least one of a_j, b_j, and thus any point on r is not contained in the wedge W_j, i.e., $r \cap W_j$ is empty. (See Fig. 6e.)

The most subtle case occurs when c is to the right of b_i. In such a case, we consider the line through a_i perpendicular to l_i, which we denote by l_i'. We first argue that both a_j and b_j must be on the same side of l_i' as b_i. Since c is to the right of b_i, at least one of a_j, b_j is on the same side of l_i' as b_i. However, we notice that $[a_j, b_j]$ cannot intersect l_i', otherwise the length of ϕ_j is (strictly) greater than that of ϕ_i, contradicting the fact that $j < i$ (recall that $\phi_1, \ldots, \phi_m$ is sorted in increasing order of their lengths). So the only possibility is that a_j, b_j, b_i are on the same side of l_i'. Now we further have two sub-cases.

- l_i' has no dual point (i.e., l_i' is vertical) or its dual point $(l_i')^*$ is not on the ray r. In this case, consider a point p moving along r from r_0 to the infinite end of r. Clearly, when p moves, the region $(l_i')^{\rightarrow} \cap (p^*)^{\uparrow}$ expands. Thus, either $r \cap W_j$ is empty or it contains the infinite end of r. (See Fig. 6, f and g.)
- $(l_i')^*$ is on r. Then $r \cap W_j$ may be a connected portion of r containing neither r_0 nor the infinite end of r. However, as $b_i \in (l_i')^{\uparrow}$ in this case, we have $a_j, b_j \in (l_i')^{\uparrow}$ (recall that a_j, b_j, b_i are on the same side of l_i'). This implies that $r \cap W_j$ contains $(l_i')^*$. (See Fig. 6h.)

In sum, we conclude that for any $j \in \{1, \ldots, i-1\}$, the intersection $r \cap W_j$ might be **(i)** empty, or **(ii)** a connected portion of r containing r_0, or **(iii)** a connected portion of r containing the infinite end of r, or **(iv)** a connected portion of r containing $(l_i')^*$ (if $(l_i')^*$ is on r). As such, the union $\bigcup_{j=1}^{i-1}(r \cap W_j)$ can have at most three connected components, among which one contains r_0, one contains the infinite end of r, and one contains $(l_i')^*$. Therefore, the complement of $\bigcup_{j=1}^{i-1}(r \cap W_j)$ in r, i.e., $r \cap F_{i-1}$, has at most two connected components. This in turn implies that $\partial W_i \cap F_{i-1}$ has only a constant number of connected components, and hence $|\Gamma_i| - |\Gamma_{i-1}| = O(1)$. Finally, since $|\Gamma_0| = O(1)$ and $m = O(n)$, we immediately have $|\Gamma| = |\Gamma_m| = O(m)$. □

With the above result in hand, we can build an optimal point-location data structure for Γ using $O(m)$ space with $O(\log m)$ query time to solve the RCP problem. Since $m = O(n)$, we obtain an $\mathcal{H}$-RCP data structure using $O(n)$ space and $O(\log n)$ query time in worst-case.

Next, we analyze the average-case bounds of the above data structure. In fact, it suffices to bound the expected number of the candidate pairs. Surprisingly, we have the following poly-logarithmic bound.

Lemma 5.2 *For a random dataset $S \propto R^n$ where R is an axes-parallel rectangle, $\mathbb{E}[|\Phi(S, \mathcal{H})|] = O(\log^2 n)$.*

5.1 Preprocessing

In this section, we consider how to construct the above data structure efficiently. The key step is to construct the subdivision Γ of the dual $\mathbb{R}^2$. Since $|\Gamma| = O(n)$, once Γ is constructed, one can build in $O(n \log n)$ time the point-location data structure for Γ, and hence our $\mathcal{H}^{\uparrow}$-RCP data structure.

Let us first consider an easier task, in which $\Phi(S, \mathcal{H}^{\uparrow})$ is already given beforehand. In this case, we show that Γ can be constructed in $O(n \log n)$ time. As in Sect. 5, suppose $\Phi(S, \mathcal{H}^{\uparrow}) = \{\phi_1, \ldots, \phi_m\}$ where $\phi_1, \ldots, \phi_m$ are sorted in increasing order of their lengths. Recall that in Sect. 5 we defined the m subdivisions $\Gamma_0, \ldots, \Gamma_m$. Our basic idea for constructing Γ is to begin with Γ_0 and iteratively construct Γ_i from Γ_{i-1} by inserting the wedge W_i dual to ϕ_i. In this process, a crucial thing is to maintain the outer face F_i (or its boundary). Note that the boundary ∂F_i of F_i (i.e., the upper envelope of F_i) is an x-monotone polygonal chain consisting of segments and two infinite rays; we call these kinds of chains *left-right polylines* and call their pieces *fractions*. Naturally, a binary search tree can be used to store a left-right polyline; the keys are its fractions in the left-right order. Therefore, we shall use a (balanced) BST $\mathcal{T}$ to maintain ∂F_i. That is, at the end of the i-th iteration, we guarantee the left-right polyline stored in $\mathcal{T}$ is ∂F_i. At each node of $\mathcal{T}$, besides storing the corresponding fraction, we also store the wedge W_j which contributes this fraction.

Suppose we are now at the beginning of the i-th iteration. We have Γ_{i-1} in hand and $\mathcal{T}$ stores ∂F_{i-1}. We need to "insert" the wedge W_i to generate Γ_i from Γ_{i-1}, and update $\mathcal{T}$. To this end, the first step is to compute $\partial W_i \cap F_{i-1}$. Now let us assume in advance that $\partial W_i \cap F_{i-1}$ is already computed in $O(\log |\mathcal{T}|)$ time; later we will explain how to achieve this. With $\partial W_i \cap F_{i-1}$ in hand, to construct Γ_i is fairly easy. By the proof of Lemma 5.1, $\partial W_i \cap F_{i-1}$ has $O(1)$ connected components. We consider these components one-by-one. Let ξ be a component, which is an x-monotone polygonal chain with endpoints (if any) on ∂F_{i-1} (indeed, ξ consists of at most two pieces as it is a portion of ∂W_i). For convenience, assume ξ has a left endpoint u and a right endpoint v. Then ξ contributes a new (inner) face to Γ_i, which is the region bounded by ξ and the portion σ of ∂F_{i-1} between u, v. We then use $\mathcal{T}$ to report all the fractions of ∂F_{i-1} that intersect σ in left-right order, using which the corresponding new face can be directly constructed. The time cost for reporting the fractions is $O(\log |\mathcal{T}| + k)$ where k is the number of the reported fractions, since they are consecutive in $\mathcal{T}$.

After all the components are considered, we can construct Γ_i by adding the new faces to Γ_{i-1} (and adjusting the involved edges/vertices if needed). As there are $O(1)$ components, the total time cost for constructing Γ_i from Γ_{i-1} is $O(\log |\mathcal{T}| + K_i)$, where K_i is the total number of the fractions reported from $\mathcal{T}$. But we can charge the reported fractions to the corresponding new faces, and the fractions charged to each face are at most as many as its edges. Therefore, $\sum_{i=1}^{m} K_i = O(m)$, and this part of the time cost is amortized $O(\log m)$ for each iteration.

The remaining task is to update the left-right polyline $\mathcal{T}$. At the beginning of the i-th iteration, $\mathcal{T}$ stores ∂F_{i-1}, and we need to modify it to ∂F_i. Clearly, ∂F_i is obtained by using the connected components of $\partial W_i \cap F_{i-1}$ to replace the corresponding portions of ∂F_{i-1}. We consider the components of $\partial W_i \cap F_{i-1}$ one-by-one (there are constant number of components to be considered by the proof of Lemma 5.1). Let ξ be a component, which must be an x-monotone polygonal chain consisting of at most two pieces. For convenience, assume ξ has a left endpoint u and a right end point v. It is clear that $u, v \in \partial F_{i-1}$. We need to replace the portion σ of ∂F_{i-1} between u, v with ξ; we call this a Replace operation. To achieve this, we first report the fractions of ∂F_{i-1} intersecting σ, by using the approach described above. Suppose the reported fractions are $\gamma_1, \ldots, \gamma_k$ sorted in the left-right order. Then $u \in \gamma_1$ and $v \in \gamma_k$. Clearly, the fractions $\gamma_2, \ldots, \gamma_{k-1}$ should be removed, as they disappear after replacing σ with ξ. This can be done by deleting the corresponding nodes from $\mathcal{T}$ via $k-2$ BST-deletion operations. Also, we need to modify γ_1 and γ_k: the portion of γ_1 (resp., γ_k) to the right (resp., left) of u (resp., v) should be "truncated". This can be done by directly updating the information stored in the two corresponding nodes. Finally, ξ should be inserted. Each piece of ξ becomes a new fraction, for which we create a new node storing the information of the fraction and insert it into $\mathcal{T}$ via a BST-insertion operation.

Now we analyze the time cost of this Replace operation. Let $|\mathcal{T}|$ be the size of $\mathcal{T}$ before the operation. The time cost for reporting is $O(\log |\mathcal{T}| + k)$. Removing $\gamma_2, \ldots, \gamma_{k-1}$ takes $O(k \log |\mathcal{T}|)$ time. Modifying γ_1, γ_k and inserting ξ takes $O(\log |\mathcal{T}|)$ time (note that ξ has at most two pieces). So the total time of this Replace operation is $O(k \log |\mathcal{T}|)$. If $k \leq 2$, then the time cost is just $O(\log |\mathcal{T}|)$. If $k > 2$, we observe that there are $\Omega(k)$ nodes deleted from $\mathcal{T}$ in this Replace operation. Note that the total number of the nodes deleted from $\mathcal{T}$ cannot exceed the total number of the nodes inserted. Over the m iterations, we have in total $O(m)$ Replace operations, each of which inserts $O(1)$ nodes into $\mathcal{T}$. Therefore, one can delete at most $O(m)$ nodes from $\mathcal{T}$ in total. It follows that the total time cost for all Replace operations is $O(m \log m)$, which is also the total time cost for updating $\mathcal{T}$. In other words, $\mathcal{T}$ can be updated in amortized $O(\log m)$ time for each iteration. As such, the overall time cost for constructing Γ is $O(m \log m)$, and thus $O(n \log n)$.

We now explain the missing part of the above algorithm: computing $\partial W_i \cap F_{i-1}$ in $O(\log |\mathcal{T}|)$ time. Let r be the left ray of ∂W_i and r_0 be the initial point of r (i.e., the vertex of W_i). It suffices to compute $r \cap F_{i-1}$. Recall that l_i is the line through a_i, b_i and l'_i is the line through a_i perpendicular to l_i. Assume $(l'_i)^* \in r$ (the case that $(l'_i)^* \notin r$ is in fact easier). The point $(l'_i)^*$ partitions r into a segment $s = [r_0, (l'_i)^*]$ and a ray r' emanating from $(l'_i)^*$, where r' is to the left of s. By the proof of Lemma 5.1, each wedge W_j for $j \in \{1, \ldots, i-1\}$ with $W_j \cap r \neq \emptyset$ satisfies at least one of the following: (1) $r_0 \in W_j$, (2) $(l'_i)^* \in W_j$, (3) W_j contains the infinite end of r. Therefore, $r \cap F_{i-1}$ can have one or two connected components; if it has two components, one should be contained in r' and the other should be contained in s. As such, r' contains at most one left endpoint and one right endpoint of (some component of) $r \cap F_{i-1}$, so does s. We show that one can find these endpoints by searching in $\mathcal{T}$.

Suppose we want to find the left endpoint z contained in r' (assume it truly exists). Let γ be a fraction of ∂F_{i-1} which is contributed by W_j for $j \in \{1, \ldots, i-1\}$. It is easy to verify that γ contains z iff γ intersects r' and W_j contains the infinite end of r.

Also, γ is to the left of z iff $\gamma \subseteq R$ and W_j contains the infinite end of r, where R is the region to the left of $(l_i')^*$ and above r'. As such, one can simply search in $\mathcal{T}$ to find the fraction γ containing z in $O(\log|\mathcal{T}|)$ time, if z truly exists. (If z does not exist, by searching in $\mathcal{T}$ we can verify its non-existence, as we can never find the desired fraction γ.) The right endpoint contained in r' and the left/right endpoints contained in s can be computed in a similar fashion. With these endpoints in hand, one can compute $r \cap F_{i-1}$ straightforwardly. The other case that $(l_i')^* \notin r$ is handled similarly and more easily, as in this case $r \cap F_{i-1}$ has at most one connected component. Therefore, $r \cap F_{i-1}$ (and thus $\partial W_i \cap F_{i-1}$) can be computed in $O(\log|\mathcal{T}|)$ time.

Next, we consider how to construct Γ if we are only given the dataset S. Abam et al. [1] showed that one can compute in $O(n\log^2 n)$ time an $\mathcal{H}^\uparrow$-local t-spanner Ψ of S for some $t < 2$ such that $|\Psi| = O(n\log n)$; see Sect. 2 for the definition of local spanners. As argued in Sect. 2, we have $\Phi(S, \mathcal{H}^\uparrow) \subseteq \Psi$. Suppose $\Psi = \{\psi_1, \ldots, \psi_M\}$ where $\psi_1, \ldots, \psi_M$ are sorted in increasing order of their lengths. The m candidate pairs $\phi_1, \ldots, \phi_m \in \Phi(S, \mathcal{H}^\uparrow)$ are among $\psi_1, \ldots, \psi_M$. Let $i_1 < \ldots < i_m$ be indices such that $\phi_1 = \psi_{i_1}, \ldots, \phi_m = \psi_{i_m}$ (note that at this point we do not know what $i_1, \ldots, i_m$ are). We shall consider $\psi_1, \ldots, \psi_M$ in order.

When considering ψ_i, we want to verify whether ψ_i is a candidate pair or not. If this can be done, the candidate pairs $\phi_1, \ldots, \phi_m$ will be found in order. Whenever a new candidate pair ϕ_k is found, we construct Γ_k from Γ_{k-1} in $O(\log m)$ time by the approach above. Now assume $\psi_1, \ldots, \psi_{i-1}$ are already considered, the candidate pairs in $\{\psi_1, \ldots, \psi_{i-1}\}$ are recognized (say they are $\phi_1, \ldots, \phi_{k-1}$), and Γ_{k-1} is constructed. We then consider ψ_i. We need to see whether ψ_i is a candidate pair, i.e., whether $\psi_i = \phi_k$. Let W be the corresponding wedge of ψ_i in the dual $\mathbb{R}^2$. Observe that $\psi_i = \phi_k$ iff $W \nsubseteq \bigcup_{j=1}^{k-1} W_j$. Indeed, if $\psi_i = \phi_k$, then $W = W_k$ and hence $W \nsubseteq \bigcup_{j=1}^{k-1} W_j$ (for ϕ_k is a candidate pair). Conversely, if $W \nsubseteq \bigcup_{j=1}^{k-1} W_j$, then their exists some halfplane $H \in \mathcal{H}^\uparrow$ such that H contains ψ_i and does not contain $\phi_1, \ldots, \phi_{k-1}$. Then the closest-pair in $S \cap H$ cannot be in $\{\psi_1, \ldots, \psi_{i-1}\}$ but must be in Ψ, hence it is nothing but ψ_i.

Based on this observation, we can verify whether $\psi_i = \phi_k$ as follows. We assume $\psi_i = \phi_k$ and try to use it to construct Γ_k from Γ_{k-1} by our above approach. If our assumption is correct, then Γ_k is successfully constructed in $O(\log m)$ time. Furthermore, in the process of constructing Γ_k, our approach allows us to find a point in $W \setminus \bigcup_{j=1}^{k-1} W_j$, which we call *witness* point. This witness point then evidences the correctness of our assumption. On the other hand, if our assumption is wrong, the process can still terminate in $O(\log m)$ time, but we can never find such a witness point because $W \subseteq \bigcup_{j=1}^{k-1} W_j$. In this case, we just discard ψ_i and continue to consider ψ_{i+1}. After considering all pairs in Ψ, we recognize all the m candidate pairs and $\Gamma = \Gamma_m$ is constructed. Since $m = O(n)$ and $M = |\Psi| = O(n\log n)$, the overall process takes $O(n\log^2 n)$ time.

Theorem 5.3 *There exists an $\mathcal{H}$-RCP data structure $\mathcal{E}$ such that*

– *For any $S \subseteq \mathbb{R}^2$ of size n,*

$$\mathsf{Space}(\mathcal{E}(S)) = O(n) \quad \textit{and} \quad \mathsf{Qtime}(\mathcal{E}(S)) = O(\log n).$$

– *For a random $S \propto R^n$ where R is the unit square or more generally an arbitrary axes-parallel rectangle,*

$$\mathbb{E}[\mathsf{Space}(\mathcal{E}(S))] = O(\log^2 n) \quad \textit{and} \quad \mathbb{E}[\mathsf{Qtime}(\mathcal{E}(S))] = O(\log\log n).$$

Furthermore, the above data structure can be constructed in $O(n \log^2 n)$ worst-case time.

5.2 Application to the $\mathcal{H}$-RSS Problem

Interestingly, our approach for solving the $\mathcal{H}$-RCP problem can also be applied to the $\mathcal{H}$-RSS problem, and leads to an optimal $\mathcal{H}$-RSS data structure for interior-disjoint (i.e., non-crossing) segments. (Recall that the $\mathcal{H}$-RSS problem is to build a data structure for a set of line segments in the plane so that the shortest one contained in a query halfplane can be reported efficiently.) This by-product is of indepedent interest.

Theorem 5.4 *There exists an $\mathcal{H}$-RSS data structure $\mathcal{F}$ such that for any set G of n interior-disjoint (i.e., non-crossing) segments in $\mathbb{R}^2$, $\mathsf{Space}(\mathcal{F}(G)) = O(n)$, $\mathsf{Qtime}(\mathcal{F}(G)) = O(\log n)$, and $\mathcal{F}(G)$ can be constructed in $O(n \log n)$ time.*

Proof The data structure is basically identical to the $\mathcal{H}$-RCP data structure given in Sect. 5. Let $\sigma_1, \ldots, \sigma_n$ be the interior-disjoint segments in G sorted in increasing order of their lengths. Suppose W_i is the wedge dual to σ_i. We successively overlay the wedges $W_1, \ldots, W_n$ to create a subdivision Γ of the dual space, as what we do in Sect. 5 for the candidate pairs. A point-location data structure on Γ is then our $\mathcal{H}$-RSS data structure for G. Note that Lemma 5.1 can be applied to show $|\Gamma| = O(n)$, because when proving Lemma 5.1 we only used the facts that the candidate pairs do not cross each other and the wedges are inserted in increasing order of the lengths of their corresponding candidate pairs (here the segments $\sigma_1, \ldots, \sigma_n$ are also non-crossing and sorted in increasing order of their lengths). As such, the space cost of the data structure is $O(n)$ and the query time is $O(\log n)$. In the previous section, we have shown that if the candidate pairs are already given, our $\mathcal{H}$-RCP data structure can be built in $O(n \log n)$ time. It follows that our $\mathcal{H}$-RSS data structure can be built in $O(n \log n)$ time, as we are directly given the segments in this case. □

6 Conclusion and Future Work

In this paper, we revisited the range closest-pair (RCP) problem, which aims to preprocess a set S of points in $\mathbb{R}^2$ into a data structure such that when a query range X is specified, the closest-pair in $S \cap X$ can be reported efficiently. We proposed new

RCP data structures for various query types (including quadrants, strips, rectangles, and halfplanes). Both worst-case and average-case analyses were applied to these data structures, resulting in new bounds for the RCP problem. See Table 1 for a comparison of our new results with the previous work.

We now list some open questions for future study. First, the RCP problem for other query types is an interesting open question. One important example is the disk query, which is usually much harder than the rectangle query and halfplane query in traditional range search. For an easier version, we can focus on the case where the query disks have a fixed radius, or equivalently, the query ranges are *translates* of a fixed disk.

Along this direction, one can also consider translation queries of some shape other than a disk. For instance, if the query ranges are translates of a fixed rectangle, can we have more efficient data structures than our rectangle RCP data structure in Sect. 4? Recently, we have achieved some results in this direction [14].

Also, the RCP problem in higher dimensions is open. To our best knowledge, the only known result for this is a simple data structure given in [6] constructed by explicitly storing all the candidate pairs, which only has guaranteed average-case performance.

Finally, one can consider a colored generalization of the RCP search, in which the given points are colored and we want to compute the closest *bichromatic* pair of points contained in a query range. We have studied the colored RCP problem in a very recent work [11], but only approximate data structures are achieved. It is interesting to see how to solve the colored RCP problem exactly.

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Appendix

A Missing Proofs

A.1. Proof of Lemma 1.1

Without loss of generality, we can assume $R = [0, \Delta] \times [0, \Delta']$ where $\Delta \leq \Delta'$. We first observe some simple facts. Let D be a disc centered at a point in R with radius δ. Then $\mathsf{Area}(D \cap R) \geq \delta^2/9$ if $\delta \leq \Delta$, and $\mathsf{Area}(D \cap R) \geq \delta\Delta/9$ if $\Delta \leq \delta \leq \Delta'$. Furthermore, we always have $\mathsf{Area}(D \cap R) \leq 4\delta^2$ and $\mathsf{Area}(D \cap R) \leq 2\delta\Delta$. With these facts in hand, we can begin our proof.

[Upper Bound] First, we prove the upper bound for $\mathbb{E}[\kappa^p(A)]$. To this end, we need to study the distribution of the random variable $\kappa(A)$. For convenience, we assume m is even and sufficiently large. We make the following claims.

(a) For any $\delta \geq 2\Delta$, we have $\Pr[\kappa(A) \geq \delta] \leq e^{-\delta/(72\Delta'/m^2)}$.
(b) For any $\delta \in (0, 2\Delta]$, we have $\Pr[\kappa(A) \geq \delta] \leq e^{-\delta^2/(144\Delta\Delta'/m^2)}$.

To prove the claims, suppose the m random points in A are $a_1, \ldots, a_m$. For any $\delta > 0$ and each $i \in \{2, \ldots, m\}$, we define $E_{\delta,i}$ as the event that $\kappa(\{a_1, \ldots, a_i\}) \geq \delta$. Note

that $E_{\delta,i}$ happens only if $E_{\delta,i-1}$ does, thus for any $\delta > 0$ we can write

$$\Pr[\kappa(A) \geq \delta] = \Pr[E_{\delta,m}] = \Pr[E_{\delta,2}] \cdot \prod_{i=3}^{m} \Pr[E_{\delta,i} \,|\, E_{\delta,i-1}].$$

We consider the probability $\Pr[E_{\delta,i} \,|\, E_{\delta,i-1}]$ for $i \in \{3, \ldots, m\}$. Let $D_1, \ldots, D_{i-1}$ denote the discs with radii $\delta/2$ centered at $a_1, \ldots, a_{i-1}$ respectively, and $U = \bigcup_{j=1}^{i-1} D_i$. If $E_{\delta,i-1}$ happens, then $D_1, \ldots, D_{i-1}$ are disjoint and hence

$$\mathsf{Area}(U \cap R) = \sum_{j=1}^{i-1} \mathsf{Area}(D_j \cap R).$$

Now assume we have a lower bound μ for all $\mathsf{Area}(D_j \cap R)$, i.e., $\mathsf{Area}(D_j \cap R) \geq \mu$ for any $j \in \{1, \ldots, i-1\}$. Then when $E_{\delta,i-1}$ happens, we always have $\mathsf{Area}(U \cap R) \geq (i-1)\mu$. This implies

$$\Pr[a_i \in U \cap R \,|\, E_{\delta,i-1}] \geq \frac{(i-1)\mu}{\mathsf{Area}(R)} = \frac{(i-1)\mu}{\Delta\Delta'}.$$

Note that $E_{\delta,i}$ happens only if $a_i \notin U \cap R$. Therefore,

$$\begin{aligned}\Pr[E_{\delta,i} \,|\, E_{\delta,i-1}] &\leq \Pr[a_i \notin U \cap R \,|\, E_{\delta,i-1}] \\ &= 1 - \Pr[a_i \in U \cap R \,|\, E_{\delta,i-1}] \leq 1 - \frac{(i-1)\mu}{\Delta\Delta'}.\end{aligned}$$

For $i \geq m/2 + 1$, we have $\Pr[E_{\delta,i} \,|\, E_{\delta,i-1}] \leq 1 - (m\mu)/(2\Delta\Delta')$. Then

$$\Pr[\kappa(A) \geq \delta] \leq \prod_{i=m/2+1}^{m} \Pr[E_{\delta,i} \,|\, E_{\delta,i-1}] \leq \left(1 - \frac{m\mu}{2\Delta\Delta'}\right)^{m/2}.$$

Using the fact $(1-x)^{1/x} < e^{-1}$ for any $x \in [0, 1]$, we deduce

$$\begin{aligned}\Pr[\kappa(A) \geq \delta] &\leq \left(1 - \frac{m\mu}{2\Delta\Delta'}\right)^{m/2} \\ &= \left(1 - \frac{m\mu}{2\Delta\Delta'}\right)^{(2\Delta\Delta'/m\mu)\cdot(m^2\mu/4\Delta\Delta')} \leq e^{m^2\mu/(4\Delta\Delta')}.\end{aligned}$$

If $\delta \geq 2\Delta$, then $\mathsf{Area}(D_j \cap R) \geq \delta\Delta/18$ for any j (as argued at the beginning of the proof), so we can set $\mu = \delta\Delta/18$. The above inequality directly implies the claim (a). If $\delta \in (0, 2\Delta]$, then $\mathsf{Area}(D_j \cap R) \geq \delta^2/36$ for any j (as argued at the beginning of the proof), so we can set $\mu = \delta^2/36$. The above inequality directly implies the

claim (b). With the two claims in hand, we now prove the lemma. We shall use the formula

$$\mathbb{E}[\kappa^p(A)] = \int_0^\infty \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t.$$

We consider two cases: $\Delta'/m^2 \geq \sqrt{\Delta\Delta'}/m$ and $\Delta'/m^2 < \sqrt{\Delta\Delta'}/m$. Assume $\Delta'/m^2 \geq \sqrt{\Delta\Delta'}/m$, i.e., $\Delta \leq \Delta'/m^2$. In this case, what we want is $\mathbb{E}[\kappa^p(A)] = O((\Delta'/m^2)^p)$ for any constant $p \geq 1$. Let $\alpha = 72\Delta'/m^2$. Then for any constant $p \geq 1$, we can write

$$\mathbb{E}[\kappa^p(A)] = \int_0^\infty \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t \leq \alpha^p + \int_{\alpha^p}^\infty \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t.$$

Set $q = 1/p$. For $t \geq \alpha^p$, we have $t^q \geq \alpha > 2\Delta'/m^2 \geq 2\Delta$. Therefore, by applying the claim (a) above we have

$$\Pr[\kappa^p(A) \geq t] = \Pr[\kappa(A) \geq t^q] \leq e^{-t^q/\alpha} = e^{-(t/\alpha^p)^q}.$$

It follows that

$$\mathbb{E}[\kappa^p(A)] \leq \alpha^p + \int_{\alpha^p}^\infty e^{-(t/\alpha^p)^q}\,\mathrm{d}t = \alpha^p + \alpha^p \int_1^\infty e^{-t^q}\,\mathrm{d}t.$$

The integral $\int_1^\infty e^{-t^q}\,\mathrm{d}t$ converges, thus $\mathbb{E}[\kappa^p(A)] = O(\alpha^p) = O((\Delta'/m^2)^p)$. Next, assume $\Delta'/m^2 < \sqrt{\Delta\Delta'}/m$, i.e., $\Delta > \Delta'/m^2$. In this case, what we want is $\mathbb{E}[\kappa^p(A)] = O((\sqrt{\Delta\Delta'}/m)^p)$ for any constant $p \geq 1$. We first claim that

$$\int_{(2\Delta)^p}^\infty \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t = O((\sqrt{\Delta\Delta'}/m)^p). \tag{2}$$

Again, let $\alpha = 72\Delta'/m^2$. By applying the claim (a) above, we have

$$\int_{(2\Delta)^p}^\infty \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t \leq \int_{(2\Delta)^p}^\infty e^{-(t/\alpha^p)^q}\,\mathrm{d}t,$$

where $q = 1/p$. Since $\Delta > \Delta'/m^2$, we further deduce

$$\int_{(2\Delta)^p}^\infty e^{-(t/\alpha^p)^q}\,\mathrm{d}t \leq \int_{(\alpha/36)^p}^\infty e^{-(t/\alpha^p)^q}\,\mathrm{d}t = \alpha^p \int_{(1/36)^p}^\infty e^{-t^q}\,\mathrm{d}t = O(\alpha^p).$$

By the assumption $\Delta'/m^2 < \sqrt{\Delta\Delta'}/m$, we have $\alpha = O(\sqrt{\Delta\Delta'}/m)$, thus (2) holds. With this in hand, we bound $\mathbb{E}[\kappa^p(A)]$ as follows. Let $\beta = 12\sqrt{\Delta\Delta'}/m$. If $\beta \geq 2\Delta$,

then

$$\mathbb{E}[\kappa^p(A)] \leq \beta^p + \int_{\beta^p}^{\infty} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t \leq \beta^p + \int_{(2\Delta)^p}^{\infty} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t$$
$$= O((\sqrt{\Delta\Delta'}/m)^p).$$

The rightmost equality above follows from (2). If $\beta \leq 2\Delta$, then

$$\mathbb{E}[\kappa^p(A)] \leq \beta^p + \int_{\beta^p}^{(2\Delta)^p} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t + \int_{(2\Delta)^p}^{\infty} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t.$$

It suffices to show $\int_{\beta^p}^{(2\Delta)^p} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t = O((\sqrt{\Delta\Delta'}/m)^p)$. By the claim (b) above,

$$\int_{\beta^p}^{(2\Delta)^p} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t = \int_{\beta^p}^{(2\Delta)^p} \Pr[\kappa(A) \geq t^q]\,\mathrm{d}t \leq \int_{\beta^p}^{(2\Delta)^p} e^{-(t/\beta^p)^{2q}}\,\mathrm{d}t.$$

Furthermore, we have

$$\int_{\beta^p}^{(2\Delta)^p} e^{-(t/\beta^p)^{2q}}\,\mathrm{d}t \leq \int_{\beta^p}^{\infty} e^{-(t/\beta^p)^{2q}}\,\mathrm{d}t = \beta^p \int_1^{\infty} e^{-t^{2q}}\,\mathrm{d}t.$$

Since the integral $\int_1^{\infty} e^{-t^{2q}}\,\mathrm{d}t$ converges, $\int_{\beta^p}^{(2\Delta)^p} \Pr[\kappa^p(A) \geq t]\,\mathrm{d}t = O((\sqrt{\Delta\Delta'}/m)^p)$. As such, $\mathbb{E}[\kappa^p(A)] = O((\sqrt{\Delta\Delta'}/m)^p)$. This proves the upper bound for $\mathbb{E}[\kappa^p(A)]$.

[Lower Bound] To prove the lower bound for $\mathbb{E}[\kappa^p(A)]$ is much easier. It suffices to show that $\mathbb{E}[\kappa^p(A)] = \Omega((\Delta'/m^2)^p)$ and $\mathbb{E}[\kappa^p(A)] = \Omega((\sqrt{\Delta\Delta'}/m)^p)$. Let $i, j \in \{1, \ldots, m\}$ such that $i \neq j$. Set $\delta_1 = \Delta'/(2m^2)$. We observe that $\Pr[\mathrm{dist}(a_i, a_j) \leq \delta_1] \leq 1/m^2$. Indeed, if D is the disc centered at a_i with radius δ_1, then we always have $\mathsf{Area}(D \cap R) \leq 2\delta_1\Delta = \Delta\Delta'/m^2$ (as argued at the beginning of the proof), and hence

$$\Pr[\mathrm{dist}(a_i, a_j) \leq \delta_1] = \Pr[a_j \in D \cap R] \leq \frac{\Delta\Delta'/m^2}{\mathsf{Area}(R)} = \frac{1}{m^2}.$$

By union bound, we have

$$\Pr[\kappa(A) \leq \delta_1] \leq \binom{m}{2} \cdot \frac{1}{m^2} < \frac{1}{2}.$$

Thus, $\Pr[\kappa(A) \leq \delta_1] \geq 1/2$ and $\mathbb{E}[\kappa^p(A)] \geq \delta_1^p/2 = \Omega((\Delta'/m^2)^p)$. Similarly, we can show $\mathbb{E}[\kappa^p(A)] = \Omega((\sqrt{\Delta\Delta'}/m)^p)$. Set $\delta_2 = \sqrt{\Delta\Delta'}/(4m)$. We again observe that $\Pr[\mathrm{dist}(a_i, a_j) \leq \delta_2] \leq 1/m^2$ for any distinct $i, j \in \{1, \ldots, m\}$. Indeed, if D is

the disc centered at a_i with radius δ_2, then we always have $\mathsf{Area}(D \cap R) \leq 4\delta_2^2 = \Delta\Delta'/m^2$ (as argued at the beginning of the proof), and hence

$$\Pr[\mathrm{dist}(a_i, a_j) \leq \delta_2] = \Pr[a_j \in D \cap R] \leq \frac{\Delta\Delta'/m^2}{\mathsf{Area}(R)} = \frac{1}{m^2}.$$

Applying the same argument as above, we can deduce $\Pr[\kappa(A) \leq \delta_2] \geq 1/2$, which implies that $\mathbb{E}[\kappa^p(A)] \geq \delta_2^p/2 = \Omega((\sqrt{\Delta\Delta'}/m)^p)$. This proves the lower bound for $\mathbb{E}[\kappa^p(A)]$.

The above proof straightforwardly applies to the special case in which R is a segment.

A.2 Proof of Lemma 2.1

Without loss of generality, assume $R = [0, 1] \times [0, \Delta]$. It suffices to show that $\mathbb{E}[|\Phi(S, \mathcal{Q}^{\swarrow})|] = O(\log^2 n)$. Suppose the n random points in S are $a_1, \ldots, a_n$. Let $E_{i,j}$ be the event $(a_i, a_j) \in \Phi(S, \mathcal{Q}^{\swarrow})$, then by the linearity of expectation,

$$\mathbb{E}[|\Phi(S, \mathcal{Q}^{\swarrow})|] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[E_{i,j}].$$

Since $a_1, \ldots, a_n$ play the same roles here, the probabilities on the right-hand side of the above equation should be the same and therefore $\mathbb{E}[|\Phi(S, \mathcal{Q}^{\swarrow})|] = O(n^2 \cdot \Pr[E_{1,2}])$. In order to bound $\Pr[E_{1,2}]$, we introduce some random variables. Let $x_{\max} = \max\{a_1.x, a_2.x\}$, $y_{\max} = \max\{a_1.y, a_2.y\}$, $x_{\min} = \min\{a_1.x, a_2.x\}$, $y_{\min} = \min\{a_1.y, a_2.y\}$. The quadrant $Q = (-\infty, x_{\max}] \times (-\infty, y_{\max}]$ is the *minimal* southwest quadrant containing both a_1 and a_2, and clearly $E_{1,2}$ happens iff (a_1, a_2) is the closest-pair in $S \cap Q$. Define $\Lambda = \{i \geq 3 : a_i \in Q\}$, which is a random subset of $\{3, \ldots, n\}$, i.e., a random variable taking value from the power set of $\{3, \ldots, n\}$. We achieve the bound for $\Pr[E_{1,2}]$ through three steps.

[Step 1] Let us first fix the values of $x_{\max}$, $y_{\max}$, Λ, and consider the corresponding conditional probability of $E_{1,2}$. Formally, we claim that, for all $\tilde{x} \in (0, 1]$, all $\tilde{y} \in (0, \Delta]$, and all nonempty $J \subseteq \{3, \ldots, n\}$,

$$\Pr\big[E_{1,2} \,|\, (x_{\max} = \tilde{x}) \wedge (y_{\max} = \tilde{y}) \wedge (\Lambda = J)\big] = O(|J|^{-2}). \tag{3}$$

For convenience, we use $C_{\tilde{x},\tilde{y},J}$ to denote the condition in the above conditional probability. Assume $|J| = m$. Let $\delta_x = x_{\max} - x_{\min}$ and $\delta_y = y_{\max} - y_{\min}$. Note that under the condition $C_{\tilde{x},\tilde{y},J}$, $E_{1,2}$ happens only if $\delta_x \leq \kappa(S_J)$ and $\delta_y \leq \kappa(S_J)$ where $S_J = \{a_j : j \in J\}$. Indeed, if $\delta_x > \kappa(S_J)$ or $\delta_y > \kappa(S_J)$, then $\mathrm{dist}(a_1, a_2) > \kappa(S_J)$, which implies $E_{1,2}$ does not happen because all the random points in S_J are contained in Q under the condition $C_{\tilde{x},\tilde{y},J}$. Therefore, it suffices to bound $\Pr[(\delta_x \leq \kappa(S_J)) \wedge (\delta_y \leq \kappa(S_J)) \,|\, C_{\tilde{x},\tilde{y},J}]$. Note that under the condition $C_{\tilde{x},\tilde{y},J}$, Q is just $(-\infty, \tilde{x}] \times (-\infty, \tilde{y}]$. Thus the condition $C_{\tilde{x},\tilde{y},J}$ is equivalent to

saying that the maximum of the x-coordinates (resp., y-coordinates) of a_1, a_2 is $\tilde{x}$ (resp., $\tilde{y}$), all a_j for $j \in J$ are contained in the rectangle $R' = [0, \tilde{x}] \times [0, \tilde{y}]$, and all a_j for $j \in J$ for $j \in \{3, \ldots, n\} \setminus J$ are in $R \setminus R'$. As such, one can easily verify that, under the condition $C_{\tilde{x},\tilde{y},J}$, the distribution of the random number δ_x (resp., δ_y) is the uniform distribution on the interval $[0, \tilde{x}]$ (resp., $[0, \tilde{y}]$), and the distributions of the m random points in S_J are the uniform distribution on R'; furthermore, these random numbers/points are independent of each other. This says, if we consider a new random experiment in which we independently generate two random numbers δ'_x, δ'_y from the uniform distributions on $[0, \tilde{x}]$, $[0, \tilde{y}]$ respectively (which correspond to δ_x, δ_y) and a random dataset $S' \propto (R')^m$ (which corresponds to S_J), then we have

$$\Pr[(\delta'_x \le \kappa(S')) \wedge (\delta'_y \le \kappa(S'))] = \Pr[(\delta_x \le \kappa(S_J)) \wedge (\delta_y \le \kappa(S_J)) \,|\, C_{\tilde{x},\tilde{y},J}].$$

So it suffices to bound $\Pr[(\delta'_x \le \kappa(S')) \wedge (\delta'_y \le \kappa(S'))]$ in the new experiment; we denote by λ this probability. We apply the formula

$$\lambda = \int_0^\infty p(t) \cdot \Pr[(\delta'_x \le t) \wedge (\delta'_y \le t)]\,\mathrm{d}t = \int_0^\infty p(t) \cdot \Pr[\delta'_x \le t] \cdot \Pr[\delta'_y \le t]\,\mathrm{d}t,$$

where $p(\cdot)$ is the probability distribution function of $\kappa(S')$. Since δ'_x (resp., δ'_y) is drawn uniformly on the interval $[0, \tilde{x}]$ (resp., $[0, \tilde{y}]$), we have $\Pr[\delta'_x \le t] = \min\{t/\tilde{x}, 1\}$ (resp., $\Pr[\delta'_y \le t] = \min\{t/\tilde{y}, 1\}$). Without loss of generality, we assume $\tilde{x} \le \tilde{y}$. Then we have

$$\Pr[\delta'_x \le t] \cdot \Pr[\delta'_y \le t] = \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}, 1\right\} \le \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}\right\}.$$

It follows that

$$\lambda \le \int_0^\infty p(t) \cdot \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}\right\}\mathrm{d}t \le \min\left\{\int_0^\infty \frac{p(t)t^2}{\tilde{x}\tilde{y}}\,\mathrm{d}t, \int_0^\infty \frac{p(t)t}{\tilde{y}}\,\mathrm{d}t\right\}.$$

Noting the fact that $\int_0^\infty p(t)t^2\,\mathrm{d}t = \mathbb{E}[\kappa^2(S')]$ and $\int_0^\infty p(t)t\,\mathrm{d}t = \mathbb{E}[\kappa(S')]$, we have

$$\lambda \le \min\left\{\frac{\mathbb{E}[\kappa^2(S')]}{\tilde{x}\tilde{y}}, \frac{\mathbb{E}[\kappa(S')]}{\tilde{y}}\right\}.$$

Since $\tilde{x} \le \tilde{y}$ by assumption, Lemma 1.1 implies that $\mathbb{E}[\kappa(S')] = O(\max\{\sqrt{\tilde{x}\tilde{y}}/m, \tilde{y}/m^2\})$ and $\mathbb{E}[\kappa^2(S')] = O(\max\{\tilde{x}\tilde{y}/m^2, \tilde{y}^2/m^4\})$. If $\sqrt{\tilde{x}\tilde{y}}/m \le \tilde{y}/m^2$, then $\mathbb{E}[\kappa(S')]/\tilde{y} = O(1/m^2)$, otherwise $\mathbb{E}[\kappa^2(S')]/(\tilde{x}\tilde{y}) = O(1/m^2)$. In either of the two cases, we have $\lambda = O(1/m^2)$. Therefore, we obtain (3). For an arbitrary nonempty $J \subseteq \{3, \ldots, n\}$, since (3) holds for all $\tilde{x} \in (0, 1]$ and $\tilde{y} \in (0, \Delta]$, we can remove the conditions $x_{\max} = \tilde{x}$ and $y_{\max} = \tilde{y}$ from (3) to deduce $\Pr[E_{1,2} \,|\, \Lambda = J] = O(1/|J|^2)$ (note that although we miss the case $\tilde{x} = 0$ or $\tilde{y} = 0$, it does not matter since the events $x_{\max} = 0$ and $y_{\max} = 0$ happen with probability 0). This further implies

that $\Pr[E_{1,2}\,||\Lambda| = m] = O(1/m^2)$ for all $m \in \{1, \ldots, n-2\}$. For $m = 0$, we have $\Pr[E_{1,2}\,||\Lambda| = m] = 1$.

[Step 2] In order to apply the result achieved in Step 1 to bound $\Pr[E_{1,2}]$, we need to bound $\Pr[|\Lambda| = m]$ for $m = \{0, \ldots, n-2\}$. This is a purely combinatorial problem, because the random variable $|\Lambda|$ only depends on the orderings of the x-coordinates and y-coordinates of $a_1, \ldots, a_n$. The ordering of the x-coordinates (x-ordering for short) of $a_1, \ldots, a_n$ can be represented as a permutation of $\{a_1, \ldots, a_n\}$; so is the y-ordering. Thus, in terms of the ordering of coordinates, there are $(n!)^2$ different configurations of S, each of which can be represented by a pair (π, π') of permutations of $\{a_1, \ldots, a_n\}$ where π (resp., π') represents the x-ordering (resp., y-ordering) of $a_1, \ldots, a_n$; that is, if $\pi = (a_{i_0}, \ldots, a_{i_n})$ and $\pi' = (a_{i'_0}, \ldots, a_{i'_n})$, then $a_{i_0}.x < \ldots < a_{i_n}.x$ and $a_{i'_0}.y < \ldots < a_{i'_n}.y$ (we can ignore the degenerate case in which two random points have the same x-coordinates or y-coordinates, because the random points in S have distinct coordinates with probability 1). Note that every configuration occurs with the same probability $1/(n!)^2$. If S has the configuration (π, π'), then Λ is just the subset of $\{3, \ldots, n\}$ consisting of all i such that $\mathsf{rk}_\pi(a_i) \le \max\{\mathsf{rk}_\pi(a_1), \mathsf{rk}_\pi(a_2)\}$ and $\mathsf{rk}_{\pi'}(a_i) \le \max\{\mathsf{rk}_{\pi'}(a_1), \mathsf{rk}_{\pi'}(a_2)\}$, where the function rk computes the *rank* of an element in a permutation (i.e., the position of the element in the permutation). Therefore, we can pass to a new random experiment in which we generate independently and uniformly the two permutations π, π' of $\{a_1, \ldots, a_n\}$ (i.e., uniformly generate a configuration of S), and study $\Pr[|\Lambda| = m]$ for $m \in \{0, \ldots, n-2\}$ in the new experiment. Let $r_i = \mathsf{rk}_\pi(a_i)$. Fixing $m \in \{0, \ldots, n-2\}$, we have the formula

$$\Pr[|\Lambda| = m] = \sum_{i=2}^{n} \Pr[\max\{r_1, r_2\} = i] \cdot \Pr[|\Lambda| = m \,|\, \max\{r_1, r_2\} = i]. \quad (4)$$

We first compute $\Pr[\max\{r_1, r_2\} = i]$. By an easy counting argument, we see that, among the $n!$ permutations of $\{a_1, \ldots, a_n\}$, there are exactly $2(i-1)(n-2)!$ permutations in which the maximum of the ranks of a_1 and a_2 is i. Therefore,

$$\Pr[\max\{r_1, r_2\} = i] = \frac{2(i-1)(n-2)!}{n!} = O\left(\frac{i}{n^2}\right).$$

We then consider $\Pr[|\Lambda| = m \,|\, \max\{r_1, r_2\} = i]$. If $i < m+2$, the probability is 0, because $|\Lambda \cup \{1, 2\}| \le \max\{r_1, r_2\}$ by definition. So assume $i \ge m+2$. Suppose the permutation π has already been generated and satisfies $\max\{r_1, r_2\} = i$. Let $A = \{a_j : r_j \le i\}$. Note that $|A| = i$ and $a_1, a_2 \in A$. Now we randomly generate the permutation π' and observe the probability of $|\Lambda| = m$. Clearly, $|\Lambda| = m$ iff $\max\{\mathsf{rk}_{\pi'|_A}(a_1), \mathsf{rk}_{\pi'|_A}(a_2)\} = m+2$, where $\pi'|_A$ is the permutation of A induced by π' (i.e., the permutation obtained by removing the points in $S \setminus A$ from π'). Using the same counting argument as above, we have

$$\Pr[\max\{\mathsf{rk}_{\pi'|_A}(a_1), \mathsf{rk}_{\pi'|_A}(a_2)\} = m+2] = O\left(\frac{m+1}{i^2}\right).$$

Therefore, $\Pr[|\Lambda| = m \,|\, \max\{r_1, r_2\} = i] = O((m+1)/i^2)$. Plugging in these results to (4), a direct calculation gives us $\Pr[|\Lambda| = m] = O((m+1)\log n/n^2)$.

[Step 3] Using the results achieved in the previous steps, to bound $\Pr[E_{1,2}]$ is quite straightforward. We apply the formula

$$\Pr[E_{1,2}] = \sum_{m=0}^{n-2} \Pr[|\Lambda| = m] \cdot \Pr[E_{1,2} \,||\Lambda| = m].$$

We use the result achieved in Step 1 to bound $\Pr[E_{1,2}\,||\Lambda| = m]$ and the result achieved in Step 2 to bound $\Pr[|\Lambda| = m]$. Then a direct calculation gives us $\Pr[E_{1,2}] = O(\log^2 n/n^2)$. As such, $\mathbb{E}[|\Phi(S, \mathcal{Q}^{\swarrow})|] = O(\log^2 n)$, and hence $\mathbb{E}[|\Phi(S, \mathcal{Q})|] = O(\log^2 n)$.

A.3 Proof of Lemma 3.1

It suffices to consider the case in which $I_1, \ldots, I_n$ are vertical aligned segments and $\mathcal{X} = \mathcal{U}^{\downarrow}$. Without loss of generality, we may assume $I_i = x_i \times [0, 1]$, $x_1 < \ldots < x_n$ are real numbers. Fix $i, j \in \{1, \ldots, n\}$ such that $i < j$. We first define some random variables. Let $y_k = a_k.y$ for all $k \in \{1, \ldots, n\}$. Define $y_{\max} = \max\{y_i, y_j\}$ and $y_{\min} = \min\{y_i, y_j\}$. The 3-sided rectangle $U = [x_i, x_j] \times (-\infty, y_{\max}]$ is the *minimal* bottom-unbounded rectangle containing both a_i and a_j, and clearly $(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})$ iff (a_i, a_j) is the closest-pair in $S \cap U$. Define $\Lambda = \{k : i < k < j \text{ and } a_k \in U\}$, which is a random subset of $\{i+1, \ldots, j-1\}$. We bound $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})]$ through three steps.

[Step 1] Let us first fix the values of $y_{\max}$ and Λ, and consider the corresponding conditional probability of the event $(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})$. Formally, we claim that, for all $\tilde{y} \in (0, 1]$ and all nonempty $K \subseteq \{i+1, \ldots, j-1\}$,

$$\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,|\, (y_{\max} = \tilde{y}) \wedge (\Lambda = K)] = O\left(\frac{1}{|K|^2}\right). \tag{5}$$

For convenience, we use $C_{\tilde{y},K}$ to denote the condition in the above conditional probability. Assume $|K| = m$. Let $\delta = y_{\max} - y_{\min}$ and $Y_K = \{y_k : k \in K\}$. We first notice that, under the condition $C_{\tilde{y},K}$, $(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})$ only if $\delta \le \kappa(Y_K)$. Indeed, if $\delta > \kappa(Y_K) = |y_{i'} - y_{j'}|$ for some distinct $i', j' \in K$, then $\mathrm{dist}(a_{i'}, a_{j'}) < \mathrm{dist}(a_i, a_j)$ since $|x_i - x_j| \ge |x_{i'} - x_{j'}|$, which implies $(a_i, a_j) \notin \Phi(S, \mathcal{U}^{\downarrow})$. Therefore, it suffices to bound $\Pr[\delta \le \kappa(Y_K) \,|\, C_{\tilde{y},K}]$. Note that under the condition $C_{\tilde{y},K}$, U is just $[x_i, x_j] \times (-\infty, \tilde{y}]$. Thus the condition $C_{\tilde{y},K}$ is equivalent to saying that the maximum of y_i, y_j is $\tilde{y}$, all y_k for $k \in K$ are in $[0, \tilde{y}]$, and all y_k for $k \in \{i+1, \ldots, j-1\} \setminus K$ are in $(\tilde{y}, 1]$. As such, one can easily verify that, under the condition $C_{\tilde{y},K}$, the distribution of δ is the uniform distribution on $[0, \tilde{y}]$, and the distributions of the m random numbers in Y_K are also the uniform distribution on $[0, \tilde{y}]$; furthermore, these random numbers are independent of each other. This says, if we consider a new random experiment in which we independently generate a random number δ' from the uniform distribution on $[0, \tilde{y}]$ (which corresponds to δ) and a random dataset $Y' \propto [0, \tilde{y}]^m$ (which

corresponds to Y_K), then we have

$$\Pr[\delta' \le \kappa(Y')] = \Pr[\delta \le \kappa(Y_K) \,|\, C_{\tilde{y},K}].$$

So it suffices to bound $\Pr[\delta' \le \kappa(Y')]$ in the new experiment. We apply the formula

$$\Pr[\delta' \le \kappa(Y')] = \int_0^{\tilde{y}} p(t) \cdot \Pr[\delta' \le t]\,\mathrm{d}t,$$

where $p(\cdot)$ is the probability distribution function of $\kappa(Y')$. Since δ' is drawn from the uniform distribution on $[0, \tilde{y}]$, $\Pr[\delta' \le t] = t/\tilde{y}$ for $t \in [0, \tilde{y}]$. Thus,

$$\Pr[\delta' \le \kappa(Y')] = \frac{1}{\tilde{y}} \int_0^{\tilde{y}} p(t)t\,\mathrm{d}t = \frac{\mathbb{E}[\kappa(Y')]}{\tilde{y}}.$$

By Lemma 1.1 (segment case), $\mathbb{E}[\kappa(Y')] = \tilde{y}/m^2$. This implies $\Pr[\delta' \le \kappa(Y')] = O(1/m^2)$, which proves (5). For an arbitrary nonempty $K \subseteq \{i+1, \ldots, j-1\}$, since (5) holds for all $\tilde{y} \in (0, 1]$, we can remove the condition $y_{\max} = \tilde{y}$ from (5) to deduce $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,|\, \Lambda = K] = O(1/|K|^2)$. This further implies $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,||\Lambda| = m] = O(1/m^2)$ for all $m = \{1, \ldots, j-i-1\}$. For $m = 0$, we have $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,||\Lambda| = m] = 1$.

[Step 2] In order to apply the result achieved in Step 1 to bound the unconditional probability of $(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})$, we need to bound $\Pr[|\Lambda| = m]$ for all $m \in \{0, \ldots, j-i-1\}$. This is a combinatorial problem, because the random variable $|\Lambda|$ only depends on the ordering of $y_i, \ldots, y_j$. There are $(j-i+1)!$ possible orderings, each of which can be represented by a permutation of $\{y_i, \ldots, y_j\}$. Every ordering occurs with the same probability $1/(j-i+1)!$. For a permutation π of $\{y_i, \ldots, y_j\}$, we write $\lambda_\pi = \max\{\mathsf{rk}_\pi(y_i), \mathsf{rk}_\pi(y_j)\}$, where the function rk computes the rank of an element in a permutation. Clearly, if the ordering is π, then $|\Lambda| = \lambda_\pi - 2$. As such, we can pass to a new random experiment in which we generate uniformly a permutation π of $\{y_i, \ldots, y_j\}$ and study $\Pr[|\Lambda| = m]$ for $m \in \{0, \ldots, j-i-1\}$ in this new experiment. Fixing $m \in \{0, \ldots, j-i-1\}$, it follows that $|\Lambda| = m$ iff $\lambda_\pi = m+2$. By an easy counting argument, we see that, among the $(j-i+1)!$ permutations of $\{y_i, \ldots, y_j\}$, there are exactly $2(m+1)(j-i-1)!$ permutations in which the maximum of the ranks of y_i and y_j is $m+2$. Therefore,

$$\Pr[|\Lambda| = m] = \Pr[\lambda_\pi = m+2] = \frac{2(m+1)(j-i-1)!}{(j-i+1)!} = O\left(\frac{m+1}{(j-i)^2}\right).$$

[Step 3] Using the results achieved in the previous steps, the lemma can be readily proved. We apply the formula

$$\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})] = \sum_{m=0}^{j-i-1} \Pr[|\Lambda| = m] \cdot \Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,||\Lambda| = m]. \tag{6}$$

We use the result achieved in Step 1 to bound $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow}) \,|\, |\Lambda| = m]$ and the result achieved in Step 2 to bound $\Pr[|\Lambda| = m]$. Then a direct calculation gives us $\Pr[(a_i, a_j) \in \Phi(S, \mathcal{U}^{\downarrow})] = O(\log(j-i)/(j-i)^2)$.

A.4 Proof of Lemma 4.4

It suffices to consider the case in which l is a vertical line. Also, we may assume the equation of l is $x = 0$, without loss of generality. Then $\mathcal{P}_l \subseteq \mathcal{P}^{\text{v}}$, so a $(\mathcal{P}^{\text{v}}, k)$-TBEP data structure is naturally a $(\mathcal{P}_l, k)$-TBEP data structure. In a dataset $S \subseteq \mathbb{R}^2$, we say a point $a \in S$ is a *candidate* point if a is one of the k topmost/bottommost points in $S \cap P$ for some $P \in \mathcal{P}_l$. We denote by $\Psi(S)$ the subset of S consisting of all candidate points in S. Note that for any $P \in \mathcal{P}_l$, the k topmost/bottommost points in $S \cap P$ are just the k topmost/bottommost points in $\Psi(S) \cap P$. As such, answering a $(\mathcal{P}_l, k)$-TBEP query on S is equivalent to answering a $(\mathcal{P}_l, k)$-TBEP query on $\Psi(S)$.

Therefore, we can define our $(\mathcal{P}_l, k)$-TBEP data structure $\mathcal{K}_l$ as $\mathcal{K}_l(S) = \mathcal{K}^{\text{v}}(\Psi(S))$, where $\mathcal{K}^{\text{v}}$ is the $(\mathcal{P}^{\text{v}}, k)$-TBEP data structure defined in Lemma 4.3. Now let $S \propto \prod_{i=1}^{n} I_i$ where $I_1, \ldots, I_n$ are distinct vertical aligned segments. Assume $I_1, \ldots, I_n$ are sorted from left to right, in which $I_1, \ldots, I_t$ are to the left of l and $I_{t+1}, \ldots, I_n$ are to the right of l. Denote by $a_i \in S$ the random point drawn on I_i. We claim that $\mathbb{E}[|\Psi(S)|] = O(\log n)$. Let $i \in \{1, \ldots, t\}$. We first notice that $a_i \in \Psi(S)$ iff a_i is one of the k topmost/bottommost points among $a_i, \ldots, a_t$. Indeed, any $P \in \mathcal{P}_l$ with $a_i \in P$ contains $a_i, \ldots, a_t$. Therefore, if $a_i \in \Psi(S)$, then it must be one of the k topmost/bottommost points among $a_i, \ldots, a_t$. Conversely, if a_i is one of the k topmost/bottommost points among $a_i, \ldots, a_t$, then $a_i \in \Psi(S)$ because it is one of the k topmost/bottommost points in $S \cap P$ for $P = [a_i.x, 0] \times \mathbb{R} \in \mathcal{P}_l$.

Since the random points are generated independently, the probability that a_i is one of the k topmost/bottommost points among $a_i, \ldots, a_t$ is exactly $\min\{2k/(t-i+1), 1\}$. As such, $\Pr[a_i \in \Psi(S)] \leq 2k/(t-i+1)$. Using the same argument, we see that for $i \in \{t+1, \ldots, n\}$, $\Pr[a_i \in \Psi(S)] \leq 2k/(i-t)$. Now

$$\mathbb{E}[|\Psi(S)|] = \sum_{i=1}^{n} \Pr[a_i \in \Psi(S)] \leq \sum_{i=1}^{t} \frac{2k}{t-i+1} + \sum_{i=t+1}^{n} \frac{2k}{t-i} = O(k \log n).$$

Since k is a constant, we have $\mathbb{E}[|\Psi(S)|] = O(\log n)$. Thus,

$$\mathbb{E}[\mathsf{Space}(\mathcal{K}_l(S))] = \mathbb{E}[\mathsf{Space}(\mathcal{K}^{\text{v}}(\Psi(S)))] = \mathbb{E}[|\Psi(S)|] = O(\log n).$$

Also, $\mathbb{E}[\mathsf{Qtime}(\mathcal{K}_l(S))] = \mathbb{E}[\mathsf{Qtime}(\mathcal{K}^{\text{v}}(\Psi(S)))] = O(\mathbb{E}[\log |\Psi(S)|])$. Note that $\mathbb{E}[\log x] \leq \log \mathbb{E}[x]$ for a random variable x, hence $\mathbb{E}[\mathsf{Qtime}(\mathcal{K}_l(S))] = O(\log \log n)$. To build this data structure, we only need to compute $\Psi(S)$ and construct $\mathcal{K}^{\text{v}}(\Psi(S))$. The latter can be done in $O(n \log n)$ time by Lemma 4.3. To compute $\Psi(S)$ is also easy. By our above observation, for $i \in \{1, \ldots, t\}$, $a_i \in \Psi(S)$ iff a_i is one of the k topmost/bottommost points among $a_i, \ldots, a_t$. Therefore, we only need to scan from a_t to a_1 and maintain the k topmost/bottommost points among $a_i, \ldots, a_t$ when we reach a_i. In this way, we can determine whether $a_i \in \Psi(S)$ for all $i \in \{1, \ldots, t\}$ in

$O(t)$ time, since k is a constant. Similarly, we can know which points of $a_{t+1}, \ldots, a_n$ are in $\Psi(S)$ in $O(n-t)$ time. It follows that $\Psi(S)$ can be computed in $O(n)$ time.

A.5 Proof of Lemma 4.7

It suffices to consider the case in which $I_1, \ldots, I_n$ are vertical aligned segments and $\mathcal{X} = \mathcal{U}^{\downarrow}$. Without loss of generality, assume $I_i = x_i \times [0,1]$ where $x_1 < \ldots < x_n$ are real numbers. We denote by $a_i \in S$ the random point drawn on I_i. Also, suppose $a_1, \ldots, a_t$ are to the left of l, while $a_{t+1}, \ldots, a_n$ are to the right of l. Let $E_{i,j}$ be the event that $(a_i, a_j) \in \Phi(A, \mathcal{U}^{\downarrow})$. Then we have the equation

$$\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|] = \sum_{i=1}^{t} \sum_{j=t+1}^{n} \Pr[E_{i,j}].$$

By applying Lemma 3.1 and the fact

$$\sum_{i=1}^{t} \sum_{j=t+1}^{n} \frac{\log(j-i)}{(j-i)^2} \leq \sum_{p=1}^{n} p \cdot \frac{\log p}{p^2} = O(\log^2 n),$$

we have $\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|] = O(\log^2 n)$.

To prove $\mathbb{E}[|\Phi_l(S, \mathcal{U}^{\downarrow})|^2] = O(\log^4 n)$ is much more difficult. Define $\Psi = \Phi_l^2(S, \mathcal{U}^{\downarrow})$, i.e., the Cartesian product of two copies of $\Phi_l(S, \mathcal{U}^{\downarrow})$. Then $|\Psi| = |\Phi_l(S, \mathcal{U}^{\downarrow})|^2$. So it suffices to bound $\mathbb{E}[|\Psi|]$. Clearly, for $i, i' \in \{1, \ldots, t\}$ and $j, j' \in \{t+1, \ldots, n\}$, $((a_i, a_j), (a_{i'}, a_{j'})) \in \Psi$ iff $E_{i,j} \wedge E_{i',j'}$. Therefore, we have

$$\mathbb{E}[|\Psi|] = \sum_{i=1}^{t} \sum_{j=t+1}^{n} \sum_{i'=1}^{t} \sum_{j'=t+1}^{n} \Pr[E_{i,j} \wedge E_{i',j'}]. \tag{7}$$

However, $\Pr[E_{i,j} \wedge E_{i',j'}] \neq \Pr[E_{i,j}] \cdot \Pr[E_{i',j'}]$ in general, as the events $E_{i,j}$ and $E_{i',j'}$ are not independent. We investigate $\Pr[E_{i,j} \wedge E_{i',j'}]$ by considering various cases.

[Case 1] We first consider the easiest case in which $i = i'$ and $j = j'$. In this case, $\Pr[E_{i,j} \wedge E_{i',j'}] = \Pr[E_{i,j}] = O(\log(j-i)/(j-i)^2)$ by Lemma 3.1. Then the sum of the terms $\Pr[E_{i,j} \wedge E_{i',j'}]$ satisfying $i = i'$ and $j = j'$ is $O(\log^2 n)$.

[Case 2] We then consider the case in which $i \neq i'$ and $j \neq j'$. Let $\delta = j - i$ and $\delta' = j' - i'$. In this case, we claim that $\Pr[E_{i,j} \wedge E_{i',j'}] = O((\log \delta \log \delta')/(\delta\delta')^2)$. To prove this, we may assume that δ' is sufficiently large, say $\delta' \geq 5$. Indeed, when $\delta' < 5$, what we want is $\Pr[E_{i,j} \wedge E_{i',j'}] = O(\log \delta/\delta^2)$, which is true as $\Pr[E_{i,j} \wedge E_{i',j'}] \leq \Pr[E_{i,j}] = O(\log \delta/\delta^2)$. For the same reason, we may also assume $\delta \geq 5$. Let S_0 (resp., S_1) be the subsets of S consisting of a_i, a_j (resp., $a_{i'}, a_{j'}$) and all the random points in $S \setminus \{a_i, a_j, a_{i'}, a_{j'}\}$ with even indices (resp., odd indices). Clearly, $S = S_0 \cup S_1$ and $S_0 \cap S_1 = \emptyset$. Define F_0 (resp., F_1) as the event $(a_i, a_j) \in \Phi(S_0, \mathcal{U}^{\downarrow})$ (resp., $(a_{i'}, a_{j'}) \in \Phi(S_1, \mathcal{U}^{\downarrow})$). Since S_0 and S_1 are subsets of S, $E_{i,j}$ (resp., $E_{i',j'}$) happens

only if F_0 (resp., F_1) happens. Besides, F_0 and F_1 are independent events, because $S_0 \cap S_1 = \emptyset$. Thus,

$$\Pr[E_{i,j} \wedge E_{i',j'}] \leq \Pr[F_0 \wedge F_1] = \Pr[F_0] \cdot \Pr[F_1].$$

To bound $\Pr[F_0]$ and $\Pr[F_1]$, we use Lemma 3.1 again. By construction, there are $\Theta(\delta)$ points in S_0 whose x-coordinates are in $[a_i.x, a_j.x]$ (recall the assumption $\delta \geq 5$). Therefore, we have $\Pr[F_0] = O(\log \delta/\delta^2)$ by Lemma 3.1. Similarly, $\Pr[F_1] = O(\log \delta'/(\delta')^2)$. Using the above inequality, we have $\Pr[E_{i,j} \wedge E_{i',j'}] = O((\log \delta \log \delta')/(\delta\delta')^2)$. The sum of the terms $\Pr[E_{i,j} \wedge E_{i',j'}]$ satisfying $i \neq i'$ and $j \neq j'$ is $O(\log^4 n)$, as one can easily verify.

[Case 3] The most subtle case is that $i = i'$ and $j \neq j'$, or symmetrically $i \neq i'$ and $j = j'$. Assume $i = i'$ and $j > j'$. Let $\delta = j - i$ and $\delta' = j' - i$. We claim that $\Pr[E_{i,j} \wedge E_{i,j'}] = O(\log \delta/(\delta^2\delta'))$. Again, we may assume δ and δ' are sufficiently large, say $\delta > \delta' \geq 5$. Let S_0 (resp., S_1) be the subsets of S consisting of a_i, a_j (resp., $a_i, a_{j'}$) and all the random points in $S \setminus \{a_i, a_j, a_{j'}\}$ with even indices (resp., odd indices). Clearly, $S = S_0 \cup S_1$ and $S_0 \cap S_1 = \{a_i\}$. Define F_0 (resp., F_1) as the event $(a_i, a_j) \in \Phi(S_0, \mathcal{U}^{\downarrow})$ (resp., $(a_i, a_{j'}) \in \Phi(S_1, \mathcal{U}^{\downarrow})$). As in Case 2, we have

$$\Pr[E_{i,j} \wedge E_{i,j'}] \leq \Pr[F_0 \wedge F_1].$$

However, $\Pr[F_0 \wedge F_1] \neq \Pr[F_0] \cdot \Pr[F_1]$ in general, because F_0 and F_1 are not independent (both of them depends on $a_i.y$). To handle this issue, we observe that

$$\Pr[F_0 \wedge F_1] = \int_0^1 \Pr[F_0 \wedge F_1 \,|\, a_i.y = t]\,\mathrm{d}t,$$

since the distribution of $a_i.y$ is the uniform distribution on $[0, 1]$. Note that under the condition $a_i.y = t$, F_0 and F_1 are in fact independent. Indeed, when $a_i.y$ is fixed, F_0 (resp., F_1) only depends on the y-coordinates of the random points in $S_0 \setminus \{a_i\}$ (resp., $S_1 \setminus \{a_i\}$). Therefore, we can write

$$\Pr[F_0 \wedge F_1] = \int_0^1 \Pr[F_0 \,|\, a_i.y = t] \cdot \Pr[F_1 \,|\, a_i.y = t]\,\mathrm{d}t,$$

We first consider $\Pr[F_1 \,|\, a_i.y = t]$ for a fixed $t \in [0, 1]$. Let $S_1' = S_1 \cap \{a_i, \ldots, a_{j'}\}$, i.e., S_1' is the subset of S_1 consisting of all the points whose x-coordinates are in $[x_i, x_{j'}]$. We notice that F_1 happens only if $a_{j'}$ is y-adjacent to a_i in S_1', i.e., there is no other point whose y-coordinate is in between $a_i.y$ and $a_{j'}.y$. Indeed, if there exists $a \in S_1' \setminus \{a_i, a_{j'}\}$ such that $a.y$ is in between $a_i.y$ and $a_{j'}.y$, then $\mathrm{dist}(a_i, a) < \mathrm{dist}(a_i, a_{j'})$ and a is in the minimal bottom-unbounded 3-sided rectangle containing $a_i, a_{j'}$, which implies F_1 does not happen.

We claim that, under the condition $a_i.y = t$, the probability that $a_{j'}$ is y-adjacent to a_i in S_1' is $O(1/\delta')$. The y-coordinates of the random points in $S_1' \setminus \{a_i\}$ are independently drawn from the uniform distribution on $[0, 1]$, so every point in $S_1' \setminus \{a_i\}$ has the same probability (say p) to be y-adjacent to a_i. Let r be the number of

the points in $S'_1 \setminus \{a_i\}$ that are y-adjacent to a_i, which is a random variable. Then $\mathbb{E}[r] = p \cdot |S'_1 \setminus \{a_i\}|$. But we always have $r \leq 2$, since there can be at most two points y-adjacent to a_i. In particular, $\mathbb{E}[r] \leq 2$ and $p = O(1/|S'_1 \setminus \{a_i\}|)$. By construction, we have $|S'_1 \setminus \{a_i\}| = \Theta(\delta')$ (recall the assumption $\delta' \geq 5$). It follows that $p = O(1/\delta')$, i.e., the probability that $a_{j'}$ is y-adjacent to a_i in S'_1 is $O(1/\delta')$. Using our previous argument, we have $\Pr[F_1 \,|\, a_i.y = t] = O(1/\delta')$. Therefore,

$$\Pr[F_0 \wedge F_1] = O(1/\delta') \cdot \int_0^1 \Pr[F_0 \,|\, a_i.y = t]\, \mathrm{d}t.$$

Note that $\int_0^1 \Pr[F_0 \,|\, a_i.y = t]\, \mathrm{d}t = \Pr[F_0]$. By construction, there are $\Theta(\delta)$ points in S_0 whose x-coordinates are in between $a_i.x$ and $a_j.x$ (recall the assumption $\delta \geq 5$). Thus, Lemma 3.1 implies $\Pr[F_0] = O(\log \delta/\delta^2)$. Plugging in this to the equation above, we have $\Pr[F_0 \wedge F_1] = O(\log \delta/(\delta^2\delta'))$. As a result, $\Pr[E_{i,j} \wedge E_{i,j'}] = O(\log \delta/(\delta^2\delta'))$. The sum of the terms $\Pr[E_{i,j} \wedge E_{i',j'}]$ satisfying $i = i'$ and $j > j'$ is $O(\log^3 n)$, as one can easily verify. For the same reason, the terms satisfying $i = i'$ and $j < j'$ also sum up to $O(\log^3 n)$. The symmetric case that $i \neq i'$ and $j = j'$ is handled in the same fashion.

Combining all the cases, we conclude that $E[|\Phi_l(S, \mathcal{U}^{\downarrow})|^2] = E[|\Psi|] = O(\log^4 n)$.

A.6 Proof of Theorem 4.9

The $\mathcal{R}$-RCP data structure $\mathcal{D}_2$ is described in Sect. 4.3.

[Query Time] We first analyze the (worst-case) query time. When answering a query, we first find the splitting nodes $\mathbf{u}$ and $\mathbf{v}$ in the 2D range tree. As argued in the proof of Theorem 4.5, this can be done in $O(\log n)$ time. Then we query the sub-structures stored at $\mathbf{v}$ to compute $\phi, \phi_\alpha, \phi_\beta$. Note that all the sub-structures have $O(\log n)$ query time and we only need constant number of queries. Therefore, this step takes $O(\log n)$ time, and hence the overall query time is also $O(\log n)$.

[Average-Case Space Cost and Preprocessing Time] We now analyze the average-case space cost and preprocessing time of $\mathcal{D}_2$. Let R be an axes-parallel rectangle and $S \propto R^n$. We denote by $a_1, \ldots, a_n$ the n random points in S. The data structure instance $\mathcal{D}_2(S)$ is essentially a 2D range tree built on S with some sub-structures stored at secondary nodes. Note that a 2D range tree built on a set of n points in $\mathbb{R}^2$ has a fixed tree structure independent of the locations of the points. This says, while $\mathcal{D}_2(S)$ is a random data structure instance depending on the random dataset S, the 2D range tree in $\mathcal{D}_2(S)$ has a deterministic structure. As such, we can view $\mathcal{D}_2(S)$ as a fixed 2D range tree with random sub-structures. Let $\mathcal{T}$ denote the primary tree of this 2D range tree and $\mathcal{T}_{\mathbf{u}}$ denote the secondary tree at the node $\mathbf{u} \in \mathcal{T}$, as in Sect. 4. To bound $\mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S))]$ and the expected time for constructing $\mathcal{D}_2(S)$, it suffices to bound the expected space and preprocessing time of the sub-structures stored at each secondary node.

For convenience of exposition, we introduce some notations. Let $\mathbf{u} \in \mathcal{T}$ be a primary node. Suppose the n leaves of $\mathcal{T}$ are $\mathbf{lf}_1, \ldots, \mathbf{lf}_n$ sorted from left to right. Then the leaves in the subtree rooted at $\mathbf{u}$ must be $\mathbf{lf}_\alpha, \ldots, \mathbf{lf}_\beta$ for some $\alpha, \beta \in \{1, \ldots, n\}$

with $\alpha \leq \beta$ We then write $\mathsf{range}(\mathbf{u}) = [\alpha : \beta]$ and $\mathsf{size}(\mathbf{u}) = \beta - \alpha + 1$. Due to the construction of a 2D range tree, we always have $|S(\mathbf{u})| = \mathsf{size}(\mathbf{u})$ no matter what the random dataset S is. Furthermore, if $\mathsf{range}(\mathbf{u}) = [\alpha : \beta]$, then $S(\mathbf{u})$ contains exactly the points in S with x-ranks $\alpha, \ldots, \beta$ (we say a point has x-rank i in S if it is the i-th leftmost point in S). Let $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$ be a secondary node. We can define $\mathsf{range}(\mathbf{v})$ and $\mathsf{size}(\mathbf{v})$ in the same way as above (just by replacing $\mathcal{T}$ with $\mathcal{T}_{\mathbf{u}}$). Also, we always have $|S(\mathbf{v})| = \mathsf{size}(\mathbf{v})$. If $\mathsf{range}(\mathbf{v}) = [\alpha : \beta]$, then $S(\mathbf{v})$ contains exactly the points in $S(\mathbf{u})$ with y-ranks $\alpha, \ldots, \beta$ (we say a point has y-rank i in $S(\mathbf{u})$ if it is the i-th bottommost point in $S(\mathbf{u})$). In what follows, we fix a secondary node $\mathbf{v} \in \mathcal{T}_{\mathbf{u}}$ and analyze the sub-structures stored at $\mathbf{v}$. Let $\mathbf{u}'$ (resp., $\mathbf{v}'$) denote the left child of $\mathbf{u}$ (resp., $\mathbf{v}$). Suppose $\mathsf{range}(\mathbf{u}) = [\alpha : \beta]$, $\mathsf{range}(\mathbf{u}') = [\alpha : \beta']$ (where $\beta' < \beta$), $\mathsf{range}(\mathbf{v}) = [\gamma : \xi]$, $\mathsf{range}(\mathbf{v}') = [\gamma : \xi']$ (where $\xi' < \xi$).

We want to use Theorem 2.2 and Lemmas 4.4 and 4.7 to bound the average-case space cost of the $\mathcal{Q}$-RCP, TBEP/LREP, $\mathcal{U}$-RSS sub-structures, respectively. For the preprocessing time, Theorem 2.2 and Lemma 4.4 already guarantee that the $\mathcal{Q}$-RCP and TBEP/LREP sub-structures can be built efficiently even in worst-case, and we want to use Lemma 4.8 to bound the average-case time for building the $\mathcal{U}$-RSS sub-structures. However, before doing this, there is a crucial issue to be handled. Recall that in Theorem 2.2, Lemmas 4.4, 4.7, and 4.8, we assume the random dataset is generated either from the uniform distribution on a rectangle ($S \propto R^n$) or from the uniform distributions on a set of aligned segments ($S \propto \prod_{i=1}^{n} I_i$). Unfortunately, here the underlying datasets of the sub-structures are $S_1(\mathbf{v}), \ldots, S_4(\mathbf{v})$ and $S_{\blacktriangle}(\mathbf{v})$, $S_{\blacktriangledown}(\mathbf{v})$, $S_{\blacktriangleleft}(\mathbf{v})$, $S_{\blacktriangleright}(\mathbf{v})$; these random point-sets are neither (independently and uniformly) generated from a rectangle nor generated from aligned segments. For instance, we cannot directly use Theorem 2.2 to deduce $\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v})))] = O(\log^2 |S_1(\mathbf{v})|)$, since $S_1(\mathbf{v})$ is not uniformly generated from a rectangle, and even its size $|S_1(\mathbf{v})|$ is not a fixed number ($|S_1(\mathbf{v})|$ varies with S). The main focus of the rest of this proof is to handle this issue.

We first consider $S_1(\mathbf{v})$. Note that $S_1(\mathbf{v}) = S(\mathbf{u}') \cap S(\mathbf{v}')$ by definition. We want to bound $\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v})))]$. Our basic idea is the following: reducing this expectation to conditional expectations in which $S_1(\mathbf{v})$ can be viewed as uniformly and independently generated from an axes-parallel rectangle so that Theorem 2.2 applies. To this end, let $\Lambda = \{i : a_i \in S_1(\mathbf{v})\}$, which is a random subset of $[n] = \{1, \ldots, n\}$, i.e., a random variable taking value from the power set of $[n]$. A *configuration* refers to a pair (J, f) where $J \subseteq [n]$ and $f : [n] \setminus J \to R$ is a *coordinate-wise injective* function, i.e., $f(i)$ and $f(i')$ have distinct x-coordinates and y-coordinates if $i \neq i'$. For a configuration (J, f), we define a corresponding event $E_{J,f}$ as

$$E_{J,f} = \left(\bigwedge_{i \in [n] \setminus J} (a_i = f(i)) \right) \wedge (\Lambda = J).$$

We say (J, f) is a *legal* configuration if $E_{J,f}$ is a possible event. We shall show that, if (J, f) is a legal configuration, then under the condition $E_{J,f}$, the $|J|$ random points in $\{a_j : j \in J\}$ can be viewed as independently drawn from the uniform distribution on an axes-parallel rectangle. Suppose (J, f) is a legal configuration. Let $F = \{f(i) : i \in [n] \setminus J\}$, and $F' \subseteq F$ be the subset consisting of the points with

x-ranks $\alpha, \ldots, \beta - |J|$ in F. Define x_1 as the x-coordinate of the $(\alpha - 1)$-th leftmost point in F, x_2 as the x-coordinate of the $(n - \beta')$-th rightmost point in F, y_1 as the y-coordinate of the $(\gamma - 1)$-th bottommost point in F', y_2 as the y-coordinate of the $(\mathsf{size}(\mathbf{u}) - \xi')$-th topmost point in F'. Set $R' = [x_1, x_2] \times [y_1, y_2]$.

We claim that $E_{J,f}$ happens iff $a_i = f(i)$ for all $i \in [n] \setminus J$ and $a_j \in R'$ for all $j \in J$. Since (J, f) is a legal configuration, there exists at least one instance of S making $E_{J,f}$ happen. Let $S^* : \{a_i = a_i^*\}_{i \in [n]}$ be such an instance, where $a_i^* \in R$ indicates the location of a_i in the instance S^*. Then $a_i^* = f(i)$ for all $i \in [n] \setminus J$, hence $\{a_1^*, \ldots, a_n^*\} = F \cup \{a_j^* : j \in J\}$. Since the points in $\{a_j^* : j \in J\}$ belong to $S(\mathbf{u}')$ (for S^* makes $\Lambda = J$), the $\alpha - 1$ leftmost points in $F \cup \{a_j^* : j \in J\}$ (which correspond to the points in S to the left of $S(\mathbf{u}')$) must be contained in F, and hence they are just the $\alpha - 1$ leftmost points in F (which we denote by F_1). This implies $a_j^*.x \geq x_1$ for all $j \in J$. Similarly, the $n - \beta'$ rightmost points in $F \cup \{a_j^* : j \in J\}$ (which correspond to the points in S to the right of $S(\mathbf{u}')$) must be the $n - \beta$ rightmost points in F (which we denote by F_2). This implies $a_j^*.x \leq x_2$ for all $j \in J$. Clearly, the points corresponding to $S(\mathbf{u})$ are exactly those in $F' \cup \{a_j^* : j \in J\}$. Since the points in $\{a_j^* : j \in J\}$ belong to $S(\mathbf{v}')$ (for S^* makes $\Lambda = J$), the $\gamma - 1$ bottommost points in $F' \cup \{a_j^* : j \in J\}$ (which correspond to the points in $S(\mathbf{u})$ below $S(\mathbf{v}')$) must be contained in F', and hence they are just the $\gamma - 1$ bottommost points in F' (which we denote by F'_1). This implies $a_j^*.y \geq y_1$ for all $j \in J$. Similarly, the $\mathsf{size}(\mathbf{u}) - \xi'$ topmost points in $F' \cup \{a_j^* : j \in J\}$ (which correspond to the points in $S(\mathbf{u})$ above $S(\mathbf{v}')$) must be the $\mathsf{size}(\mathbf{u}) - \xi'$ topmost points in F' (which we denote by F'_2). This implies $a_j^*.y \leq y_2$ for all $j \in J$. Now we already see $a_j^* \in R'$ for all $j \in J$. It follows that $E_{J,f}$ happens only if $a_i = f(i)$ for all $i \in [n] \setminus J$ and $a_j \in R'$ for all $j \in J$.

Furthermore, we note that $F_1 \cup F_2 \cup F'_1 \cup F'_2$ corresponds to $S \setminus S_1(\mathbf{v})$. Since S^* makes $\Lambda = J$, we must have $F = F_1 \cup F_2 \cup F'_1 \cup F'_2$ (this argument relies on the existence of such an instance S^* making $E_{J,f}$ happen, i.e., it may fail if (J, f) is not a legal configuration). We then use this fact to show the "if" part. Let $S^* : \{a_i = a_i^*\}_{i \in [n]}$ be an instance of S satisfying $a_i^* = f(i)$ for all $i \in [n] \setminus J$ and $a_j^* \in R'$ for all $j \in J$. Then $\{a_1^*, \ldots, a_n^*\} = F \cup \{a_j^* : j \in J\}$. We look at the subsets F_1, F_2, F'_1, F'_2 of F. Since $a_j^*.x \in [x_1, x_2]$ for all $j \in J$, F_1 (resp., F_2) contains exactly the $\alpha - 1$ leftmost points (resp., $n - \beta'$ rightmost points) in $F \cup \{a_j^* : j \in J\}$, which correspond to the points to the left (resp., right) of $S(\mathbf{u}')$. Similarly, since $a_j^*.y \in [y_1, y_2]$ for all $j \in J$, F'_1 (resp., F'_2) contains exactly the $\gamma - 1$ bottommost points (resp., $\mathsf{size}(\mathbf{u}) - \xi'$ topmost points) in $F' \cup \{a_j^* : j \in J\}$, which correspond to the points in $S(\mathbf{u})$ below (resp., above) $S(\mathbf{v})$. Then $F = F_1 \cup F_2 \cup F'_1 \cup F'_2$ corresponds to $S \setminus S_1(\mathbf{v})$. The remaining points, which correspond to $S_1(\mathbf{v})$, are exactly those in $\{a_j^* : j \in J\}$. Therefore, $\Lambda = J$ and S^* makes $E_{J,f}$ happen. Now we see that $E_{J,f}$ happens iff $a_i = f(i)$ for all $i \in [n] \setminus J$ and $a_j \in R'$ for all $j \in J$, i.e.,

$$E_{J,f} = \left(\bigwedge_{i \in [n] \setminus J} (a_i = f(i)) \right) \wedge \left(\bigwedge_{j \in J} (a_j \in R') \right).$$

As such, under the condition $E_{J,f}$, the random points in $S_J = \{a_j : j \in J\}$ can be viewed as independently drawn from the uniform distribution on R'. Applying Theorem 2.2, we have

$$\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v}))) \,|\, E_{J,f}] = \mathbb{E}[\mathsf{Space}(\mathcal{A}(S_J)) \,|\, E_{J,f}] = O(\log^2|J|).$$

Noting that $|J| \le \mathsf{size}(\mathbf{v}') \le \mathsf{size}(\mathbf{v})$ if (J, f) is a legal configuration, we can deduce

$$\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v}))) \,|\, E_{J,f}] = O(\log^2\mathsf{size}(\mathbf{v})) \quad \text{for any } E_{J,f} \in \mathcal{E}, \tag{8}$$

where $\mathcal{E} = \{E_{J,f} : (J, f) \text{ is a legal configuration}\}$. Using this result, we further show that $\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v})))] = O(\log^2\mathsf{size}(\mathbf{v}))$. Clearly, $\mathcal{E}$ is a collection of *mutually disjoint* (or *mutually exclusive*) events. Furthermore, we notice that whenever $a_1, \ldots, a_n$ have distinct x-coordinates and y-coordinates, some $E_{J,f} \in \mathcal{E}$ happens. That says, $\mathcal{E}$ is a collection of *almost collectively exhaustive* events in the sense that with probability 1 some $E_{J,f} \in \mathcal{E}$ happens. Since the events in $\mathcal{E}$ are mutually disjoint and almost collectively exhaustive, $\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_1(\mathbf{v})))] = O(\log^2\mathsf{size}(\mathbf{v}))$ follows directly from the law of total expectation and (8). Clearly, the same idea applies to bound $\mathbb{E}[\mathsf{Space}(\mathcal{A}(S_i(\mathbf{v})))]$ for all $i \in \{1, \ldots, 4\}$. For the preprocessing time, Theorem 2.2 shows that $\mathcal{A}(S_i(\mathbf{v}))$ can be built in $O(\mathsf{size}(\mathbf{v}) \log^2\mathsf{size}(\mathbf{v}))$ time even in worst case.

Next, we consider $S_{\blacktriangle}(\mathbf{v})$. We want to bound $\mathbb{E}[\mathsf{Space}(\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v})))]$ and $\mathbb{E}[\mathsf{Space}(\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v})))]$ where $\Phi_{\blacktriangle}(\mathbf{v}) = \Phi_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}), \mathcal{U}^{\downarrow})$ by definition. The idea is totally the same as in the last paragraph: reducing to conditional expectations in which $S_{\blacktriangle}(\mathbf{v})$ can be viewed as independently generated from a set of (vertical) aligned segments so that Lemmas 4.4 and 4.7 apply. We change the definition of Λ in the last paragraph to $\Lambda = \{i : a_i \in S_{\blacktriangle}(\mathbf{v})\}$, and again define

$$E_{J,f} = \left(\bigwedge_{i \in [n] \setminus J} (a_i = f(i)) \right) \wedge (\Lambda = J),$$

based on the new definition of Λ. As we see in the last paragraph, it suffices to bound the conditional expectations $\mathbb{E}[\mathsf{Space}(\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}))) \,|\, E_{J,f}]$, $\mathbb{E}[\mathsf{Space}(\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))) \,|\, E_{J,f}]$ for all legal configuration (J, f). Suppose (J, f) is a legal configuration. Let $F = \{f(i) : i \in [n] \setminus J\}$, and $F' \subseteq F$ be the subset consisting of the points with x-ranks $\alpha, \ldots, \beta - |J|$ in F. Define x_1 as the x-coordinate of the $(\alpha - 1)$-th leftmost point in F, x_2 as the x-coordinate of the $(n - \beta)$-th rightmost point in F, y_1 as the y-coordinate of the $(\gamma - 1)$-th bottommost point in F', y_2 as the y-coordinate of the $(\mathsf{size}(\mathbf{u}) - \xi')$-th topmost point in F'. Set $R' = [x_1, x_2] \times [y_1, y_2]$. Using the same argument as in the last paragraph, one can easily verify that $E_{J,f}$ happens iff $a_i = f(i)$ for all $i \in [n] \setminus J$ and $a_j \in R'$ for all $j \in J$. For an injective function $g : J \to (x_1, x_2)$, we further define

$$E_{J,f,g} = E_{J,f} \wedge \left(\bigwedge_{j \in J} (a_j.x = g(j)) \right).$$

Now $E_{J,f,g}$ happens iff $a_i = f(i)$ for all $i \in [n] \setminus J$ and $a_j \in \{g(j)\} \times [y_1, y_2]$ for all $j \in J$. Thus, under $E_{J,f,g}$, the $|J|$ random points in $S_J = \{a_j : j \in J\}$ can be viewed as independently drawn from the $|J|$ vertical aligned segments in $\{\{g(j)\} \times [y_1, y_2] : j \in J\}$. To apply Lemmas 4.4 and 4.7, we still need to consider one thing: the line $l_{\mathbf{u}}$. The line $l_{\mathbf{u}}$ is a random vertical line depending on S. However, we notice that under $E_{J,f,g}$, $l_{\mathbf{u}}$ is fixed. Indeed, under $E_{J,f,g}$, $S(\mathbf{u})$ corresponds to $F' \cup \{a_j : j \in J\}$. Thus, the x-coordinates of the points in $S(\mathbf{u})$ are fixed under $E_{J,f,g}$, and hence $l_{\mathbf{u}}$ is fixed. As such, we are able to apply Lemma 4.4 to deduce

$$\begin{aligned}\mathbb{E}[\mathsf{Space}(\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}))) \,|\, E_{J,f,g}] &= \mathbb{E}[\mathsf{Space}(\mathcal{K}_{l_{\mathbf{u}}}(S_J)) \,|\, E_{J,f,g}] \\ &= O(\log |J|) = O(\log \mathsf{size}(\mathbf{v})),\end{aligned}$$

and apply Lemma 4.7 to deduce

$$\mathbb{E}[\mathsf{Space}(\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))) \,|\, E_{J,f,g}] = \mathbb{E}[|\Phi_{l_{\mathbf{u}}}(S_J, \mathcal{U}^{\downarrow})|^2] = O(\log^4 |J|) = O(\log^4 \mathsf{size}(\mathbf{v})).$$

Note that, if $E_{J,f}$ happens, then with probability 1 some $E_{J,f,g}$ happens. Therefore, the collection $\mathcal{E} = \{E_{J,f,g}\}$, which consists of all $E_{J,f,g}$ where (J, f) is a legal configuration and $g : J \to (x_1, x_2)$ is an injective function with range (x_1, x_2) depending on (J, f), is a collection of mutually disjoint and almost collectively exhaustive events. By the law of total expectation, we immediately have $\mathbb{E}[\mathsf{Space}(\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v})))] = O(\log \mathsf{size}(\mathbf{v}))$ and $\mathbb{E}[\mathsf{Space}(\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v})))] = O(\log^2 \mathsf{size}(\mathbf{v}))$.

For the preprocessing time, Lemma 4.4 shows that $\mathcal{K}_{l_{\mathbf{u}}}(S_{\blacktriangle}(\mathbf{v}))$ can be constructed in $O(\mathsf{size}(\mathbf{v}) \log^2 \mathsf{size}(\mathbf{v}))$ time even in worst-case. By applying Lemma 4.8 and our above argument, the expected time for constructing $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$ under each event $E_{J,f,g}$ is $O(\mathsf{size}(\mathbf{v}) \log^4 \mathsf{size}(\mathbf{v}))$. As before, using the law of total expectation, we then conclude that the expected time for constructing $\mathcal{C}(\Phi_{\blacktriangle}(\mathbf{v}))$ is $O(\mathsf{size}(\mathbf{v}) \log^4 \mathsf{size}(\mathbf{v}))$. The expected space cost and preprocessing time of the sub-structures built on $S_{\blacktriangledown}(\mathbf{v})$ can be bounded using the same argument. Also, one can handle $S_{\blacktriangleleft}(\mathbf{v})$ and $S_{\blacktriangleright}(\mathbf{v})$ in a similar way. The only difference is that, in the event $E_{J,f,g}$, the g function should indicate the y-coordinates of the points in $\{a_j : j \in J\}$ instead of the x-coordinates.

Once we know that the expected space cost of all the sub-structures stored at $\mathbf{v}$ is poly-logarithmic in $\mathsf{size}(\mathbf{v})$, we can deduce that the expected space cost of each secondary tree $\mathcal{T}_{\mathbf{u}}$ (with the sub-structures) is $O(\mathsf{size}(\mathbf{u}))$. As a result, $\mathbb{E}[\mathsf{Space}(\mathcal{D}_2(S))] = O(n \log n)$. Also, we know that we can construct the sub-structures stored at $\mathbf{v}$ in $O(\mathsf{size}(\mathbf{v}) \log^4 \mathsf{size}(\mathbf{v}))$ average-case time. Therefore, each secondary tree $\mathcal{T}_{\mathbf{u}}$ (with the sub-structures) can be constructed in $O(\mathsf{size}(\mathbf{u}) \log^5 \mathsf{size}(\mathbf{u}))$ average-case time and the entire data structure $\mathcal{D}_2(S)$ can be constructed in $O(n \log^6 n)$ average-case time.

A.7 Proof of Lemma 5.2

Without loss of generality, assume $R = [0, 1] \times [0, \Delta]$. It suffices to show that $\mathbb{E}[|\Phi(S, \mathcal{H}^{\downarrow})|] = O(\log^2 n)$. This can be further reduced to showing $\mathbb{E}[|\Phi(S, \mathcal{H}')|] = O(\log^2 n)$ where

$$\mathcal{H}' = \{l^{\downarrow} : l \text{ is a non-vertical line whose slope is non-positive}\} \subseteq \mathcal{H}^{\downarrow}.$$

Suppose the n random points in S are $a_1, \ldots, a_n$. Let $E_{i,j}$ be the event that $(a_i, a_j) \in \Phi(S, \mathcal{H}')$, and observe

$$\mathbb{E}[|\Phi(S, \mathcal{H}')|] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[E_{i,j}].$$

Note that all $\Pr[E_{i,j}]$ in the above equation are the same, which implies $\mathbb{E}[|\Phi(S, \mathcal{H}')|] = O(n^2 \cdot \Pr[E_{1,2}])$. Thus, it suffices to bound $\Pr[E_{1,2}]$. As in the proof of Lemma 2.1, we define random variables $x_{\max} = \max\{a_1.x, a_2.x\}$, $y_{\max} = \max\{a_1.y, a_2.y\}$, $x_{\min} = \min\{a_1.x, a_2.x\}$, $y_{\min} = \min\{a_1.y, a_2.y\}$, $Q = (-\infty, x_{\max}] \times (-\infty, y_{\max}]$, and $\Lambda = \{i \geq 3 : a_i \in Q\}$. We also define $Q' = (-\infty, x_{\max}/2] \times (-\infty, y_{\max}/2]$ and $\Lambda' = \{i \geq 3 : a_i \in Q'\}$. We achieve the bound for $\Pr[E_{1,2}]$ through four steps.

[Step 1] We begin with establishing the following key observation: for any $H \in \mathcal{H}'$, $a_1, a_2 \in H$ implies $Q' \subseteq H$. To see this, let $H \in \mathcal{H}'$ and assume $a_1, a_2 \in H$. If $\{a_1, a_2\} = \{(x_{\min}, y_{\min}), (x_{\max}, y_{\max})\}$, then H contains the point $(x_{\max}, y_{\max})$. This implies that H contains the point $(x_{\max}/2, y_{\max}/2)$ and hence contains Q', because $H = l^{\downarrow}$ for a line l of non-positive slope. If $\{a_1, a_2\} = \{(x_{\min}, y_{\max}), (x_{\max}, y_{\min})\}$, then H contains the 5-polygon P whose vertices are $(0, 0)$, $(x_{\max}, 0)$, $(x_{\max}, y_{\min})$, $(x_{\min}, y_{\max})$, $(0, y_{\max})$. Note that P contains the point $(x_{\max}/2, y_{\max}/2)$, which implies that H also contains the point $(x_{\max}/2, y_{\max}/2)$ and hence contains Q'.

[Step 2] Based on the observation in Step 1, we prove a result which is similar to (3) in the proof of Lemma 2.1. We claim that for all $\tilde{x} \in (0, 1]$, all $\tilde{y} \in (0, \Delta]$, and all nonempty $J' \subseteq \{3, \ldots, n\}$,

$$\Pr\left[E_{1,2} \,|\, (x_{\max} = \tilde{x}) \wedge (y_{\max} = \tilde{y}) \wedge (\Lambda' = J')\right] = O(|J'|^{-2}). \tag{9}$$

The argument for proving this is similar to that for proving (3). We use $C'_{\tilde{x},\tilde{y},J'}$ to denote the condition in the above conditional probability. Assume $|J'| = k$. Let $\delta_x = x_{\max} - x_{\min}$ and $\delta_y = y_{\max} - y_{\min}$. Since any halfplane $H \in \mathcal{H}'$ containing a_1, a_2 must contain Q', $E_{1,2}$ happens only if $\delta_x \leq \kappa(S_{J'})$ and $\delta_y \leq \kappa(S_{J'})$, where $S_{J'} = \{a_j : j \in J'\}$. So it suffices to bound

$$\Pr\left[(\delta_x \leq \kappa(S_{J'})) \wedge (\delta_y \leq \kappa(S_{J'})) \,|\, C'_{\tilde{x},\tilde{y},J'}\right].$$

Under the condition $C'_{\tilde{x},\tilde{y},J'}$, Q' is just $(-\infty, \tilde{x}/2] \times (-\infty, \tilde{y}/2]$. Thus the condition $C'_{\tilde{x},\tilde{y},J'}$ is equivalent to saying that the maximum of the x-coordinates (resp.,

y-coordinates) of a_1, a_2 is $\tilde{x}$ (resp., $\tilde{y}$), all a_j for $j \in J'$ are contained in the rectangle $R' = [0, \tilde{x}/2] \times [0, \tilde{y}/2]$, and all a_j for $j \in \{3, \ldots, n\} \setminus J'$ are contained in $R \setminus R'$. As such, one can easily verify that, under the condition $C'_{\tilde{x},\tilde{y},J'}$, the distribution of the random number δ_x (resp., δ_y) is the uniform distribution on the interval $[0, \tilde{x}]$ (resp., $[0, \tilde{y}]$) and the distributions of the k random points in $S_{J'}$ are the uniform distribution on R'; furthermore, these random numbers/points are independent of each other. This says, if we consider a new random experiment in which we independently generate two random numbers δ'_x, δ'_y from the uniform distributions on $[0, \tilde{x}]$, $[0, \tilde{y}]$ respectively (which correspond to δ_x, δ_y) and a random dataset $S' \propto (R')^k$ (which corresponds to $S_{J'}$), then we have

$$\Pr\left[(\delta'_x \le \kappa(S')) \wedge (\delta'_y \le \kappa(S'))\right] = \Pr\left[(\delta_x \le \kappa(S_{J'})) \wedge (\delta_y \le \kappa(S_{J'})) \,|\, C'_{\tilde{x},\tilde{y},J'}\right].$$

So it suffices to bound $\Pr\left[(\delta'_x \le \kappa(S')) \wedge (\delta'_y \le \kappa(S'))\right]$ in the new experiment; we denote by λ this probability. We apply the formula

$$\lambda = \int_0^\infty p(t) \cdot \Pr\left[(\delta'_x \le t) \wedge (\delta'_y \le t)\right] \mathrm{d}t = \int_0^\infty p(t) \cdot \Pr\left[\delta'_x \le t\right] \cdot \Pr\left[\delta'_y \le t\right] \mathrm{d}t,$$

where $p(\cdot)$ is the probability distribution function of $\kappa(S')$. Since δ'_x (resp., δ'_y) is uniformly drawn from the interval $[0, \tilde{x}]$ (resp., $[0, \tilde{y}]$), we have $\Pr\left[\delta'_x \le t\right] = \min\{t/\tilde{x}, 1\}$ (resp., $\Pr\left[\delta'_y \le t\right] = \min\{t/\tilde{y}, 1\}$). Without loss of generality, we assume $\tilde{x} \le \tilde{y}$. Then we have

$$\Pr\left[\delta'_x \le t\right] \cdot \Pr\left[\delta'_y \le t\right] = \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}, 1\right\} \le \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}\right\}.$$

It follows that

$$\lambda \le \int_0^\infty p(t) \cdot \min\left\{\frac{t^2}{\tilde{x}\tilde{y}}, \frac{t}{\tilde{y}}\right\} \mathrm{d}t = \min\left\{\int_0^\infty \frac{p(t)t^2}{\tilde{x}\tilde{y}} \mathrm{d}t, \int_0^\infty \frac{p(t)t}{\tilde{y}} \mathrm{d}t\right\}.$$

Noting the fact that $\int_0^\infty p(t)t^2 \,\mathrm{d}t = \mathbb{E}[\kappa^2(S')]$ and $\int_0^\infty p(t)t \,\mathrm{d}t = \mathbb{E}[\kappa(S')]$, we have

$$\lambda \le \min\left\{\frac{\mathbb{E}[\kappa^2(S')]}{\tilde{x}\tilde{y}}, \frac{\mathbb{E}[\kappa(S')]}{\tilde{y}}\right\}.$$

Since $\tilde{x} \le \tilde{y}$ by assumption, Lemma 1.1 implies $\mathbb{E}[\kappa(S')] = O(\max\{\sqrt{\tilde{x}\tilde{y}}/k, \tilde{y}/k^2\})$ and $\mathbb{E}[\kappa^2(S')] = O(\max\{\tilde{x}\tilde{y}/k^2, \tilde{y}^2/k^4\})$. If $\sqrt{\tilde{x}\tilde{y}}/k \le \tilde{y}/k^2$, then $\mathbb{E}[\kappa(S')]/\tilde{y} = O(1/k^2)$, otherwise $\mathbb{E}[\kappa^2(S')]/(\tilde{x}\tilde{y}) = O(1/k^2)$. In either of the two cases, we have $\lambda = O(1/k^2)$. Therefore, we obtain (9). For an arbitrary nonempty $J' \subseteq \{3, \ldots, n\}$, since (9) holds for all $\tilde{x} \in (0, 1]$ and $\tilde{y} \in (0, \Delta]$, we can remove the conditions $x_{\max} = \tilde{x}$ and $y_{\max} = \tilde{y}$ from (9) to deduce $\Pr\left[E_{1,2} \,|\, \Lambda' = J'\right] = O(|J'|^{-2})$ (note that although we miss the case $\tilde{x} = 0$ or $\tilde{y} = 0$ for (9), it does not matter since the events $x_{\max} = 0$ and $y_{\max} = 0$ happen with probability 0). This further implies

that $\Pr[E_{1,2}\,|\,|\Lambda'|=k]=O(1/k^2)$ for all $k\in\{1,\dots,n-2\}$. For $k=0$, we have $\Pr[E_{1,2}\,|\,|\Lambda'|=k]=1$.

[Step 3] Let m be a sufficiently large integer, and $m'=\lfloor m/8\rfloor$. Our goal in this step is to bound $\Pr[|\Lambda'|\le m'\,|\,|\Lambda|=m]$. Again, we reduce to conditional probability. We claim that for all $\tilde{x}\in(0,1]$, all $\tilde{y}\in(0,\Delta]$, and all $J\subseteq\{3,\dots,n\}$ with $|J|=m$,

$$\Pr\left[|\Lambda'|\le m'\,|\,(x_{\max}=\tilde{x})\wedge(y_{\max}=\tilde{y})\wedge(\Lambda=J)\right]\le e^{-m/32}. \tag{10}$$

We use $C_{\tilde{x},\tilde{y},J}$ to denote the condition in the above conditional probability. Under the condition $C_{\tilde{x},\tilde{y},J}$, $Q=(-\infty,\tilde{x}]\times(-\infty,\tilde{y}]$ and $Q'=(-\infty,\tilde{x}/2]\times(-\infty,\tilde{y}/2]$. Since $\Lambda'\subseteq\Lambda$ by definition, we have, under the condition $C_{\tilde{x},\tilde{y},J}$,

$$|\Lambda'|=\sum_{j\in J}\mathbf{1}_{a_j\in Q'},\ \text{where}\ \mathbf{1}_{a_j\in Q'}=\begin{cases}1 & a_j\in Q'\\ 0 & a_j\notin Q'\end{cases}\ \text{is the indicator function.}$$

As we have seen when proving (3) in the proof of Lemma 2.1, under the condition $C_{\tilde{x},\tilde{y},J}$, the m random points in S_J can be viewed as independently drawn from the uniform distribution on the rectangle $[0,\tilde{x}]\times[0,\tilde{y}]$. Note that a random point drawn from the uniform distribution on $[0,\tilde{x}]\times[0,\tilde{y}]$ has probability $1/4$ to be contained in Q'. Therefore, under the condition $C_{\tilde{x},\tilde{y},J}$, $\{\mathbf{1}_{a_j\in Q'}:j\in J\}$ is a set of i.i.d. random variables each of which equals 1 with probability $1/4$ and equals 0 with probability $3/4$. It follows that $\mathbb{E}[|\Lambda'|\,|\,C_{\tilde{x},\tilde{y},J}]=m/4$. By Hoeffding's inequality, we have $\Pr[m/4-|\Lambda|\ge m/8\,|\,C_{\tilde{x},\tilde{y},J}]\le e^{-2(m/8)^2/m}=e^{-m/32}$, which implies (10). Since (10) holds for all $\tilde{x}\in(0,1]$, all $\tilde{y}\in(0,\Delta]$, and all $J\subseteq\{3,\dots,n\}$ with $|J|=m$, we can deduce that $\Pr[|\Lambda'|\le m'\,|\,|\Lambda|=m]\le e^{-m/32}$.

[Step 4] Finally, we try to bound $\Pr[E_{1,2}]$ using the results obtained in the previous steps. We apply the formula

$$\Pr[E_{1,2}]=\sum_{k=0}^{n-2}\Pr[|\Lambda'|=k]\cdot\Pr[E_{1,2}\,|\,|\Lambda'|=k].$$

Since

$$\Pr[|\Lambda'|=k]=\sum_{m=k}^{n-2}\Pr[|\Lambda|=m]\cdot\Pr[|\Lambda'|=k\,|\,|\Lambda|=m],$$

we further deduce

$$\Pr[E_{1,2}]=\sum_{m=0}^{n-2}\left(\Pr[|\Lambda|=m]\cdot\sum_{k=0}^{m}g_{m,k}\right). \tag{11}$$

where $g_{m,k}=\Pr[E_{1,2}\,|\,|\Lambda'|=k]\cdot\Pr[|\Lambda'|=k\,|\,|\Lambda|=m]$. We claim that $\sum_{k=0}^{m}g_{m,k}=O(1/m^2)$ for all $m\in\{1,\dots,n-2\}$. To prove this, we may assume i is sufficiently

large. Set $m' = \lfloor m/8 \rfloor$. Using the result of Step 3, we can deduce that

$$\sum_{k=0}^{m'} g_{m,k} \leq \sum_{k=0}^{m'} \Pr\left[|\Lambda'| = k \,||\Lambda| = m\right] = \Pr\left[|\Lambda'| \leq m' \,||\Lambda| = m\right] \leq e^{-m/32}.$$

On the other hand, by the choice of m' and the result of Step 2, $\Pr\left[E_{1,2} \,||\Lambda'| = k\right] = O(1/m^2)$ for all $k \in \{m'+1, \ldots, m\}$. As such, we have

$$\sum_{k=m'+1}^{m} g_{m,k} = \sum_{k=m'+1}^{m} O(m^{-2}) \cdot \Pr\left[|\Lambda'| = k \,||\Lambda| = m\right] = O(m^{-2}).$$

It follows that

$$\sum_{k=0}^{m} g_{m,k} = \sum_{k=0}^{m'} g_{m,k} + \sum_{k=m'+1}^{m} g_{m,k} \leq e^{-m/32} + O(m^{-2}) = O(m^{-2}).$$

For $m = 0$, we have the trivial bound $\sum_{k=0}^{m} g_{m,k} = g_{m,0} \leq 1$. Thanks to (11) and the bounds for $\sum_{k=0}^{m} g_{m,k}$, the only thing remaining for bounding $\Pr[E_{1,2}]$ is to bound $\Pr\left[|\Lambda| = m\right]$. Recall that, in the proof of Lemma 2.1, we have shown $\Pr\left[|\Lambda| = m\right] = O((m+1)\log n/n^2)$ for all $m \in \{0, \ldots, n-2\}$. Plugging in this and the bounds for $\sum_{k=0}^{m} g_{m,k}$ to (11), a direct calculation gives us $\Pr[E_{1,2}] = O(\log^2 n/n^2)$. As such, $\mathbb{E}[|\Phi(S, \mathcal{H}')|] = O(\log^2 n)$ and thus $\mathbb{E}[|\Phi(S, \mathcal{H})|] = O(\log^2 n)$.

References

1. Abam, M.A., Carmi, P., Farshi, M., Smid, M.: On the power of the semi-separated pair decomposition. Comput. Geom. **46**(6), 631–639 (2013)
2. Agarwal, P.K., Erickson, J.: Geometric range searching and its relatives. In: Advances in Discrete and Computational Geometry (South Hadley 1996). Contemporary Mathematics, vol. 223, pp. 1–56. American Mathematical Society, Providence (1999)
3. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M.: Computational Geometry. Springer, Berlin (2008)
4. Edelsbrunner, H., Guibas, L.J., Stolfi, J.: Optimal point location in a monotone subdivision. SIAM J. Comput. **15**(2), 317–340 (1986)
5. Gupta, P.: Range-aggregate query problems involving geometric aggregation operations. Nordic J. Comput. **13**(4), 294–308 (2006)
6. Gupta, P., Janardan, R., Kumar, Y., Smid, M.: Data structures for range-aggregate extent queries. Comput. Geom. **47**(2C), 329–347 (2014)
7. Sarnak, N., Tarjan, R.E.: Planar point location using persistent search trees. Commun. ACM **29**(7), 669–679 (1986)
8. Shan, J., Zhang, D., Salzberg, B.: On spatial-range closest-pair query. In: 8th International Symposium on Spatial and Temporal Databases (Santorini 2003). Lecture Notes in Computer Science, vol. 2750, pp. 252–269 (2003)
9. Sharathkumar, R., Gupta, P.: Range-aggregate proximity queries. Technical Report IIIT/TR/2007/80. IIIT Hyderabad, Telangana, # 500032 (2007)
10. Smid, M.: Closest point problems in computational geometry. In: Handbook of Computational Geometry, pp. 877–935. North-Holland, Amsterdam (2000)
11. Xue, J.: Colored range closest-pair problem under general distance functions (2018). arXiv:1807.09977

12. Xue, J., Li, Y., Janardan, R.: Approximate range closest-pair queries. Comput. Geom. **90**, # 101654 (2020)
13. Xue, J., Li, Y., Rahul, S., Janardan, R.: New bounds for range closest-pair problems. In: 34th International Symposium on Computational Geometry (Budapest 2018). Leibniz International Proceedings in Informatics, vol. 99, # 73. Leibniz-Zent. Inform., Wadern (2018)
14. Xue, J., Li, Y., Rahul, S., Janardan, R.: Searching for the closest-pair in a query translate. J. Comput. Geom. **11**(2), 26–61 (2020)