Completeness for the Complexity Class ∀∃ℝ and Area-Universality

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Abstract

Exhibiting a deep connection between purely geometric problems and real algebra, the complexity class ∃ℝ plays a crucial role in the study of geometric problems. Sometimes ∃ℝ is referred to as the 'real analog' of NP. While NP is a class of computational problems that deals with existentially quantified *boolean* variables, ∃ℝ deals with existentially quantified *real* variables. In analogy to Π_2^p and Σ_2^p in the famous polynomial hierarchy, we study the complexity classes ∀∃ℝ and ∃∀ℝ with *real* variables. Our main interest is the AREA UNIVERSALITY problem, where we are given a plane graph G, and ask if for each assignment of areas to the inner faces of G, there exists a straight-line drawing of G realizing the assigned areas. We conjecture that AREA UNIVERSALITY is ∀∃ℝ-complete and support this conjecture by proving ∃ℝ- and ∀∃ℝ-completeness

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of two variants of AREA UNIVERSALITY. To this end, we introduce tools to prove
∀∃ℝ-hardness and membership. Finally, we present geometric problems as candi-
dates for ∀∃ℝ-complete problems. These problems have connections to the concepts
of imprecision, robustness, and extendability.

1 Introduction

Size is a very intuitive visual variable. Therefore, statistics are frequently illustrated by
distorted maps where regions are scaled according to the population, number of births,
average income or some other parameter of interest. Such distorted maps, known as
cartograms, inspire the investigation of problems related to face areas in straight-line
drawings of planar graphs.

A plane graph is a graph together with a *planar drawing*, in which edges may
intersect only in common vertices. Two planar drawings of a graph are *equivalent* if
they have the same outer face and the same set of inner faces; i.e., walking on the
face boundaries (in counter clockwise direction) yields the same (directed) cycles.
Let G be a plane graph and let F be the set of inner faces of G. An *area assignment*
is a real-valued function $\mathcal{A} \colon F \to \mathbb{R}^+$. We say that a drawing G' is $\mathcal{A}$-*realizing*,
if G' is a straight-line drawing equivalent to G in which the area of each $f \in F$ is
exactly $\mathcal{A}(f)$. If G has an $\mathcal{A}$-realizing drawing, we say that $\mathcal{A}$ is *realizable*. A plane
graph G is *area-universal* if every area assignment is realizable. In this paper, we study
the computational problem of deciding whether a given plane graph is area-universal.
We denote this problem by AREA UNIVERSALITY.

AREA UNIVERSALITY
Input: A plane graph $G = (V, E)$.
Question: Does every positive area assignment $\mathcal{A}$ of G have a planar $\mathcal{A}$-realizing
drawing?

With cartograms in mind, it is also natural to allow face areas of 0. Note that for
a planar realizing drawing, the area assignment must be *positive*, i.e., all assigned
areas are positive. In order to be able to realize face areas of 0, we slightly relax
the planarity condition for the set of realizing drawings: We consider a straight-line
drawing as a *crossing-free* drawing of a plane graph G if for every $\varepsilon > 0$, the vertex
coordinates can be perturbed within a ball of radius ε such that the result is a planar
drawing equivalent to G. In other words, we extend the set of planar drawings by the
(non-planar) *degenerate* drawings that can be obtained as the limit of a sequence of
planar drawings. In the context of weakly simple polygons, this notion is also known
under the term *weak embedding*; see also [5]. Because all considered drawings are

crossing-free and straight-line, we simply call them drawings from now on; we use the terms degenerate or non-degenerate to distinguish between planar and crossing-free drawings if necessary. This results in the following closely related decision problem.

AREA UNIVERSALITY$_{\geq 0}$
Input: A plane graph $G = (V, E)$.
Question: Does every (non-negative) area assignment $\mathcal{A}$ of G have a crossing-free $\mathcal{A}$-realizing drawing?

Interestingly, for a large family of plane graphs, the problems AREA UNIVERSALITY and AREA UNIVERSALITY$_{\geq 0}$ coincide: A compactness argument shows that for the area-universality of a triangulation (even more generally, for every plane graph with a triangular outer face) it makes in fact no difference whether or not faces of 0 area are allowed, i.e., all positive area assignments of a triangulation are realizable if and only if all non-negative area assignments are realizable. For more details, we refer to [27, Cor. 5] and [28, Cor. 3]. As we will also point out in Sect. 1.3, allowing face areas of 0 turned out to be a useful tool to significantly simplify proofs and enable (more) elegant proofs, e.g., in the context of disproving area-universality [27].

1.1 Introduction to the Complexity Class ∀∃ℝ

When investigating geometric problems, one often discovers that their instances can be described by a system of polynomial equations and inequalities Φ, so that real-valued variable assignments that satisfy Φ correspond to solutions of the original geometric problem. The variables encode the configuration of the geometric objects and the quantifier-free formula Φ describes the relations between them. EXISTENTIAL THEORY OF THE REALS (ETR) is a computational problem that takes a first-order formula containing only existential quantifiers: $\exists X = (X_1, X_2, \ldots, X_n) : \Phi(X)$ where Φ may contain the symbols

$$0, 1, +, \cdot, =, <, \wedge, \neg, (,), X_1, \ldots, X_n$$

and asks whether it is true or not over the reals. An example of such an instance is the following formula[1]: $\exists(X_1, X_2) : (X_1^2 + X_2^2 > 1) \wedge (3X_1 + 2X_2 = 10)$.

The complexity class $\exists\mathbb{R}$ consists of all decision problems that are many-one reducible to ETR by a Turing machine in a polynomial number of steps. Many natural geometric problems appear to be $\exists\mathbb{R}$-complete, i.e., they are in $\exists\mathbb{R}$ and ETR is reducible to them. Prominent examples include separability of pseudoline arrangements [32, 36, 46], recognizing unit disk graphs [25] and intersection graphs of segments [32, 44], the art gallery problem [1], geometric packing [4], or training neural networks [2].

Despite tremendous work in real algebraic geometry, we do not know any simple algorithm to decide ETR. Consequently, the $\exists\mathbb{R}$-completeness of any of the problems

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[1] We use the notation of Matoušek [32] where capital letters denote the names of variables, while lower-case letters denote the values of variables (numbers or logic values).

mentioned above implies that there is little hope that they could be solved using some simple algorithms, and most known algorithms have some algebraic flavor. In fact, we usually need to employ certain algebraic tools even to show that these problems are *decidable*. In conclusion, the $\exists\mathbb{R}$-completeness of many geometric problems reflects deep algebraic connections between these problems and real algebra.

   While geometric problems that are $\exists\mathbb{R}$-complete usually ask for the existence of certain objects, satisfying some semi-algebraic properties, the nature of area-universality seems to be different. We therefore define the complexity class $\forall\exists\mathbb{R}$ as the set of all problems that reduce in polynomial time to UNIVERSAL EXISTENTIAL THEORY OF THE REALS (UETR). The input of UETR is a first order formula over the reals in prenex form, which starts with a block of universal quantifiers followed by a block of existential quantifiers and is otherwise quantifier-free. We ask if the formula is true. An example of such an instance is:

$$\forall Y_1, Y_2\ \exists X_1 : (Y_1^2 + X_1^2 \geq 1) \wedge (3Y_1 + 2Y_2 = 10X_1).$$

UNIVERSAL EXISTENTIAL THEORY OF THE REALS (UETR)
Input: A formula $(\forall Y = (Y_1, Y_2, \ldots, Y_m))\,(\exists X = (X_1, X_2, \ldots, X_n))$: $\Phi(X, Y)$ over the reals where $\Phi(X, Y)$ is quantifier-free.
Question: Is the input formula true?

Relation of the Complexity Classes.   The complexity class $\exists\forall\mathbb{R}$ is defined analogously. Clearly, $\exists\mathbb{R}$ is contained in both, $\exists\forall\mathbb{R}$ and $\forall\exists\mathbb{R}$. It is easy to observe and well-known that NP is contained in $\exists\mathbb{R}$. Highly non-trivial is the containment of $\exists\forall\mathbb{R}$ and $\forall\exists\mathbb{R}$ in PSPACE, which follows from a more general result that deciding first-order formulae over the reals with bounded number of quantifier blocks is in PSPACE [6]. For all we know, all these complexity classes could collapse, as it is open whether NP and PSPACE constitute two different or the same complexity class, see Fig. 1. However, $\exists\mathbb{R} \neq \forall\exists\mathbb{R}$ can be believed with similar confidence as NP $\neq \Pi_2^P$.
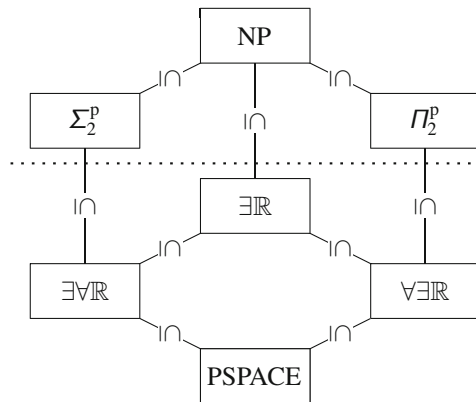

Fig. 1 Containment diagram of the complexity classes

In addition, it is known that the algebraic expressibility of ∀∃ℝ-formulae is larger than ∃ℝ-formulae [16].

It is worth mentioning that Blum et. al. [10] also introduced a hierarchy of complexity classes analogous to the complexity class NP, but over the reals (or other rings). Their canonical model of computation is the so-called Blum–Shub–Smale machine (BSS). The main difference between these approaches is that BSS accepts real numbers as input and the arithmetic operations over the reals can be performed at unit cost, while the classes discussed in our paper (∃ℝ, ∀ℝ, ∃∀ℝ) work with ordinary Turing machines, accepting only finite strings over finite alphabets.

1.2 Our Results

First of all, we show that both variants of area-universality belong to ∀∃ℝ.

Proposition 1.1 Area Universality *and* Area Universality$_{\geq 0}$ *are contained in* ∀∃ℝ.

The idea of the proof (presented in Sect. 2.3) is to use a block of universal quantifiers to describe the area assignment and the block of existential quantifiers to describe the placement of the vertices of the drawing of G. While it is straightforward to show containment for Area Universality, the challenge for proving Proposition 1.1 for Area Universality$_{\geq 0}$ lies in the fact that we allow for degenerate realizing drawings. Theorem 1.2 enables us to exploit the fact that testing whether a drawing is crossing-free lies in P. Interestingly, this is far more intricate than deciding whether a drawing is planar.

Recently, Erickson et al. showed a result analogous to the Cook–Levin Theorem for ∃ℝ-membership [20]. It is straightforward to generalize their proof to ∀∃ℝ. The gist of the theorem is that we can can use real RAM algorithms to establish ∀∃ℝ-membership. (See Sect. 2.2 for precise definitions.)

Theorem 1.2 *For every discrete decision problem Q, there exists a real challenge-- verification algorithm if and only if Q ∈* ∀∃ℝ.

In view of Proposition 1.1, we believe that a stronger statement holds.

Conjecture Area Universality *is* ∀∃ℝ-*complete.*

While this conjecture, if true, would show that Area Universality is a difficult problem in an algebraic and combinatorial sense, it would also give the first known natural geometric problem that is complete for ∀∃ℝ.

Unfortunately, in contrast to most other algorithmic problems, studying area-universality seems very difficult even on small instances. This is particularly unfortunate since most hardness results are based on constructing small gadgets that are well understood and then combined in a very controlled and clever way. As discussed in more detail in Sect. 1.3, there exist graphs on nine vertices for which we do not yet know whether or not they are area-universal. This is due to (1) the continuous nature of possible area assignments and the vertex placements, and (2) the doubly quantified interplay between the two. Due to those difficulties, we focus on restricted variants

that give us some control over parts of the final drawing. Although we are still some steps away from resolving our conjecture, we believe that our results bring us a big step forward. Even more, many of the presented results and tools are interesting by themselves especially in the context of existing ∃ℝ-hardness results. We consider two variants of AREA UNIVERSALITY, each approaching the conjecture from a different direction.

Firstly, a natural relaxation is to drop the planarity restriction. For a plane graph G with vertex set V, the *face hypergraph* of G has vertex set V, and its edges correspond to sets of vertices forming the faces (such hypergraphs were studied e.g. by Dvořák et al. [18]). Recall that a *triangulation* is a maximal planar graph, i.e., every face is incident to three edges. Therefore, the face hypergraph of a plane triangulation is 3-uniform, i.e., each hyperedge has three vertices. Clearly, AREA UNIVERSALITY can be equivalently formulated in the language of face hypergraphs. This relation motivates the following version of the problem in which we are fixing some of the areas, i.e., we consider instances *with a partial fixed assignment* (*wPFA*).

AREA UNIVERSALITY FOR TRIPLES WPFA

Input: A set V of vertices, a collection of vertex-triples $F \subseteq \binom{V}{3}$, and a partial area assignment $\mathcal{A}' \colon F' \to \mathbb{R}^+$ for some $F' \subseteq F$.

Question: For every $\mathcal{A} \colon F \to \mathbb{R}^+$ with $\mathcal{A}(f) = \mathcal{A}'(f)$ for all $f \in F'$, does there exist a placement of V in the plane, such that the triangle of each $f \in F$ has area $\mathcal{A}(f)$?

Note that AREA UNIVERSALITY FOR TRIPLES WPFA differs from AREA UNIVERSALITY in two aspects: Firstly, we consider arbitrary triples in contrast to interior disjoint faces, and secondly, some areas are fixed. However, we are able to restrict to a linear number of triples and show the following hardness result.

Theorem 1.3 AREA UNIVERSALITY FOR TRIPLES WPFA *is ∀∃ℝ-complete, even if the number of triples is linear in the number of vertices.*

For the proof of Theorem 1.3 we use gadgets similar to the *von Staudt constructions* used to show the ∃ℝ-hardness of order-types (we refer the reader to the paper of Matoušek [32] for more information). The proof can be found in Sect. 3.

Our second result concerns a variant, where we investigate the complexity of realizing a specific area assignment. PRESCRIBED AREA denotes the following problem:

PRESCRIBED AREA

Input: A plane graph $G = (V, E)$ with an area assignment $\mathcal{A}$.

Question: Does there exist an $\mathcal{A}$-realizing drawing of G?

We believe that PRESCRIBED AREA is ∃ℝ-complete. However, we are only able to show hardness of an *extension* version of the problem, where some vertex positions are fixed and we seek a placement of the remaining vertices realizing the prescribed areas. We call this problem PRESCRIBED AREA EXTENSION.

> **PRESCRIBED AREA EXTENSION**
>
> **Input:** A plane graph $G = (V, E)$, a (positive) area assignment $\mathcal{A}$, fixed positions for $V' \subseteq V$.
>
> **Question:** Does there exist a (planar) $\mathcal{A}$-realizing drawing of G respecting the positions of all vertices in V'?

We show the following hardness result.

Theorem 1.4 PRESCRIBED AREA EXTENSION *is ∃ℝ-complete.*

The proof of Theorem 1.4 can be found in Sect. 4. Let us point out that, along with the recent result of Lubiw et al. [31], Theorem 1.4 is one of the *first* ∃ℝ-hardness results concerning drawings of planar graphs in the plane.

To conclude and to motivate further research, we present problems that are interesting candidates for ∀∃ℝ-complete problems in Sect. 5. These problems are linked to notions such as robustness, imprecision, and extendability. To the best of our knowledge, no further problems are known to be ∀∃ℝ-complete.

1.3 Area-Universality — State of the Art & Related Work

In this section, we present the state of the art of area-universality. We present some ideas and challenges of how to prove and disprove area-universality.

Area-Universal Graph Families.   Despite the fact that area-universality seems to be a strong property, it is straightforward to observe that it holds for *stacked triangulations*, also known as *plane 3-trees* or *Apollonian networks*. A stacked triangulation T is defined recursively by subdividing a triangle t of a stacked triangulation T' into three smaller triangles. An area assignment of T can be realized by first realizing T' so that t has the total area of the three smaller triangles, and then subdividing t accordingly. Moreover, it is easy to see that if a graph is area-universal, then each of its subgraphs is also area-universal. Thus, subgraphs of stacked triangulations are area-universal. Biedl and Velázquez [9] additionally studied the grid size of realizing drawings of subgraphs of stacked triangulations.

The first graphs that were shown to be area-universal were plane cubic graphs, as proven already in 1992 by Thomassen [47]. However, unlike for stacked triangulations, this inductive argument is highly technical. The proof makes extensive use of the fact that if a vertex v of degree 3 is placed collinearly with two of its neighbors u and w, then v may be freely moved on the segment between u and w. This fact allows to lay out a path in a straight-line fashion in a step of the induction. Alternatively, an edge is contracted. Consequently, the resulting realizing drawings are highly degenerate.

More recently, Kleist [26, 27] showed that the 1-subdivision of every plane graph is area-universal. In other words, every area assignment of a plane graph is realizable if the straight-line property is relaxed so that each edge may have one bend. This result builds on the fact that every rectangular layout (dissection of a rectangle into rectangles) has a weakly-equivalent rectangular layout realizing prescribed areas [19,
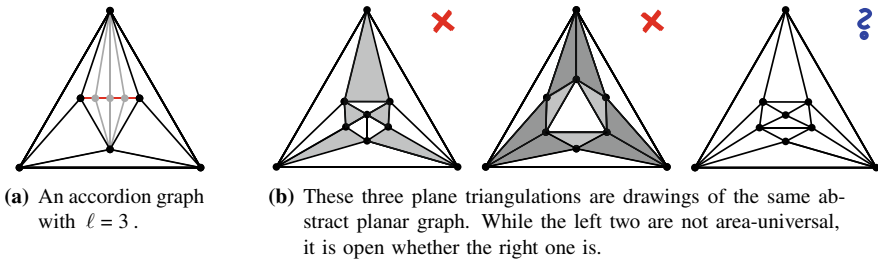
(a) An accordion graph with $\ell = 3$.

(b) These three plane triangulations are drawings of the same abstract planar graph. While the left two are not area-universal, it is open whether the right one is.

Fig. 2 Illustration of interesting triangulations

23, 48]. In this setting, positive area assignments are guaranteed to have non-degenerate (planar) realizing drawings.

For straight-line drawings, however, deciding if a graph is area-universal seems to be a challenging problem even for very small triangulations. [The interested reader may play with 4-connected triangulations on seven or eight vertices or, alternatively, with the graph on nine vertices in Fig. 2b, see our discussion later.] Kleist [29] developed a method to prove the area-universality of triangulations with special vertex orderings. The method is based on a sufficient criterion for area-universality that only requires the investigation of one area assignment. Among others, the criterion uses the fact that for a triangulation, the realizability of an area assignment can be characterized by a real solution of an equation system that describes the area of each triangular face with a determinant equation. The machinery can be used to characterize the area-universality of a graph family, called accordion graphs: An accordion graph $\mathcal{K}_\ell$ can be obtained from the plane octahedron graph by subdividing an edge of the central triangle by ℓ vertices and introducing 2ℓ additional edges such that the new vertices are of degree 4. For an example consider Fig. 2a. Obviously, accordion graphs are structurally very similar. However, in terms of area-universality there is a surprising distinction: An accordion graph is area-universal if and only if ℓ is odd. This shows that there exists a very fine line between graphs that are and those that are not area-universal. In particular, this exhibits the sensitivity of area-universality to small local changes. Thus, area-universality might be a *global* property.

As a further interesting candidate, it is conjectured that plane bipartite graphs are area-universal. Evans et al. [22, 28] developed the first tools to tackle this problem, and used them to confirm the conjecture for large subclasses of plane bipartite graphs, as well as for small plane bipartite graphs on up to 13 vertices.

Disproving Area-Universality.   In terms of negative results, it was already known to Ringel [42] in 1990 that the octahedron graph is not area-universal. For a long time, the octahedron and its supergraphs remained the only known non-area-universal graphs. Kleist [26, 27] introduced the first infinite family of non-area-universal graphs, that is not defined as the supergraphs of a single non-area-universal graph. More precisely, she presented a simple combinatorial argument to show that all Eulerian triangulations (different from the triangle) are not area-universal. Note that the octahedron is also an Eulerian triangulation. Among other things, the elegance of the argument relies on the

fact of using face areas of 0. Recall that for the area-universality of a triangulation, it makes no difference whether or not faces of 0 area are allowed.

Moreover, the above results have several interesting consequences. For instance, there exist plane graphs and area assignments such that no drawing approximates the area assignment by a constant factor c, i.e., there is no drawing such that for every inner face f with assigned area A it holds that $1/c \cdot A \leq \mathrm{AREA}(f) \leq c \cdot A$. Combined with geometric arguments, the developed technique can be used to disprove area-universality of other graphs such as the icosahedron graph [27] or other small triangulations [24, 28]. The fact that the icosahedron is not area-universal shows that high connectivity of a graph does not imply area-universality. Furthermore, area-universality is not a minor-closed property, as every grid graph is area-universal [21, 22, 28], but the octahedron is not area-universal, although it is a minor of the grid.

Understanding Small Graphs. As mentioned before, understanding properties of realizing drawings of small graphs can be very useful; in particular in order to construct gadgets for hardness proofs. However, deciding whether a graph is area-universal is already a challenge for quite small triangulations. Using the tools developed by Kleist [27–29] one can characterize the area-universality of triangulations with special vertex orderings on up to ten vertices. In fact, it suffices to study 4-connected triangulations. The most interesting and smallest graph with unknown status is the triangulation depicted in the right of Fig. 2b; this triangulation does not have one of the special vertex orderings mentioned above. The underlying planar graph has three distinct embeddings which are depicted in Fig. 2b. While the leftmost plane graph has a non-realizable area assignment consisting of 0's and 1's (called 01-assignments for simplicity), all 01-assignments of the middle graph are realizable. Nevertheless, the middle graph is not area-universal. If the rightmost triangulation was area-universal, then this would show that area-universality is a property of plane rather than planar graphs.

2 Preliminaries

In this section, we firstly introduce restricted but hard variants of ETR and UETR. These variants will be the base problems for our reductions. Secondly, we present a Cook–Levin analog which we then use in order to prove the containment of AREA UNIVERSALITY and AREA UNIVERSALITY$_{\geq 0}$ in $\forall \exists \mathbb{R}$.

2.1 Toolbox: Hard variants of ETR and UETR

Abrahamsen et al. showed that the following problem is $\exists \mathbb{R}$-complete [1].

> **ETR-INV**
> **Input:** A formula over the reals of the form $(\exists\ X_1, X_2, \ldots, X_n) : \Phi$, where Φ is a conjunction of constraints of the following form: $X = 1$ (introducing a constant 1), $X + Y = Z$ (addition), $X \cdot Y = 1$ (inversion), with $X, Y, Z \in \{X_1, \ldots, X_n\}$. Additionally, Φ is either unsatisfiable or has a solution where each variable is within $[1/2, 2]$.
> **Question:** Is the input formula true?

In order to define some even more restricted variant of ETR-INV, we need one more definition. Consider a formula Φ of the form $\Phi = \Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_m$, where each Φ_i is a quantifier-free formula of the first-order theory of the reals with variables $X_1, X_2, \ldots, X_n$, which uses arithmetic operators and comparisons $(=, <, \leq)$ but no logic symbols. The *incidence graph* of Φ is the bipartite graph with vertex set $\{X_1, X_2, \ldots, X_n\} \cup \{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ that has an edge $X_i \Phi_j$ if and only if the variable X_i appears in the subformula Φ_j. By PLANAR-ETR-INV we denote the variant of ETR-INV where the incidence graph of Φ is planar and Φ is either unsatisfiable or has a solution with all variables within $[1/2, 4]$.

Note that Lubiw, Mondal, and Miltzow introduced another version of PLANAR-ETR-INV, which has inequality instead of equality constraints [31].

Theorem 2.1 PLANAR-ETR-INV *is $\exists\mathbb{R}$-complete.*

Proof Consider an instance $(\exists\ X_1, X_2, \ldots, X_n) : \Phi$ of ETR-INV. Let G be some embedding of $G(\Phi)$ in $\mathbb{R}^2$. Suppose that G is not crossing-free and consider a pair of crossing edges. Let X and Y denote the variables corresponding to (one endpoint of) these edges as in Fig. 3.

We introduce three new existential variables X', Y', Z and three constraints: $X + Y = Z$, $X + Y' = Z$, and $X' + Y = Z$. Observe that these constraints ensure that $X = X'$ and $Y = Y'$. Moreover, the embedding of G can be modified so that the new incidence graph G' has strictly fewer crossings: G' loses the considered crossing and no new crossing is introduced. We repeat this procedure until the incidence graph of the obtained formula is planar. Finally, note that $1 \leq Z = X + Y \leq 2 + 2 = 4$ whenever $1/2 \leq X, Y \leq 2$, and the number of new variables and constraints is polynomial in $|\Phi|$, since the number of variables in each constraint in ETR-INV is at most three. □
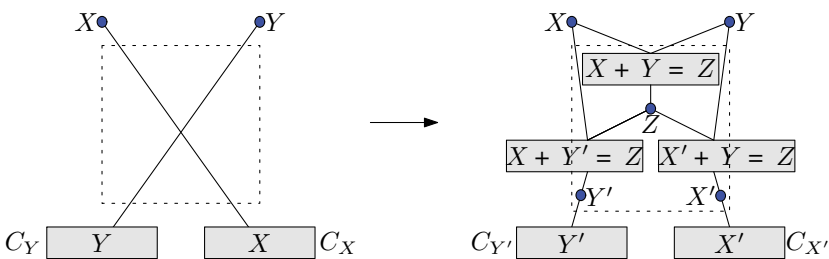

Fig. 3 Eliminating crossings

Now we introduce a restricted variant of UETR.

CONSTRAINED-UETR

Input: A formula $(\forall \, Y_1, \ldots, Y_m \in \mathbb{R}^+)(\exists \, X_1, \ldots, X_n \in \mathbb{R}^+) : \Phi(X, Y)$ over the reals, where Φ is a conjunction of constraints of the form: $X = 1$ (introducing constant 1), $X + Y = Z$ (addition), $X \cdot Y = Z$ (multiplication), with $X, Y, Z \in \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$.

Question: Is the input formula true?

CONSTRAINED-UETR can be seen as a variant of $\forall \exists \mathbb{R}$ that is simplified in a way analogous to a $\exists \mathbb{R}$-complete variant of ETR called INEQ [32, 46]. Similarly, we will show that CONSTRAINED-UETR is $\forall \exists \mathbb{R}$-complete.

Theorem 2.2 CONSTRAINED-UETR *is* $\forall \exists \mathbb{R}$-*complete.*

The proof of this theorem relies on the toolbox by Abrahamsen and Miltzow [3]. Note that their reduction can be considered folklore. However, Abrahamsen and Miltzow have carefully described all the details and pointed out some important properties, which are crucial to us. They describe several reductions, two of which are relevant to us. In particular, they show that there is a reduction from ETR to ETR-AMI, which runs in linear time. We need the following definitions as a preparation, see [3].

ETR-AMI

Input: A formula $\exists \, X_1, \ldots, X_n \in \mathbb{R} : \Phi(X_1, \ldots, X_n)$ over the reals, where Φ is a conjunction of constraints of the form: $X = 1$ (introducing constant 1), $X + Y = Z$ (addition), $X \cdot Y = Z$ (multiplication), $Z \geq 0$ (inequality), with $X, Y, Z \in \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$.

Question: Is the input formula true?

We define UETR-AMI analogous to ETR-AMI, except with universal and existential quantifiers, as in UETR.

Moreover, given two sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$, we say that W is a *linear extension* of V if there is an orthogonal projection $\pi \colon W \to V$ and two vectors $a, b \in \mathbb{Q}^n$ such that the mapping

$$x \mapsto a \odot \pi(x) + b$$

is a continuous bijection. Here, $c \odot d$ denotes the *dot product* $(c_1 d_1, \ldots, c_n d_n)$, for $c, d \in \mathbb{R}^n$. (Note that we require $a_i \neq 0$, for all i.) In this case we write $V \leq_{\text{lin}} W$.

Given a quantifier-free formula $\Phi(X_1, \ldots, X_n)$, we define the set

$$V(\Phi) := \{x \in \mathbb{R}^n : \Phi(x) \text{ is true}\}.$$

In our context, it is important that a linear extension also implies that the reduction can be applied to different quantifiers. This is very useful, as it allows us to transfer reductions between $\exists \mathbb{R}$-hard problems to the analogous $\forall \exists \mathbb{R}$-hard problems.

Lemma 2.3 *Let* $\Psi = (\forall Y = (Y_1, Y_2, \ldots, Y_m))(\exists X = (X_1, X_2, \ldots, X_n))$: $\Phi(X, Y)$, *and* $\Psi' = (\forall Y = (Y_1, Y_2, \ldots, Y_m))(\exists X = (X_1, X_2, \ldots, X_{n'}))$: $\Phi'(X, Y)$. *If* $V(\Phi')$ *is a linear extension of* $V(\Phi)$ *then* Ψ *and* Ψ' *have the same truth value.*

Proof Let us assume that Ψ is true. We have to show that Ψ' is true as well. The reverse direction is similar. Let us denote the bijection f from $V(\Phi')$ to $V(\Phi)$, by

$$f : z \mapsto a \odot \pi(z) + b,$$

for some $a, b \in \mathbb{R}^{m+n}$. (Recall that π denotes an orthogonal projection.) Note that f can be extended to a *surjective* mapping $g : \mathbb{R}^{m+n'} \to \mathbb{R}^{m+n}$.

Now, let $y' \in \mathbb{R}^m$ be some arbitrary vector. We have to show that there is a vector $x' \in \mathbb{R}^{n'}$ such that $\Phi'(x', y')$ is true. Let $(x^*, y) = g(0, y')$. As we assume that Ψ is true, there exists an x such that $\Phi(x, y)$ is true. Because f is a bijection, there exists $(x', y'') \in V(\Phi')$ such that $f(x', y'') = (x, y)$. Note that the projection π is only acting on the dimension of the existentially quantified variables. The vectors a and b as in the definition of linear extension are just scaling and shifting. Thus $y' = y''$. Consequently, it holds that $\Phi'(x', y')$ is true. $\square$

Theorem 2.4 [3] *For every instance* Φ *of* ETR, *we can construct in* $O(|\Phi|)$ *time an instance* Ψ *of* ETR-AMI *such that* $V(\Psi)$ *is a linear extension of* $V(\Phi)$.

Proof This follows immediately from the proofs of Lemmas A and C in [3]. (Note that Lemma C guarantees to preserve compactness. However, compactness is not used in any other way. Therefore, we do not need to require compactness.) $\square$

2.1.1 The Proof of Theorem 2.2

We are now ready to prove Theorem 2.2:

Proof of Theorem 2.2 Membership of CONSTRAINED-UETR to $\forall\exists\mathbb{R}$ follows from the definition of UETR, as CONSTRAINED-UETR is a special case of UETR. In order to show hardness, we reduce UETR to CONSTRAINED-UETR. By Theorem 2.4 and Lemma 2.3, it is sufficient to reduce from UETR-AMI. The two algorithmic problems UETR-AMI and CONSTRAINED-UETR are different in two aspects. Firstly, CONSTRAINED-UETR has no constraint of the form $X \geq 0$. Secondly, CONSTRAINED-UETR has only positive variables.

Removing Inequalities. Let $X \geq 0$ be some inequality-constraint. It is obvious that X is existentially quantified, as the formula otherwise, is trivially false. For every inequality, we do the following standard trick. We add another existentially quantified variable V and replace the old constraint by the new constraint $X = V^2$. It is obvious that this transformation can be done in linear time and is a proper reduction.

Changing Ranges of Quantifiers. Next, we want to exchange quantifiers ranging over all reals with quantifiers ranging over positive reals. For each variable Z, we introduce two positive variables Z^+ and Z^-. If Z is universally quantified, then so are both Z^+ and Z^-; analogously in the case if Z is existentially quantified. Every

appearance of Z is substituted by $(Z^+ - Z^-)$. It is easy to observe that the constructed formula is equivalent. However, the form of the constraints, as defined above, is lost.

Restoring the Form of Constraints. Now, we want to restore the constraints. We introduce new variables and, by following Abrahamsen and Miltzow [3], denote them by multi-character symbols. For example, in order to introduce the square of an already established variable X, we represent the new variable by $[\![X^2]\!]$. The value of $[\![X^2]\!]$ will be forced by appropriate constraints.

We introduce the existential variable $[\![1]\!]$ with the constraint $[\![1]\!] = 1$. Then, every constraint is transformed in the following way: A constraint to introduce a constant $(Z^+ - Z^-) = 1$ is transformed into $Z^+ = Z^- + [\![1]\!]$.

An addition constraint $(X^+ - X^-) + (Y^+ - Y^-) = (Z^+ - Z^-)$ is equivalent to the following expression: $X^+ + Y^+ + Z^- = X^- + Y^- + Z^+$. We introduce new positive, existentially quantified variables and the constraints:

$$
\begin{aligned}
X^+ + Y^+ &= [\![X^+ + Y^+]\!] , \\
X^- + Y^- &= [\![X^- + Y^-]\!] , \\
[\![X^- + Y^-]\!] + Z^+ &= [\![X^- + Y^- + Z^+]\!] , \\
[\![X^+ + Y^+]\!] + Z^- &= [\![X^- + Y^- + Z^+]\!] .
\end{aligned}
$$

A multiplication constraint $(X^+ - X^-) \cdot (Y^+ - Y^-) = (Z^+ - Z^-)$ is equivalent to the expression $X^+ Y^+ + X^- Y^- + Z^- = X^+ Y^- + X^- Y^+ + Z^+$. We introduce new positive, existentially quantified variables and the constraints for each pair $\circ, \times \in \{+, -\}$

$$
X^\circ \cdot Y^\times = [\![X^\circ Y^\times]\!]
$$

as well as the constraints:

$$
\begin{aligned}
[\![X^+ Y^+]\!] + [\![X^- Y^-]\!] &= [\![X^+ Y^+ + X^- Y^-]\!] , \\
[\![X^+ Y^-]\!] + [\![X^- Y^+]\!] &= [\![X^+ Y^- + X^- Y^+]\!] , \\
[\![X^+ Y^- + X^- Y^+]\!] + Z^+ &= [\![X^+ Y^- + X^- Y^+ + Z^+]\!] , \\
[\![X^+ Y^- + X^- Y^+]\!] + Z^- &= [\![X^+ Y^- + X^- Y^+ + Z^+]\!] .
\end{aligned}
$$

Note that now all constraints are of desired forms and all variables can be assumed to be strictly positive. Specifically, newly introduced variables are the sum or product of other variables. Furthermore, the last three steps can all be done in linear time. □

2.2 A Cook–Levin Type Theorem

Complexity classes are defined in various ways. Some complexity classes are defined by an algorithmic problem and a notion of equivalence or reduction; other complexity classes are described in terms of models of computation. Most famously, the complexity class NP is originally defined in terms of a non-deterministic Turing machine.

Namely, the complexity class NP consists of all problems that can be solved in polynomial time on a non-deterministic Turing machine. Alternatively, NP can be defined as the set of all problems that reduce to boolean satisfiability in polynomial time. Cook and Levin independently showed that the two definitions are equivalent [14, 30]. Boolean satisfiability plays a crucial role in showing NP-hardness. On the other hand, the formulation in terms of non-deterministic Turing machines plays a crucial role to show NP-membership.

The history of $\exists\mathbb{R}$ is reversed. The complexity-class $\exists\mathbb{R}$ was defined in terms of the algorithmic problem ETR, which is the real analog to boolean satisfiability. Recently, Erickson et al. [20] gave a new definition of the complexity class $\exists\mathbb{R}$ in terms of a model of computation, which is analogous to non-deterministic Turing machines. They showed that the two definitions are equivalent. In this way their result can be seen as a Cook–Levin type theorem for $\exists\mathbb{R}$.

In this section, we prove the Cook–Levin-type result for $\forall\exists\mathbb{R}$ yielding an alternative way to establish $\forall\exists\mathbb{R}$-membership. Roughly speaking, we show that UETR-formulas and *real challenge-verification* algorithms are equally powerful. Simultaneously, this gives an alternative perspective and definition on $\forall\exists\mathbb{R}$.

Our proof strongly relies on the result by Erickson et al. [20]. To describe our results, we follow their terminology. Their definition of the real-RAM corresponds to the intuitive definition most researchers use. The real-RAM consists of real registers, word registers, a program counter and a central processing unit (CPU). An algorithm is a list of supported instructions, see [20] for details. Inputs are denoted by $(x, y) \in \mathbb{R}^n \times \mathbb{Z}^m$. A *discrete decision problem* is a function Q from arbitrary integer vectors to the booleans {TRUE, FALSE} (or equivalently, any language over the alphabet $\{0, 1\}$). An integer vector I is a *yes-instance* of Q if $Q(I) = $ TRUE and a *no-instance* of Q if $Q(I) = $ FALSE. Let $\circ$ denote the concatenation operator. A *real challenge-verification algorithm* (CV algorithm) for Q is a real-RAM program A that satisfies the following conditions, for some constant $c \geq 1$:

- A halts after at most N^c time steps, using word size $\lceil c \log_2 N \rceil$, given any input of total length N.
- For every yes-instance $I \in \mathbb{Z}^n$, and for every real vector y, there is a real vector x and an integer vector z, each of length at most n^c, such that A accepts input $(y \circ x, I \circ z)$.
- For every no-instance I, there is a real vector y such that for every real vector x and an integer vector z, each of length at most n^c, A rejects input $(y \circ x, I \circ z)$.

Note that we assume that the algorithm A knows, which parts belong to x, y, I, and z due to their length.

There exists a clear analogy to UETR-formulas. The instance I represents the non-quantified part of the formula, y represents values of universally quantified variables and x represents existentially quantified variables. We call y the *challenge* and x the *witness*. We are now ready to state the main theorem of the section.

Theorem 2.5 (Cook–Levin analog) *For any discrete decision problem Q there is a real challenge-verification algorithm if and only if $Q \in \forall\exists\mathbb{R}$.*

Sketch of proof We have to show how an UETR-formula can be transformed into an instance I and a CV algorithm A and vice versa. Given a UETR-formula φ, it is very easy to simulate it using a CV algorithm, by merely evaluating the formula. The challenge and witness represent the universal and existential variables respectively.

Given an instance I and a CV algorithm A, we construct an equivalent UETR-formula φ. Interestingly, this can be done in the same way as in Theorem 10 of Erickson et al. [20]. To explain this, first note that the proof idea of the mentioned [20, Thm. 10] is to simulate every execution step of the algorithm by an ETR-formula. However, the simulation does not mind how the variables are quantified. Thus, even if some of the variables are universally quantified, the simulation is still identical. □

In the next section, we use Theorem 2.5 to prove the containment of area-universality in ∀∃ℝ (Proposition 1.1). By the above result, this proof does not require to construct a formula, but solely a real challenge-verification algorithm. This allows to build on existing algorithms without reformulating them into the language of ∀∃ℝ-formulas.

2.3 Containment of AREA UNIVERSALITY$_{\geq 0}$ in ∀∃ℝ

In this section, we present a proof of Proposition 1.1.

Proposition 1.1 AREA UNIVERSALITY *and* AREA UNIVERSALITY$_{\geq 0}$ *are contained in* ∀∃ℝ.

Proof By Theorem 2.5, it suffices to show that there exists a real challenge-verification algorithm for testing both AREA UNIVERSALITY and AREA UNIVERSALITY$_{\geq 0}$. Given a plane graph G, we encode it by its rotation systems (that is the cyclic order of incident edges at each vertex) and its outer face, i.e., by a vector of words. This defines our input I. Universally quantified areas define the challenge y. The witness d consists of the vertex coordinates of a realizing straight-line drawing D of G. The algorithm checks that the drawing D is a realizing drawing.

For AREA UNIVERSALITY this is straightforward. In particular, we ensure that D is

(i)   planar,
(ii) equivalent to G, and
(iii) realizes the face areas defined in y.

In order to test planarity, we check for every pair of disjoint edges, whether the intersection of their segments is empty. To test the equivalence, we check if the rotation system of G and D coincide, i.e., if the cyclic order of incident edges are equal. Here we use the fact that the rotation system uniquely defines the 2-cell embedding of a planar graph [37, Thm. 3.2.4].

In order to check the face areas, we exploit the fact that after checking planarity, we know that every face corresponds to a simple polygon; in the presence of cut-vertices, the polygon is degenerate. The area of a (possibly degenerate) polygon can be computed with the *shoelace formula* which first described by Meister [34]. Denote the vertex coordinates of a simple polygon $\mathcal{P}$ in counterclockwise order by $(x_1, y_1), \ldots, (x_n, y_n)$. Then the area A of $\mathcal{P}$ is given by

$$2 \cdot A(\mathcal{P}) = \sum_{i=2}^{n-1} \det \begin{pmatrix} x_1 & x_i & x_{i+1} \\ y_1 & y_i & y_{i+1} \\ 1 & 1 & 1 \end{pmatrix}. \tag{1}$$

Proving containment of AREA UNIVERSALITY$_{\geq 0}$ is more intricate. We check that D

(i) is crossing-free,
(ii) realizes the face areas defined in y, and
(iii) the complement of the outer face has the correct total area, namely the sum of the areas in y.

Furthermore, we show that the properties (i), (ii), and (iii) imply that D is equivalent to G, i.e., the sets of inner faces coincide.   □

We show how the three steps can be computed with a real-RAM algorithm. For (i), the difficulty arises from the fact that we allow for degenerate drawings. To check this property, we exploit a result of Akitaya et al. [5]; restated in the terminology of this paper it reads as follows.

Claim [5, Cor. 1.3] *Given an abstract graph with n vertices and a vector of vertex coordinates d, we can decide in $O(n^2 \log n)$ time whether the straight-line drawing induced by d is crossing-free.*

For the remainder, we exploit the fact that after checking (i), we know that every face corresponds to a (possibly degenerate) simple polygon. Similarly to the above, we check properties (ii) and (iii) by computing the area of a (possibly degenerate) polygon with (1).

It remains to show how (i), (ii), and (iii) imply that D and G have the same set of faces, i.e., that D is a crossing-free drawing of G. Observe that we verify (ii), using (1). In other words, we ensure that the *signed areas* are correct. In particular, this implies that the *orientation* of each face D is correct, i.e., when walking in counter-clockwise direction on the boundary of a face in D, the bounded part lies on the left side. If the bounded part were to lie on the right side, the determinant in (1) would have a negative sign.

For each inner face f of G, we consider the corresponding face polygon in D defined by the bounded region of the face cycle. Due to the correct orientation of all face polygons, every inner edge of G is covered from both sides in D. Moreover, every outer edge of G is covered from the inside in D. In particular, the region obtained by the union of all face polygons has no holes and is bounded by the outer edges of G forming a closed curve. We claim that the face polygons are interior-disjoint; otherwise some parts of the plane are covered twice – a contradiction to the fact that the area of the face polygons sums to the area of the complement of the outer face. Because the face polygons are interior-disjoint, they are faces of D and hence, the drawing D is a crossing-free drawing for G. This completes the proof.   □

3 Hardness of AREA UNIVERSALITY FOR TRIPLES wPFA

In this section, we prove the following theorem.

Theorem 1.3 AREA UNIVERSALITY FOR TRIPLES wPFA *is* $\forall\exists\mathbb{R}$-*complete, even if the number of triples is linear in the number of vertices.*

Proof The membership follows from the fact that we can easily express the area of a triangle by a polynomial equation, see (1) in Sect. 2.3. So an instance of AREA UNIVERSALITY FOR TRIPLES wPFA can be expressed as an UETR-formula, which is a conjunction of equations of the above form for each triple.

For the hardness, we reduce from CONSTRAINED-UETR. For every instance Ψ of CONSTRAINED-UETR, we give a set of points V and unordered triples T, along with a partial area assignment $\mathcal{A}'$. Let Ψ be a formula of the form:

$$\Psi = (\forall Y_1, \ldots, Y_m \in \mathbb{R}^+)(\exists X_1, \ldots, X_n \in \mathbb{R}^+) : \Phi(Y_1, \ldots, Y_m, X_1, \ldots, X_n).$$

Recall that Φ is a conjunction of constraints of the form $X = 1$, $X + Y = Z$, and $X \cdot Y = Z$. First, we show how to express Φ. Our gadgets are similar to the *von Staudt constructions*, used for showing $\exists\mathbb{R}$-hardness of ORDER TYPE (see Mnëv [36] or Richter-Gebert [40]). All variables are represented by points on one line which we denote by ℓ for the rest of the proof. First, we enforce points to be on ℓ. Afterwards, we construct gadgets for mimicking addition and multiplication. Finally, we describe how to represent universally quantified variables.

We introduce three points p_0, p_1, and r and define $\mathcal{A}'(p_0, p_1, r) := 1$. The positive area ensures that the points are not collinear and pairwise different.

Denoting a line through two points a and b by $\ell_{a,b}$, we set $\ell := \ell_{p_0, p_1}$. The points on ℓ will correspond to real numbers, where p_0 and p_1 are interpreted as 0 and 1, respectively. Each variable X is represented by a point x on ℓ. Additionally, since all variables X are non-zero, we introduce a triangle forcing x to be different from p_0. In general, we can ensure that two points x_1, x_2 are distinct, by introducing a point q and adding a triangle (x_1, x_2, q) with $\mathcal{A}'(x_1, x_2, q) := 1$. The absolute value of X is defined by $\|p_0 x\|$; if x and p_1 lie on the same side of p_0, then the value of X is
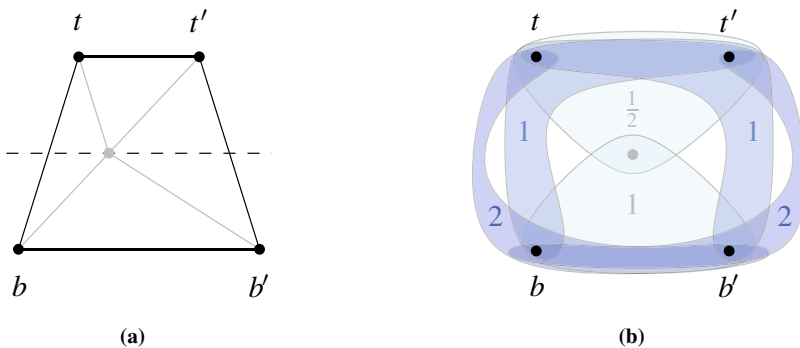

Fig. 4 The gadget to force parallel segments

positive, otherwise it is negative. Here, we allow negative values, but later we force the original variables to be positive.

A simple way to force a point x on ℓ, is to set $\mathcal{A}'(x, p_0, p_1) := 0$. However, as we want to avoid prescribing area 0, we introduce a more sophisticated way. As an important building block, we show how to enforce on four pairwise different points t, t', b, b' that the segment tt' is parallel to the segment bb' and has half its length, see Fig. 4a; besides these two properties no additional constraint on any of the four points is imposed. Thus, we define $\mathcal{A}'(t, t', b) = \mathcal{A}'(t, t', b') = 1$ and $\mathcal{A}'(b, b', t) = \mathcal{A}'(b, b', t') = 2$, see Fig. 4b.

We show that b and b' lie on the same side of the line $\ell_{t,t'}$: Suppose for contradiction that $\ell_{t,t'}$ separates b and b'. If, additionally, t and t' are on the same side of $\ell_{b,b'}$ then the triangle (b, b', t) is contained in or contains the triangle (b, b', t'), see Fig. 5a. However, both triangles have the same area and $t \neq t'$, which is a contradiction. Consequently, $\ell_{b,b'}$ separates t and t' and the quadrangle $tbt'b'$ can be partitioned by either diagonal bb' or tt' as illustrated in Fig. 5b. Thus, $2 = \mathcal{A}(t, t', b) + \mathcal{A}(t, t', b') = \mathcal{A}(b, b', t') + \mathcal{A}(b, b', t) = 4$, which is again a contradiction. Therefore b and b' lie on the same side of $\ell_{t,t'}$.

By the prescribed area, b and b' have the same distance to $\ell_{t,t'}$, i.e., the segments tt' and bb' are parallel, and bb' has twice the length of tt'. Moreover, no further constraints are imposed on t, t', b, b'.

We use this building block in two ways. First, the gadget is used to force points on a specific line defined by two distinct points. Note that a point x with $\mathcal{A}'(t, t', x) := 1/2$, $\mathcal{A}'(b, b', x) := 1$ is forced to lie on the line cutting the trapezoid at half height as illustrated in Fig. 4. In order to force points on the line ℓ, we use a copy of this building block as follows: Introduce four new distinct points c, c', d, d'. Construct a trapezoid with parallel segments cc' and dd', where the length of the second one doubles the length of the first one. Let P denote a set of points to be enforced on ℓ, then we define for every $x \in P \cup \{p_0, p_1\}$ the areas $\mathcal{A}'(c, c', x) := 1/2$ and $\mathcal{A}'(d, d', x) := 1$. This introduces no other constraints on the position of $x \in P$. Likewise, we can introduce points on any given line defined by two distinct points. We refer to this as the *line construction*.

The second way to use the trapezoid construction is to enforce two lines to be parallel. For this, we first introduce new and distinct points on the two lines, as described in the previous paragraph. Afterwards, we build the trapezoid gadget on the four points. We refer to this as the *parallel-line construction*.
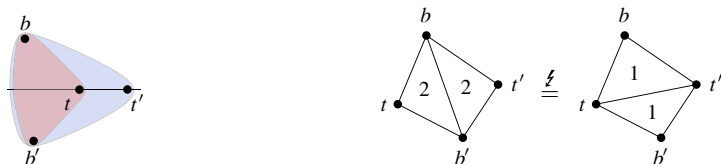
(a) Case 1: t, t' are on the same side of $\ell_{b,b'}$.　　**(b)** Case 2: t, t' are separated by $\ell_{b,b'}$.

Fig. 5 The situation that $\ell_{t,t'}$ separates b and b'

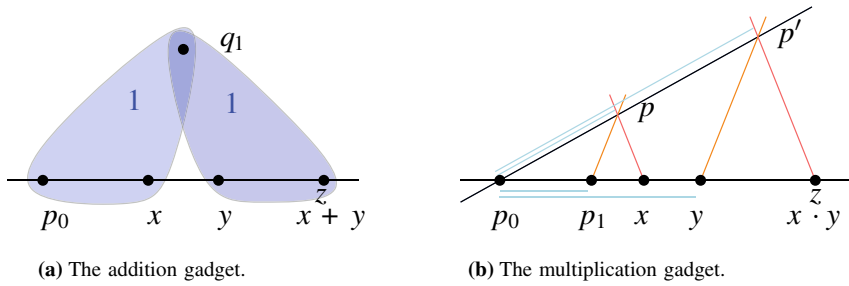**(a)** The addition gadget.   **(b)** The multiplication gadget.

Fig. 6 The constraint gadgets

Now, we describe the *addition gadget* for a constraint $X + Y = Z$. Let x, y, z be the points encoding the values of X, Y, Z, respectively. Recall that $x, y, z \in \ell$ and $x, y, z \neq p_0$. We introduce a point q_1 and prescribe the areas $\mathcal{A}'(p_0, x, q_1) = \mathcal{A}'(y, z, q_1) = 1$, see Fig. 6a. Because the two triangles have the same height and same area, it holds that $\|yz\| = \|p_0x\|$. Thus, the value of Z is either $X + Y$ or $Y - X$. Analogously, we introduce a point q_2 and define $\mathcal{A}'(p_0, y, q_2) = \mathcal{A}'(x, z, q_2) = 1$, implying that Z is either $Y + X$ or $X - Y$. Therefore either $Z = X + Y$ (the intended solution) or $Z = X - Y = Y - X$. The latter case implies $X = Y$ and thus $Z = 0$. This contradicts the fact that $z \neq p_0$.

To construct a *multiplication gadget* for a constraint $X \cdot Y = Z$, let $x, y, z (\neq p_0)$ be the points encoding the values of X, Y, Z, respectively. We introduce two distinct points p, p' whose supporting line contains p_0, i.e., p_0, p, p' lie on a common line by the line-construction. Furthermore, we make sure that p and p' are not on ℓ. To do so, we introduce two new points q, q', force them to lie on ℓ by the line construction, and prescribe the areas $\mathcal{A}'(p_0, p, q) := 1$ and $\mathcal{A}'(p_0, p', q') := 1$. Note that this does not introduce any additional constraints on the positions of p and p'. By the parallel-line construction, we enforce the line $\ell_{p_1, p}$ to be parallel to $\ell_{y, p'}$ and the line $\ell_{x, p}$ to be parallel to $\ell_{z, p'}$, see Fig. 6b. The intercept theorem, which is also known as Thales' theorem or the basic proportionality theorem [49], ensures that the following ratios coincide:

$$\frac{|p_0 p|}{|p_0 p'|} = \frac{|p_0 p_1|}{|p_0 y|} = \frac{|p_0 x|}{|p_0 z|}.$$

This also holds for negative values. By definition of x, y, z, we obtain $1/Y = X/Z$, and hence $X \cdot Y = Z$. Recall that $p_1 = 1$.

For every universally quantified variable Y_i, $i \in \{1, \ldots, m\}$, let y_i be the point encoding the value of Y_i with $y_i \in \ell$, $y_i \neq p_0$. We introduce a triple $f_i = (p_0, r, y_i)$, whose area is universally quantified. Recall that r is a point with $\mathcal{A}'(p_0, p_1, r) = 1$. To enforce each original variable X to be positive, we add an existentially quantified variable S_X and the constraint $X = S_X \cdot S_X$ where S_X may or may not be positive. This finishes the polynomial time reduction.

It remains to argue that Ψ is true if and only if for our constructed instance of AREA UNIVERSALITY FOR TRIPLES WPFA, where $\mathcal{A}'$ is the partial assignment of areas, every

assignment $\mathcal{A}$ consistent with $\mathcal{A}'$ is realizable. Suppose Ψ is true, let $\mathcal{A}'$ be as above, and consider an assignment $\mathcal{A}$ that is consistent with $\mathcal{A}'$. Let $V(Y_i)$ be the values assigned to the triples f_i, and let $V(X_i)$ be the values of the variables X_i in some satisfying assignment for Φ. Let $y_1, \ldots, y_m, x_1, \ldots, x_n$ be points positioned on a line at distances from a point p_0 corresponding to these values. Since each addition and multiplication relation specified by Φ holds, the corresponding gadgets can be realized, so $\mathcal{A}$ is realizable. Suppose now that every assignment $\mathcal{A}$ that is consistent with $\mathcal{A}'$ is realizable, and consider values $V(Y_1), \ldots, V(Y_m) \in \mathbb{R}^+$ of the universally quantified variables of Ψ. Then, there is a realization of $\mathcal{A}$ where each triple f_i has area $V(Y_i)$, in this realization $p_0 \neq p_1$, and $V(X_i) = \|x_i - p_0\| / \|p_1 - p_0\|$ is a satisfying assignment for Φ. Thus Ψ is true.

Finally, let us point out that for each variable and for each gadget we introduced a constant number of vertices and a constant number of triangles, so the number of triangles is linear in the number of vertices. □

Let us point out that if we start the reduction in Theorem 1.3 from a variant of CON-STRAINED-UETR, where universally quantified variables are non-negative (instead of positive), we can obtain the hardness of a variant of AREA UNIVERSALITY FOR TRIPLES WPFA, where we allow degenerate triangles (i.e., of area 0). Establishing ∀∃ℝ-hardness of such a variant of CONSTRAINED-UETR can be done in a way analogous to Theorem 2.2.

We conclude this section with a small discussion of how to strengthen this result. An interesting step towards establishing our conjecture would be to prove an analog of Theorem 1.3, where the realizing point set and the triples induce a plane triangulation. However, the construction in our proof of Theorem 1.3 is highly non-planar, and obtaining such a result seems to require a very different approach.

4 Hardness of PRESCRIBED AREA EXTENSION

In this section we prove Theorem 1.4 by presenting a reduction from PLANAR-ETR-INV.

Theorem 1.4 PRESCRIBED AREA EXTENSION *is* ∃ℝ-*complete.*

Proof Let $\Psi = \exists X_1 \ldots X_n : \Phi(X_1, \ldots, X_n)$ be an instance of PLANAR-ETR-INV. Recall that we can assume that if Ψ is a YES-instance, then it has a solution, in which the values of variables are in the interval $(0, \lambda)$ with $\lambda = 5$. We construct a plane graph $G_\Psi = (V, E)$, a (positive) area assignment $\mathcal{A}$ of inner faces of G_Ψ, and fixed positions of a subset of vertices, such that G_Ψ has a realizing drawing respecting pre-drawn vertices if and only if Φ is satisfiable by real values from the interval $(0, \lambda)$.

Consider the incidence graph of Φ. It is straightforward to verify that a planar drawing of this graph can be modified to an embedding in which

- each edge is represented by a polyline consisting of vertical and horizontal segments,
- the polylines of edges sharing a vertex might partially overlap—the common part, called a bundle, is a polyline that contains the common vertex,
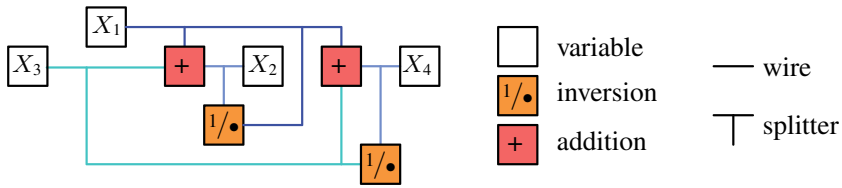
Fig. 7 The embedding of the incidence graph of the formula $\Psi = (X_1 + X_2 = X_3) \wedge (X_1 \cdot X_2 = 1) \wedge (X_1 + X_4 = X_3) \wedge (X_4 \cdot X_3 = 1)$
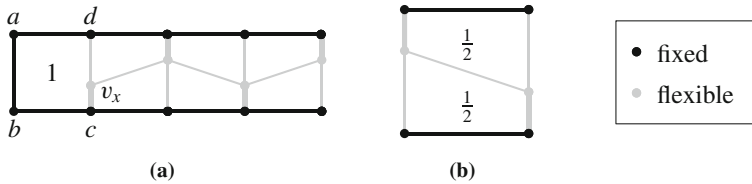

Fig. 8 **a** The variable gadget with an attached wire and **b** a wire gadget

- each vertex representing a variable is incident to one bundle of edges.

Figure 7 depicts an example of such an embedding. We will construct the instance of PRESCRIBED AREA EXTENSION, based on the formula Ψ and its incidence graph.

We design several types of gadgets: *variable gadgets* representing variables, as well as *inversion* and *addition gadgets*, realizing the corresponding constraints. Moreover, we construct *wires* and *splitters* in order to copy and transport information. Some vertices in our gadgets will have prescribed positions. We call such vertices *fixed* and we call all other vertices *flexible*.

The *variable gadget* for a variable X consists of four fixed vertices a, b, c, d and one flexible v_x, see the left image of Fig. 8a. The fixed vertices a, b, c, d form a square of area 1. Observe that the prescribed area of the face a, d, v_x, c, b and planarity constraint force v_x to be on the segment cd. Now the length of the segment cv_x specifies the value of X, divided by the scaling factor λ. We symbolized this segment by a bold gray line in Fig. 8a.

Now, we introduce a constant 1 with a constraint $X = 1$ by using a variable gadget. Instead of making vertex v_x free, we make it fixed and place it in such a way that the distance from c to v_x is $1/\lambda$. We will sometimes assume that a flexible vertex is forced to lie on a specified segment because this property can be induced by a variable gadget.

The *wire gadget* consists of several box-like fragments: four fixed vertices positioned as the corners of an axis-parallel unit square and two opposite fixed edges, see Fig. 8b. Each of the other two sides of the square is subdivided by a flexible vertex, these two vertices are joined by an edge. Each of the two quadrangular faces has a prescribed area of $1/2$.

Note that if one of the flexible vertices is collinear with its fixed neighbors, so is the other one. The crossing-freeness constraint ensures that each flexible vertex lies between the corresponding fixed vertices, as otherwise the edge that joins them would cross one of the edges on the boundary of the gadget. Moreover, in a sequence of such
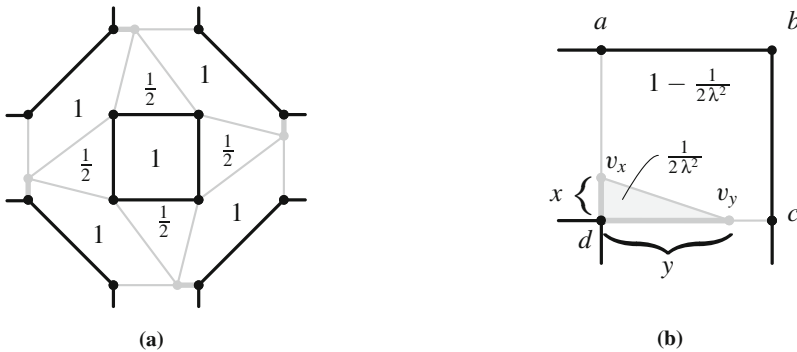
Fig. 9 **a** The splitter gadget and **b** the inversion gadget

squares, the segment representing the value of the variable alternates between top and bottom. Hence, if necessary, we may use an odd number of fragments in order to invert the side where the value is represented. Wires are used to connect variable gadgets to inversion and addition gadgets, see Fig. 7. Let us point out that a variable gadget may be connected to other gadgets only using a wire. This ensures that the property that we stated when introducing a variable gadget: the flexible vertex does not lie outside the square forming the gadget, see Fig. 8a.

If one variable participates in several constraints, we need to be able to split the wire, in order to provide a connection to all necessary gadgets. The *splitter gadget* contains a central fixed square of area 1, where each side is adjacent to a triangle of area 1/2, see Fig. 9a.

Each triangle fixes a flexible vertex on a line. These flexible vertices and their neighbors on the boundary of the splitter gadget are identified with the appropriate vertices in a wire; this is how we connect the splitter with other gadgets. Observe that the value of one variable fixes the values of all variables. Indeed, since the non-fixed faces of the gadget are arranged in a circular way, it is straightforward to verify that all triangular faces must have exactly the same shape (up to translation and rotation). Note that each face area 1 has exactly two flexible vertices which are forced to lie on specific segments. Thus if one of them is determined, clearly the other is also uniquely determined in a realizing drawing. Note that we may use the splitter gadget not only for splitting wires, but also for realizing turns.

Now let us discuss gadgets for inversion and addition constraints. First consider an inversion constraint $X \cdot Y = 1$. The *inversion gadget* consists of four fixed vertices a, b, c, d, two flexible vertices v_x and v_y, and the edge $v_x v_y$, see Fig. 9b. The fixed vertices belong to a unit square. By linking the inversion gadget with wires, we can ensure that v_x belongs to the segment ad and v_y belongs to the segment cd. Let x be the distance from d to v_x, and y be the distance of d to v_y.

Suppose that these distances represent, respectively, the values of variables X and Y as described in the paragraph about variable gadgets; this is obtained by identifying them with appropriate vertices of such a gadget or a wire. Namely, the value of X is $\lambda \cdot x$, and the value of Y is $\lambda \cdot y$. To realize the area $1/(2\lambda^2)$ of the triangle $v_x v_y d$, the
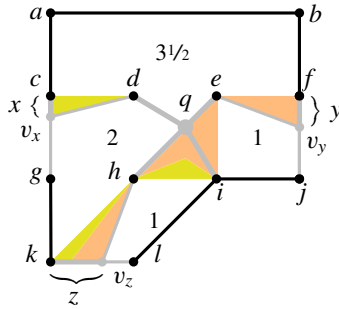
Fig. 10 The addition gadget

lengths x and y are forced to satisfy $xy/2 = 1/(2\lambda^2)$. This implies that the value of $X \cdot Y$ is $\lambda x \cdot \lambda y = 1$. Consequently, the inversion constraint is satisfied.

Finally, we define an *addition gadget* for an addition constraint $X + Y = Z$. We introduce fixed vertices $a, b, c, d, e, f, g, h, i, j, k, l$ placed on an integer grid, as depicted in Fig. 10. Moreover, flexible vertices v_x, v_y, v_z encode the values of variables X, Y, Z, respectively; they are identified by appropriate flexible vertices in wire gadgets. This ensures that v_x, v_y, v_z are forced to lie on the segments cg, fj, and kl, respectively. Finally, we introduce the flexible vertex q, joined to d, e, i, and h. The areas of faces are set as follows:

$$
\begin{array}{ll}
f_1 = v_x, c, a, b, f, v_y, e, q, d, & \mathcal{A}(f_1) = 3.5, \\
f_2 = v_x, d, q, h, v_z, k, g, & \mathcal{A}(f_2) = 2, \\
f_3 = e, v_y, j, i, q, & \mathcal{A}(f_3) = 1, \\
f_4 = v_z, l, i, q, h, & \mathcal{A}(f_4) = 1.
\end{array}
$$

Let x be the distance from c to v_x, let y be the distance from f to v_y, and let z be the distance from k to v_z. Since v_x, v_y, v_z encode the values of X, Y, and Z, respectively, we know that $X = \lambda x$, $Y = \lambda y$, and $Z = \lambda z$. We aim to show that the areas of faces of the gadget force that $x + y = z$. Observe that the area of the triangle c, v_x, d is $a_x := x/2$, the area of the triangle e, f, v_y is $a_y := y/2$ and the area of k, h, v_z is $a_z := z/2$. Thus is it sufficient to show that $a_x + a_y = a_z$.

The area of the triangle d, e, q is forced to be $1/2 - a_x - a_y$, because of the constraint on the area of f_1. Note if Ψ is a positive instance, we may assume that $Z = X + Y < \lambda$, thus the area of the triangle d, e, q is positive. On the other hand, if $X + Y > \lambda$, it is easy to see that there is no way to put flexible points to realize the prescribed areas. Since the triangles d, e, q and h, i, q have parallel bases and share the third vertex, we observe that the area of the triangle h, i, q is forced to be $a_x + a_y$. It is easy to see that the area of f_4 equals $1 = A(h, k, l, i) + A(h, q, i) - A(h, k, v_z) = 1 + a_x + a_y - a_z$. Thus we enforce $a_x + a_y = a_z$.

It remains to show that all faces have the correct area given $a_x + a_y = a_z$. Observe that the area of the triangle e, q, i is forced to be equal a_y, because of the constraint

on the area of f_3. Consequently, the triangle d, q, h has area $1/2 - a_y$. It holds that

$$\mathcal{A}(f_2) = A(c, d, h, k) - A(c, v_x, d) + A(d, h, q) + A(h, k, v_z)$$
$$= 3/2 - a_x + (1/2 - a_y) + a_z = 2.$$

Finally, observe that by connecting the gadgets we might have introduced some inner faces, which do not have specified areas. However, the area of each gadget is fixed, so we exactly know what the areas of these newly introduced faces are. We finish the construction by fixing the area for each remaining face, according to the way it is drawn in the plane.

Hence we created a planar graph G_Ψ where the positions of some vertices are fixed, and a positive area assignment $\mathcal{A}$, such that G_Ψ has an $\mathcal{A}$-realizing drawing that respects the fixed vertices if and only if Ψ is true. The correctness of the construction follows easily from the correctness of individual gadgets. Let us point out that we may assume that all variables are in $(0, \lambda)$ and thus the flexible vertex of each variable gadget is not superimposed on another vertex; this property propagates to wires and other gadgets. Consequently, the realizing drawing is planar. The size of G is clearly polynomial in $|\Psi|$, so the proof is complete. □

5 Discussion and Open Problems

In this section, our aim is two-fold. Firstly, we present further variants of AREA UNIVERSALITY that might pave the path on the journey to prove ∀∃ℝ-hardness. Secondly, we want to motivate further research on the complexity classes ∀∃ℝ and ∃∀ℝ by presenting candidates of complete problems. In particular, we highlight connections to the concepts of imprecision, robustness, and extendability.

5.1 Open problems related to AREA UNIVERSALITY

Our first open problem aims at strengthening Theorem 1.3.

Open Problem 1 *Given a plane inner triangulation T and an area assignment $\mathcal{A}'$ for some inner faces of T. Is it ∀∃ℝ-complete to decide whether there exist realizing drawings for all area assignments extending $\mathcal{A}'$?*

Similarly, it is interesting to investigate the complexity of deciding whether or not an area assignment is realizable.

Open Problem 2 *Is PRESCRIBED AREA ∃ℝ-complete?*

Potentially, answering Open Problem 1 also implies an answer for Open Problem 2.

5.2 Candidates for ∀∃ℝ- and ∃∀ℝ-complete problems

Complexity classes become interesting if they contain interesting algorithmic problems. To motivate the research on ∀∃ℝ and ∃∀ℝ, we present some candidates of
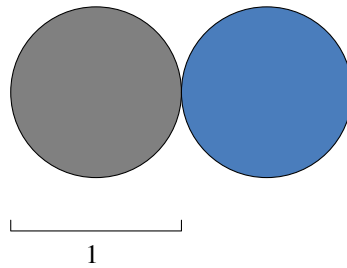
Fig. 11 The two disks have Hausdorff distance 1, but d-distance 0

problems that might be complete for these classes. We draw connections to existing concepts in computer science, like robustness and imprecision. Although $\forall\exists\mathbb{R}$ and $\exists\forall\mathbb{R}$ are different complexity classes, it is worth mentioning that the complement of every language in $\exists\forall\mathbb{R}$ belongs to $\forall\exists\mathbb{R}$. Thus from an algorithmic point of view they are equally difficult.

The aim of this section is to point at potentially interesting future problems rather than giving technical insights. We start with the problem we find most natural, besides AREA UNIVERSALITY$_{\geq 0}$. Afterwards we come to the notions of universal extension problem, imprecision, and robustness. We conclude with a problem that has a similar flavor as area-universality but turns out to be polynomial time solvable.

A very natural metric for point sets is the so-called Hausdorff distance. For two sets $A, B \subseteq \mathbb{R}^d$, the Hausdorff distance $d_{\mathrm{H}}(A, B)$ is defined as

$$d_{\mathrm{H}}(A, B) = \max\left\{\sup_{a \in A}\inf_{b \in B}\|ab\|,\ \sup_{b \in B}\inf_{a \in A}\|ab\|\right\},$$

where $\|ab\|$ denotes the Euclidean distance between the two points a and b. We define the corresponding algorithmic problem as follows:

HAUSDORFF DISTANCE
Input: Quantifier-free formulas Φ and Ψ in the first-order theory of the reals, $t \in \mathbb{Q}$.
Question: Is $d_{\mathrm{H}}(S_\Phi, S_\Psi) \leq t$?

Recall that S_Φ is defined as $\{x \in \mathbb{R}^n : \Phi(x)\}$. Marcus Schaefer pointed out[2] the following interesting question:

Open Problem 3 *Is computing the Hausdorff distance of semi-algebraic sets $\forall\exists\mathbb{R}$-complete?*

Schaefer and Štefankovič [46] studied the following very common notion of distance, which we call here *d-distance* for clarity. The *d*-distance between two sets A and B in $\mathbb{R}^d$ is defined as $d(A, B) = \inf\{\|ab\| : a \in A,\ b \in B\}$. See Fig. 11 to see

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[2] Personal communication.

a simple example on how those two distances are different. They show the following lemma.

Lemma 5.1 [46] *Deciding if two semi-algebraic sets have d-distance zero is* $\exists\mathbb{R}$-*complete.*

Note that containment in $\exists\mathbb{R}$ is not clear for the Hausdorff distance. However, it is contained in $\forall\exists\mathbb{R}$. An encoding of Hausdorff distance as a formula of the first order theory of the reals looks as follows:

$$(\forall\, a \in A,\ \varepsilon > 0\, \exists\, b \in B : \|ab\| < \varepsilon + t)$$
$$\wedge\, (\forall\, b' \in B,\ \varepsilon > 0\, \exists\, a' \in A : \|a'b'\| < \varepsilon + t).$$

The formula holds true if and only if $d_{\mathrm{H}}(A, B) \leq t$. Note that we need the ε in the formula, because semi-algebraic sets can be open. This can be easily reformulated as a formula in prenex form with two blocks of quantifiers. By using the following logical equivalences:

$$(\forall\, x\, \exists\, y : \Phi(x, y)) \wedge (\forall\, x'\, \exists\, y' : \Phi'(x', y')) \equiv \forall\, x, x'\, \exists\, y, y' : \Phi(x, y) \wedge \Phi'(x', y')$$
$$\text{and } \forall\, x \in X\, \exists\, y \in X : \Phi(x, y) \equiv \forall\, x\, \exists\, y : x \in X \Rightarrow (y \in Y \wedge \Phi(x, y)),$$

where Φ and Φ' are two quantifier free formulas, and $\equiv$ indicates that the two formulas are logically equivalent.

By definition, a semi-algebraic set is a subset S of $\mathbb{R}^n$ defined by a finite sequence of polynomial equations and strict inequalities or any finite union of such sets.

Universal Extension Problems. Before we explain the general concept of an Universal Extension problem, we take a look at a specific example: the ART GALLERY PROBLEM. In this problem we are given a simple polygon P and we say that a point p *sees* another point q if the line segment pq is fully contained in P. The ART GALLERY PROBLEM asks for a smallest set of points, which are called guards, such that every point inside the polygon is seen by at least one guard. It was recently shown that ART GALLERY PROBLEM is $\exists\mathbb{R}$-complete [1]. In an extension version of ART GALLERY PROBLEM, we are given a polygon and a partial set of guards G_1. The task is to find a set of guards G_2 such that every point inside the polygon is seen by at least one guard from the set $G_1 \cup G_2$. So the spirit of an extension problem is to give a partial solution as an additional input and ask if it is extendable to a full solution. Now we define the universal extension variant of the ART GALLERY PROBLEM as follows. The input consists of a simple polygon P and a set of regions $R_1, \ldots, R_t$ inside P. We ask whether for every guard placement of $G_1 = \{g_1, \ldots, g_t\}$ such that $g_i \in R_i$ it holds that there exists a second set of guards G_2 of some given size such that $G_1 \cup G_2$ guard the entire polygon. We denote this as the UNIVERSAL GUARD EXTENSION problem. For an example consider Fig. 12.
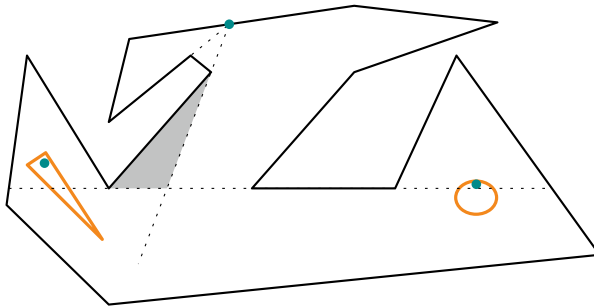
Fig. 12 If two guards in the orange regions are badly placed it is impossible to guard the remaining polygon with just one guard

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UNIVERSAL GUARD EXTENSION

Input: A polygon P, regions $R_1, \ldots, R_t$ inside P, and a number $k \in \mathbb{N}$.
Question: Is it true that for every placement of guards $G_1 = \{g_1, \ldots, g_t\}$ with $g_i \in R_i$, there exists a set G_2 of k guards, such that $G_1 \cup G_2$ guard P completely?

Open Problem 4 *Is* UNIVERSAL GUARD EXTENSION $\forall \exists \mathbb{R}$-*complete?*

The spirit is that we do not want to extend one given partial solution (as in the extension variant), but all possible partial solutions. In this example we require the guards to lie in specific regions. This is necessary, as otherwise we had to consider such extreme cases as a partial solution with all the guards in a single point, which would not be meaningful. As we will later see it seems to be a common theme that we have to impose some extra conditions on partial solutions and there is usually a choice which conditions we want to impose.

Many problems in combinatorics, computational geometry, and computer science ask for the realization or existence of a certain object, for instance: a triangulation, an independent set, a plane drawing of a given graph, a plane drawing of a graph with prescribed areas, or a set of points in the plane realizing a certain combinatorial structure. These questions become extension questions as soon as a part of the solution is already created and the question is, whether there is a way to finish it. For instance: can a given plane matching be extended to a perfect matching, can a given plane graph be extended to a 3-regular graph. When going to an extension question the difficulty might increase. This is the case for matching extensions: Given a set with an even number of points in the plane there always exists a perfect, straight-line, crossing-free matching on that set of points. However, it is NP-hard to decide if a given partial matching can be extended to a perfect one [35]. This is a good reason why our results must be considered with care as Theorem 1.3 is by its nature an extension result.

We take this one step further and ask whether there is a way to complete *every* "reasonable" partial realization. As explained before, we might need to impose some problem-specific conditions, for the question to become meaningful.

Without giving explicit definitions, we mention here a few more problems that are worth considering:
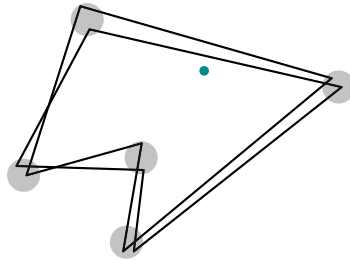
Fig. 13 The cyan guard sees every point of the imprecisely given polygon. Therefore, it is a universal guard (set)

- Order-Type Extension [32, 36],
- Extension variant of the Steinitz problem [41],
- Graph Metric Extension [38],
- Simultaneous Graph Embedding Extension [11].

Imprecision. Again, we use the ART GALLERY PROBLEM as an example. We introduce the IMPRECISE GUARDING and UNIVERSAL GUARDING problems. The underlying idea is to guard a polygon, but we know the polygon only in an imprecise way. One might think of two different scenarios. In the first scenario, we want to know whether it is always possible to guard the polygon with k guards, no matter how the actual polygon behaves. In the second scenario, we want to find a set of *universal guards* that will guard every possible polygon. Figure 13 depicts an example of a universal guard set.

GUARDING UNDER IMPRECISION
Input: A set of unit disks $d_1, \ldots, d_n$ and a number $k \in \mathbb{N}$.
Question: Is it true that for every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ the polygon described by $p_1, \ldots, p_n$ can be guarded by k guards?

UNIVERSAL GUARD SET
Input: A set of unit disks $d_1, \ldots, d_n$ and a number $k \in \mathbb{N}$.
Question: Is there a set G of k guards such that for every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ the polygon described by $p_1, \ldots, p_n$ is guarded by G?

Both solution concepts seem sensible as they are able to deal with the situation that the input is not known precisely, nevertheless, they guarantee a precise solution. Algorithms solving this problem must be more fault-tolerant in the sense that they are forgiving towards small errors in the input. It is easy to see that GUARDING UNDER IMPRECISION is contained in $\forall\exists\mathbb{R}$ and that UNIVERSAL GUARD SET is contained in $\exists\forall\mathbb{R}$. Using the Cook–Levin analog (Theorem 1.2) may be the simplest way to see this. Therefore, we wonder:

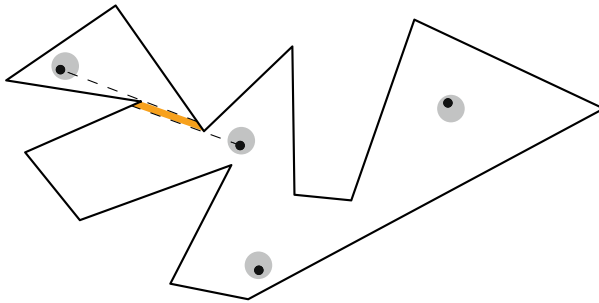**Open Problem 5** *Is* GUARDING UNDER IMPRECISION $\forall\exists\mathbb{R}$-*complete?*

Fig. 14 This guarding is not robust. The orange region might not be guarded

Open Problem 6 *Is* UNIVERSAL GUARD SET ∃∀ℝ-*complete?*

For any geometric problem, we can ask whether there is a universal solution that still works under any small perturbation. However, most problems with geometric input are contained in NP. On the other hand, problems that are known to be ∃ℝ-hard have usually combinatorial input. The ART GALLERY PROBLEM has geometric input and is ∃ℝ-complete. These features make it distinct from many other geometric problems. This means there might not be too many natural problems, which become ∀∃ℝ-complete if the algorithm is required to deal with imprecision. Another exception to this rule of thumb are problems related to linkages [13].

Let us give in this context another example to clarify these observations. Consider a unit-disk intersection graph G given explicitly by a set of disks in the plane. We can ask for a dominating set. Now the problem becomes more challenging, if we ask if the graph can be dominated even after some small perturbation of the disks. The perturbations of the input can also be understood as the inherent imprecision of the input.

DOMINATION OF IMPRECISE UNIT DISKS

Input: A set of unit disks $d_1, \ldots, d_n$, a number $k \in \mathbb{N}$, and a number $\delta \in \mathbb{Q}$.
Question: Is it true that for every translation of each disk by a vector of length at most δ, it is possible to dominate the resulting disk intersection graph by k disks?

Note that the problem of finding a dominating set in a graph is contained in NP. However, it is unclear if this remains the case even with the perturbation. Additionally, the perturbation might be captured best with real-valued variables. However, as a dominating set can be described in a discrete way, it looks unlikely to be ∀∃ℝ-complete. However the following weaker question might have a positive answer.

Open Problem 7 *Is* IMPRECISE DOMINATION OF UNIT DISKS Σ_2^p *and* ∃ℝ-*complete?*

Indeed, we think that it might be true that there is a plausible class between Σ_2^p and ∀∃ℝ capturing imprecision issues.

Robustness. As above, we use the ART GALLERY PROBLEM as an illustrative example and we define the ROBUST GUARDING problem as follows. For an example consider Fig. 14.

> **ROBUST GUARDING**
>
> **Input:** A simple polygon P and a number $k \in \mathbb{N}$.
> **Question:** Does there exist a set of unit disks $D = \{d_1, \ldots, d_k\}$, each fully contained inside the polygon, such that for every placement $G = \{g_1, \ldots, g_k\}$ of guards with $g_i \in d_i$, the placement G guards the whole polygon P?

It can be easily seen that this problem is contained in $\exists\forall\mathbb{R}$. The positions of the disks are existentially quantified, the positions of the guards are universally quantified, and the remaining formula enforces the guards to be inside the disks, the disks inside the polygon, and the condition that the guards are indeed guarding the whole polygon.

Open Problem 8 *Is* ROBUST GUARDING $\exists\forall\mathbb{R}$*-complete?*

In more general terms we employ the notion that a solution is robust, if it remains a valid solution also if it gets slightly perturbed. Another example is ROBUST ORDER TYPE REALIZABILITY. Given a set of points P in the plane, each ordered triple is either in clockwise or counter clockwise direction. In ORDER TYPE REALIZABILITY, we are given an orientation for each triple of some abstract set and we are asked to find a set of points in the plane with that given order type. Robustness asks if the solution remains correct, even after some small perturbation. As order types stay fixed under uniform scaling it makes sense to restrict the points to lie in a unit square.

> **ROBUST ORDER TYPE REALIZABILITY**
>
> **Input:** An abstract order type of n points and a rational number $w > 0$.
> **Question:** Does there exists a set of disks $d_1, \ldots, d_n$ with radius w in the unit square $[0, 1]^2$ such that every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ has the given order type?

Open Problem 9 *Is* ROBUST ORDER TYPE REALIZABILITY $\exists\forall\mathbb{R}$*-complete?*

We assume here that w is encoded in binary. Furthermore, if an order type is realizable for some w_1 then it is also robustly realizable for $w_2 < w_1$. Thus this robustness problem becomes an optimization problem with respect to the parameter w. It is conceivable that at least an approximate solution to the problem can be found by restricting the centers of the disks to a fine grid. This would imply that we can find a polynomial-sized witness for an approximate solution and thus the robust problem might become easier than the basic version for all these problems. Recall that for most of the mentioned recognition problems are $\exists\mathbb{R}$-complete [12, 32, 33, 45].

In a similar spirit, we can define the robust variants for all kinds of recognition problems of intersections graphs, such as intersection graph of unit disks, segments, rays, unit segments, or of your favorite geometric object, see also Fig. 15.

Open Problem 10 *Is* ROBUST RECOGNITION OF INTERSECTION *graphs* $\exists\forall\mathbb{R}$*-complete?*

Environmental Nash-Equilibria. Finding Nash equilibria [39] is one of the most important problems in game theory and it is known to be PPAD-complete [15]. As
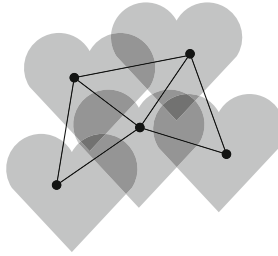
Fig. 15 An intersection graph of hearts

every game has a Nash equilibrium, there is no point in formulating a corresponding decision problem. However, if we restrict our attention to a small region of the strategy space, the problem becomes $\exists\mathbb{R}$-complete already for two players [46]. Restricting the strategy spaces makes sense, as one might be looking for an exact solution "close" to a given approximate one. A natural question we could ask is whether in a three-player game the first two players can find a Nash equilibrium for any fixed behavior of the third player. The underlying idea is that the third player simulates a potentially changing environment.

5.3 A False Candidate—Universal Graph Metric

Given a graph $G = (V, E)$ together with some edge weights $w \colon E \to \mathbb{R}^+$, we can ask for an embedding φ of G in the plane such that for each edge u, v the distance $\mathrm{dist}(u, v)$ in the plane equals the edge weight $w(uv)$. In this case we say φ realizes the edge-weight w. Figure 16 presents an example. Interestingly, it is $\exists\mathbb{R}$-complete to decide if an edge weight with all weights equal 1 can be realized [45].

Many edge weights, are trivially not attainable, as they might not satisfy the triangle inequality. For the purpose of concreteness let us say that a metric is *reasonable* if there exists a Euclidean space of some dimension, into which the graph is embeddable. It is easy to see that the dimension can be upper bounded by the number of vertices.

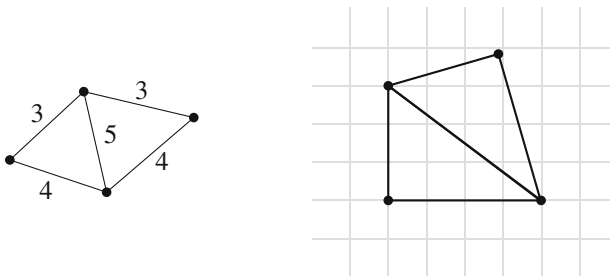We are ready to define the Universal Graph Metric problem.

Fig. 16 A graph with given edge weights (on the left) and an embedding in the Euclidean plane realizing the edge weights (on the right)
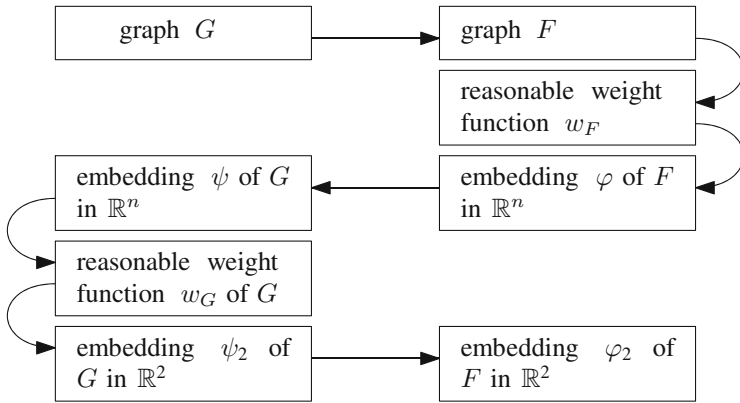
Fig. 17 UNIVERSAL GRAPH METRIC is a minor closed property

UNIVERSAL GRAPH METRIC
Input: A graph G.
Question: Is it true that for every reasonable edge weight function, there exists a realizing embedding in the Euclidean plane?

UNIVERSAL GRAPH METRIC clearly has a similar spirit as area-universality. The edge lengths are universally quantified and the drawing is existentially quantified. However, a simple argument shows the following (surprising), but known, fact:

Proposition 5.2 (Belk, Connelly [8]) UNIVERSAL GRAPH METRIC *is in P.*

Proof The argument builds on the powerful theory of minor-closed graph classes. Recall that a graph F is a *minor* of a graph G (denoted by $F \prec G$), if we can attain F from G by deleting edges, vertices, or contracting edges. A graph class $\mathcal{G}$ is minor-closed, if $G \in \mathcal{G}$ and $F \prec G$ imply $F \in \mathcal{G}$. A celebrated and deep theorem of Robertson and Seymour [43] implies that every minor-closed graph class can be characterized by a finite number of forbidden minors. Since testing if a graph G contains a fixed graph F as a minor can be tested in polynomial time, the membership of a graph in any minor-closed graph class is checkable in polynomial time.

It is easy to observe that the implicitly defined graph class of YES-instances of the UNIVERSAL GRAPH METRIC problem is indeed minor-closed: Let $\mathcal{G}$ denote the class of YES-instances for all *reasonable* weight function. For the following argument refer to Fig. 17. Let $G \in \mathcal{G}$ be a graph with a minor F. Let w_F be a reasonable weight function of F. We need to show that F has an embedding respecting w_F into $\mathbb{R}^2$. (Here n denotes the number of vertices of G, we can assume that the embedding is w.l.o.g. into $\mathbb{R}^n$ as we have at most n vertices.) For that purpose consider an embedding φ of F into $\mathbb{R}^n$ realizing w_F. This exists by definition since w_F is reasonable. We can extend φ to an embedding ψ of G. For contracted edges, we use weight 0, deleted vertices are placed arbitrarily, and weights of deleted edge are inferred by the embedding. This

defines a weight function w_G of G, which is, as we have just shown, reasonable. Thus there is an embedding ψ_2 of G into $\mathbb{R}^2$ as $G \in \mathcal{G}$. This embedding restricted to the vertices of F gives an embedding of w_F into $\mathbb{R}^2$. Thus $F \in \mathcal{G}$.   □

It is easy to see that if we require just the triangle inequality, then K_4 is a forbidden minor and it is known to be the only one [7, 8]. If we choose *reasonable* to be all weight functions, then due to the triangle inequality, the YES-instances are exactly the set of trees and the forbidden minor is the triangle. In a recent master thesis by Muller [38] other meanings of *reasonable* are discussed.

It is worth noting that the same proof does not work for AREA UNIVERSALITY$_{\geq 0}$. It is easy to see that if G is a graph and we remove either an edge or a vertex, then the resulting graph is still area-universal. The same does not hold for edge contractions. To see this, let G denote a graph that is not area-universal. Due to Kleist [26], we know that the 1-subdivision F of G is area-universal. We contract the subdivided edges of F one by one until we obtain G. At some point in this process, the contraction of an edge destroys the area-universality.

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