Discrete & Computational Geometry (2022) 68:608–617
https://doi.org/10.1007/s00454-021-00342-z

Graded Cohen–Macaulay Domains and Lattice Polytopes with Short *h*-Vector

Lukas Katthän[1] · Kohji Yanagawa[2]

Received: 2 September 2020 / Revised: 30 September 2021 / Accepted: 1 October 2021 /
Published online: 9 November 2021

Abstract
Let P be a lattice polytope with the h^*-vector $(1, h_1^*, \dots, h_s^*)$. In this note we show that if $h_s^* \le h_1^*$, then the Ehrhart ring $\Bbbk[P]$ is generated in degrees at most $s-1$ as a $\Bbbk$-algebra. In particular, if $s = 2$ and $h_2^* \le h_1^*$, then P is IDP. To see this, we show the corresponding statement for semi-standard graded Cohen–Macaulay domains over algebraically closed fields.

Keywords Lattice polytope · h^*-Vector · Semi-standard graded ring · Cohen–Macaulay domain

Mathematics Subject Classification 13H10 · 52B20 · 05E40

1 Introduction

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a noetherian graded commutative ring. Throughout the paper, we assume that $\Bbbk := R_0$ is a field. If $R = \Bbbk[R_1]$, that is, R is generated by R_1 as a $\Bbbk$-algebra, we say R is *standard graded*. If R is finitely generated as a $\Bbbk[R_1]$-module, we say R is *semi-standard graded*.

Editor in Charge: János Pach

This work was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 19K03456.

Lukas Katthän
katthaen@math.uni-frankfurt.de

Kohji Yanagawa
yanagawa@kansai-u.ac.jp

1 Göttingen, Germany

2 Department of Mathematics, Kansai University, Suita, Osaka 564-8680, Japan

If R is a semi-standard graded ring of Krull dimension d, its Hilbert series is of the form

$$\sum_{i\in\mathbb{N}}(\dim_{\Bbbk} R_i)\cdot t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

for some integers $h_0, h_1, \ldots, h_s$ with $\sum_{i=0}^{s} h_i \neq 0$ and $h_s \neq 0$. We call the vector $(h_0, h_1, \ldots, h_s)$ the *h-vector* of R. We always have $h_0 = 1$ and $\deg R = \sum_{i=0}^{s} h_i$. If R is Cohen–Macaulay, then $h_i \geq 0$ for all i. We have the following.

Theorem 1.1 *Let R be a semi-standard graded Cohen–Macaulay domain with the h-vector (h_0, h_1, h_2). Assume that $\Bbbk$ is algebraically closed. If $h_2 \leq h_1$, then R is standard graded. If further $h_2 < h_1$ and $\operatorname{char}\Bbbk = 0$, then R is Koszul.*

In fact, Theorem 2.1 states that, under the same situation, if the h-vector of R is $(h_0, h_1, \ldots, h_s)$ with $h_s \leq h_1$, then R is generated in degrees at most $s-1$ as a $\Bbbk$-algebra. So the first statement of Theorem 1.1 follows. To see the latter statement, we can use [5, Thm. 5.2 (1)], since we know R is standard graded now.

An important class of semi-standard graded Cohen–Macaulay domains are the Ehrhart rings of lattice polytopes, which we now recall. Let $P \subset \mathbb{R}^d$ be a lattice polytope. Its *Ehrhart ring* $\Bbbk[P]$ is the monoid algebra of the monoid of lattice points in the cone $C = \operatorname{cone}(\{1\} \times P) \subset \mathbb{R}^{d+1}$ over P. The additional coordinate in the construction of C yields a natural grading on $\Bbbk[P]$, such that $\Bbbk[P]$ is semi-standard graded, and its Hilbert series is the Ehrhart series of P. In particular, the h-vector of $\Bbbk[P]$ is the *h^*-vector* of P (for general information about this notion and its background, see [1]). Hence the Krull dimension $\dim \Bbbk[P]$ equals $\dim P + 1$.

It is well known that $\Bbbk[P]$ is a normal domain, and by Hochster's Theorem [12, Thm. 1], it is Cohen–Macaulay. We refer the reader to the monograph by Bruns and Gubeladze [2] for more information on Ehrhart rings. The index of the last non-zero entry of the h^*-vector is called the *degree* of P. We always have $\deg P \leq \dim P$. The h^*-vector of P is sometimes denoted by $(h_0^*, h_1^*, \ldots, h_{\dim P}^*)$, even if $\deg P < \dim P$. In this case, $h_i^* = 0$ for all $i > \deg P$. We also remark that there is no direct relation between $\deg P$ and $\deg \Bbbk[P] = \sum_{i=0}^{d} h_i^*$, the latter being the multiplicity of $\Bbbk[P]$, which also equals the normalized volume $\operatorname{Vol} P$ of P.

A lattice polytope P is called *IDP* (an abbreviation for "integer decomposition property") if for every $k \in \mathbb{N}$ and every lattice point $p \in kP \cap \mathbb{Z}^d$, there exist k lattice points $p_1, \ldots, p_k \in P \cap \mathbb{Z}^d$ with $p = \sum_i p_i$. Clearly, P is IDP if and only if $\Bbbk[P]$ is standard graded. Hence we obtain the following (here we do not have to assume that $\Bbbk$ is algebraically closed, since we can replace $\Bbbk[P]$ by $\overline{\Bbbk}[P] \cong \overline{\Bbbk} \otimes_{\Bbbk} \Bbbk[P]$):

Corollary 1.2 *Let $P \subset \mathbb{R}^d$ be a lattice polytope of degree 2 with h^*-vector $(1, h_1^*, h_2^*)$. If $h_2^* \leq h_1^*$, then P is IDP. If further $h_2^* < h_1^*$, the Ehrhart ring $\Bbbk[P]$ is Koszul for a filed $\Bbbk$ of characteristic 0.*

Note that if $P \subset \mathbb{R}^2$ is a lattice polygon, then it has degree at most 2, and it always satisfies $h_2^* \leq h_1^*$. Therefore, the former half of this corollary can be seen as an extension of the well-known fact that lattice polygons are IDP. See also Remark 3.3 (i) below. We give an example to show that the bound in Corollary 1.2 is sharp:

Example 1.3 Let P be the 3-simplex with vertices

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\2\end{pmatrix}.$$

It is Reeve's simplex (cf. [1, Exam. 3.22]) and its h^*-vector is $(1, 0, 1)$. It is not IDP, and hence the bound $h^*_2 \le h^*_1$ is sharp.

In the latter part of the present paper, we take a more combinatorial approach. Especially, keeping Corollary 1.2 in mind, we analyze the results of [9], and give several (counter)examples of the related statements.

2 Proofs of the Main Results

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a noetherian graded commutative ring such that $\Bbbk := R_0$ is an algebraically closed field. We are going to regard R as a module over $S := \mathrm{Sym}_{\Bbbk} R_1$. Note that S is isomorphic to the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ with $n = \dim_{\Bbbk} R_1$. Moreover, R is standard graded if and only if R is a quotient ring of S, and R is semi-standard graded if and only if R is finitely generated as an S-module. For a finitely generated graded S-module M, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, set

$$\beta^S_{i,j}(M) := \dim_{\Bbbk}[\mathrm{Tor}^S_i(\Bbbk, M)]_j.$$

In particular, $\beta^S_{0,j}(M)$ is the number of S-module generators for M in degree j.

2.1 A Bound on the Degrees of the Generators

Assume that R is semi-standard graded, and has the h-vector $(h_0, h_1, \ldots, h_s)$. The goal of this section is to obtain a bound on the degrees of the generators of R as an S-module. If R is Cohen–Macaulay, it is well known that the generators have degree at most s. Our result is a sufficient criterion when this bound can be improved by one:

Theorem 2.1 *Let R be a semi-standard graded Cohen–Macaulay domain with h-vector $(h_0, h_1, h_2, \ldots, h_s)$, and $S := \mathrm{Sym}_{\Bbbk} R_1$. Then it holds that*

$$\beta^S_{p,p+s}(R) = 0 \quad \textit{for} \quad 0 \le p \le h_1 - h_s.$$

In particular, if $h_s \le h_1$, then R is generated by elements of degree $\le s - 1$ as an S-module, and hence as a $\Bbbk$-algebra.

Note that Theorem 1.1 amounts to the special case $s = 2$ and $p = 0$. This result and its proof have been inspired by Green's Theorem of the Top Row, [7, Thm. 4.a.4]. For the proof of Theorem 2.1, we are going to use the following version of Green's vanishing theorem:

Theorem 2.2 ([6, Thm. 1.1]) *Let $\mathfrak{p} \subset S$ be a homogeneous prime ideal, which does not contain any linear forms. Let M be a torsion-free finitely generated graded $S/\mathfrak{p}$-module and let $q \in \mathbb{Z}$ be the minimal integer such that $M_q \neq 0$. Then it holds that*

$$\beta^S_{p,p+q}(M) = 0 \quad \textit{for} \quad p \geq \dim_{\Bbbk} M_q.$$

In addition, we need the following result:

Theorem 2.3 ([2, Thm. 6.18], [3, Thm. 4.4.5]) *Let M be a finitely generated graded Cohen–Macaulay module over $S = \Bbbk[x_1, \ldots, x_n]$ with $d = \dim M$. Define $M' := \mathrm{Ext}^{n-d}_S(M, \omega_S)$. Then*

(i) *M' is also Cohen–Macaulay,* $\mathrm{Ann}\, M' = \mathrm{Ann}\, M$*, and $M'' \cong M$;*
(ii) $\beta^S_{p,q}(M') = \beta^S_{n-d-p,n-q}(M)$*;*
(iii) $H_{M'}(t) = (-1)^d H_M(t^{-1})$.

Here, $\omega_S := S(-n)$ denotes the canonical module of S, and $H_M(t)$ denotes the Hilbert series $\sum_{i \in \mathbb{Z}} (\dim_{\Bbbk} M_i) \cdot t^i$ of M.

Proof of Theorem 2.1 Let $d := \dim R$. By [3, Prop. 3.6.12], $\omega_R := \mathrm{Ext}^{n-d}_S(R, \omega_S)$ is a canonical module for R. Using Theorem 2.3 with $M = R$ and $M' = \omega_R$, we have

$$\beta^S_{p,p+q}(R) = \beta^S_{n-d-p,n-(p+q)}(\omega_R) = \beta^S_{n-d-p,(n-d-p)+(d-q)}(\omega_R).$$

and the Hilbert series of ω_R is

$$\frac{h_s t^{d-s} + h_{s-1} t^{d-s+1} + \cdots + h_0 t^d}{(1-t)^d}.$$

In particular, ω_R has no elements in degrees below $d - s$ and we have that $\dim_{\Bbbk} (\omega_R)_{d-s} = h_s$. Now, since R is a domain and ω_R is a canonical module, it is torsion-free over R by Theorem 2.3 (i), and thus it satisfies the hypothesis of Theorem 2.2. Applying that result to ω_R (note that $q = d - s$ in the notation there) yields that

$$\beta^S_{p,p+s}(R) = \beta^S_{n-d-p,(n-d-p)+(d-s)}(\omega_R) = 0 \quad \text{if} \quad (n-d) - p \geq h_s.$$

Finally, note that $h_1 = n - d$, and the proof is completed. □

Remark 2.4 Another important example of a semi-standard graded ring appearing in combinatorial commutative algebra is the *face ring* A_P of a simplicial poset P. See [15] for details. For the simplicial poset P given in Fig. 1, we have

$$A_P \cong \frac{\Bbbk[x, y, z, u, v]}{(xz, uz, vz, uv, xy - u - v)},$$

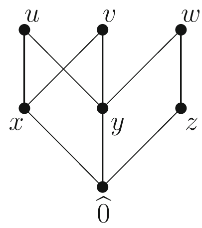


Fig. 1 The poset P of Remark 2.4

where $\deg x = \deg y = \deg z = 1$ and $\deg u = \deg v = 2$, and $\widehat{0}, w \in P$ correspond to $1, yz \in A_P$, respectively. It is easy to see that A_P is a 2-dimensional Cohen–Macaulay reduced semi-standard graded ring with the h-vector $(1, 1, 1)$, but it is *not* standard graded. It means that Theorem 2.1 indeed requires the assumption that R is a domain.

3 Further Discussion on Ehrhart Rings

3.1 Direct Applications of Theorem 2.1

We now apply Theorem 2.1 in the setting of Ehrhart theory. Let $P \subset \mathbb{R}^n$ be a lattice polytope. We write $M(P) \subset \mathbb{Z}^{n+1}$ for the affine monoid generated by the lattice points in $\{1\} \times P \subset \mathbb{R}^{n+1}$, and $\widehat{M}(P) \subset \mathbb{Z}^{n+1}$ for its integral closure inside $\mathbb{Z}^{n+1}$. Let $R = \Bbbk[P]$ be the Ehrhart ring of P and $\Bbbk[R_1]$ its subalgebra generated by R_1. Then R and $\Bbbk[R_1]$ are the monoid algebras of the monoids $\widehat{M}(P)$ and $M(P)$, respectively. It is well known that $\widehat{M}(P)$ is generated by elements of degree at most $\min\{\deg P, \dim P - 1\}$ as a module over $M(P)$ (cf. [2, Thm. 2.52]). Equivalently, $R = \Bbbk[P]$ is generated by elements of at most that degree as a $\Bbbk[R_1]$-module, and hence in particular as a $\Bbbk$-algebra.

Since $\Bbbk[P]$ is always Cohen–Macaulay, Theorem 2.1 allows us to improve this bound under an additional assumption as follows. Clearly, this generalizes Corollary 1.2.

Corollary 3.1 *Suppose $P \subset \mathbb{R}^n$ is a lattice polytope of degree s with h^*-vector $(1, h_1^*, \dots, h_s^*)$. If $h_s^* \le h_1^*$, then $\widehat{M}(P)$ is generated by elements of degrees $\le s - 1$ as an $M(P)$-module.*

Let P° be the relative interior of P. The lattice points in P° are closely related to the canonical module of $\Bbbk[P]$ (cf. [3, Thm. 6.3.5 (b)]). In general, it holds that $h^*_{\dim P} \le h_1^*$ (because $h^*_{\dim P} = \#(P^\circ \cap \mathbb{Z}^n) \le \#(P \cap \mathbb{Z}^n) - (\dim P + 1) = h_1^*$), therefore this corollary extends the above mentioned fact that $\widehat{M}(P)$ is generated in degrees at most $\min\{\deg P, \dim P - 1\}$.

If P is IDP, then $R = \Bbbk[P]$ is the quotient ring of $S = \mathrm{Sym}_\Bbbk R_1$ by a certain prime ideal $I \subseteq S$, which is called the *toric ideal* of P. It is known that I is generated by polynomials of degree at most $\deg P + 1 \le \dim P + 1$ (Sturmfels, cf. [2, Cor. 7.27]), and again we can improve these bounds by one:

Corollary 3.2 *Let P be an IDP lattice polytope and let $I \subset S$ be its toric ideal.*

(i) *If $h^*_s \leq h^*_1 - 1$, then I is generated in degrees $\leq \deg P$.*
(ii) *If P is not a clean simplex, then I is generated in degrees $\leq \dim P$.*

Recall that a *clean simplex* is a lattice simplex where the only lattice points on its boundary are the vertices.

Proof

(i) Apply Theorem 2.1 to $R = \Bbbk[P]$ with $p = 1$.
(ii) If I has a generator in degree $\dim P + 1$, then it holds that $\deg P = \dim P$ by the above mentioned fact that I is generated in degrees at most $\deg P + 1 \leq \dim P + 1$. Moreover, the hypothesis of part (i) needs to be violated, hence it holds that $h^*_{\dim P} \geq h^*_1$. It follows that $h^*_{\dim P} = h^*_1$, which is equivalent to P being a clean simplex. □

Remark 3.3

(i) Let $P \subset \mathbb{R}^2$ be a lattice polygon. Then we have $\deg P \leq 2$ and P is IDP. Moreover, Koelman [13] showed that the toric ideal of P is generated by quadrics if and only if $h^*_2 < h^*_1$. Hence Corollary 3.2 (i) is an extension of one implication of his result. In particular, the result of [13] shows that the bound $h^*_2 < h^*_1$ is sharp.
(ii) In [14], Schenck also applied the theory of Green to the study of Ehrhart rings $\Bbbk[P]$. However, the focus of [14] is different from ours. More precisely, he always assumed that $\Bbbk[P]$ is standard graded (i.e., P is IDP), and treated the case the toric ideal is generated by quadrics.

3.2 Combinatorial Proofs

Corollary 1.2 is a purely combinatorial statement, and hence one might hope for a combinatorial proof. As a first step, we prove a weak variant of Corollary 1.2 which admits an elementary proof.

We remind the reader that a lattice polytope $P \subset \mathbb{R}^n$ is called *spanning* [11], if $(\{1\} \times P) \cap \mathbb{Z}^{n+1}$ generates the lattice $\mathbb{Z}^{n+1}$. Every IDP polytope is spanning, but the converse is far from being true. Algebraically, for the Ehrhart ring $R = \Bbbk[P]$, P is spanning if and only if the field of fractions of R coincides with that of $\Bbbk[R_1]$.

Proposition 3.4 *Let $P \subset \mathbb{R}^n$ be a d-dimensional lattice polytope with h^*-vector $(h^*_0, h^*_1, \ldots)$. If $h^*_1 + h^*_d \geq \sum_{i=2}^{d-1} h^*_i$, then P is spanning.*

Proof Let L be the lattice generated by the lattice points in P, and q the index of L in $\mathbb{Z}^n$. Further, let $\widetilde{P}$ be the polytope P considered in the lattice L (see [11]). We write $\widetilde{h}^*$ for the h^*-vector of $\widetilde{P}$. It holds that

$$\sum_{i=0}^{d} h^*_i = \operatorname{Vol} P = q \operatorname{Vol} \widetilde{P} = q \sum_{i=0}^{d} \widetilde{h}^*_i. \tag{1}$$

Moreover, it holds that $h_1^* = \widetilde{h}_1^*$, $h_d^* = \widetilde{h}_d^*$, and $h_i^* \geq \widetilde{h}_i^*$ for $1 \leq i \leq d$ (see [11, Sect. 3.2]). Now (1) and the assumption that $\sum_{i=2}^{d-1} h_i^* \leq h_1^* + h_d^*$ imply that

$$\begin{aligned} 0 \leq q \sum_{i=2}^{d-1} \widetilde{h}_i^* &= \sum_{i=2}^{d-1} h_i^* - (q-1)(1 + h_1^* + h_d^*) \\ &\leq h_1^* + h_d^* - (q-1)(1 + h_1^* + h_d^*) = (2-q)(1 + h_1^* + h_d^*) - 1. \end{aligned}$$

Since $1 + h_1^* + h_d^* \geq 1$, we have $q = 1$, and it means that P is spanning. □

In the next corollary, (i) is just a weak version of Corollary 1.2, but (ii) and (iii) are new.

Corollary 3.5 *With the above notation, the following hold:*

(i) *If* $\deg P = 2$ *and* $h_1^* \geq h_2^*$, *then* P *is spanning.*
(ii) *If* $\dim P = 3$ *and* $h_1^* + h_3^* \geq h_2^*$, *then* P *is spanning.*
(iii) *If* $\dim P = 4$, $\deg P \geq 3$, *and* $h_1^* + h_4^* \geq h_2^* + h_3^*$, *then* P *is spanning. In this case, it holds that* $h_1^* = h_2^* = h_3^* = h_4^*$.

Proof Only the very last statement is not immediate from Proposition 3.4. If $\dim P = 4$, then $h_4^* \leq h_1^*$. By assumption and Proposition 3.4, P is spanning, and hence by [10, Thm. 1.4] it holds that $h_1^* \leq h_i^*$ for $1 \leq i < \deg P$. As $\deg P \geq 3$ it holds that $h_1^* \leq h_2^*$ and $h_3^* > 0$, and thus $h_1^* < h_2^* + h_3^*$. It follows that $h_4^* \neq 0$, so $\deg P = 4$. Hence we have that $h_4^* \leq h_1^* \leq h_2^*$ and $h_1^* \leq h_3^*$. These inequalities, together with $h_1^* + h_4^* \geq h_2^* + h_3^*$, imply that $h_1^* = h_2^* = h_3^* = h_4^*$. □

Unfortunately, these are the only cases where Proposition 3.4 can be applied, due to the following observation:

Proposition 3.6 *If* $d := \dim P \geq 5$ *and* $\deg P \geq 3$, *then* $h_1^* + h_d^* < \sum_{i=2}^{d-1} h_i^*$.

Proof Set $s := \deg P$. If $h_d^* = 0$ (i.e., $s < d$), then we have

$$h_1^* + h_d^* = h_1^* < h_0^* + h_1^* \leq h_s^* + h_{s-1}^* \leq \sum_{i=2}^{d-1} h_i^*,$$

where the first $\leq$ follows from [16, Prop. 4.1] and the second follows from the assumption that $d \geq 5$ and $s \geq 3$. If $h_d^* \neq 0$, we have $h_i^* \geq h_1^*$ for all $1 \leq i \leq d-1$ by [8, Thm. 1.1]. Hence

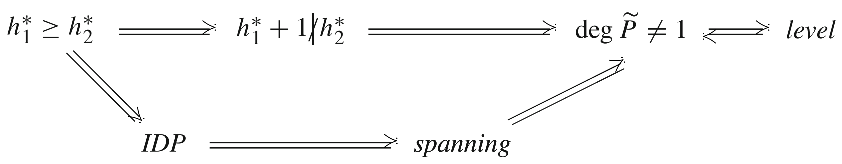
$$h_1^* + h_d^* \leq 2h_1^* < h_2^* + h_3^* + h_4^* \leq \sum_{i=2}^{d-1} h_i^*.$$

Here we use the assumption that $d \geq 5$. □

3.3 About Polytopes of Degree 2

One can combine our Corollary 1.2 with the results of [9] to obtain the following web of implications for lattice polytopes of degree 2:

Theorem 3.7 *Let $P \subset \mathbb{R}^n$ be a lattice polytope of degree 2 with h^*-vector $(1, h_1^*, h_2^*)$, and let $\widetilde{P}$ denote the polytope P considered as a lattice polytope inside the lattice generated by the lattice points in P. Then the following implications hold:*



Here, we say that a lattice polytope P is *level* if its Ehrhart ring $R = \Bbbk[P]$ is level, that is, its canonical module ω_R is generated in a single degree as an R-module. The levelness of P is a combinatorial property of the monoid $\widehat{M}(P)$ (c.f. [9, Prop. 4.3]), and does not depend on the base field $\Bbbk$.

Proof

- $h_1^* \geq h_2^* \implies h_1^* + 1 \nmid h_2^*$**:** This is elementary.
- $1 + h_1^* \nmid h_2^* \implies \deg \widetilde{P} \neq 1$**:** We show the contrapositive. Assume that $\deg \widetilde{P} = 1$. Denote the h^*-vector of $\widetilde{P}$ by $\tilde{h}^*$. The volume of $\widetilde{P}$ divides the volume of P, since the latter is normalized with respect to a finer lattice. Thus we have that

$$(1 + \tilde{h}_1^* + \tilde{h}_2^*) \mid (1 + h_1^* + h_2^*).$$

  On the other hand, we have that $\tilde{h}_1^* = h_1^*$ and by assumption, $\tilde{h}_2^* = 0$. It follows that $(1 + h_1^*) \mid h_2^*$.
- $h_1^* \geq h_2^* \implies$ **IDP:** This is Corollary 1.2.
- **IDP** $\implies$ **spanning:** This is well known (and elementary), and it does not need the assumption $\deg P \leq 2$.
- **spanning** $\implies \deg \widetilde{P} \neq 1$**:** $\deg \widetilde{P} = \deg P = 2 \neq 1$.
- $\deg \widetilde{P} \neq 1 \implies$ **level:** Under this hypothesis, the degree of $\widetilde{P}$ is either 0 or 2. We distinguish those cases:

$\deg \widetilde{P} = 0$: In this case $h_1^* = 0$, the claim follows from [9, Lem. 2.1].
$\deg \widetilde{P} = 2$: Let R and $\widetilde{R}$ denote the Ehrhart rings of P and $\widetilde{P}$, respectively. Then $\deg \widetilde{R} = 1 + h_1^* + \tilde{h}_2^* > 1 + h_1^*$ by assumption, and thus R is level by [9, Prop. 3.4].

- **level** $\implies \deg \widetilde{P} \neq 1$**:** This follows from more general Lemma 3.8 below. □

Lemma 3.8 *Let $P \subset \mathbb{R}^n$ be a lattice polytope. If P is level, then* $\deg \widetilde{P} \neq \deg P - 1$.

Proof Let $c(P) := \min\{\ell \in \mathbb{Z}_{>0} : \ell P^\circ \cap \mathbb{Z}^n \neq \emptyset\}$. It is well known that $\deg P = \dim P + 1 - c(P)$.

We are going to use [9, Prop. 4.3], which we recall for convenience: If P is level, then for any $k \geq c(P)$ and $\alpha \in kP^\circ \cap \mathbb{Z}^n$, there exist $\beta \in c(P)P^\circ \cap \mathbb{Z}^n$ and $\gamma \in (k - c(P))P \cap \mathbb{Z}^n$ such that

$$\alpha = \beta + \gamma.$$

Now, assume that $\deg \widetilde{P} = \deg P - 1$, and note that this implies $c(\widetilde{P}) = c(P) + 1$. Let $L \subset \mathbb{Z}^n$ be the sublattice spanned by the lattice points in P. As $P \neq \widetilde{P}$, this is a proper sublattice of $\mathbb{Z}^n$. Choose $\alpha \in c(\widetilde{P})P^\circ \cap L$. Then, if P were level, there would exist β and γ as above. As $\beta \in c(P)P^\circ \cap \mathbb{Z}^n$, it follows that $\beta \notin L$ (because $c(P)\widetilde{P}$ has no interior lattice points). Further, γ lies in $(c(\widetilde{P}) - c(P))P = P$ and thus $\gamma \in L$. But this contradicts $\beta + \gamma = \alpha \in L$. □

We provide some examples to show that all the implications are strict and that there are no other implications. In each example, the claimed properties can conveniently be verified using `normaliz` [4].

Example 3.9 ($h_1^* + 1 \nmid h_2^* \not\Longrightarrow$ spanning, IDP, $h_1^* \geq h_2^*$). Consider the 4-polytope P with vertices

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

Its h^*-vector is $(1, 2, 5)$, so it satisfies $h_1^* + 1 \nmid h_2^*$, but $h_1^* \not\geq h_2^*$. To see that it is not spanning (and thus not IDP), consider the vector

$$v := \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right).$$

It lies in $2P \cap \mathbb{Z}^4$, but the sum of its coordinates is odd, while the coordinate sum of each vertex of P is even. Hence v cannot lie in the lattice spanned by them.

Example 3.10 (IDP and spanning $\not\Longrightarrow h_1^* + 1 \nmid h_2^*, h_1^* \geq h_2^*$). Let P be the 3-simplex with vertices

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\4\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\0\\3\end{pmatrix}.$$

It is IDP and its h^*-vector is $(1, 5, 6)$, so $h_1^* + 1 \mid h_2^*$ and $h_1^* \not\geq h_2^*$.

Example 3.11 (spanning $\not\Longrightarrow$ IDP). It is well known that this implication does not hold in general. For an example with degree 2, see [2, Exe. 2.24]. This is a very-ample (and thus spanning) 3-polytope which is not IDP. Its h^*-vector is $(1, 4, 5)$, so it has degree 2.

Example 3.12 ($\deg \widetilde{P} \neq 1 \quad \not\Longrightarrow \quad h_1^* + 1 \nmid h_2^*$, spanning). Let P be the polytope of Example 1.3 with h^*-vector $(1, 0, 1)$. In this case $\widetilde{P}$ is a unit simplex and thus $\deg \widetilde{P} = 0 \neq 1$. However, P is not spanning and it holds that $h_1^* + 1 \mid h_2^*$.

Example 3.13 ($h_1^* + 1 \nmid h_2^*$ and IDP $\not\Longrightarrow h_1^* \geq h_2^*$). Let P be the 3-simplex with vertices

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\4\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\0\\4\end{pmatrix}.$$

Its h^*-vector is $(1, 6, 9)$, so it satisfies $h_1^* + 1 \nmid h_2^*$, but $h_1^* \ngeq h_2^*$. Moreover, it is IDP.

Acknowledgements The authors thank Kazuma Shimomoto for many inspiring discussions throughout this project. They also thank anonymous reviewers for their careful reading and valuable comments.

References

1. Beck, M., Robins, S.: Computing the Continuous Discretely. Undergraduate Texts in Mathematics. Springer, New York (2007)
2. Bruns, W., Gubeladze, J.: Polytopes, Rings, and K-Theory. Springer Monographs in Mathematics. Springer, Dordrecht (2009)
3. Bruns, W., Herzog, J.: Cohen–Macaulay Rings. Cambridge Studies in Advanced Mathematics, vol. 39. Cambridge University Press, Cambridge (1993)
4. Bruns, W., Ichim, B., Römer, T., Sieg, R., Söger, Ch.: Normaliz. Algorithms for Rational Cones and Affine Monoids. https://www.normaliz.uni-osnabrueck.de
5. Conca, A., Iyengar, S.B., Nguyen, H.D., Römer, T.: Absolutely Koszul algebras and the Backelin–Roos property. Acta Math. Vietnam. **40**(3), 353–374 (2015)
6. Eisenbud, D., Koh, J.: Some linear syzygy conjectures. Adv. Math. **90**(1), 47–76 (1991)
7. Green, M.L.: Koszul cohomology and the geometry of projective varieties. J. Differ. Geom. **19**(1), 125–171 (1984)
8. Hibi, T.: A lower bound theorem for Ehrhart polynomials of convex polytopes. Adv. Math. **105**(2), 162–165 (1994)
9. Higashitani, A., Yanagawa, K.: Non-level semi-standard graded Cohen–Macaulay domain with h-vector (h_0, h_1, h_2). J. Pure Appl. Algebra **222**(1), 191–201 (2018)
10. Hofscheier, J., Katthän, L., Nill, B.: Spanning lattice polytopes and the uniform position principle (2017). arXiv:1711.09512
11. Hofscheier, J., Katthän, L., Nill, B.: Ehrhart theory of spanning lattice polytopes. Int. Math. Res. Not. IMRN **2018**(19), 5947–5973 (2018)
12. Hochster, M.: Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes. Ann. Math. **96**, 318–337 (1972)
13. Koelman, R.J.: A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics. Beiträge Algebra Geom. **34**(1), 57–62 (1993)
14. Schenck, H.: Lattice polygons and Green's theorem. Proc. Am. Math. Soc. **132**(12), 3509–3512 (2004)
15. Stanley, R.P.: f-vectors and h-vectors of simplicial posets. J. Pure Appl. Algebra **71**(2–3), 319–331 (1991)
16. Stanley, R.P.: On the Hilbert function of a graded Cohen–Macaulay domain. J. Pure Appl. Algebra **73**(3), 307–314 (1991)