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Flexible Placements of Periodic Graphs in the Plane

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Abstract
Given a periodic graph, we wish to determine via combinatorial methods whether it has periodic embeddings in the plane that—via motions that preserve edge-lengths and periodicity—can be continuously deformed into another non-congruent embedding of the graph. By introducing NBAC-colourings for the corresponding quotient gain graphs, we identify which periodic graphs have flexible embeddings in the plane when the lattice of periodicity is fixed. We further characterise with NBAC-colourings which 1-periodic graphs have flexible embeddings in the plane with a flexible lattice of periodicity, and characterise in special cases which 2-periodic graphs have flexible embeddings in the plane with a flexible lattice of periodicity.

Keywords Periodic frameworks · Flexibility · Linkages · Gain graphs

Mathematics Subject Classification 52C25 · 13A18

1 Introduction

A (*bar-joint*) *framework in the plane* is a pair $(\mathcal{G}, \mathcal{P})$, where $\mathcal{G}$ is a simple graph and $\mathcal{P}$ (the *placement* of $\mathcal{G}$) is a map from $V(\mathcal{G})$ to $\mathbb{R}^2$.[1] By considering each edge vw as a rigid bar that restricts the distance between v and w, a natural question to ask is whether or not the structure is *flexible*, i.e., does there exist a continuous path in

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[1] Although (G, p) is the standard notation for a framework, we shall instead reserve this for the quotient frameworks that we use throughout the majority of this paper.

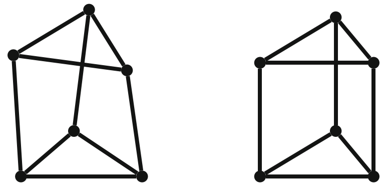

Fig. 1 (Left): A rigid placement of $K_2 \times K_3$ in the plane. As $K_2 \times K_3$ is a Laman graph, almost all placements will give a rigid framework. (Right): A flexible placement of the same graph

the space of placements of $\mathcal{G}$ that preserves the edge distances but is not a rigid body motion? If the vertex set of $\mathcal{G}$ is finite and the coordinates of the vector $(\mathcal{P}(v))_{v\in V(\mathcal{G})}$ are algebraically independent over $\mathbb{Q}$, then it has been proved (first by Pollaczek–Geiringer [18] and later by Laman independently [12]) that $(\mathcal{G}, \mathcal{P})$ is *rigid* (i.e., not flexible) in the plane if and only if $\mathcal{G}$ contains a (somewhat erroneously named) *Laman graph*; a graph $\mathcal{H}$ where $|E(\mathcal{H})| = 2|V(\mathcal{H})| - 3$ and $|E(\mathcal{H}')| \leq 2|V(\mathcal{H}')| - 3$ for all subgraphs $\mathcal{H}'$ of $\mathcal{H}$ with $|V(\mathcal{H}')| \geq 2$. Given a graph that contains a Laman graph, there can however still exist non-generic placements that are flexible; see Fig. 1.

This raises a new question; can we use combinatorial methods to determine if a graph $\mathcal{G}$ has *any* placement that defines a flexible framework $(\mathcal{G}, \mathcal{P})$? This question was answered in the positive in [7], where it was proved that a finite simple graph will have flexible placements in the plane if and only if it has an *NAC-colouring*, a surjective red-blue edge colouring where no cycle has exactly one red edge or exactly one blue edge. Detecting whether graphs have flexible placements via NAC-colourings is a very recent area of research which utilises many different areas of algebraic geometry, including valuation theory [5–8].

We now wish to extend the method using NAC-colourings to frameworks in the plane with *k-periodic symmetry*, i.e., frameworks $(\mathcal{G}, \mathcal{P})$ where there exist a matrix $L \in M_{2\times k}(\mathbb{R})$ and a free group action θ of $\mathbb{Z}^k$ on $\mathcal{G}$ via graph automorphisms, such that $\mathcal{G}$ has a finite set of vertex orbits under θ and $\mathcal{P}(\theta(\gamma)v) = \mathcal{P}(v) + L \cdot \gamma$ for all $v \in V(G)$ and $\gamma \in \mathbb{Z}^k$; we call L the *lattice* of $\mathcal{P}$, θ the *symmetry* of $\mathcal{G}$, and $\mathcal{P}$ a *k-periodic placement* of $(\mathcal{G}, \theta)$. Specifically, we wish to be able to determine if a graph $\mathcal{G}$ with symmetry θ has a k-periodic placement $\mathcal{P}$ where $(\mathcal{G}, \mathcal{P})$ can be deformed by a motion that preserves the periodic structure of $(\mathcal{G}, \mathcal{P})$, and if such a placement does exist, be able to also determine in advance whether the motion will preserve the lattice structure of $(\mathcal{G}, \mathcal{P})$.

Research into the rigidity of periodic frameworks has seen much interest in the last decade. Some of the main areas of research include combinatorial characterisations of rigid periodic graphs [2,3,13,16,21], periodic graphs with unique realisations [11], rigid unit modes of periodic frameworks [17,19], and rigidity under infinitesimal motions where the periodicity is relaxed somewhat [1,4,10,14,23].

Each k-periodic framework $(\mathcal{G}, \mathcal{P})$ in the plane with a given symmetry θ defines a family of *gain-equivalent* triples (G, p, L), where G is a $\mathbb{Z}^k$-*gain graph* and $p\colon V(G) \to \mathbb{R}^2$ is a *placement* of G (see Sects. 2.2 and 2.3 for definitions), and likewise, each such triple (G, p, L) will define a framework $(\mathcal{G}, \mathcal{P})$ with k-periodic symmetry; see [21, Sect. 2.2] for more details. As $\mathbb{Z}^k$-gain graphs have a finite amount

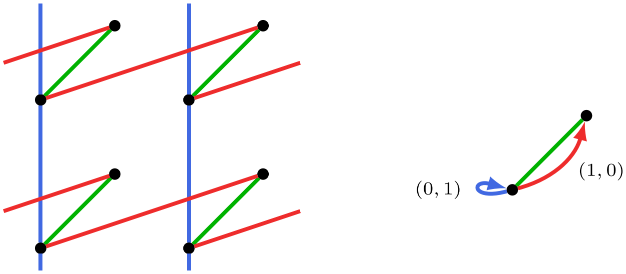

Fig. 2 (Left): A framework $(\mathcal{G}, \mathcal{P})$ with 2-periodic symmetry. (Right): A corresponding triple (G, p, L) with $L := 2I_2$, where I_2 is the 2×2 identity matrix

of vertices but still encode all the required information needed for working with motions that preserve periodicity, we shall define a *k-periodic framework in the plane* to be a triple (G, p, L) for some $\mathbb{Z}^k$-gain graph G, and the pair (p, L) to be a *placement-lattice* of G; for example, see Fig. 2.

Using the gain graph description of k-periodic frameworks, our question is now the following; can we use combinatorial methods to determine if a $\mathbb{Z}^k$-gain graph G has *any* placement-lattice that defines a flexible k-periodic framework (G, p, L)? We shall answer this in the positive for 1-periodic frameworks where the lattice is allowed to deform (see Theorem 5.1) and k-periodic frameworks where the lattice is fixed (see Theorem 4.1). We also obtain partial results for the more difficult case of 2-periodic frameworks where the lattice is allowed to deform (see Lemma 6.4, Theorems 7.5 and 7.10). To do this we shall introduce *NBAC-colourings* ("NBAC" being an acronym for "No Balanced Almost Circuit"), an analogue of NAC-colourings for $\mathbb{Z}^k$-gain graphs. We shall also characterise the various types of NBAC-colourings that are generated by different motions of a given k-periodic framework.

The outline of the paper is as follows. In Sect. 2, we shall layout some background on valuation theory, gain graphs, and periodic frameworks in both $\mathbb{R}^d$ and $\mathbb{C}^d$. In Sect. 3, we shall define NBAC-colourings and their various sub-types, including active NBAC-colourings, and utilise valuations to prove that flexibility will imply the existence of an NBAC-colouring. In Sects. 4–6, we shall apply our methods using NBAC-colourings to fixed lattice k-periodic frameworks, flexible lattice 1-periodic frameworks, and flexible lattice 2-periodic frameworks respectively, with partial results in the latter case. In Sect. 7, we shall prove that a full characterisation of $\mathbb{Z}^2$-gain graphs with a flexible placement-lattice is possible if we assume that our graph has at least a single loop.

2 Preliminaries

2.1 Function Fields and Valuations

We shall refer to all affine algebraic sets over $\mathbb{C}$ as algebraic sets, and we shall call any irreducible algebraic set a variety. For an algebraic set V in $\mathbb{C}^n$, we define $I(V)$

to be the ideal of $\mathbb{C}^n$ that defines V. We recall that the dimension of an algebraic set is the maximal length of chains of distinct nonempty subvarieties of A. An *algebraic curve* is an affine variety of dimension 1.

Definition 2.1 Let V be a variety in the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. We define the *coordinate ring* of V to be the quotient $\mathbb{C}[V] := \mathbb{C}[X_1, \ldots, X_n]/I(V)$ and the *function field* of V to be the field of fractions of $\mathbb{C}[V]$, denoted by $\mathbb{C}(V)$. Each $\hat{f}/\hat{g} \in \mathbb{C}(V)$ can, for any $f \in \hat{f}$ and $g \in \hat{g}$, be considered to be a partially defined function

$$f/g \colon V \to \mathbb{C}, \quad x \mapsto f(x)/g(x),$$

and this function is independent of the choice of f, g.

We recall that for a field extension K/k, an element $a \in K$ is *transcendental over* k if there is no polynomial $p \in k[X]$ with $p(a) = 0$, and *algebraic over* k otherwise. The following useful result stems from the observation that any rational function must either be constant on a variety or take an infinite amount of values; indeed if this was not true, we would be able to construct a non-invertible element of the function field.

Lemma 2.2 *Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}[X_1, \ldots, X_n]$ and let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then one of the following holds:*

(i) *f takes an infinite amount of values on $\mathcal{C}$ and is transcendental over $\mathbb{C}$ when considered as an element of $\mathbb{C}(\mathcal{C})$.*
(ii) *f is constant on $\mathcal{C}$.*

Definition 2.3 For a function field $\mathbb{C}(C)$, a function $\nu \colon \mathbb{C}(C) \to \mathbb{Z} \cup \{\infty\}$ is a *valuation* if

(i) $\nu(x) = \infty$ if and only if $x = 0$;
(ii) $\nu(xy) = \nu(x) + \nu(y)$;
(iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$, with equality if $\nu(x) \neq \nu(y)$;
(iv) $\nu(x) = 0$ if $x \in \mathbb{C} \setminus \{0\}$.

The following is a useful rewording of [22, Cor. 1.1.20].

Proposition 2.4 *Let $\mathbb{C}(C)$ be a function field and suppose $f \in \mathbb{C}(\mathcal{C})$ is transcendental over $\mathbb{C}$. Then there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ with $\nu(f) > 0$.*

2.2 Gain Graphs

We shall briefly cover the topic of *gain graphs*. For a more in depth analysis of the topic for general groups, we refer the reader to [9]. We will be mainly be interested in the case when the group is an abelian free group; for more discussion on techniques often used for this specific topic, we refer the reader to [20].

Definition 2.5 A Γ-*gain graph* is a triple $G := (V(G), E(G), \Gamma)$, where:

(i) $V(G)$ is a finite set of *vertices*.
(ii) Γ is an additive abelian group with identity 0.

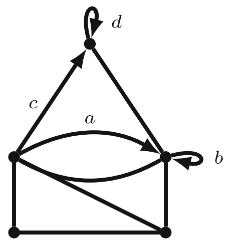

Fig. 3 A Γ-gain graph with $a, b, c, d \in \Gamma$. We represent any edge (v, w, γ) by an arrow from v to w with a label γ, and we represent any edge $(v, w, 0)$ by an undirected and unlabelled edge from v to w

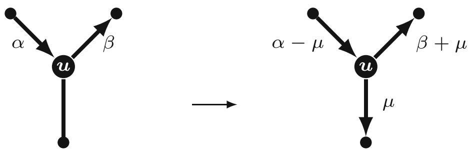

Fig. 4 A switching operation at u by μ

(iii) $K(V(G)) := (V(G)^2 \times \Gamma)/R$, where R is the equivalence relation with $(a, b, \gamma)\, R\, (c, d, \mu)$ if and only if either $a = c$, $b = d$, and $\gamma = \mu$, or $a = d$, $b = c$, and $\gamma = -\mu$.

(iv) $E(G) \subset K(V(G))$ is a set of *edges*. We shall assume that there is no edge of the form $(v, v, 0)$; we shall, however, allow $E(G)$ to be an infinite set.

While the edges of a gain graph are not orientated, we often find it easier to assume that there is some orientation on the edges, i.e., G is directed. We may then define the *gain* of an edge (v, w, γ) to be γ. We refer the reader to Fig. 3 for an example.

A *switching operation at u by* μ is the map $\phi_u^\mu\colon K(V(G)) \to K(V(G))$ where

$$\phi_u^\mu(v, w, \gamma) = \begin{cases} (u, w, \gamma + \mu) & \text{if } v = u,\ w \neq u, \\ (v, u, \gamma - \mu) & \text{if } v \neq u,\ w = u, \\ (v, w, \gamma) & \text{if } v, w \neq u \text{ or } v = w = u. \end{cases}$$

See Fig. 4 for an example of a gain switching operation at a vertex.

Given the switching operations $\phi_{u_1}^{\mu_1}, \ldots, \phi_{u_n}^{\mu_n}$ (where the vertices $u_1, \ldots, u_n$ and elements $\mu_1, \ldots, \mu_n$ need not be distinct), we define $\phi := \phi_{u_n}^{\mu_n} \circ \cdots \circ \phi_{u_1}^{\mu_1}$ to be a *gain equivalence*. We say Γ-gain graphs G, G' are *gain-equivalent* (or $G \approx G'$) if G and G' are Γ-gain graphs with the same vertex set and $G' = \phi(G) := (V(G), \phi(E(G)), \Gamma)$ for some gain equivalence ϕ. If $H \subset G$ and $H' := \phi(H)$, then we say H' is the *corresponding subgraph* of H in G'. The relation $\approx$ is an equivalence relation for gain graphs.

A *walk* in G is an ordered set $C := (e_1, \ldots, e_n)$ of edges of G where $e_i = (v_i, v_{i+1}, \gamma_i)$ (with $v_{n+1} = v_1$) for some γ_i; we note that we orientate each edge so we have a *directed walk* from v_1 to v_n. The *length* of a walk is the amount of edges it contains (including any repetitions). If $v_1 = v_n$ then C is a *circuit*. Unless specified otherwise, all walks and circuits of length n will be of the form described above. For a circuit C, we define

$$\psi(C) := \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

to be the *gain* of C. A circuit is *balanced* if $\psi(C) = 0$, and *unbalanced* otherwise. For a connected subgraph $H \subset G$, we define the *span of* H to be the subgroup

$$\operatorname{span} H := \{\psi(C) : C \text{ is a circuit in } H\}.$$

If $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$, then we define rank H to be the rank of span H. A connected subgraph H is *balanced* if span H is the trivial group, and *unbalanced* otherwise; likewise, a subgraph is *balanced* if every connected component is balanced and *unbalanced* otherwise.

Proposition 2.6 *Let G, G' be gain-equivalent Γ-gain graphs and $H \subset G$ be a connected subgraph. If H' is the corresponding subgraph of H in G, then* span $H' =$ span H.

Proof This follows from noting that switching operations will not change the span of a circuit. □

Proposition 2.7 *Let G be a Γ-gain graph and $\{H_1, \ldots, H_n\}$ a set of connected subgraphs with pairwise disjoint vertex sets. Then there exists $G' \approx G$ such that for each $i \in \{1, \ldots, n\}$, all the edges of the corresponding subgraph H'_i of H_i in G' have gain in* span H_i.

Proof Choose a spanning tree T_i for each $i \in \{1, \ldots, n\}$. We note that we may choose $G' \approx G$ so that each corresponding subgraph T'_i of T_i in G' has only trivial gain for its edges; see [21, Sect. 2.4] for a description of the method. Fix $i \in \{1, \ldots, n\}$ and choose any $e = (v, w, \gamma) \in E(H'_i)$. Let W be the unique walk from w to v in T_i, and define C to be the circuit formed by the travelling along the edge e and then following the walk W. As $\psi(C) = \gamma$, then $\gamma \in$ span H'_i. By Proposition 2.6, span $H_i =$ span H'_i, hence $\gamma \in$ span H_i as required. □

2.3 Rigidity and Flexibility for *k*-Periodic Frameworks

Let $d \in \mathbb{N}$ and $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. We shall define $\|\cdot\|^2 : \mathbb{K}^d \to \mathbb{K}$ to be the quadratic form with

$$\left\|(x_i)_{i=1}^d\right\|^2 := \sum_{i=1}^d x_i^2$$

for all $(x_i)_{i=1}^d \in \mathbb{K}^d$. For $\mathbb{K} = \mathbb{R}$, the quadratic form $\|\cdot\|^2$ is in fact the square of the Euclidean norm, however this is not true for $\mathbb{K} = \mathbb{C}$. The isometries of $(\mathbb{K}^d, \|\cdot\|^2)$ are exactly the affine maps $x \mapsto Mx + y$, where $y \in \mathbb{K}^d$ and $M \in M_n(\mathbb{K})$ is a $d \times d$ matrix where $M^T M = I_d$.

Remark 2.8 For any matrix $M \in M_{m\times n}(\mathbb{K})$ and $x := (x_1, \dots, x_n) \in \mathbb{K}^n$, we shall denote by $M \cdot x$ the matrix multiplication $M[x_1 \dots x_n]^T$.

We shall be using the definition of periodic frameworks, originally stated by Ross, which utilises gain graphs (see [20]), although many of our results can be adapted to fit the terminology used by Borcea and Streinu (see [2]). These two differing definitions can be seen to be identical; we refer the reader to [20, Sect. 3.1] for more details.

Definition 2.9 Let $d \in \mathbb{N}$ and G be a $\mathbb{Z}^k$-gain graph for some $1 \le k \le d$. A *k-periodic framework in* $\mathbb{K}^d$ is a triple (G, p, L) such that G is a $\mathbb{Z}^k$-gain graph, $p\colon V(G) \to \mathbb{K}^d$, and $L \in M_{d\times k}(\mathbb{K})$, with the assumption that if $(v, w, \gamma) \in E(G)$ then $p(v) \neq p(w) + L\cdot\gamma$. We shall define p to be a *placement*, L to be a *lattice*, and the pair (p, L) to be a *placement-lattice*. If L is also injective then (G, p, L) is *full*, and if $\mathbb{K} = \mathbb{R}$ then we simply refer to (G, p, L) as a k-periodic framework.

For a given $\mathbb{Z}^k$-gain graph G, we define $\mathcal{V}^d_{\mathbb{K}}(G)$ to be the space of placement-lattices of G, which we shall consider to be a subspace of $\mathbb{K}^{d|V(G)|+dk}$. We immediately note that $\mathcal{V}^d_{\mathbb{K}}(G)$ is an open non-empty subset in the Zariski topology, and if G has an edge, it is a proper subset.

Definition 2.10 Let (G, p, L) and (G, p', L') be k-periodic frameworks in $\mathbb{K}^d$. Then $(G, p, L) \sim (G, p', L')$ (or (G, p, L) and (G, p', L') are *equivalent*) if for all $(v, w, \gamma) \in E(G)$,

$$\|p(v) - p(w) - L\cdot\gamma\|^2 = \|p'(v) - p'(w) - L'\cdot\gamma\|^2, \tag{1}$$

and $(p, L) \sim (p', L')$ (or (p, L) and (p', L') are *congruent*) if (1) holds for all $v, w \in V(G)$ and $\gamma \in \mathbb{Z}^k$; equivalently, we may define $(p, L) \sim (p', L')$ if and only if there exist a linear isometry $M \in M_d(\mathbb{K})$ and $y \in \mathbb{K}^d$ such that $p'(v) = M\cdot p(v) + y$ for all $v \in V(G)$ and $L' = ML$. For any $L, L' \in M_{d\times k}(\mathbb{K})$, we define L and L' to be *orthogonally equivalent* (or $L \sim L'$) if for any $\gamma, \mu \in \mathbb{Z}^k$,

$$(L\cdot\gamma)\cdot(L\cdot\mu) = (L'\cdot\gamma)\cdot(L'\cdot\mu). \tag{2}$$

We note that, by linearity, if (2) holds for all pairs of some basis of $\mathbb{Z}^k$, then it holds for all $\gamma, \mu \in \mathbb{Z}^k$. Furthermore, if $(p, L) \sim (p', L')$ then $(G, p, L) \sim (G, p', L')$ and $L \sim L'$.

Definition 2.11 For a k-periodic framework (G, p, L) we define the algebraic subsets

$$\mathcal{V}_{\mathbb{K}}(G, p, L) := \{(p', L') \in \mathcal{V}^d_{\mathbb{K}}(G) : (G, p', L') \sim (G, p, L)\},$$
$$\mathcal{V}^f_{\mathbb{K}}(G, p, L) := \{(p', L') \in \mathcal{V}^d_{\mathbb{K}}(G) : (G, p', L') \sim (G, p, L),\ L' \sim L\}.$$

Definition 2.12 Let (G, p, L) be a k-periodic framework in $\mathbb{K}^d$. A *flex* of (G, p, L) is a continuous path $t \mapsto (p_t, L_t), t \in [0, 1]$, in $\mathcal{V}_{\mathbb{K}}(G, p, L)$. If $(p_t, L_t) \in \mathcal{V}^f_{\mathbb{K}}(G, p, L)$ for all $t \in [0, 1]$ then (p_t, L_t) is a *fixed lattice flex*. If $(p_t, L_t) \sim (p, L)$ for all $t \in [0, 1]$ then (p_t, L_t) is *trivial*.

Remark 2.13 An equivalent definition for a trivial finite flex is as follows: (p_t, L_t) is a trivial flex of (G, p, L) if and only (p_t, L_t) is a trivial flex of (K, p, L), where K is $\mathbb{Z}^k$-gain graph with vertex set $V(G)$ and edge set $K(V(G)) \setminus \{(v, v, 0) : v \in V(G)\}$.

Definition 2.14 Let (G, p, L) be a k-periodic framework. Then we define the following:

(i) (G, p, L) is *rigid* if all flexes of (G, p, L) are trivial, and *flexible* otherwise.
(ii) (G, p, L) is *fixed lattice rigid* if all fixed lattice flexes of (G, p, L) are trivial, and *fixed lattice flexible* otherwise.

Let ϕ_u^μ be a switching operation of G. We define the *framework switching operation at u by μ* to be (by abuse of notation) the linear map $\phi_u^\mu : \mathbb{K}^{d|V(G)|+dk} \to \mathbb{K}^{d|V(G)|+dk}$, where, given $(p', L') = \phi_u^\mu(p, L)$, we have $L' = L$ and

$$p'(v) = \begin{cases} p(u) + L \cdot \mu & \text{if } v = u, \\ p(v) & \text{otherwise,} \end{cases}$$

for all $v \in V_{\mathbb{K}}^d(G)$. We define any composition $\phi := \phi_{u_n}^{\mu_n} \circ \cdots \circ \phi_{u_1}^{\mu_1}$ to be a *gain equivalence*, and define $\phi(G, p, L) := (\phi_u^\mu(G), \phi_u^\mu(p, L))$. If there exists a gain equivalence such that $(G', p', L) = \phi(G, p, L)$, then we say (G, p, L) and (G', p', L) are *gain-equivalent*; we denote that two k-periodic frameworks (G, p, L) and (G', p', L) are gain equivalent by $(G, p, L) \approx (G', p', L)$.

As each gain equivalence ϕ is a linear isomorphism and $\phi(\mathcal{V}_{\mathbb{K}}^d(G, p, L)) = \mathcal{V}_{\mathbb{K}}^d(\phi(G, p, L))$, then the sets $\mathcal{V}_{\mathbb{K}}(G, p, L)$ and $\mathcal{V}_{\mathbb{K}}(\phi(G, p, L))$ are isomorphic as algebraic sets. It follows that, given $(G, p, L) \approx (G', p', L)$, we have that (G, p, L) is (fixed lattice) rigid if and only if (G', p', L) is (fixed lattice) rigid.

3 NBAC-Colourings and Flexibility in the Plane

3.1 NBAC-Colourings

Definition 3.1 Let G be a Γ-gain graph with edge colouring $\delta\colon E(G) \to \{\text{red}, \text{blue}\}$. We define the following:

(i) $G_{\text{red}}^\delta := (V(G), \{e \in E(G) : \delta(e) = \text{red}\})$.
(ii) A *red component* is a connected component of G_{red}^δ.
(iii) A *red walk* (respectively, *red circuit*) is a walk (respectively, circuit) where every edge is red.
(iv) An *almost red circuit* is a circuit with exactly one blue edge.
(v) G_{blue}^δ, *blue components*, *blue walks*, *blue circuits*, and *almost blue circuits* are defined analogously.
(vi) We define a component/walk/circuit to be *monochromatic* if it is either red or blue, and we define an *almost monochromatic circuit* to be any circuit that is either almost red or almost blue.

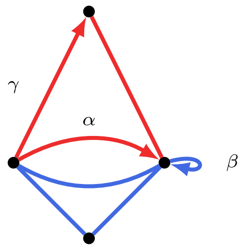

Fig. 5 A surjective colouring δ of a Γ-gain graph. If $\alpha \notin \langle\beta\rangle$, $\beta \notin \langle\alpha-\gamma\rangle$, and $\gamma \neq 0$, then δ is an NBAC-colouring

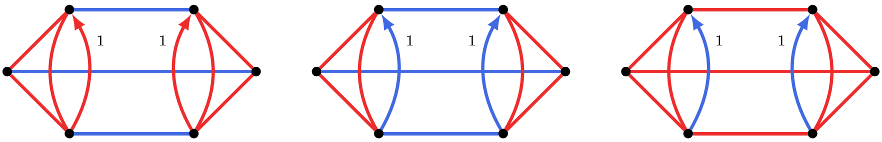

Fig. 6 All three possible NBAC-colourings of a $\mathbb{Z}$-gain graph up to switching the colours red and blue. The left is a fixed lattice NBAC-colouring but not a flexible 1-lattice NBAC-colouring, while the middle and right are flexible 1-lattice NBAC-colourings but not fixed lattice NBAC-colourings

A colouring δ is an *NBAC-colouring* (*No Balanced Almost Circuits*) if it is surjective, and there are no balanced almost red circuits and no balanced almost blue circuits; see Fig. 5 for an example of an NBAC-colouring.

If δ is a colouring of G and $G' \approx G$, then by abuse of notation we shall also define δ to be a colouring for G'. We note that if δ is an NBAC-colouring of G, then δ is an NBAC-colouring of G'.

Definition 3.2 Let G be a $\mathbb{Z}^k$-gain graph for some $k \in \{1,2\}$, with an NBAC-colouring δ. If either G^δ_{red} is balanced and G has no almost blue circuits, or G^δ_{blue} is balanced and G has no almost red circuits, then δ is a *fixed lattice NBAC-colouring*.

Definition 3.3 Let G be a $\mathbb{Z}$-gain graph with an NBAC-colouring δ. If both G^δ_{red} and G^δ_{blue} are balanced, then δ is a *flexible* 1-*lattice NBAC-colouring*.

Remark 3.4 We note that if G is a $\mathbb{Z}$-gain graph with NBAC-colouring δ, then δ can be either both a fixed lattice NBAC-colouring and a flexible 1-lattice NBAC-colouring, one or the other, or neither. We can even have that G has no NBAC-colouring that is both, but has both fixed lattice and flexible 1-lattice NBAC-colourings; see Fig. 6 for an example.

Definition 3.5 Let G be a $\mathbb{Z}^2$-gain graph with an NBAC-colouring δ. We define the following (see Fig. 7 for examples of each colouring):

(i) If both G^δ_{red} and G^δ_{blue} are balanced, then δ is a *type* 1 *flexible* 2-*lattice NBAC-colouring*.
(ii) If there exist $\alpha, \beta \in \mathbb{Z}^2$ such that

– either α, β are linearly independent or exactly one of α, β is equal to $(0,0)$,

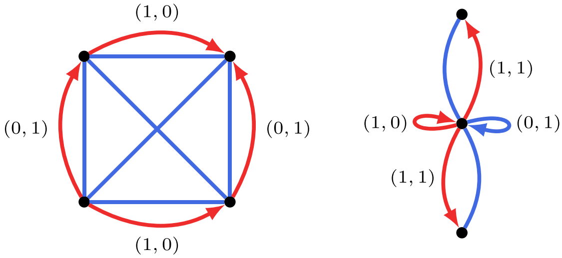

Fig. 7 (Left): A $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. (Middle): A $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring ($\alpha = (1, 0)$, $\beta = (0, 1)$). (Right): A $\mathbb{Z}^2$-gain graph with a type 3 flexible 2-lattice NBAC-colouring ($\alpha = (1, 0)$)

- span G^δ_{red} is a non-trivial subgroup of $\mathbb{Z}\alpha$, or $\alpha = (0, 0)$ and G^δ_{red} is balanced,
- span G^δ_{blue} is a non-trivial subgroup of $\mathbb{Z}\beta$, or $\beta = (0, 0)$ and G^δ_{blue} is balanced,
- there are no almost red circuits with gain in $\mathbb{Z}\alpha$, and
- there are no almost blue circuits with gain in $\mathbb{Z}\beta$,

then δ is a *type* 2 *flexible* 2-*lattice NBAC-colouring*.

(iii) If there exists $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that

- span G^δ_{red} and span G^δ_{red} are non-trivial subgroups of $\mathbb{Z}\alpha$, and
- there are no almost monochromatic circuits with gain in $\mathbb{Z}\alpha$,

then δ is a *type* 3 *flexible* 2-*lattice NBAC-colouring*; see Fig. 7.

Remark 3.6 We note that if G is a $\mathbb{Z}^2$-gain graph with NBAC-colouring δ, then the following holds:

- For distinct $i, j \in \{1, 2, 3\}$, δ cannot be both a type i and type j flexible 2-lattice NBAC-colouring.
- Similarly, δ cannot be both a fixed lattice NBAC-colouring and type 3 flexible 2-lattice NBAC-colouring.
- The colouring δ can, however, be both a fixed lattice NBAC-colouring and either a type 1 or 2 flexible 2-lattice NBAC-colouring; see Fig. 8 for an example of an NBAC-colouring that is both fixed lattice and type 2.
- If $H \subset G$ is not monochromatic and δ is a type k flexible 2-lattice NBAC-colouring for some $k \in \{1, 2, 3\}$, then δ restricted to H is a type k' flexible 2-lattice NBAC-colouring for some $1 \leq k' \leq k$; furthermore, if $k' = 1 < k$ then δ restricted to H will also be a fixed lattice NBAC-colouring.

3.2 *k*-Periodic Frameworks in the Plane

Let G be a $\mathbb{Z}^k$-gain graph for $k \in \{1, 2\}$, with placement $p\colon V(G) \to \mathbb{R}^2$ and lattice $L \in M_{2\times k}(\mathbb{R})$; if $k = 1$ we shall define $L_1 := L \cdot 1$ and if $k = 2$ we shall define $L_1 := L \cdot (1, 0)$ and $L_2 := L \cdot (0, 1)$. For each $e = (v, w, \gamma)$ with $\gamma := (\gamma_j)_{j=1}^k$, we

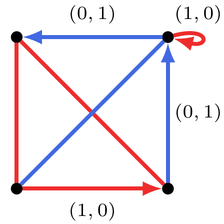

Fig. 8 A $\mathbb{Z}^2$-gain graph with a colouring that is both a fixed lattice NBAC-colouring and a type 2 flexible 2-lattice NBAC-colouring ($\alpha = (1,0)$, $\beta = (0,0)$)

define

$$\lambda(e) := \left\| p(v) - p(w) - \sum_{j=1}^{k} \gamma_j L_j \right\|$$

(we note that this is well defined as (G, p, L) is a k-periodic framework in $\mathbb{R}^2$). We further define for each $1 \leq j, l \leq k$,

$$\lambda(j,l) := L_j \cdot L_l.$$

We shall consider each point $(q, M) \in \mathcal{V}^2_{\mathbb{C}}(G)$ to be a point

$$\big((x_v, y_v)_{v \in V(G)}, (x_j, y_j)_{j=1}^{k}\big),$$

where $x_v, y_v, x_j, y_j \in \mathbb{C}$; the points (x_v, y_v) will correspond to the coordinates of q_v, and the points (x_j, y_j) will correspond to the coordinates of L_j. To help simplify things later on, we will first wish to quotient out $\mathcal{V}^2_{\mathbb{C}}(G)$ by the orientation-preserving isometries by fixing an edge $\tilde{e} = (\tilde{v}, \tilde{w}, \tilde{\gamma})$. To do so, we define the algebraic set $\mathcal{V}_{\tilde{e}}(G, p, L) \subset \mathcal{V}^2_{\mathbb{C}}(G)$ of all points where

$$x_{\tilde{v}} = y_{\tilde{v}} = 0, \qquad y_{\tilde{w}} + \sum_{j=1}^{k} \tilde{\gamma}_j y_j = 0,$$

and for all $e = (v, w, \gamma) \in E(G)$,

$$\left(x_v - x_w - \sum_{j=1}^{k} \gamma_j x_j\right)^2 + \left(y_v - y_w - \sum_{j=1}^{k} \gamma_j y_j\right)^2 = \lambda(e)^2. \qquad (3)$$

We further define $\mathcal{V}^f_{\tilde{e}}(G, p, L)$ to be the algebraic subset of $\mathcal{V}_{\tilde{e}}(G, p, L)$ where $x_j x_l + y_j y_l = \lambda(j,l)^2$, for each $1 \leq j, l \leq k$.

We note that the placement-lattice (p, L) may not be contained in $\mathcal{V}_{\tilde{e}}(G, p, L)$. However, the unique k-periodic framework obtained by translating and rotating

(G, p, L) so that $p_{\tilde{v}}$ lies at the origin and $p_{\tilde{w}} + L \cdot \tilde{\gamma}$ lies on the x-axis, will be contained in $\mathcal{V}_{\tilde{e}}(G, p, L)$. Hence, the set $\mathcal{V}_{\mathbb{C}}(G, p, L)$ is homeomorphic to

$$\mathcal{V}_{\tilde{e}}(G, p, L) \times \mathrm{SO}(2, \mathbb{C}) \times \mathbb{C}^2$$

as $\mathcal{V}_{\tilde{e}}(G, p, L)$ is the set of frameworks equivalent to (G, p, L) in $\mathbb{C}^2$ where the edge $\tilde{e}$ is fixed to lie on the x-axis. Similarly, $\mathcal{V}^f_{\mathbb{C}}(G, p, L)$ is homeomorphic to

$$\mathcal{V}^f_{\tilde{e}}(G, p, L) \times \mathrm{SO}(2, \mathbb{C}) \times \mathbb{C}^2.$$

It follows that, if we require it, we may assume $(p, L) \in \mathcal{V}_{\tilde{e}}(G, p, L)$.

Given an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ and any $v, w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we define the maps

$$W^{\gamma}_{v,w,}|_{\mathcal{C}}, Z^{\gamma}_{v,w}|_{\mathcal{C}} \colon \mathcal{C} \to \mathbb{C}$$

by the polynomials

$$W^{\gamma}_{v,w}|_{\mathcal{C}} := \left(x_v - x_w - \sum_{j=1}^{k} \gamma_j x_j \right) + i \left(y_v - y_w - \sum_{j=1}^{k} \gamma_j y_j \right),$$
$$Z^{\gamma}_{v,w}|_{\mathcal{C}} := \left(x_v - x_w - \sum_{j=1}^{k} \gamma_j x_j \right) - i \left(y_v - y_w - \sum_{j=1}^{k} \gamma_j y_j \right).$$

We further define the maps $W_j|_{\mathcal{C}}, Z_j|_{\mathcal{C}} \colon \mathcal{C} \to \mathbb{C}$ for $1 \le j \le k$ as the polynomials

$$W_j|_{\mathcal{C}} := x_j + i y_j, \qquad Z_j|_{\mathcal{C}} := x_j - i y_j.$$

For the case of $k = 2$, we shall define for each $\gamma := (a, b) \in \mathbb{Z}^2$ the maps

$$\gamma W|_{\mathcal{C}} := a W_1|_{\mathcal{C}} + b W_2|_{\mathcal{C}}, \qquad \gamma Z|_{\mathcal{C}} := a Z_1|_{\mathcal{C}} + b Z_2|_{\mathcal{C}}.$$

When there is no ambiguity regarding which algebraic curve we are observing, we shall for brevity drop the notation "$|_{\mathcal{C}}$"; for example, $W^{\gamma}_{v,w}|_{\mathcal{C}}$ shall be shortened to $W^{\gamma}_{v,w}$.

We first observe that $W^{-\gamma}_{w,v} = -W^{\gamma}_{v,w}$ and $Z^{-\gamma}_{w,v} = -Z^{\gamma}_{v,w}$. Furthermore, we note that if $e = (v, w, \gamma) \in E(G)$,

$$W^{\gamma}_{v,w} Z^{\gamma}_{v,w} = \lambda(e)^2,$$

and if $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ then

$$W_j \cdot Z_j = \lambda(j, j)^2, \qquad W_j \cdot Z_l + W_l \cdot Z_j = 2\lambda(j, l)^2,$$

for all $1 \le j, l \le k$.

3.3 Active NBAC-Colourings

Active NAC-colourings for finite simple graphs were first introduced in [8]. We shall now give an analogue of them for $\mathbb{Z}^k$-gain graphs.

Definition 3.7 Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve, and δ an NBAC-colouring of G. We define δ to be an *active NBAC-colouring of* $\mathcal{C}$ if there exist a valuation ν of $\mathbb{C}(\mathcal{C})$ and $\alpha \in \mathbb{R}$ such that for each $e \in E(G)$,

$$\delta(e) = \begin{cases} \text{red} & \text{if } \nu(W^{\gamma}_{v,w}) > \alpha, \\ \text{blue} & \text{if } \nu(W^{\gamma}_{v,w}) \leq \alpha; \end{cases}$$

if this is the case, we shall say that δ is the NBAC-colouring *generated by* ν *and* α. For a k-periodic framework (G, p, L) in $\mathbb{R}^2$, we define δ to be an *active NBAC-colouring of* (G, p, L) if it is an active NBAC-colouring of an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$. We define δ to be an *active NBAC-colouring of* G if it is an active NBAC-colouring of a full k-periodic framework (G, p, L) in $\mathbb{R}^2$.

Remark 3.8 If δ is an active NBAC-colouring of an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ and δ' is an NBAC-colouring with $\delta'(e) \neq \delta(e)$ for all $e \in E(G)$, then δ' is also an active NBAC-colouring of $\mathcal{C}$; this can be shown in a similar way to the proof of [8, Lem. 1.13].

Lemma 3.9 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ and $e_1, e_2 \in E(G)$, with $e_1 = (v_1, w_1, \gamma_1)$ and $e_2 = (v_2, w_2, \gamma_2)$. Then the map*

$$\begin{aligned} f_{e_1,e_2} \colon \mathcal{V}_{e_1}(G, p, L) &\to \mathcal{V}_{e_2}(G, p, L), \\ (q, M) &\mapsto \big((R_{e_2} \cdot (q(v) - q(v_2)))_{v \in V(G)}, R_{e_2} M\big) \end{aligned}$$

is biregular, where

$$R_{e_2} := \frac{1}{\lambda(e_2)} \begin{bmatrix} x_{w_2} - x_{v_2} & y_{w_2} - y_{v_2} \\ -(y_{w_2} - y_{v_2}) & x_{w_2} - x_{v_2} \end{bmatrix}.$$

Furthermore, for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{e_1}(G, p, L)$ and any $v, w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we have that $\mathcal{C}' := f_{e_1,e_2}(\mathcal{C})$ is an algebraic curve and

$$\begin{aligned} W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2} &= \frac{1}{\lambda(e_2)} W^{\gamma}_{v,w}|_{\mathcal{C}}\, Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}, \\ Z^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2} &= \frac{1}{\lambda(e_2)} Z^{\gamma}_{v,w}|_{\mathcal{C}}\, W^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}. \end{aligned} \tag{4}$$

Proof We note that the transform $z \mapsto R_{e_2} \cdot (z - q(v_2))$ will preserve distance under $\|\cdot\|^2$ in $\mathbb{C}^2$. It follows that $(G, f_{e_1,e_2}(q, M))$ will be an equivalent framework to (G, q, M), except now the edge e_2 (not e_1) has been fixed, with v_2 at the origin and

w_2 on the y-axis, Hence $f_{e_1,e_2}(q, M) \in \mathcal{V}_{e_2}(G, p, L)$ for all $(q, M) \in \mathcal{V}_{e_1}(G, p, L)$, i.e., the map f_{e_1,e_2} is well defined. It is clear that the map f_{e_1,e_2} is regular. To see that f_{e_1,e_2} is biregular, we note that the map f_{e_2,e_1} is the inverse of f_{e_1,e_2}. Since f_{e_1,e_2} is biregular, $\mathcal{C}'$ will be an algebraic curve. Equation (4) now holds by direct computation. □

Proposition 3.10 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$, $e_1, e_2 \in E(G)$ with $e_1 = (v_1, w_1, \gamma_1)$ and $e_2 = (v_2, w_2, \gamma_2)$, and $\mathcal{C} \subset \mathcal{V}_{e_1}(G, p, L)$. If δ is an active NBAC-colouring of $\mathcal{C}$ then there exists an algebraic curve $\mathcal{C}' \subset \mathcal{V}_{e_2}(G, p, L)$ such that δ is an active NBAC-colouring of $\mathcal{C}'$.*

Proof Let $\mathcal{C}' := f_{e_1,e_2}(\mathcal{C})$, where f_{e_1,e_2} is the map defined in Lemma 3.9. Let ν be the valuation of $\mathbb{C}(\mathcal{C})$ and $\alpha \in \mathbb{R}$ be chosen so that they generate δ. Define ν' to be the valuation of $\mathbb{C}(\mathcal{C}')$ where $\nu'(f) := \nu(f \circ f_{e_1,e_2})$ for each $f \in \mathbb{C}(\mathcal{C}')$. By Lemma 3.9,

$$\begin{aligned}\nu'\big(W^{\gamma}_{v,w}|_{\mathcal{C}'}\big) &= \nu\big(W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2}\big) = \nu\bigg(\frac{1}{\lambda(e_2)} W^{\gamma}_{v,w}|_{\mathcal{C}}\, Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\bigg)\\ &= \nu\big(W^{\gamma}_{v,w}|_{\mathcal{C}}\big) + \nu\big(Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\big).\end{aligned}$$

If we define $\alpha' := \alpha + \nu\big(Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\big)$, then ν' and α' will generate δ. □

Lemma 3.11 *Let (G, p, L) and (G', p', L) be gain equivalent frameworks with gain equivalence $\phi\colon \mathcal{V}^d_{\mathbb{K}}(G) \to \mathcal{V}^d_{\mathbb{K}}(G')$. If $\tilde{e} \in E(G)$ and $\tilde{e}' := \phi(\tilde{e})$, then ϕ is a biregular map with $\phi(\mathcal{V}_{\tilde{e}}(G, p, L)) = \mathcal{V}_{\tilde{e}'}(G', p', L)$. Furthermore, for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{e_1}(G, p, L)$ and any $v, w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we have that $\mathcal{C}' := \phi(\mathcal{C})$ is an algebraic curve and*

$$W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi = W^{\gamma}_{v,w}|_{\mathcal{C}}, \qquad Z^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi = Z^{\gamma}_{v,w}|_{\mathcal{C}}. \tag{5}$$

Proof As ϕ is a bijective map that is the restriction of an invertible linear map, it is a biregular map; hence, $\phi(\mathcal{C})$ is an algebraic curve. Equation (5) now follows by direct computation. □

Proposition 3.12 *Let G and G' be gain equivalent $\mathbb{Z}^k$-gain graphs. Then δ is an active NBAC-colouring of G if and only if δ is an active NBAC-colouring of G'.*

Proof Let δ be an active NBAC-colouring of $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ generated by the valuation ν of $\mathbb{C}(\mathcal{C})$ and $\alpha \in \mathbb{R}$. Let ϕ be the gain equivalence from G to G'. We define the gain equivalent framework $(G', p', L) := \phi(G, p, L)$, the algebraic curve $\mathcal{C}' := \phi(\mathcal{C})$ (Lemma 3.11), and the valuation ν' of $\mathbb{C}(\mathcal{C}')$ where $\nu'(f) := \nu(f \circ \phi)$ for each $f \in \mathbb{C}(\mathcal{C}')$. By Lemma 3.11,

$$\nu'\big(W^{\gamma}_{v,w}|_{\mathcal{C}'}\big) = \nu\big(W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi\big) = \nu\big(W^{\gamma}_{v,w}|_{\mathcal{C}}\big),$$

thus ν' and α generate δ for G'. □

3.4 Key Tools

We are now ready to outline the key tools that shall help us throughout the rest of the paper.

Lemma 3.13 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$. Then the following holds:*

(i) *If (G, p, L) is flexible, there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$.*
(ii) *If (G, p, L) is fixed lattice flexible, there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$.*

Proof (i): If (G, p) is flexible then $\mathcal{V}_{\tilde{e}}(G, p, L)$ cannot be finite. As every algebraic set that is not finite contains a variety with positive dimension and every variety with positive dimension contains an algebraic curve, the result holds. (ii): This follows by a similar method. □

Lemma 3.14 *Let (G, p, L) be k-periodic and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve. Suppose G contains a spanning tree T that contains $\tilde{e}$ and has trivial gain for all of its edges. If* rank $G = k$*, then there exists $(v, w, \gamma) \in E(G)$ such that $W^\gamma_{v,w}$ takes an infinite amount of values on $\mathcal{C}$.*

Proof Suppose that for each $(v, w, \gamma) \in E(G)$, the map $W^\gamma_{v,w}$ takes a finite amount of values. By Lemma 2.2, each map $W^\gamma_{v,w}$ is constant. As $W^\gamma_{v,w} Z^\gamma_{v,w}$ is constant, $Z^\gamma_{v,w}$ is also constant. Choose any two vertices $v, w \in V(G)$ with $v \neq w$. Then there exists a unique walk $v_1, \ldots, v_n$ from v to w in T. As

$$W^0_{v,w} = \sum_{j=1}^{n-1} W^0_{v_j,v_{j+1}}, \qquad Z^0_{v,w} = \sum_{j=1}^{n-1} Z^0_{v_j,v_{j+1}},$$

both $W^0_{v,w}$ and $Z^0_{v,w}$ are constant; furthermore, as

$$x_v - x_w = \frac{1}{2}(W^0_{v,w} + Z^0_{v,w}), \qquad y_v - y_w = \frac{i}{2}(Z^0_{v,w} - W^0_{v,w}),$$

then $x_v - x_w$ and $y_v - y_w$ are also constant on $\mathcal{C}$. Since $x_{\tilde{v}}, y_{\tilde{w}}, x_{\tilde{w}}, y_{\tilde{v}}$ are constant on $\mathcal{C}$ and both $\tilde{v}$ and $\tilde{w}$ are contained in T, both x_v, y_v are constant on $\mathcal{C}$ for every $v \in V$ also.

Suppose $k = 1$ and let $e = (v, w, \gamma)$ be any edge with $\gamma \neq 0$. By observing the maps $W^\gamma_{v,w}$ and $Z^\gamma_{v,w}$, we note that x_1 and y_1 are constant on $\mathcal{C}$ (since x_v, x_w, y_v, y_w are all constant on $\mathcal{C}$). It now follows that $\mathcal{C}$ is a single point, contradicting that $\dim \mathcal{C} > 0$.

Now suppose $k = 2$. As rank $G = k$, there exist edges (v, w, γ) and (v', w', γ') such that γ, γ' are independent. By observing the maps $W^\gamma_{v,w}, Z^\gamma_{v,w}, W^{\gamma'}_{v',w'}, Z^{\gamma'}_{v',w'}$, we note that the polynomials

$$\begin{aligned} f &:= \gamma_1 x_1 + \gamma_2 x_2, \qquad & g &:= \gamma_1 y_1 + \gamma_2 y_2, \\ f' &:= \gamma'_1 x_1 + \gamma'_2 x_2, \qquad & g' &:= \gamma'_1 y_1 + \gamma'_2 y_2 \end{aligned}$$

are constant on $\mathcal{C}$. As both x_1 and x_2 can be formed by linear combinations of f, f', both are constant on $\mathcal{C}$; similarly, as both y_1 and y_2 can be formed by linear combinations of g, g' then both y_1 and y_2 are also constant on $\mathcal{C}$. It now follows that $\mathcal{C}$ is a single point, contradicting that $\dim \mathcal{C} > 0$. □

Lemma 3.15 *Let (G, p, L) be a full k-periodic framework in $\mathbb{R}^2$,* rank $G = k$ *for $k \in \{1, 2\}$, and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve. Suppose there exists a $:= (a_1, a_2, \alpha) \in E(G)$ such that $W^{\alpha}_{a_1,a_2}$ takes an infinite amount of values on $\mathcal{C}$. Then there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ such that the colouring $\delta\colon E(G) \to \{red, blue\}$ given by*

$$\delta(e) := \begin{cases} red & \text{if } \nu(W^{\gamma}_{v,w}) > 0, \\ blue & \text{if } \nu(W^{\gamma}_{v,w}) \leq 0, \end{cases}$$

for each $e = (v, w, \gamma)$, is an NBAC-colouring of G; furthermore, $\delta(\tilde{e}) = blue$ and $\delta(a) = red$.

Proof By Lemma 2.2, $W^{\alpha}_{a_1,a_2}$ is transcendental over $\mathbb{C}$, thus, by Proposition 2.4, there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ such that $\nu(W^{\alpha}_{a_1,a_2}) > 0$. As $\tilde{e}$ is fixed and $\lambda(\tilde{e}) \neq 0$, we have $\nu\big(W^{\tilde{\gamma}}_{\tilde{v},\tilde{w}}\big) = 0$. We note that $\nu\big(W^{\gamma}_{v,w} Z^{\gamma}_{v,w}\big) = 0$ for each $(v, w, \gamma) \in E(G)$ since $W^{\gamma}_{v,w} Z^{\gamma}_{v,w}$ is constant, hence $\nu(W^{\gamma}_{v,w}) = -\nu(Z^{\gamma}_{v,w})$.

Let $\delta\colon E(G) \to \{\text{red}, \text{blue}\}$ be as described in the statement of the lemma for the valuation ν. It follows that a is red and $\tilde{e}$ is blue, thus δ is surjective. Suppose there exists a balanced almost red circuit C of length n in G with $\delta(e_n) = \text{blue}$. Then

$$\nu(W^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big) : j = 1, \ldots, n-1\right\} > 0,$$

however this contradicts that $\nu\big(W^{\gamma_n}_{v_1,v_n}\big) \leq 0$. Now suppose instead that C is a balanced almost blue circuit with $\delta(e_n) = \text{red}$. Then

$$\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) = \nu\left(\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : j = 1, \ldots, n-1\right\} \geq 0,$$

however this contradicts that $\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) < 0$. □

Definition 3.16 For any two edges e_1, e_2 of a k-periodic framework (G, p, L) in $\mathbb{R}^2$ with $e_i := (v_i, w_i, \gamma_i)$ for each $i \in \{1, 2\}$, we define the *angle function of* e_1, e_2 to be the map

$$\begin{aligned} A_{e_1,e_2}\colon & \mathcal{V}^2_{\mathcal{C}}(G) \to \mathbb{C}, \\ & (p', L') \mapsto (p'(v_1) - p'(w_1) - L' \cdot \gamma_1) \cdot (p'(v_2) - p'(w_2) - L' \cdot \gamma_2). \end{aligned}$$

Remark 3.17 For any $\tilde{e} \in E(G)$ and any algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$,

$$A_{e_1,e_2}|_{\mathcal{C}} = \frac{1}{2}\big(W^{\gamma_1}_{v_1,w_1} Z^{\gamma_2}_{v_2,w_2} + Z^{\gamma_1}_{v_1,w_1} W^{\gamma_2}_{v_2,w_2}\big).$$

Furthermore, if $(p, L) \sim (p', L')$, then $A_{e_1,e_2}(p, L) = A_{e_1,e_2}(p', L')$; this is since linear isometries of $(\mathbb{C}^2, \|\cdot\|^2)$ will preserve the bilinear form associated to $\|\cdot\|^2$.

Lemma 3.18 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve, and $e_1, e_2 \in E(G)$, with $e_j := (v_j, w_j, \gamma_j)$ for $j \in \{1, 2\}$. If $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings of $\mathcal{C}$, then $A_{e_1,e_2}|_{\mathcal{C}}$ is constant.*

Proof As A_{e_1,e_2} is invariant for congruent placement-lattices, by Proposition 3.10, we may assume $\tilde{e} = e_1$. We note the map

$$(p', L') \mapsto p'(v_1) - p'(w_1) - L' \cdot \gamma_1 \tag{6}$$

is constant on $\mathcal{C}$, and $W^{\gamma_1}_{v_1,w_1}$ is constant also. Suppose $A_{e_1,e_2}|_{\mathcal{C}}$ is not constant, then as (6) is constant,

$$(p', L') \mapsto p'(v_2) - p'(w_2) - L' \cdot \gamma_2$$

is not constant on $\mathcal{C}$. This in turn implies that $W^{\gamma_2}_{v_2,w_2}$ takes an infinite amount of values over $\mathcal{C}$. By Lemma 3.15, there exists an active NBAC-colouring δ of $\mathcal{C}$ with $\delta(e_1) \neq \delta(e_2)$. □

Lemma 3.19 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$ and $\tilde{e}, e_1, e_2 \in E(G)$. If A_{e_1,e_2} takes an infinite amount of values on $\mathcal{V}_{\tilde{e}}(G, p, L)$ then there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ such that $A_{e_1,e_2}|_{\mathcal{C}}$ is not constant.*

Proof As A_{e_1,e_2} takes an infinite amount of values on $\mathcal{V}_{\tilde{e}}(G, p, L)$, there exists a variety $\mathcal{V} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ and points $(p', L'), (p'', L'') \in \mathcal{V}_{\tilde{e}}(G, p, L)$ such that $A_{e_1,e_2}(p', L') \neq A_{e_1,e_2}(p'', L'')$. By [15, Lem., p. 56], there exists an algebraic curve $\mathcal{C}$ that contains (p', L') and (p'', L''). □

Proposition 3.20 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$ and $\tilde{e}, e_1, e_2 \in E(G)$. Then $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings δ of (G, p, L) if and only if A_{e_1,e_2} takes only finitely many values on $\mathcal{V}_{\tilde{e}}(G, p, L)$.*

Proof Suppose $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings δ of (G, p, L). By Lemma 3.18, $A_{e_1,e_2}|_{\mathcal{C}}$ is constant for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$. By Lemma 3.19, it follows that A_{e_1,e_2} takes only a finite amount of values on $\mathcal{V}_{\tilde{e}}(G, p, L)$.

Suppose there exist an algebraic curve $\mathcal{C}$ and active NBAC-colouring δ of $\mathcal{C}$ generated by ν, α, such that $\delta(e_1) \neq \delta(e_2)$. Let $e_j = (v_j, w_j, \gamma_j)$ for $j \in \{1, 2\}$. Without loss of generality we may assume $\nu\big(W^{\gamma_1}_{v_1,w_1}\big) \leq \alpha < \nu\big(W^{\gamma_2}_{v_2,w_2}\big)$. We now note

$$\nu(A_{e_1,e_2}|_{\mathcal{C}}) = \nu\big(W^{\gamma_1}_{v_1,w_1} Z^{\gamma_2}_{v_2,w_2} + Z^{\gamma_1}_{v_1,w_1} W^{\gamma_2}_{v_2,w_2}\big) = \nu\big(W^{\gamma_1}_{v_1,w_1}\big) - \nu\big(W^{\gamma_2}_{v_2,w_2}\big) < 0,$$

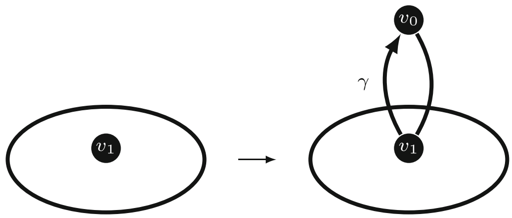

Fig. 9 A vertex addition of (G, p, L) at v_1 by γ

thus $A_{e_1,e_2}|_{\mathcal{C}}$ must be transcendental over $\mathbb{C}$ when considered as an element of $\mathcal{C}(\mathbb{C})$. It follows from Proposition 2.4 that A_{e_1,e_2} takes an infinite amount of values on $\mathcal{V}_{\tilde{e}}(G, p, L)$ as required. □

We shall end this section by defining a graph operation we shall use later in Lemmas 5.4 and 6.4.

Definition 3.21 Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ and $\gamma \in \mathbb{Z}^k$ be a non-zero element. We define a k-periodic framework (G', p', L) in $\mathbb{R}^2$ to be a *vertex addition of* (G, p, L) *at* v_1 *by* γ if

$$V(G') := V(G) \cup \{v_0\}, \qquad E(G') := E(G) \cup \{(v_0, v_1, 0), (v_0, v_1, \gamma)\}$$

and $p'(v) = p(v)$ for all $v \in V(G)$; see Fig. 9.

Remark 3.22 The graph operation that takes G to G' in the vertex addition described above is the first of the two *gain-preserving Henneberg moves*; we refer the reader to [16] for more information.

Lemma 3.23 *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$ with non-trivial flex (p_t, L_t), $t \in [0, 1]$. Assume that $\|L_t \cdot \gamma\| \neq 0$ for all $t \in [0, 1]$. Then there exists a vertex addition (G', p', L) of (G, p, L) at v_1 by γ with non-trivial flex (p'_t, L_t) such that p'_t restricted to $V(G)$ is the placement p_t for each $t \in [0, 1]$.*

Proof As $[0, 1]$ is compact, we may choose $r > 0$ such that $r > \|L_t \cdot \gamma\|/2$ for all $t \in [0, 1]$. By our choice of r, there exist for each $t \in [0, 1]$ exactly two points that satisfy the equation

$$\|z - p_t(v_1)\|^2 = \|z - p_t(v_1) + L \cdot \gamma\|^2 = r^2. \tag{7}$$

As (p_t, L_t) is continuous, it follows that there exists a continuous path $z_t : [0, 1] \to \mathbb{R}^2$ that satisfies (7). We now set $p'_t(v) := p_t$ for all $v \in V(G)$ and $p'_{v_0} := z_t(v_0)$. □

4 Characterising Fixed Lattice Flexible Frameworks

In this section we shall prove the following result.

Theorem 4.1 *Let G be a connected $\mathbb{Z}^k$-gain graph for $k \in \{1, 2\}$. Then there exists a placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a fixed lattice flexible full k-periodic framework if and only if either*

(i) *G has a fixed lattice NBAC-colouring, or*
(ii) *G is balanced.*

We shall first need to prove four results: Lemma 4.3 for $k = 1$, Lemma 4.6 for $k = 2$, and Lemmas 4.7 and 4.8 for any $k \in \{1, 2\}$. The latter two will also explicitly show how to construct a fixed lattice flexible framework when either G has a fixed lattice NBAC-colouring or is balanced.

4.1 Necessary Conditions for Fixed Lattice Flexibility

Lemma 4.2 *Let (G, p, L) be a full 1-periodic framework in $\mathbb{R}^2$ where G is connected and unbalanced, and let $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ be an algebraic curve. Then every active NBAC-colouring of $\mathcal{C}$ is a fixed lattice NBAC-colouring.*

Proof Let δ be an active NBAC-colouring of $\mathcal{C}$ generated by the valuation ν and $\alpha \in \mathbb{R}$. As $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$, we have $W_1 Z_1 = \|L \cdot 1\|^2$. Since $W_1 Z_1$ is constant, then $\nu(W_1) = -\nu(Z_1)$. We shall assume $\nu(W_1) > \alpha$ as the proof for the case $\nu(W_1) \leq \alpha$ follows by a similar method.

Suppose there exists an almost red circuit C of length n in G with $\delta(e_n) = \text{blue}$. As δ is an NBAC-colouring, we must have that $\gamma := \psi(C) \neq 0$. It then follows that

$$\begin{aligned}\nu(W^{\gamma_n}_{v_1,v_n}) &= \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + \gamma W_1\right)\\ &\geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big), \nu(W_1) : j = 1, \ldots, n-1\right\} > \alpha,\end{aligned}$$

however this contradicts that $\nu\big(W^{\gamma_n}_{v_1,v_n}\big) \leq \alpha$. Now suppose there exists an unbalanced blue circuit C of length n in G with $\gamma := \psi(C)$. We note

$$\nu(-\gamma Z_1) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : j = 1, \ldots, n\right\} \geq \alpha,$$

contradicting that $\nu(Z_1) < \alpha$. □

We are now ready to prove our first necessity lemma.

Lemma 4.3 *Let (G, p, L) be a full 1-periodic framework in $\mathbb{R}^2$. If (G, p, L) is fixed lattice flexible then either G has an active fixed lattice NBAC-colouring, G is balanced, or G is disconnected.*

Proof Suppose G is unbalanced and connected. It follows from Proposition 2.7 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$, since by Proposition 3.12, if an equivalent graph to G has an active NBAC-colouring then so does G. By Lemma 3.13 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$. By Lemma 3.14, there exists $a := (a_1, a_2, \alpha) \in E(G)$ such that $W^{\alpha}_{a_1,a_2}$ is not constant on $\mathcal{C}$. By Lemma 3.15, there exists an active NBAC-colouring δ of $\mathcal{C}$, thus by Lemma 4.2, δ is a fixed lattice NBAC-colouring as required. □

Lemma 4.4 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$, $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ be an algebraic curve, and suppose the function field $\mathbb{C}(\mathcal{C})$ has valuation ν. Then the following holds:*

(i) $\nu(W_1) = -\nu(Z_1)$, $\nu(W_2) = -\nu(Z_2)$, *and* $\nu(W_1 \cdot Z_2 + W_2 \cdot Z_1) = 0$.
(ii) $\nu(W_1) = \nu(W_2)$ *and* $\nu(Z_1) = \nu(Z_2)$.
(iii) *For all* $\gamma \in \mathbb{Z}^2$, $\nu(\gamma Z) = -\nu(\gamma W)$.
(iv) *For any* $\gamma \in \mathbb{Z}^2$ *and* $\alpha \in \mathbb{R}$, *if* $\nu(W_1) > \alpha$, *then* $\nu(\gamma W) > \alpha$, *and if* $\nu(W_1) \leq \alpha$, *then* $\nu(\gamma W) \leq \alpha$.

Proof (i): As $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ then

$$W_1 Z_1 = \lambda(1,1)^2, \quad W_2 Z_2 = \lambda(2,2)^2, \quad W_1 \cdot Z_2 + W_2 \cdot Z_1 = 2\lambda(1,2)^2,$$

thus all are non-zero and constant. Since $\nu(f) = 0$ for all non-zero and constant $f \in \mathbb{C}(\mathcal{C})$, the result follows.

(ii): We see that

$$\nu(W_1 \cdot Z_2 + W_2 \cdot Z_1) \geq \min\{\nu(W_1) - \nu(W_2), \nu(W_2) - \nu(W_1)\}$$

with equality if $\nu(W_1) \neq \nu(W_2)$. If $\nu(W_1) \neq \nu(W_2)$, then $\nu(W_1 \cdot Z_2 + W_2 \cdot Z_1) < 0$, contradicting that $\nu(W_1 \cdot Z_2 + W_2 \cdot Z_1) = 0$, thus $\nu(W_1) = \nu(W_2)$ (and similarly $\nu(Z_1) = \nu(Z_2)$).

(iii): Let $\gamma := (\gamma_1, \gamma_2)$ and define

$$\begin{aligned} g := (\gamma_1 W_1 + \gamma_2 W_2)(\gamma_1 Z_1 + \gamma_2 Z_2) &= \gamma_1^2 W_1 Z_1 + \gamma_2^2 W_2 Z_2 + \gamma_1\gamma_2(W_1 Z_2 + W_2 Z_1) \\ &= (\gamma_1 x_1 + \gamma_2 x_2)^2 + (\gamma_1 y_1 + \gamma_2 y_2)^2. \end{aligned}$$

As $W_1 Z_1$, $W_2 Z_2$, and $W_1 Z_2 + W_2 Z_1$ are all constant (since $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$), then g is constant. We further note that if $g = 0$ then the vectors (x_1, y_1) and (x_2, y_2) are linearly dependent for all points in $\mathcal{C}$. As this would contradict that (G, p, L) is full, we have $\nu(g) = 0$. The required equality will now follow.

(iv): Let $\gamma := (\gamma_1, \gamma_2)$. By (i) and (ii), $\nu(W_1) = \nu(W_2)$. If $\nu(W_1) > \alpha$, then

$$\nu(\gamma_1 W_1 + \gamma_2 W_2) \geq \min\{\nu(W_1), \nu(W_2)\} > \alpha,$$

while if $\nu(W_1) \leq \alpha$, then by (iii),

$$\begin{aligned} \nu(\gamma_1 W_1 + \gamma_2 W_2) &= -\nu(\gamma_1 Z_1 + \gamma_2 Z_2) \leq -\min\{\nu(Z_1), \nu(Z_2)\} \\ &= \max\{\nu(W_1), \nu(W_2)\} \leq \alpha. \end{aligned}$$

□

Lemma 4.5 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$ where G is connected graph with* rank $G = 2$*, and let $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ be an algebraic curve. Then every active NBAC-colouring of $\mathcal{C}$ is a fixed lattice NBAC-colouring.*

Proof Let δ be an active NBAC-colouring of $\mathcal{C}$ with corresponding valuation ν and non-zero $\alpha \in \mathbb{R}$. By Lemma 4.4, (i) and (ii), $\nu(W_1) = \nu(W_2)$, $\nu(Z_1) = -\nu(W_1)$, and $\nu(Z_2) = -\nu(W_2)$. We shall assume $\nu(W_1) > \alpha$ as the proof for the case $\nu(W_1) \leq \alpha$ follows by a similar method.

Suppose there exists an almost red circuit C of length n in G with $\gamma := \psi(C)$ and $\delta(e_n) =$ blue. Then

$$W^{\gamma_n}_{v_1,v_n} = \sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + \gamma W.$$

By Lemma 4.4 (iv),

$$\nu(W^{\gamma_n}_{v_1,v_n}) \geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big), \gamma W : j = 1, \dots, n-1\right\} > \alpha,$$

however this contradicts that $\nu\big(W^{\gamma_n}_{v_1,v_n}\big) \leq \alpha$. Now suppose there exists an unbalanced blue circuit C of length n in G with $\gamma := \psi(C)$. We note

$$\nu(-\gamma Z) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : j = 1, \dots, n\right\} \geq \alpha.$$

However, by Lemma 4.4, (iii) and (iv), we have $\nu(-\gamma Z) < \alpha$, a contradiction. □

We are now ready to prove our final necessity lemma.

Lemma 4.6 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$. If (G, p, L) is fixed lattice flexible then either G has an active fixed lattice NBAC-colouring, G is balanced, or G is disconnected.*

Proof Suppose rank $G = 1$ and G is connected. We note that any 2-periodic framework with rank 1 is fixed lattice flexible if and only if it is fixed lattice flexible when considered as a 1-periodic framework. By Lemma 4.3, G has an active fixed lattice NBAC-colouring.

Suppose rank $G = 2$ and G is connected. It follows from Propositions 2.7 and 3.12 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. By Lemma 3.13 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$. By Lemma 3.14, there exists $a := (a_1, a_2, \alpha) \in E(G)$ such that $W^{\alpha}_{a_1,a_2}$ is not constant on $\mathcal{C}$. By Lemma 3.15, there exists an active NBAC-colouring δ of $\mathcal{C}$, and by Lemma 4.5, δ is a fixed lattice NBAC-colouring as required. □

4.2 Constructing Fixed Lattice Flexible Frameworks

Lemma 4.7 *Let G be a connected $\mathbb{Z}^k$-gain graph for $k \in \{1, 2\}$. If G has a fixed lattice NBAC-colouring δ, then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is fixed lattice flexible.*

Proof The proof for $k = 1$ is identical to that for $k = 2$ except we have $L := [c\ 0]^T$ for some irrational $c > 0$. Due to this, we shall only prove the case for $k = 2$.

We may assume without loss of generality that G^δ_{red} is balanced; furthermore, by Proposition 2.7, we may assume all edges of G^δ_{red} have trivial gain. Let $R_1, \ldots, R_n$ be the red connected components and $B_1, \ldots, B_m$ be the blue connected components. As δ is an NBAC-colouring, there exists a blue edge $\tilde{e} \in E(G)$; by reordering the blue components we may assume the end points of $\tilde{e}$ lie in B_1.

Choose any two points $c_1, c_2 > 0$ so that $Ac_1 + Bc_2 \notin \mathbb{Z}$ for all $A, B \in \mathbb{Z} \setminus \{0\}$; it is sufficient that the set $\{c_1, c_2\}$ is algebraically independent over $\mathbb{Q}$. We define the placement-lattice (p, L) of G with

$$p(v) := (x, y), \qquad L := \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

for $v \in V(R_x) \cap V(B_y)$. We shall now prove (G, p, L) is a well-defined k-periodic framework.

Suppose there exists a red edge $e := (v, w, \gamma) \in E(G)$ such that $p(v) = p(w) + L \cdot \gamma$. As e is red then $\gamma = (0, 0)$, thus $p(v) = p(w)$. It follows that for some $1 \leq x \leq n$ and $1 \leq y \leq m$, we have $v, w \in V(R_x) \cap V(B_y)$, thus there exists a blue path $(e_1, \ldots, e_n)$ that starts at w and ends at v. We note, however, that $(e_1, \ldots, e_n, e)$ is an almost blue circuit, contradicting that δ is a fixed-lattice NBAC-colouring.

Now suppose there exists a blue edge $e := (v, w, \gamma) \in E(G)$ with $\gamma = (\gamma_1, \gamma_2)$ such that $p(v) = p(w) + L \cdot \gamma$, then $p(v) = p(w) + (\gamma_1 c_1, \gamma_2 c_2)$. By our choice of c_1, c_2 we must have $\gamma_1 = \gamma_2 = 0$, thus $p(v) = p(w)$. This implies that for some $1 \leq x \leq n$ and $1 \leq y \leq m$, we have $v, w \in V(R_x) \cap V(B_y)$, and there exists a red path $(e_1, \ldots, e_n)$ that starts at w and ends at v. We note, however, that $(e_1, \ldots, e_n, e)$ is a balanced almost red circuit (since all red edges have trivial gain), contradicting that δ is an NBAC-colouring. It now follows that (G, p, L) is a full k-periodic framework.

Define the motion (p_t, L_t), $t \in [0, 1]$, where for $p(v) = (x, y)$,

$$p_t(v) := (x + y \sin t, y \cos t),$$

and $L_t = L$. Choose any $t \in [0, 1]$ and $e = (v, w, \gamma) \in E(G)$, with $\gamma = (\gamma_1, \gamma_2)$, $p(v) = (x, y)$ and $p(w) = (x', y')$. Suppose $\delta(e) = \text{red}$. Then $x' = x$ and $\gamma = (0, 0)$ (as all red edges have trivial gain), and it follows that

$$\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 = ((y - y') \sin t)^2 + ((y - y') \cos t)^2 = (y - y')^2.$$

Now suppose $\delta(e) = \text{blue}$. Then $y' = y$ and we note that

$$\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 = (x - x' + \gamma_1 c_1)^2 + (\gamma_2 c_2)^2.$$

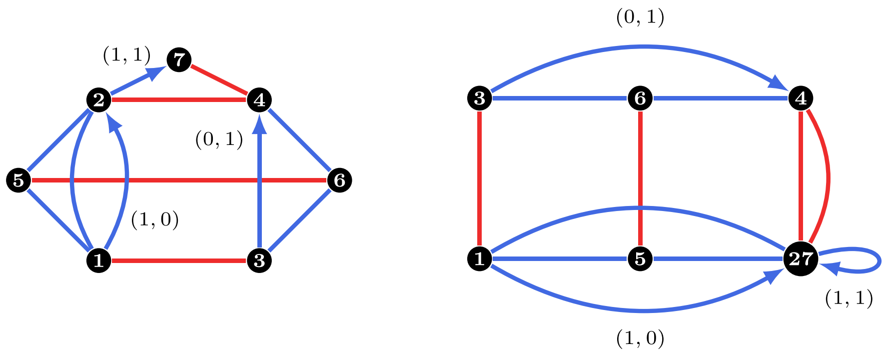

Fig. 10 (Left): A $\mathbb{Z}^2$-gain graph G with a fixed lattice NBAC-colouring. (Right): The constructed full 2-periodic framework (G, p, L) in $\mathbb{R}^2$. We note that even though we place (2) and (7) at the same point in $\mathbb{R}^2$, $p(2) \neq p(7) + L \cdot (1, 1)$

It follows that $(G, p_t, L_t) \sim (G, p, L)$ for all $t \in [0, 1]$, thus (p_t, L_t) is a fixed lattice flex of (G, p, L). As the edge $\tilde{e}$ is fixed then (p_t, L_t) is non-trivial, thus (G, p, L) is fixed lattice flexible as required. We refer the reader to Fig. 10 for an example of the construction described. □

Lemma 4.8 *Let G be a* $\mathbb{Z}^k$*-gain graph for* $k \in \{1, 2\}$*. If G is balanced, then there exists a full placement-lattice* (p, L) *of G in* $\mathbb{R}^2$ *such that* (G, p, L) *is fixed lattice flexible.*

Proof By Proposition 2.7, we may assume every edge of G has trivial gain. Choose any injective map p and any full lattice L. We may now define the fixed lattice flex (p_t, L_t) for $t \in [0, 1]$, where $p_t = p$ and

$$L_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} L.$$ □

We may now combine the results of this section to prove Theorem 4.1

Proof of Theorem 4.1 If (G, p, L) is a fixed lattice flexible full k-periodic framework, then by Lemma 4.3 if $k = 1$ or Lemma 4.6 if $k = 2$, either G has a fixed lattice NBAC-colouring or G is balanced.

If G has a fixed lattice NBAC-colouring, then by Lemma 4.7, there exists a fixed lattice flexible full k-periodic framework (G, p, L) in $\mathbb{R}^2$. If G is balanced, then by Lemma 4.8, there exists a fixed lattice flexible full k-periodic framework (G, p, L) in $\mathbb{R}^2$. □

5 Characterising Flexible 1-Periodic Frameworks

In this section we shall prove the following theorem.

Theorem 5.1 *Let G be a connected* $\mathbb{Z}$*-gain graph. Then there exists a full placement-lattice* (p, L) *of G in* $\mathbb{R}^2$ *such that* (G, p, L) *is a flexible full* 1*-periodic framework if and only if either:*

(i) *G has a fixed lattice NBAC-colouring,*
(ii) *G has a flexible* 1*-lattice NBAC-colouring, or*
(iii) *G is balanced.*

Fortunately, much of the required work has been dealt with in Sect. 4, since fixed lattice flexible 1-periodic frameworks are a subclass of flexible 1-periodic frameworks. Due to this, we only need to prove two results: a necessity lemma that proves a flexible 1-periodic framework will have one of the required properties (see Lemma 5.2), and a construction lemma to prove that we can construct a flexible 1-periodic framework given a graph with a flexible 1-lattice NBAC-colouring (see Lemma 5.7).

5.1 Necessary Conditions for 1-Periodic Flexibility

Lemma 5.2 *Let* (G, p, L) *be a* 1*-periodic framework in* $\mathbb{R}^2$ *with edge* $(v, w, \gamma) \in E(G)$ *for some* $\gamma \neq 0$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ *be an algebraic curve, and* ν *a valuation of* $\mathbb{C}(\mathcal{C})$*. Suppose* $x_v - x_w$ *and* $y_v - y_w$ *are constant on* $\mathcal{C}$*. Then* $W^{\gamma}_{v,w}$ *is constant if and only if* $\mathcal{C} \subset \mathcal{V}^{f}_{\tilde{e}}(G, p, L)$*.*

Proof We note that $W^{\gamma}_{v,w}$ is constant if and only if $Z^{\gamma}_{v,w}$ is also constant as $W^{\gamma}_{v,w} Z^{\gamma}_{v,w}$ is constant. As $x_v - x_w$ and $y_v - y_w$ are constant then $W^{\gamma}_{v,w}$ and $Z^{\gamma}_{v,w}$ are constant if and only if both $x_1 + iy_1$ and $x_1 - iy_1$ are constant, which in turn is equivalent to both x_1, y_1 being constant. The result now follows. □

Lemma 5.3 *Let* (G, p, L) *be a full* 1*-periodic framework in* $\mathbb{R}^2$*. Suppose that* (G, p, L) *is flexible, G is connected and unbalanced, and G contains a pair of parallel edges* $\tilde{e}, \tilde{f}$*. Then G either has an active fixed lattice NBAC-colouring where* $\tilde{e}, \tilde{f}$ *are of the same colour, or G has an active flexible* 1*-lattice NBAC-colouring where* $\tilde{e}, \tilde{f}$ *are of opposite colours.*

Proof We may assume $\tilde{e}$ and $\tilde{f}$ are the pair of parallel edges on $\tilde{v}, \tilde{w}$, with $\psi(\tilde{f}) = \mu \neq 0$. It follows from Propositions 2.7 and 3.12 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. By Lemma 3.13 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$.

Suppose $\mathcal{C} \subset \mathcal{V}^{f}_{\tilde{e}}(G, p, L)$. By Lemma 4.3, G has an active fixed lattice NBAC-colouring δ. By Lemma 5.2, we note that we must have $\delta(\tilde{e}) = \delta(\tilde{f})$. Now suppose $\mathcal{C} \not\subset \mathcal{V}^{f}_{\tilde{e}}(G, p, L)$. By Lemma 5.2, $W^{\mu}_{\tilde{v},\tilde{w}}$ is not constant on $\mathbb{C}(\mathcal{C})$. Let ν be the valuation of $\mathbb{C}(\mathcal{C})$ and δ the NBAC-colouring given by Lemma 3.15 with $a := \tilde{f}$. By our choice of valuation, $\nu(W^{0}_{\tilde{v},\tilde{w}}) = 0$ and $\nu(W^{\mu}_{\tilde{v},\tilde{w}}) > 0$; it follows immediately that $\nu(Z^{0}_{\tilde{v},\tilde{w}}) = 0$ and $\nu(Z^{\mu}_{\tilde{v},\tilde{w}}) < 0$ as both $W^{0}_{\tilde{v},\tilde{w}} Z^{0}_{\tilde{v},\tilde{w}}$ and $W^{\mu}_{\tilde{v},\tilde{w}} Z^{\mu}_{\tilde{v},\tilde{w}}$ are constant. As $\mu W_1 = W^{0}_{\tilde{v},\tilde{w}} - W^{\mu}_{\tilde{v},\tilde{w}}$ then $\nu(W_1) = \nu(W^{0}_{\tilde{v},\tilde{w}}) = 0$. Similarly, as $\mu Z_1 = Z^{0}_{\tilde{v},\tilde{w}} - Z^{\mu}_{\tilde{v},\tilde{w}}$ then $\nu(Z_1) = \nu(Z^{\mu}_{\tilde{v},\tilde{w}}) < 0$.

Suppose G has an unbalanced monochromatic circuit C of length n. If C is red, then

$$\nu(W_1) = \nu(-\psi(C) W_1) = \nu\left(\sum_{j=1}^{n} W^{\gamma_j}_{v_j, v_{j+1}}\right) \geq \min\left\{\nu\left(W^{\gamma_j}_{v_j, v_{j+1}}\right) : 1 \leq j \leq n\right\} > 0,$$

contradicting that $\nu(W_1) = 0$. If C is blue, then

$$\nu(Z_1) = \nu(-\psi(C)Z_1) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} \geq 0,$$

contradicting that $\nu(Z_1) < 0$. It now follows that δ is an active flexible 1-lattice NBAC-colouring. □

We are now ready to state our necessity lemma.

Lemma 5.4 *Let (G, p, L) be a full 1-periodic framework in $\mathbb{R}^2$. If (G, p, L) is flexible then G either has an active fixed lattice NBAC-colouring, an active flexible 1-lattice NBAC-colouring, G is balanced, or G is disconnected.*

Proof We may suppose G is connected and unbalanced. If G contains a pair of parallel edges then the result holds by Lemma 5.3, thus we shall also assume that G does not contain a pair of parallel edges.

By Lemma 3.23, there exists a vertex addition (G', p', L) of (G, p, L) at v_1 by 1 such that (G', p', L) has a non-trivial not fixed lattice flex; we shall define these new edges by $\tilde{e}, \tilde{f}$, with $\psi(\tilde{e}) = 0$ and $\psi(\tilde{f}) = 1$. As G' contains a pair of parallel edges then by Lemma 5.3, either G' has an active flexible 1-lattice NBAC-colouring δ' with $\delta'(\tilde{e}) =$ blue and $\delta'(\tilde{f}) =$ red, or G' has an active fixed lattice NBAC-colouring δ'' with $\delta''(\tilde{e}) = \delta''(\tilde{f}) =$ blue.

Suppose G' has a colouring δ' as described above. Let δ be the colouring of G with $\delta(e) := \delta'(e)$ for all $e \in E(G)$. We note that δ is a flexible 1-lattice NBAC-colouring if and only if δ' is not monochromatic on the subgraph G of G'. As G is unbalanced, δ' cannot be monochromatic on G, thus δ is a flexible 1-lattice NBAC-colouring of G.

Now suppose G' has a colouring δ'' as described above. Let δ be the colouring of G with $\delta(e) := \delta''(e)$ for all $e \in E(G)$. We note that δ is a fixed lattice NBAC-colouring if and only if δ' is not monochromatic on the subgraph G of G'. If δ' is monochromatic on G, then as $\delta'(\tilde{e}) = \delta'(\tilde{f}) =$ blue and G is unbalanced, we must have $\delta(G) =$ blue, however this would contradict that $\delta'(G') = \{$red, blue$\}$. It now follows that δ is a fixed lattice NBAC-colouring of G. □

5.2 Constructing Flexible Frameworks from Flexible 1-Lattice NBAC-Colourings

Lemma 5.5 *Let G be a $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring. Then there exists $G' \approx G$ such that each blue edge has trivial gain and no red edge has trivial gain.*

Proof As G^{δ}_{blue} is balanced, by Proposition 2.7, we may suppose all blue edges of G have trivial gain. Let $B_1, \ldots, B_n$ be the blue components of G and choose $\mu \in \mathbb{N}$ such that $\mu > |\gamma|$ for all $(v, w, \gamma) \in E(G)$. We now define

$$G' := \left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{i\mu}\right)(G).$$

We first note that any blue edge of G' will have trivial gain since both of its ends will lie in the same blue component. Choose a red edge $(v, w, \gamma) \in E(G)$ and suppose $v \in B_i$ and $w \in B_j$. We note that

$$\left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{i\mu}\right)(v, w, \gamma) = \phi_v^{i\mu} \circ \phi_w^{j\mu}(v, w, \gamma) = (v, w, \gamma + (i-j)\mu).$$

As $\mu > |\gamma|$ and $i - j \in \mathbb{Z}$, then $\gamma + (i-j)\mu = 0$ if and only if $\gamma = 0$ and $i = j$. If $v, w \in B_i$ and $\gamma = 0$ then there would exist a balanced almost blue circuit as v, w are connected by a blue path and all blue edges of G have trivial gain, thus $\gamma + (i-j)\mu \neq 0$ as required. □

Lemma 5.6 *Let H be a balanced $\mathbb{Z}$-gain graph. Then there exists a placement q of H in $\mathbb{Z}$ such that for all $(v, w, \gamma) \in E(H)$, $q(w) - q(v) = 2\gamma$.*

Proof We may suppose without loss of generality that H is connected. Choose a spanning tree T of H. It is immediate that we may choose a placement q of T that satisfies the condition $q(w) - q(v) = 2\gamma$ for all $(v, w, \gamma) \in E(T)$. Choose an edge $e = (a, b, \mu) \in E(H) \setminus E(T)$, then there exists a path $(e_1, \ldots, e_{n-1})$ in T with $e_i = (v_i, v_{i+1}, \gamma_i)$, $v_1 = b$ and $v_n = a$. As H is balanced, $\psi(e_1, \ldots, e_{n-1}) = -\mu$, thus by our choice of q,

$$q(b) - q(a) = -\left(\sum_{i=1}^{n-1} q(v_{i+1}) - q(v_i)\right) = -2\psi(e_1, \ldots, e_{n-1}) = 2\mu.$$ □

We our now ready to prove our construction lemma.

Lemma 5.7 *Let G be a $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 1-periodic framework.* □

Proof By Lemma 5.5, we may assume all blue edges of G have trivial gain and all red edges have non-trivial gain. Let $R_1, \ldots, R_n$ be the red components of G and define E_j to be the set of edges (v, w, γ) in G^{δ}_{red} with $v, w \in R_j$. By Lemma 5.6, for each R_j there exists a placement q_j in $\mathbb{R}$ where $q_j(w) - q_j(v) = 2\gamma$ for all $(v, w, \gamma) \in E_j$. We now define for each $t \in [0, 2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$, with

$$p_t(v) := (q_j(v), j), \qquad L_t \cdot 1 := (-2 + \cos t, \sin t)$$

for $v \in R_j$ and $t \in [0, 2\pi]$. We shall denote $(p, L) := (p_0, L_0)$.

To see that (p, L) is a well-defined placement-lattice, choose any $e = (v, w, \gamma)$ and suppose that $p(v) = p(w) + L \cdot \gamma$. It follows that $v, w \in R_j$ and $q_j(v) - q_j(w) = \gamma$. If $\delta(e) = \text{red}$ then $\gamma \neq 0$, however this contradicts that $q_j(v) - q_j(w) = -2\gamma$. Suppose $\delta(e) = \text{blue}$. Since every blue edge has trivial gain, $\gamma = 0$. As $v, w \in R_j$, there exists a red path $(e_1, \ldots, e_{n-1})$ with $e_j = (v_j, v_{j+1}, \gamma_j) \in E_j$, $v_1 = w$ and $v_n = v$. Since

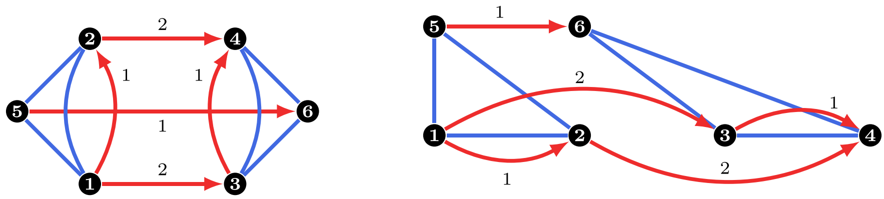

Fig. 11 (Left): A $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring. (Right): The constructed full 1-periodic framework in $\mathbb{R}^2$

$q_j(v) = q_j(w)$, we have $\sum_{j=1}^{n-1} \gamma_j = 0$. However, this implies $(e_1, \ldots, e_{n-1}, e)$ is a balanced almost red circuit, contradicting that δ is an NBAC-colouring.

Choose any $e = (v, w, \gamma)$. If $\delta(e) =$ blue then $\gamma = 0$. As $p_t = p$ then for each $t \in [0, 2\pi]$,

$$\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 = \|p(v) - p(v)\|^2.$$

If $\delta(e) =$ red then $v, w \in R_j$, thus for each $t \in [0, 2\pi]$,

$$\begin{aligned}\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 &= (-(q_j(w) - q_j(v)) + 2\gamma - \gamma \cos t)^2 + (\gamma \sin t)^2 \\ &= \gamma^2.\end{aligned}$$

It follows that (p_t, L_t) is a flex of (G, p, L) as required. We refer the reader to Fig. 11 for an example of the construction. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.1 Suppose (G, p, L) is flexible. By Lemma 5.4, either G is balanced, G has a fixed lattice NBAC-colouring, or G has a flexible 1-periodic NBAC-colouring. If G is balanced, then by Lemma 4.8, G has a flexible full placement-lattice in $\mathbb{R}^2$. If G has a fixed lattice NBAC-colouring, then by Lemma 4.7, G has a flexible full placement-lattice in $\mathbb{R}^2$. If G has a flexible 1-lattice NBAC-colouring, then by Lemma 5.7, G has a flexible full placement-lattice in $\mathbb{R}^2$. □

6 Characterising Flexible 2-Periodic Frameworks

Unlike with 1-periodic frameworks, a full characterisation of $\mathbb{Z}^2$-gain graphs with flexible 2-periodic full placements in the plane via NBAC-colourings is unknown. We would conjecture the following.

Conjecture 1 Let G be a connected $\mathbb{Z}^2$-gain graph. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:

(i) G has a type 1 flexible 2-lattice NBAC-colouring,
(ii) G has a type 2 flexible 2-lattice NBAC-colouring,

(iii) G has a type 3 flexible 2-lattice NBAC-colouring,
(iv) G has a fixed lattice NBAC-colouring, or
(v) $\operatorname{rank} G < 2$.

We are able to obtain the required necessity lemma and most of the required construction lemmas, however a construction of a flexible full 2-periodic framework from a type 3 flexible 2-lattice NBAC-colouring is still currently unknown. In this section we shall, however, outline some partial results regarding $\mathbb{Z}^2$-gain graphs, in particular, Lemmas 6.4, 6.5, 6.8, and 6.11. We shall discuss some other possible conjectures at the end of the section, and later in Sect. 7 we shall obtain analogues of Theorem 5.1 for certain types of graphs; see Theorems 7.5 and 7.8.

6.1 Necessary Conditions for 2-Periodic Flexibility

For any $\gamma = (a, b) \in \mathbb{Z}^2$, we recall the notation $\gamma W := aW_1 + bW_2$ and $\gamma Z := aZ_1 + bZ_2$.

Lemma 6.1 *Let (G, p, L) be a 2-periodic framework in $\mathbb{R}^2$ with edge $(v, w, \gamma) \in E(G)$ for some $\gamma = (\gamma_1, \gamma_2) \neq (0, 0)$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve, and ν a valuation of $\mathbb{C}(\mathcal{C})$. Suppose $x_v - x_w$ and $y_v - y_w$ are constant on $\mathcal{C}$. If $W^{\gamma}_{v,w}$ is constant then*

$$(\gamma_1 x_1 + \gamma_2 x_2)^2 + (\gamma_1 y_1 + \gamma_2 y_2)^2$$

is constant.

Proof We note that $W^{\gamma}_{v,w}$ is constant if and only if $Z^{\gamma}_{v,w}$ is also constant as $W^{\gamma}_{v,w} Z^{\gamma}_{v,w}$ is constant. As $x_v - x_w$ and $y_v - y_w$ are constant, both $(\gamma_1 x_1 + \gamma_2 x_2) + i(\gamma_1 y_1 + \gamma_2 y_2)$ and $(\gamma_1 x_1 + \gamma_2 x_2) - i(\gamma_1 y_1 + \gamma_2 y_2)$ are constant. The result now follows from the observation that $(a + ib)(a - ib) = a^2 + b^2$. □

Lemma 6.2 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$ and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve. Suppose the function field $\mathbb{C}(\mathcal{C})$ has valuation ν and for some $\mu \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,*

$$\nu(\mu W) = 0, \qquad \nu(\mu Z) < 0.$$

Then one of the following cases holds:

(i) *For all $\gamma \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,*

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0.$$

(ii) *There exist $\alpha, \beta \in \mathbb{Z}^2$, at least one non-zero, such that for all $\gamma \in \mathbb{Z}^2 \setminus (\mathbb{Z}\alpha \cup \mathbb{Z}\beta)$,*

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0,$$

for all $\gamma \in \mathbb{Z}\alpha \setminus \{(0,0)\}$,

$$\nu(\gamma W) > 0, \qquad \nu(\gamma Z) < 0,$$

and for all $\gamma \in \mathbb{Z}\beta \setminus \{(0,0)\}$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) \geq 0.$$

(iii) *There exists* $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ *such that for all* $\gamma \in \mathbb{Z}^2 \setminus \mathbb{Z}\alpha$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0,$$

and for all $\gamma \in \mathbb{Z}\alpha \setminus \{(0,0)\}$,

$$\nu(\gamma W) > 0, \qquad \nu(\gamma Z) \geq 0.$$

Proof Choose $\lambda \in \mathbb{Z}^2$ so that μ and λ are linearly independent.

If $\nu(\mu W) \neq \nu(\lambda W)$, then we note that for all $\gamma \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $\gamma = a\mu + b\lambda$,

$$\nu(\gamma W) = \nu((a\mu + b\lambda)W) = \min\{\nu(\mu W), \nu(\lambda W)\} \leq 0;$$

similarly, if $\nu(\mu Z) \neq \nu(\lambda Z)$, then $\nu(\gamma Z) < 0$ for all $\gamma \in \mathbb{Z}^2$.

If $\nu(\mu W) = \nu(\lambda W)$, then there can exist $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ that is pairwise independent with μ and λ such that $\nu(\alpha W) > 0$. We note that α is unique up to scalar multiplication, as if there exists $\gamma \in \mathbb{Z}^2 \setminus \mathbb{Z}\alpha$ such that $\nu(\gamma W) > 0$ also, then we may choose $A, B \in \mathbb{R}$ such that $A\alpha + B\gamma = \mu$, and note

$$\nu(\mu W) \geq \min\{\nu(\alpha W), \nu(\gamma W)\} > 0,$$

contradicting that $\nu(\mu W) = 0$. Likewise, if $\nu(\mu Z) = \nu(\lambda Z)$, then there can exist at most one $\beta \in \mathbb{Z}^2 \setminus \{0\}$ such that $\nu(\beta Z) \geq 0$.

We now check the cases:

- Suppose $\nu(\mu W) \neq \nu(\lambda W)$ and $\nu(\mu Z) \neq \nu(\lambda Z)$.
  - Case (i) holds if $\nu(\lambda W), \nu(\lambda Z) < 0$.
  - Case (ii) holds if $\nu(\lambda W) < 0 < \nu(\lambda Z)$ or $\nu(\lambda Z) < 0 < \nu(\lambda W)$.
  - Case (iii) holds if $\nu(\lambda W), \nu(\lambda Z) > 0$.
- Suppose $\nu(\mu W) = \nu(\lambda W)$ and $\nu(\mu Z) \neq \nu(\lambda Z)$.
  - Case (i) holds if α does not exist and $\nu(\mu Z) < 0$.
  - Case (ii) holds otherwise.
- Suppose $\nu(\mu W) \neq \nu(\lambda W)$ and $\nu(\mu Z) = \nu(\lambda Z)$.
  - Case (i) holds if $\nu(\mu W) < 0$ and β does not exist.

  – Case (ii) holds otherwise.

– Suppose $\nu(\mu W) = \nu(\lambda W)$ and $\nu(\mu Z) = \nu(\lambda Z)$.

  – Case (i) holds if α, β do not exist.
  – Case (ii) holds if α exists and β does not exist, α does not exist and β exists, or if α, β exist and $\alpha \neq \beta$.
  – Case (iii) holds if α, β exist and $\alpha = \beta$. □

Lemma 6.3 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$ and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve. Further, suppose G contains a pair of parallel edges (v, w, γ) and (v, w, γ') such that $\gamma - \gamma' = (\lambda_1, \lambda_2)$ and*

$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$

is not constant on $\mathcal{C}$. Then one of the following holds:

(i) *G has an active type 1 flexible 2-lattice NBAC-colouring,*
(ii) *G has an active type 2 flexible 2-lattice NBAC-colouring, or*
(iii) *G has an active type 3 flexible 2-lattice NBAC-colouring.*

Proof By our choice of $\tilde{e}$, we may assume $\tilde{e}$ and $\tilde{f}$ are the pair of parallel edges on $\tilde{v}, \tilde{w}$, with $\psi(\tilde{f}) = \mu$ for some $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. It follows from Propositions 2.7 and 3.12 that we also may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. Since μ is the difference in gains of $\tilde{e}, \tilde{f}$, then

$$(\mu_1 x_1 + \mu_2 x_2)^2 + (\mu_1 y_1 + \mu_2 y_2)^2$$

is not constant. By Lemma 6.1, $W^{\mu}_{\tilde{v},\tilde{w}}$ is not constant on $\mathbb{C}(\mathcal{C})$. Let ν be the valuation of $\mathbb{C}(\mathcal{C})$ and δ be the active NBAC-colouring given by Lemma 3.15 with $a := \tilde{f}$.

We note that $\nu(W^0_{\tilde{v},\tilde{w}}) = 0$ and $\nu(W^{\mu}_{\tilde{v},\tilde{w}}) > 0$. As $\mu W = W^0_{\tilde{v},\tilde{w}} - W^{\mu}_{\tilde{v},\tilde{w}}$,

$$\nu(\mu W) = \nu(W^0_{\tilde{v},\tilde{w}}) = 0.$$

Similarly, as $\mu Z = Z^0_{\tilde{v},\tilde{w}} - Z^{\mu}_{\tilde{v},\tilde{w}}$ then

$$\nu(\mu Z) = \nu(Z^{\mu}_{\tilde{v},\tilde{w}}) < 0.$$

Let case (i), case (ii), and case (iii) refer to the three possibilities given by Lemma 6.2. We shall now proceed to prove that case (i) implies G has a type 1 flexible 2-lattice NBAC-colouring, case (ii) implies G has either a type 1 or type 2 flexible 2-lattice

NBAC-colouring, and case (iii) implies G has either a type 1, type 2, or a type 3 flexible 2-lattice NBAC-colouring.

(Case (i) holds): Suppose G has an unbalanced monochromatic circuit C of length n and define $\gamma := \psi(C)$. If C is red, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. If C is blue, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$. It now follows that δ is a type 1 flexible 2-lattice NBAC-colouring.

(Case (ii) holds): Let C be an unbalanced monochromatic circuit of length n with $\gamma := \psi(C)$. If C is red and $\gamma \notin \mathbb{Z}\alpha$, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. Likewise, if C is blue and $\gamma \notin \mathbb{Z}\beta$, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$.

Now let C be an almost monochromatic circuit of length n where $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \dots, n-1\}$. If C is almost red and $\psi(C) = c\alpha$ for some $c \in \mathbb{Z}$, then

$$\begin{aligned}\nu\big(W^{\gamma_n}_{v_1,v_n}\big) &= \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + c\alpha W\right)\\ &\geq \min\left\{\nu\big(\alpha W\big),\, \nu(W^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n-1\right\} > 0,\end{aligned}$$

contradicting that $\nu\big(W^{\gamma_n}_{v_1,v_n}\big) \leq 0$. Similarly, if C is almost blue and $\psi(C) = c\beta$ for some $c \in \mathbb{Z}$, then

$$\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) = \nu\left(\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}} + c\beta Z\right)$$

$$\geq \min\left\{\nu(\beta Z),\, \nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n-1\right\} > 0,$$

contradicting that $\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) \leq 0$.

It now follows that if G has no unbalanced monochromatic circuits then δ is a type 1 flexible 2-lattice NBAC-colouring, and if G has an unbalanced monochromatic circuit then δ is a type 2 flexible 2-lattice NBAC-colouring.

(Case (iii) holds): Let C be an unbalanced monochromatic circuit of length n with $\gamma := \psi(C) \notin \mathbb{Z}\alpha$. If C is red, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. Likewise, if C is blue, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n\right\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$.

Now let C be an almost monochromatic circuit of length n where $\psi(C) := c\alpha$ for some $c \in \mathbb{Z}$ and $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \ldots, n-1\}$. If C is almost red, then

$$\begin{aligned}\nu\big(W^{\gamma_n}_{v_1,v_n}\big) &= \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + c\alpha W\right)\\ &\geq \min\left\{\nu(\alpha W),\, \nu\big(W^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n-1\right\} > 0,\end{aligned}$$

contradicting that $\nu\big(W^{\gamma_n}_{v_1,v_n}\big) \leq 0$. Similarly, if C is almost blue, then

$$\begin{aligned}\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) &= \nu\left(\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}} + c\alpha Z\right)\\ &\geq \min\left\{\nu(\alpha Z),\, \nu\big(Z^{\gamma_j}_{v_j,v_{j+1}}\big) : 1 \leq j \leq n-1\right\} > 0,\end{aligned}$$

contradicting that $\nu\big(Z^{\gamma_n}_{v_1,v_n}\big) \leq 0$.

It now follows that if G has no unbalanced monochromatic circuits then δ is a type 1 flexible 2-lattice NBAC-colouring, if G only has unbalanced monochromatic circuits for a single colour then δ is a type 2 flexible 2-lattice NBAC-colouring, and if G has unbalanced monochromatic circuits for both colours then δ is a type 3 flexible 2-lattice NBAC-colouring. ◻

We are now ready for our necessity lemma.

Lemma 6.4 *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$. If (G, p, L) is flexible then one of the following holds:*

(i) *G has an active type 1 flexible 2-lattice NBAC-colouring,*
(ii) *G has an active type 2 flexible 2-lattice NBAC-colouring,*
(iii) *G has an active type 3 flexible 2-lattice NBAC-colouring,*
(iv) *G has an active fixed lattice NBAC-colouring,*
(v) $\operatorname{rank} G < 2$*, or*
(vi) *G is disconnected.*

Proof Suppose $\operatorname{rank} G = 2$ and G is connected. Choose any $\tilde{e} \in E(G)$. By Lemma 3.13 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$. We now have three possible outcomes:

(a) $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}^{f}(G, p, L)$.
(b) G contains a pair of parallel edges (v, w, γ) and (v, w, γ') such that $\gamma - \gamma' = (\lambda_1, \lambda_2)$ and

$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$

is not constant on $\mathcal{C}$.
(c) Possibilities (a) and (b) do not hold.

(Possibility (a) holds): If $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}^{f}(G, p, L)$ then by Lemma 4.3, G has an active fixed lattice NBAC-colouring.
(Possibility (b) holds): By Lemma 6.3, G has either an active type 1, type 2, or type 3 flexible 2-lattice NBAC-colouring.
(Possibility (c) holds): As $\mathcal{C} \not\subset \mathcal{V}_{\tilde{e}}^{f}(G, p, L)$, we may choose $\mu := (\mu_1, \mu_2) \in \mathbb{Z}^2$ such that

$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$

is not constant. By Lemma 3.23, there exists a vertex addition (G', p', L) of (G, p, L) at v_1 by λ such that (G', p', L) has a non-trivial not fixed lattice flex. As (b) holds for (G', p', L), then by Lemma 6.3, G' has an active type k flexible 1-lattice NBAC-colouring δ' for some $k \in \{1, 2, 3\}$.

Suppose G' has a colouring δ' as described above. Let δ be the colouring of G with $\delta(e) := \delta'(e)$ for all $e \in E(G)$. We note that δ is an active type k' flexible 2-lattice NBAC-colouring for some $k' \in \{1, 2, 3\}$ if and only if δ' is not monochromatic on the subgraph G of G'. As $\operatorname{rank} G = 2$ and δ' is a type k flexible 2-lattice NBAC-colouring, δ' is not monochromatic on G, thus G has an active type k' flexible 2-lattice NBAC-colouring for some $k' \in \{1, 2, 3\}$. □

6.2 Constructing Flexible Frameworks: Low Rank Graphs

Our first construction lemma is the simplest one, as the framework is not connected.

Lemma 6.5 *Let G be a $\mathbb{Z}^2$-gain graph. If* rank $G < 2$ *then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is flexible.*

Proof If rank $G = 0$ then this holds by Lemma 4.7, so we may suppose rank $G = 1$, i.e., span $G = \mathbb{Z}\alpha$ for some non-zero $\alpha \in \mathbb{Z}^2$. By Proposition 2.7, we may assume every edge of G has gain in $\mathbb{Z}\alpha$. Choose any injective map p, any full lattice L, and any element $\beta \in \mathbb{Z}^2$ that is linearly independent of α. We may now define the fixed lattice flex (p_t, L_t) for $t \in [0, 2\pi]$, where $p_t = p$ and

$$L_t \cdot \alpha := L \cdot \alpha, \qquad L_t \cdot \beta := (1+t)L \cdot \beta. \qquad \square$$

6.3 Constructing Flexible Frameworks: Type 1 Flexible 2-Lattice NBAC-Colourings

We recall that a type 1 flexible 2-lattice NBAC-colouring is an NBAC-colouring δ where all monochromatic circuits are balanced.

Lemma 6.6 *Let G be a $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. Then there exists $G' \approx G$ such that each blue edge has trivial gain and no red edge has trivial gain.*

Proof The proof follows a similar method as Lemma 5.5. $\square$

Lemma 6.7 *Let H be a balanced $\mathbb{Z}^2$-gain graph with no multiple edges and no loops. Then there exists a placement q of H in $\mathbb{Z}^2$ such that for all $(v, w, \gamma) \in E(H)$, $q(w) - q(v) = 2\gamma$.*

Proof The proof follows the same method as Lemma 5.6. $\square$

We are now ready for our construction lemma for type 1 flexible 2-lattice NBAC-colourings. We note that it is essentially the same as the construction given in Lemma 5.7.

Lemma 6.8 *Let G be a $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.*

Proof By Lemma 6.6, we may assume all blue edges of G have trivial gain and all red edges have non-trivial gain. Let $R_1, \ldots, R_n$ be the red components of G and define E_j to be the set of edges (v, w, γ) in G^δ_{red} with $v, w \in R_j$. By Lemma 6.7, for each R_j there exists a placement q_j in $\mathbb{R}^2$ where $q_j(w) - q_j(v) = 2\gamma$ for all $(v, w, \gamma) \in E_j$. By applying translations to each of the placements q_j, we may assume that for any blue edge $(v, w, 0) \in E(G)$ with $v \in R_j$, $w \in R_k$ and $j \neq k$, we have $q_j(v) \neq q_k(w)$. We now define for each $t \in [0, 2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$, with

$$L_t \cdot (1, 0) := (-2 + \cos t, \sin t), \qquad L_t \cdot (0, 1) := (\sin t, -2 - \cos t)$$

and $p_t(v) := q_j(v)$ for $v \in R_j$. We shall denote $(p, L) := (p_0, L_0)$.

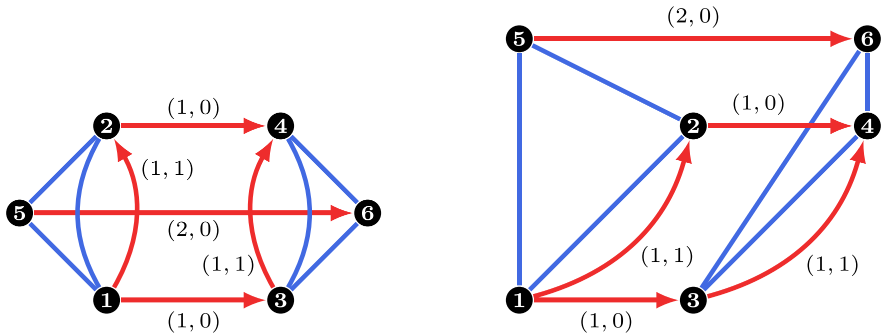

Fig. 12 (Left): A $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. (Right): The constructed full 2-periodic framework in $\mathbb{R}^2$

To see that (p, L) is a well-defined placement-lattice, choose any $e = (v, w, \gamma)$ with $\gamma = (\gamma_1, \gamma_2)$ and suppose that $p(v) = p(w) + L \cdot \gamma$. If $\delta(e) = $ red, then $\gamma \neq (0, 0)$ and $v, w \in R_j$ for some j. It follows that

$$-L \cdot \gamma = q_j(w) - q_j(v) = 2\gamma.$$

However as $-L \cdot \gamma = (\gamma_1, 3\gamma_2)$, then $-L \cdot \gamma = 2\gamma$ if and only if $\gamma = (0, 0)$, contradicting that all red edges have non-trivial gain. If $\delta(e) = $ blue, then $\gamma = (0, 0)$. By our choice of placements $\{q_i : 1 \leq i \leq n\}$, we must have $v, w \in R_j$ for some j; furthermore, as $\gamma = (0, 0)$ then $q_j(v) = q_j(w)$. Let $(e_1, \ldots, e_{n-1})$ be a red path from w to v with $e_j = (v_j, v_{j+1}, \gamma_j) \in E_j$, $v_1 = w$ and $v_n = v$. Since $q_j(v) = q_j(w)$, we have $\sum_{j=1}^{n-1} \gamma_j = 0$. However, this implies $(e_1, \ldots, e_{n-1}, e)$ is a balanced almost red circuit, contradicting that δ is a type 1 flexible 2-lattice NBAC-colouring.

Choose any edge $e = (v, w, \gamma)$ with $\gamma = (\gamma_1, \gamma_2)$. If $\delta(e) = $ blue then $\gamma = 0$. As $p_t = p$ then for each $t \in [0, 2\pi]$,

$$\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 = \|p(v) - p(v) - L \cdot \gamma\|^2.$$

If $\delta(e) = $ red then $v, w \in R_j$, thus for each $t \in [0, 2\pi]$,

$$\begin{aligned} \|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 &= (\gamma_1 \cos t + \gamma_2 \sin t)^2 + (\gamma_1 \sin t - \gamma_2 \cos t)^2 \\ &= \gamma_1^2 + \gamma_2^2 \end{aligned}$$

It follows that (p_t, L_t) is a flex of (G, p, L), as required. We refer the reader to Fig. 12 for an example of the construction. □

6.4 Constructing Flexible Frameworks: Type 2 Flexible 2-Lattice NBAC-Colourings

We recall that a type 2 flexible 2-lattice NBAC-colouring is an NBAC-colouring δ where there exist $\alpha, \beta \in \mathbb{Z}^2$ such that:

- either α, β are linearly independent or exactly one of α, β is equal to $(0, 0)$,
- span G^δ_{red} is a non-trivial subgroup of $\mathbb{Z}\alpha$, or $\alpha = (0, 0)$ and G^δ_{red} is balanced,
- span G^δ_{blue} is a non-trivial subgroup of $\mathbb{Z}\beta$, or $\beta = (0, 0)$ and G^δ_{blue} is balanced,
- there are no almost red circuits with gain in $\mathbb{Z}\alpha$, and
- there are no almost blue circuits with gain in $\mathbb{Z}\beta$.

Lemma 6.9 *Let G be a $\mathbb{Z}^2$-gain graph and δ a type 2 flexible 2-lattice NBAC-colouring of G with α, β as described previously. Suppose $\alpha \neq (0, 0)$. Then there exists $G' \approx G$ such that each red edge has gain $a\alpha + b\beta$ for some $a, b \in \mathbb{Z}$ with $a \neq 0$, and each blue edge has gain $c\beta$ for some $c \in \mathbb{Z}$.*

Proof As span $G^\delta_{\text{blue}} = \mathbb{Z}\beta$, by Proposition 2.7, we may suppose all blue edges of G have gain in $\mathbb{Z}\beta$. Let $B_1, \ldots, B_n$ be the blue components of G and choose $N \in \mathbb{N}$ such that $N > |a|$ for all $(v, w, \gamma) \in E(G)$ with $\gamma = a\alpha + b\beta$. We now define the gain equivalent graph

$$G' := \left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{iN\alpha} \right) (G).$$

We first note that any blue edge of G' will have gain in $\mathbb{Z}\beta$ since both of its ends will lie in the same blue component. Choose a red edge $(v, w, \gamma) \in E(G)$ with $\gamma = a\alpha + b\beta$ and suppose $v \in B_i$ and $w \in B_j$. We note that

$$\begin{aligned}\left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{iN\alpha} \right) (v, w, \gamma) &= \phi_v^{iN\alpha} \circ \phi_w^{jN\alpha} (v, w, \gamma) \\ &= (v, w, (N(i - j) + a)\alpha + b\beta).\end{aligned}$$

As $N > |a|$ and $i - j \in \mathbb{Z}$, we have $N(i - j) + a = 0$ if and only if $a = 0$ and $i = j$. If this holds, then as $v, w \in B_i$, we can define an almost blue circuit containing v with red edge $(v, w, b\beta)$ and gain in $\mathbb{Z}\beta$ (as every blue edge has gain in $\mathbb{Z}\beta$), contradicting that δ is a type 2 flexible 2-lattice NBAC-colouring. It now follows that $a \neq 0$ as required. □

Lemma 6.10 *Let $\alpha, \beta \in \mathbb{Z}^2$ be linearly independent and let H be a $\mathbb{Z}^2$-gain graph where* span H *is a subgroup of $\mathbb{Z}\alpha$. Then there exists a placement q of H in $\mathbb{Z}$ such that for all $(v, w, a\alpha + b\beta) \in E(H)$, $q(v) - q(w) = b$.*

Proof Define the $\mathbb{Z}$-gain graph H' with vertex set $V(H') := V(H)$ and edge set

$$E(H') := \{(v, w, b\beta) : (v, w, a\alpha + b\beta) \in E(H)\};$$

we delete any loops with trivial gain that may arrive, and note that multiple edges may become a single edge. By Lemma 5.6, we may define a placement q' of H' in $\mathbb{Z}$ such that $q'(v) - q'(w) = -2b$ for all $(v, w, b\beta) \in E(H')$. We now define q to be the placement of H where $q(v) := -q'(v)/2$. □

We are now ready for our construction lemma for type 2 flexible 2-lattice NBAC-colourings.

Lemma 6.11 *Let G be a $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.*

Proof Without loss of generality we may assume span $G^\delta_{\text{red}} = \mathbb{Z}\alpha$ and span $G^\delta_{\text{blue}} = \mathbb{Z}\beta$, with $\alpha \neq (0, 0)$. If $\beta = (0, 0)$ then δ is a fixed-lattice NBAC-colouring and the result holds by Lemma 4.6, thus we may assume α, β are linearly independent.

By Lemma 6.9, we may assume all red edges have gain $a\alpha + b\beta$ for some $a, b \in \mathbb{Z}$ with $a \neq 0$, and all blue edges have gain $c\beta$ for some $c \in \mathbb{Z}$. Let $R_1, \ldots, R_n$ be the red components of G and define E_j to be the set of edges (v, w, γ) in G^δ_{red} with $v, w \in R_j$. By Lemma 6.10, for each R_j there exists a placement q_j in $\mathbb{R}$ where $q_j(v) - q_j(w) = b$ for all $(v, w, \gamma) \in E_j$ with $\gamma = a\alpha + b\beta$. We now define for each $t \in [0, 2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$ with

$$L_t \cdot \alpha := (\sin t, \cos t), \qquad L_t \cdot \beta := (1, 0)$$

and $p_t(v) := (q_j(v), j)$ for $v \in R_j$. We shall denote $(p, L) := (p_0, L_0)$.

To see that (p, L) is a well-defined placement-lattice, choose any $e = (v, w, \gamma)$ and suppose that $p(v) = p(w) + L \cdot \gamma$. If $\delta(e) = \text{red}$, then $\gamma = a\alpha + b\beta$ for some $a, b \in \mathbb{Z} \setminus \{0\}$ and $v, w \in R_j$ for some j. We note

$$p(v) = (q_j(v), j) = (q_j(w) + b, j + a) = p(w) + L \cdot \gamma,$$

which implies $a = 0$, a contradiction. If $\delta(e) = \text{blue}$, then $\gamma = b\beta$ for some $b \in \mathbb{Z}$. If $v \in R_j$ and $w \in R_k$ then

$$p(v) = (q_j(v), j) = (q_k(w) + b, k) = p(w) + L \cdot \gamma,$$

therefore $j = k$. Let $P := (e_1, \ldots, e_{n-1})$ be a red path from w to v with $e_i = (v_i, v_{i+1}, \gamma_i) \in E_j$, $v_1 = w$, $v_n = v$, $\gamma_i = a_i\alpha + b_i\beta$. Define $C := (e_1, \ldots, e_{n-1}, e)$. As

$$b = q_j(v) - q_j(w) = \sum_{i=1}^{n-1} (q_j(v_{i+1}) - q_j(v_i)) = -\sum_{i=1}^{n-1} b_i,$$

we have $\psi(C) = a\alpha$ for some $a \in \mathbb{Z}$. This contradicts that δ is a type 2 flexible 2-lattice NBAC-colouring, as C is an almost red circuit with $\psi(C) \in \mathbb{Z}\alpha$.

Choose any edge $e = (v, w, \gamma)$ with $\gamma = a\alpha + b\beta$. If $\delta(e) = \text{blue}$ then $a = 0$. As $p_t = p$ and $L_t \cdot \beta = (1, 0)$, $\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2$ is constant. If $\delta(e) = \text{red}$ then $v, w \in R_j$, thus for each $t \in [0, 2\pi]$,

$$\|p_t(v) - p_t(w) - L_t \cdot \gamma\|^2 = (q_j(v) - q_j(w) - b - a \cos t)^2 + (a \sin t)^2 = a^2.$$

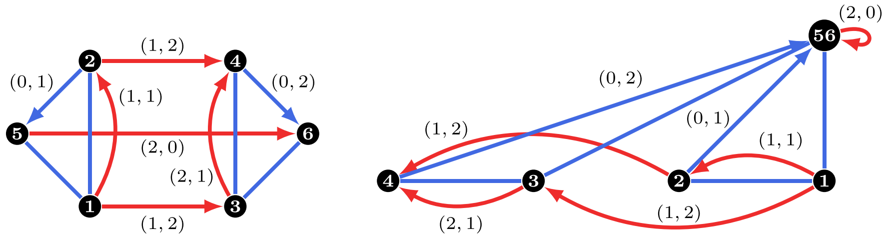

Fig. 13 (Left): A $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring ($\alpha = (1, 0)$, $\beta = (0, 1)$). (Right): The constructed full 2-periodic framework in $\mathbb{R}^2$

It follows that (p_t, L_t) is a flex of (G, p, L) as required. We refer the reader to Fig. 13 for an example of the construction. □

6.5 Conjectures Regarding Type 3 Flexible 2-Lattice NBAC-Colourings

We recall that an NBAC-colouring δ of a $\mathbb{Z}^2$-gain graph G is a type 3 flexible 2-lattice NBAC-colouring if there exists $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that

- span G^δ_{red} and span G^δ_{red} are non-trivial subgroups of $\mathbb{Z}\alpha$, and
- there are no almost monochromatic circuits with gain in $\mathbb{Z}\alpha$.

It is an open question whether the existence of a type 3 flexible 2-lattice NBAC-colouring implies the existence of a flexible placement of a $\mathbb{Z}^2$-gain graph in $\mathbb{R}^2$. As this is the case for all other types of flexible 2-lattice NBAC-colourings, we would conjecture the following.

Conjecture 2 Let G be a $\mathbb{Z}^2$-gain graph with type 3 flexible 2-lattice NBAC-colouring. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.

All examples of $\mathbb{Z}^2$-gain graphs with a type 3 flexible 2-lattice NBAC-colouring discovered so far will also have either a type 1 or type 2 flexible 2-lattice NBAC-colouring, a fixed lattice NBAC-colouring, or have a low rank. Due to this, we would also conjecture the following.

Conjecture 3 Let G be a $\mathbb{Z}^2$-gain graph with type 3 flexible 2-lattice NBAC-colouring. Then G has either a type 1 or type 2 flexible 2-lattice NBAC-colouring, G has a fixed lattice NBAC-colouring, or rank $G < 2$.

If Conjecture 2 is true, then by Lemmas 6.4, 6.8, 6.11, 4.7, and 6.5, we can deduce that Conjecture 1 would be also true. If Conjecture 3 is true, then we obtain the slightly stronger result.

Conjecture 4 Let G be a connected $\mathbb{Z}^2$-gain graph. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:

(i) G has a type 1 flexible 2-lattice NBAC-colouring,

(ii) G has a type 2 flexible 2-lattice NBAC-colouring,
(iii) G has a fixed lattice NBAC-colouring, or
(iv) rank $G < 2$.

7 Special Cases of Flexible 2-Periodic Frameworks

We shall now focus on 2-periodic frameworks with loops. With this added assumption, we can fully characterise whether a $\mathbb{Z}^2$-gain graph has a flexible placement-lattice by observing the graph's NBAC-colourings.

7.1 2-Periodic Frameworks with Loops

Lemma 7.1 *Let (G, p, L) be a k-periodic framework in $\mathbb{K}^d$ and suppose G has a loop (w, w, α). If G' is the $\mathbb{Z}^k$-gain graph with*

$$V(G') := V(G), \qquad E(G') := E(G) \cup \{(v, v, c\alpha) : v \in V(G),\ c \in \mathbb{N}\},$$

then

$$\mathcal{V}_{\mathbb{K}}(G', p, L) = \mathcal{V}_{\mathbb{K}}(G, p, L).$$

Proof First note that $\mathcal{V}_{\mathbb{K}}(G', p, L) \subset \mathcal{V}_{\mathbb{K}}(G, p, L)$. Choose any placement-lattice $(p', L') \in \mathcal{V}_{\mathbb{K}}(G, p, L)$. As $(w, w, \alpha) \in E(G)$, then

$$\|L' \cdot \alpha\|^2 = \|p'(w) - p'(w) - L' \cdot \alpha\|^2 = \|p(w) - p(w) - L \cdot \alpha\|^2 = \|L \cdot \alpha\|^2.$$

We note that for any $v \in V(G)$ and non-zero $c \in \mathbb{Z}$,

$$\begin{aligned}\|p'(v) - p'(v) - L' \cdot c\alpha\|^2 &= c^2\|L' \cdot \alpha\|^2 = c^2\|L \cdot \alpha\|^2 \\ &= \|p(v) - p(v) - L \cdot c\alpha\|^2,\end{aligned}$$

thus $(p', L') \in \mathcal{V}_{\mathbb{K}}(G', p, L)$ as required. □

Lemma 7.2 *Let G be a connected $\mathbb{Z}^2$-gain graph and suppose that there exists some $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, we have $(v, v, c\alpha) \in E(G)$. If δ is an NBAC-colouring of G, then every loop with gain $c\alpha$ for some $c \in \mathbb{N}$ is of the same colour.*

Proof We first note that every loop at a vertex must have the same colour. To see this, suppose there exists a loop $(v, v, c\alpha)$ for some integer $c > 1$, where $\delta(v, v, n\alpha) \neq \delta(v, v, \alpha)$. This would imply the circuit

$$\big(\overbrace{(v, v, -\alpha), \ldots, (v, v, -\alpha)}^{c \text{ times}}, (v, v, c\alpha)\big)$$

is balanced and almost monochromatic, contradicting that δ is an NBAC-colouring.

Suppose not all loops are of the same colour. As G is connected, there must exist distinct vertices $v, w \in V(G)$ connected by an edge (v, w, γ) where $\delta(v, v, \alpha) \neq \delta(w, w, \alpha)$. Without loss of generality we may assume $\delta(v, v, \alpha) = \delta(v, w, \gamma)$. The circuit

$$((v, w, \gamma), (w, w, \alpha), (w, v, -\gamma), (v, v, -\alpha))$$

is balanced and almost monochromatic, contradicting that δ is an NBAC-colouring. □

Lemma 7.3 *Let G be a connected $\mathbb{Z}^2$-gain graph. Suppose that there exists some $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, we have $(v, v, c\alpha) \in E(G)$. Then there are no type 1 or type 3 flexible 2-lattice NBAC-colourings of G.*

Proof Suppose there exists an NBAC-colouring δ of G that is either a type 1 or type 3 flexible 2-lattice NBAC-colouring. As one of G^δ_{red} or G^δ_{blue} must contain a loop (and thus be unbalanced), δ must be a type 3 flexible 2-lattice NBAC-colouring. By Lemma 7.3, every loop of the form $(v, v, c\gamma)$ has the same colour, and without loss of generality we shall assume they are all red. We note immediately that there cannot be any unbalanced blue circuits; indeed, if C was a blue circuit containing v with gain $c\alpha$, then the circuit formed by C followed by $(v, v, -c\alpha)$ will be balanced and almost blue. However this implies rank $G^\delta_{\text{blue}} = 0$, contradicting that δ is a type 3 flexible 2-lattice NBAC-colouring. □

Lemma 7.4 *Let G be a connected $\mathbb{Z}^2$-gain graph with a loop. Then there are no active type 1 or type 3 flexible 2-lattice NBAC-colourings of G.*

Proof Let (w, w, α) be a loop of G. By Lemma 7.1, we may assume that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, we have $(v, v, c\alpha) \in E(G)$. The result now follows from Lemma 7.3. □

We may now prove a special case of Conjecture 2.

Theorem 7.5 *Let G be a connected $\mathbb{Z}^2$-gain graph with a loop. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:*

(i) *G has a type 2 flexible 2-lattice NBAC-colouring,*
(ii) *G has a fixed lattice NBAC-colouring,*
(iii) rank $G = 1$.

Proof Suppose there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework. Since G contains a loop, rank $G \geq 1$. By Lemmas 6.4 and 7.4, either G has an active type 2 flexible 2-lattice NBAC-colouring, G has an active fixed lattice NBAC-colouring, or rank $G = 1$.

Now suppose that either G has a type 2 flexible 2-lattice NBAC-colouring, G has a fixed lattice NBAC-colouring, or rank $G = 1$. Then by Lemmas 6.11, 4.7, or 6.5, there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework. □

7.2 Scissor Flexes

We now define a special class of flexes.

Definition 7.6 Let (p_t, L_t) be a flex of a 2-periodic framework (G, p, L) in $\mathbb{R}^2$. If there exist linearly independent $\alpha, \beta \in \mathbb{Z}^2$ such that $\|L_t \cdot \alpha\|^2$ and $\|L_t \cdot \beta\|^2$ are constant but $(L_t \cdot \alpha) \cdot (L_t \cdot \beta)$ is not constant, then (p_t, L_t) is a *scissor flex*.

If rank $G < 2$, then it can be seen that some placement-lattice of G will have a scissor flex. We shall show in Theorem 7.10 that we can characterise the $\mathbb{Z}^2$-gain graphs with scissor flexes by their NBAC-colouring. We first prove the following lemmas.

Lemma 7.7 *Let G be a connected $\mathbb{Z}^2$-gain graph and $\alpha, \beta \in \mathbb{Z}^2$ be linearly independent. Suppose that $(v, v, c\alpha), (v, v, c\beta) \in E(G)$ for all $v \in V(G)$ and $c \in \mathbb{N}$. If δ is an NBAC-colouring of G then either:*

(i) *All loops of G are in the same colour.*
(ii) *All loops of G with gain in $\mathbb{Z}\alpha$ are red (respectively, blue), all loops of G with gain in $\mathbb{Z}\beta$ are blue (respectively, red), and all loops of G have gain in $\mathbb{Z}\alpha \cup \mathbb{Z}\beta$.*

Proof By Lemma 7.2 we have (without loss of generality) two possibilities:

(a) All loops with gain in $\mathbb{Z}\alpha \cup \mathbb{Z}\beta$ are red.
(b) All loops with gain in $\mathbb{Z}\alpha$ are red and all loops with gain in $\mathbb{Z}\beta$ are blue.

Suppose (a) holds. If G only has loops with gain $\gamma \in \mathbb{Z}\alpha \cup \mathbb{Z}\beta$, then (i) holds. Suppose G has a loop $l := (v, v, \gamma)$ with $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$. Let $a, b \in \mathbb{Z} \setminus \{0\}$ be any pair where $c\gamma = a\alpha + b\beta$ for some $c > 0$. If $\delta(l) =$ blue then we note that the circuit

$$\big((v, v, -a\alpha), (v, v, -a\beta), \overbrace{l, \dots, l}^{c \text{ times}}\big)$$

is balanced and almost red, contradicting that δ is an NBAC-colouring. Hence $\delta(l) =$ red and (i) holds.

Suppose (b) holds but G has a loop $l := (v, v, \gamma)$ with $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$. Choose $a, b \in \mathbb{Z}$ such that $c\gamma = a\alpha + b\beta$ for some $c > 0$. If $\delta(l) =$ blue then the circuit

$$C := \big((v, v, -a\alpha), (v, v, -a\beta), \overbrace{l, \dots, l}^{c \text{ times}}\big)$$

is balanced and almost blue, while if $\delta(l) =$ red then C is balanced and almost red. As both possibilities contradict that δ is an NBAC-colouring, then no such loop may exist and (ii) holds. □

Lemma 7.8 *Let G be a connected $\mathbb{Z}^2$-gain graph with loops $l_\alpha := (v, v, \alpha)$, $l_\beta := (v, v, \beta)$, where α and β are linearly independent. Then all active NBAC-colourings of G with $\delta(l_\alpha) \neq \delta(l_\beta)$ are type 2 flexible 2-lattice NBAC-colourings.*

Proof Let δ be an active NBAC-colouring of G with $\delta(l_\alpha) \neq \delta(l_\beta)$. Without loss of generality, we may assume $\delta(l_\alpha) = $ red and $\delta(l_\beta) = $ blue. By Lemma 7.1, we may assume that $(v, v, c\gamma) \in E(G)$ for all $v \in V(G)$ and $c \in \mathbb{N}$, where $\gamma \in \{\alpha, \beta\}$. By Lemma 7.7, all loops with gain in $\mathbb{Z}\alpha$ are red, all loops with gain in $\mathbb{Z}\beta$ are blue, and there are no loops with gain $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$.

Suppose there exists a red circuit C containing v with $\psi(C) = a\alpha + b\beta$. If $a, b \neq 0$, then the circuit formed from C followed by $(v, v, -a\alpha)$, $(v, v, -a\beta)$ is balanced and almost red, while if $a = 0$, $b \neq 0$, then the circuit formed from C followed by $(v, v, -a\beta)$ is balanced and almost red. As both contradict that δ is an NBAC-colouring of G, then $\psi(C) \in \mathbb{Z}\alpha$. We similarly note that for any blue circuit C', $\psi(C') \in \mathbb{Z}\beta$.

Let C be an almost monochromatic circuit of length n where $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \dots, n-1\}$. If C is almost red and $\psi(C) = c\alpha$ for some $c \in \mathbb{Z}$, then

$$(e_1, \dots, e_n, (v_1, v_1, -c\alpha))$$

is a balanced almost red circuit, contradicting that δ is an NBAC-colouring. If C is almost blue and $\psi(C) = c\beta$ for some $c \in \mathbb{Z}$, then

$$(e_1, \dots, e_n, (v_1, v_1, -c\beta))$$

is a balanced almost blue circuit, contradicting that δ is an NBAC-colouring. It now follows that δ is a type 2 flexible 2-lattice as required. □

Lemma 7.9 *Let G be a connected $\mathbb{Z}^2$-gain graph with loops $l_\alpha := (v, v, \alpha)$, $l_\beta := (v, v, \beta)$, where α and β are linearly independent. If $\delta(l_\alpha) = \delta(l_\beta)$ for all active NBAC-colourings of G, then for any 2-periodic framework (G, p, L), every non-trivial flex of (G, p, L) is a fixed-lattice flex.*

Proof Let (G, p, L) be a flexible 2-periodic framework with a flex (p_t, L_t), $t \in [0, 1]$. Since we have loops $l_\alpha, l_\beta \in E(G)$, it follows that $\|L_t \cdot \alpha\|^2 = \|L \cdot \alpha\|^2$ and $\|L_t \cdot \beta\|^2 = \|L \cdot \beta\|^2$ for all $t \in [0, 1]$. For each $t \in [0, 1]$ we have

$$(L \cdot \alpha) \cdot (L \cdot \beta) = (p(v) - p(v) - L \cdot \alpha) \cdot (p(v) - p(v) - L \cdot \beta),$$
$$(L_t \cdot \alpha) \cdot (L_t \cdot \beta) = (p_t(v) - p_t(v) - L_t \cdot \alpha) \cdot (p_t(v) - p_t(v) - L_t \cdot \beta).$$

By Proposition 3.20 and the continuity of the flex (p_t, L_t), we must have $(L \cdot \alpha) \cdot (L \cdot \beta) = (L_t \cdot \alpha) \cdot (L_t \cdot \beta)$ for all $t \in [0, 1]$. It now follows that $L_t \sim L$ for all $t \in [0, 1]$ as required. □

Theorem 7.10 *Let G be a connected $\mathbb{Z}^2$-gain graph with* rank $G = 2$. *Then there exists a full 2-periodic framework (G, p, L) in $\mathbb{R}^2$ with a scissor flex if and only if either:*

(i) *G has a type 2 flexible 2-lattice NBAC-colouring, or*
(ii) *G has a type 1 flexible 2-lattice NBAC-colouring where, for some linearly independent pair $\alpha, \beta \in \mathbb{Z}^2$, there are no almost red circuits of G with gain in $\mathbb{Z}\alpha$ and there are no almost blue circuits of G with gain in $\mathbb{Z}\beta$.*

Proof If G has a type 2 flexible 2-lattice NBAC-colouring, then there exists a full 2-periodic framework (G, p, L) with a scissor flex by Lemma 6.11. Suppose G has a type 1 flexible 2-lattice NBAC-colouring δ with no almost red circuits with gain in $\mathbb{Z}\alpha$ and no almost blue circuits with gain in $\mathbb{Z}\beta$. We note that if we add the loops (v, v, α) and (v, v, β) to G to form the graph G', then we can extend δ to a type 2 flexible 2-lattice NBAC-colouring δ' of G' by setting $\delta'(v, v, \alpha) =$ red and $\delta'(v, v, \beta) =$ blue. A full 2-periodic framework (G', p, L) with a scissor flex can now be constructed by Lemma 6.11. We finish by noting that (G, p, L) will also have a scissor flex as required.

Now suppose there exists a full 2-periodic framework (G, p, L) with a scissor flex (p_t, L_t); we shall assume that $\alpha, \beta \in \mathbb{Z}^2$ are linearly independent gains where $\|L_t \cdot \alpha\|^2$ and $\|L_t \cdot \beta\|^2$ are both constant. Choose any $v \in V(G)$ and define G' to be the $\mathbb{Z}^2$-gain graph formed from G by adding the loops $l_\alpha := (v, v, \alpha)$ and $l_\beta := (v, v, \beta)$. We note that (p_t, L_t) is a flex of (G', p, L) also, hence as rank $G' = 2$, we have that G' has an active NBAC-colouring by Lemma 6.4.

If all active NBAC-colourings δ' of G' have $\delta'(l_\alpha) = \delta'(l_\beta)$, then by Lemma 7.9, (G', p, L) is fixed lattice flexible, a contradiction. It follows that G' has an active NBAC-colouring δ' with $\delta'(l_\alpha) \neq \delta'(l_\beta)$. By Lemma 7.8, δ' is a type 2 flexible 2-lattice NBAC-colouring. If we define δ to be the restriction of δ' to G, then δ is a type k flexible 2-lattice NBAC-colouring for some $k \in \{1, 2\}$, as rank $G = 2$ and δ' cannot be monochromatic on any subgraph of rank 2. We finish by noting that if δ is a type 1 flexible 2-lattice NBAC-colouring, then (with respect to δ) there are no almost red circuits of G with gain in $\mathbb{Z}\alpha$ and there are no almost blue circuits of G with gain in $\mathbb{Z}\beta$. □

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References

1. Badri, G., Kitson, D., Power, S.C.: The almost periodic rigidity of crystallographic bar-joint frameworks. Symmetry **6**(2), 308–328 (2014)
2. Borcea, C.S., Streinu, I.: Periodic frameworks and flexibility. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **466**(2121), 2633–2649 (2010)
3. Borcea, C.S., Streinu, I.: Minimally rigid periodic graphs. Bull. Lond. Math. Soc. **43**(6), 1093–1103 (2011)
4. Borcea, C., Streinu, I.: Geometric auxetics. Proc. A. **471**(2184), # 20150033 (2015)
5. Dewar, S., Grasegger, G., Legerský, J.: Flexible placements of graphs with rotational symmetry (2020). arXiv:2003.09328
6. Gallet, M., Grasegger, G., Legerský, J., Schicho, J.: On the existence of paradoxical motions of generically rigid graphs on the sphere. SIAM J. Discrete Math. **35**(1), 325–361 (2021)
7. Grasegger, G., Legerský, J., Schicho, J.: Graphs with flexible labelings. Discrete Comput. Geom. **62**(2), 461–480 (2019)
8. Grasegger, G., Legerský, J., Schicho, J.: Graphs with flexible labelings allowing injective realizations. Discrete Math. **343**(6), # 111713 (2020)

9. Gross, J.L., Tucker, T.W.: Topological Graph Theory. Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York (1987)
10. Kastis, E., Power, S.C.: Algebraic spectral synthesis and crystal rigidity. J. Pure Appl. Algebra **223**(11), 4954–4965 (2019)
11. Kaszanitzky, V.E., Schulze, B., Tanigawa, S.: Global rigidity of periodic graphs under fixed-lattice representations. J. Comb. Theory Ser. B **146**, 176–218 (2021)
12. Laman, G.: On graphs and rigidity of plane skeletal structures. J. Engrg. Math. **4**(4), 331–340 (1970)
13. Malestein, J., Theran, L.: Generic combinatorial rigidity of periodic frameworks. Adv. Math. **233**, 291–331 (2013)
14. Malestein, J., Theran, L.: Ultrarigid periodic frameworks (2015). arXiv:1404.2319v4
15. Mumford, D.: Abelian Varieties. Tata Institute of Fundamental Research Studies in Mathematics, vol. 5. Oxford University Press, London (1970)
16. Nixon, A., Ross, E.: Periodic rigidity on a variable torus using inductive constructions. Electron. J. Comb. **22**(1), # P1.1 (2015)
17. Owen, J.C., Power, S.C.: Infinite bar-joint frameworks, crystals and operator theory. New York J. Math. **17**, 445–490 (2011)
18. Pollaczek-Geiringer, H.: Über die Gliederung ebener Fachwerke. Z. Angew. Math. Mech. **7**(1), 58–72 (1927)
19. Power, S.C.: Polynomials for crystal frameworks and the rigid unit mode spectrum. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **372**(2008), # 20120030 (2014)
20. Ross, E.: The rigidity of periodic frameworks as graphs on a fixed torus. Contrib. Discrete Math. **9**(1), 11–45 (2014)
21. Ross, E.: Inductive constructions for frameworks on a two-dimensional fixed torus. Discrete Comput. Geom. **54**(1), 78–109 (2015)
22. Stichtenoth, H.: Algebraic Function Fields and Codes. Graduate Texts in Mathematics, vol. 254. Springer, Berlin (2009)
23. Whiteley, W.: Fragmentary and incidental behaviour of columns, slabs, and crystals. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **372**(2008), # 20120032 (2014)