



Approximate Carathéodory's Theorem in Uniformly Smooth Banach Spaces

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Abstract

We study the ‘no-dimension’ analogue of Carathéodory’s theorem in Banach spaces. We prove such a result together with its colorful version for uniformly smooth Banach spaces. It follows that uniform smoothness leads to a greedy de-randomization of Maurey’s classical lemma Pisier (in: Séminaire Analyse fonctionnelle (dit “Maurey-Schwartz”), 1980), which is itself a ‘no-dimension’ analogue of Carathéodory’s theorem with a probabilistic proof. We find the asymptotically tight upper bound on the deviation of the convex hull from the k -convex hull of a bounded set in L_p with $1 < p \leq 2$ and get asymptotically the same bound as in Maurey’s lemma for L_p with $1 < p < \infty$.

Keywords Carathéodory’s theorem · Uniform smoothness · Supporting hyperplane · Type of a Banach space

Mathematics Subject Classification 52A05 · 52A27 · 46B20 · 52A35

1 Introduction

Carathéodory’s theorem [4] is a classical result in Convex Geometry. It states that every point in the convex hull $\text{conv } S$ of a set $S \subset \mathbb{R}^d$ is a convex combination of

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at most $d + 1$ points of S . There are many different generalizations of this theorem. Recently, several authors studied so-called approximate versions of Carathéodory’s theorem, where the distance between a point of the convex hull $\text{conv } S$ of a bounded set S and the k -convex hull $\text{conv}_k S$ were investigated. The latter, the k -convex hull of S , is the set of all convex combinations of at most k points of S . For example, [1, Thm. 2.2] the following optimal result in the Euclidean case was proven:

The distance between any point p in the convex hull of a bounded set S of a Euclidean space and the k -convex hull $\text{conv}_k S$ is at most $\frac{\text{diam } S}{\sqrt{2k}}$.

However, not only the Euclidean case was studied. Probably, the most significant result in the area is Maurey’s lemma [16], in which an approximate version of Carathéodory’s theorem is proven for spaces that have (Rademacher) type $p > 1$. We explain the definitions and formulate Maurey’s lemma and explain its relation to our results in the next section. We note here that Maurey’s lemma is a more general statement than our results. However, Maurey’s proof uses Khintchine’s inequality and it is probabilistic.

We think that a simple geometric property of Banach spaces hides behind such a sophisticated technique. In this paper, we prove the approximate Carathéodory’s theorem for uniformly smooth Banach spaces, and bound the distance between a point of the convex hull of a bounded set and its k -convex hull in terms of the modulus of smoothness of a Banach space. In fact, we provide a greedy algorithm for the approximation of a point of the convex hull of a bounded set in a uniformly smooth Banach space. We note here that an L_p space for $1 < p < \infty$ is uniformly smooth and its modulus of smoothness is well known. We give all the required definitions in Sect. 2 and briefly discuss there some properties of Banach spaces that are connected to the uniform smoothness of Banach spaces. The following statement is the main result of the paper.

Theorem 1.1 *Let S be a bounded set in a uniformly smooth Banach space X , $a \in \text{conv } S$. Then there exists a sequence $\{x_i\}_{i=1}^\infty \subset S$ such that for vectors $a_k = \frac{1}{k} \sum_{i \in [k]} x_i$ the following inequality holds:*

$$\|a - a_k\| \leq \frac{2e^2}{k\rho_X^{-1}(1/k)} \text{diam } S, \tag{1.1}$$

where $\rho_X(\cdot)$ is the modulus of smoothness of X .

In the proof we provide a greedy algorithm for constructing such a sequence, which directly implies the colorful version of approximate Carathéodory’s theorem.

Corollary 1.2 *Let $\{S_i\}_{i=1}^\infty$ be a family of sets of a Banach space X such that $a \in \bigcap_{i=1}^\infty \text{conv } S_i$ and $D = \sup_i \text{diam } S_i < \infty$. Then there exists a transversal sequence $\{x_i\}_{i=1}^\infty$ (that is, $x_i \in S_i$) such that for vectors $a_k = \frac{1}{k} \sum_{i \in [k]} x_i$ the following inequality holds:*

$$\|a - a_k\| \leq \frac{2e^2}{k\rho_X^{-1}(1/k)} D, \tag{1.2}$$

where $\rho_X(\cdot)$ is the modulus of smoothness of X .

At the end of the next section, we show that the bound in inequality (1.1) is tight for all L_p spaces with $1 \leq p \leq 2$ and coincides with the bound obtained in [2] for $2 \leq p < \infty$ up to a reasonable constant factor.

There are some other “measures of non-convexity” of the k -convex hull. We refer the reader to the survey [8]. However, they were mostly studied in the Euclidean case.

2 Modulus of Smoothness and Its Properties

In this section, we give definitions from the Banach space theory and provide a certain reformulation of the main result using this language.

The modulus of smoothness (or Lindenstrauss’ modulus) of a Banach space X is the function $\rho_X : [0, \infty] \rightarrow [0, \infty]$ given by

$$\rho_X(\tau) = \sup \left\{ \frac{1}{2} (\|x + \tau y\| + \|x - \tau y\|) - 1 \mid \|x\| = \|y\| \leq 1 \right\}.$$

A Banach space X is called *uniformly smooth* if $\rho_X(t) = o(t)$ as $t \rightarrow 0$. It is known that uniform smoothness is equivalent to the uniform differentiability of the norm. As a good reference with simple geometric explanations of different properties of the modulus of smoothness and uniformly smooth spaces we refer to Chapter 2 of [6].

It is known that the modulus of smoothness $\rho_X(\cdot)$ of a Banach space X is a convex strictly increasing function that satisfies the following inequality of Day–Nordlander type [13] for all positive τ :

$$\sqrt{1 + \tau^2} - 1 = \rho_H(\tau) \leq \rho_X(\tau),$$

where $\rho_H(\tau)$ is the modulus of smoothness of Hilbert space. The latter inequality implies the following technical observations, which we use in the sequel:

$$\frac{2e^2}{\rho_X^{-1}(1)} \geq 1; \quad \frac{2e^2}{\rho_X^{-1}(1/2)} \geq 2; \tag{2.1}$$

and

$$\frac{1}{\rho_X^{-1}(1/k)} \geq 1 \quad \text{for } k \geq 3. \tag{2.2}$$

Clearly, uniform smoothness of the norm is not stable under small perturbations of the norm. However, equivalent renormalization of a space just adds a constant factor to inequalities (1.1), (1.2). That is, we can approximate a point of the convex hull by a point of the k -convex hull of a bounded set in every Banach space, which admits an equivalent uniformly smooth norm. Such spaces are well-studied; moreover, due to Enflo [7] and Pisier [15], we know that these spaces are exactly the so-called super-reflexive Banach spaces. Summarizing their results, the following assertions are equivalent

- a Banach space X admits an equivalent uniformly smooth norm;

- a Banach space X admits an equivalent uniformly smooth norm with modulus of smoothness of *power type* p ;
- a Banach space X is a super-reflexive space.

It is said that a uniformly smooth Banach space X has modulus of smoothness of *power type* p if, for some $0 < C < \infty$, $\rho_X(\tau) \leq C\tau^p$. Now we can reformulate our results in a norm renormalization invariant way. We use $\text{dist}(a, B)$ to denote the distance between a point a and a set B . We define the *deviation* of a set A from a set B as follows:

$$h^+(A, B) = \sup_{a \in A} \text{dist}(a, B).$$

We say that a Banach space X has *Carathéodory’s approximation property* if there is a function $f: \mathbb{N} \rightarrow [0, \infty)$ with $\lim_{k \rightarrow \infty} f(k) = 0$ such that $h^+(\text{conv } S, \text{conv}_k S) \leq f(k)$ for an arbitrary subset S of the unit ball of X . Theorem 1.1 and the discussion above imply the following.

Corollary 2.1 *A super-reflexive Banach space X has Carathéodory’s approximation property: that is, there exists $p \in (1, 2]$ and constant C such that*

$$h^+(\text{conv } S, \text{conv}_k S) \leq \varepsilon,$$

for all $\varepsilon > 0$ and $k \geq C\left(\frac{\text{diam } S}{\varepsilon}\right)^{p/(p-1)}$.

A Banach space X is said to be of *type* p for some $1 < p \leq 2$, if there exists a constant $T_p(X) < \infty$ such that, for every finite set of vectors $\{x_j\}_{j=1}^n$ in X , we have

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\| dt \leq T_p(X) \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p},$$

where $\{r_j\}_{j=1}^\infty$ denotes the sequence of the Rademacher functions.

We can formulate Maurey’s lemma as follows (see also [3, Lem. D]).

Let S be a bounded set in a Banach space X which is of type p , $a \in \text{conv } S$. Set $q = p/(p - 1)$. Then there exists a sequence $\{x_i\}_{i=1}^\infty \subset S$ such that for vectors $a_k = \frac{1}{k} \sum_{i \in [k]} x_i$ the following inequality holds:

$$\|a - a_k\| \leq T_p(X)k^{-1/q} \text{diam } S.$$

Since a Banach space with modulus of smoothness of power type p implies type p , then Theorem 1.1 follows from Maurey’s lemma (up to a constant term in the inequalities). The converse is not true in general (see [12,17]). However, type p with some additional not very restrictive property (see table on p. 101 in [14]) implies modulus of smoothness of power type p . It is known [13] that L_p , $1 < p < \infty$, is of type $q = \max\{p, 2\}$ and has modulus of smoothness of power $\max\{p, 2\}$. More precisely,

$$\rho_{L_p}(t) = \left(\frac{(1+t)^p + (1-t)^p}{2} \right)^{1/p} - 1 < \frac{p-1}{2} t^2 \quad \text{for } 2 \leq p < \infty; \quad (2.3)$$

$$\rho_{L_p}(t) = (1+t^p)^{1/p} - 1 \leq \frac{t^p}{p} \quad \text{for } 1 < p \leq 2. \quad (2.4)$$

Since p in Corollary 2.1 can be chosen to be the order of the modulus of smoothness, we see that $k = O\left(\frac{\text{diam}^2 S}{\varepsilon^2}\right)$ in L_p , $2 \leq p < \infty$, which coincides with the rate of convergence in Maurey’s lemma. Moreover, as was shown in Barman’s paper [2, Thm. 3.2], the constants in both inequalities are close enough to one another in this case.

In case of an L_p space with $1 \leq p \leq 2$, the bound in inequality (1.1) is the best possible up to a constant: when $n = 2k$, $S = \{e_1, \dots, e_{2n}\}$ and $a = \{1/n, \dots, 1/n\}$, then $\text{dist}(a, \text{conv}_k S) \geq \frac{1}{4} k^{1/p-1}$. Moreover, this implies that L_1 and L_∞ have no Carathéodory’s approximation property.

3 Geometrical Idea Behind the Proof

We begin with a sketch of the proof of the following folklore analogue of the main theorem in a Euclidean space (see [5,18]).

Let S be a subset of the unit ball of a Euclidean space such that $0 \in \text{conv } S$. Then there exists a sequence $\{x_i\}_{i=1}^\infty \subset S$ such that

$$\left\| \sum_{i=1}^k x_i \right\| \leq \sqrt{k}$$

for all $k \in \mathbb{N}$.

We apply induction on k . Assume that we have chosen x_1, \dots, x_n that satisfy the inequality for all $k \in [n]$. Since $0 \in \text{conv } S$, there exists x_{n+1} such that $\langle x_{n+1}, \sum_{i=1}^n x_i \rangle \leq 0$. Then, by the law of cosines,

$$\left\| \sum_{i=1}^n x_i + x_{n+1} \right\| \leq \sqrt{\left\| \sum_{i=1}^n x_i \right\|^2 + \|x_{n+1}\|^2} \leq \sqrt{n+1}.$$

The statement is proven.

In fact, the law of cosines can be reformulated in terms of the deviation of the unit sphere from its supporting hyperplane as follows.

We use u^* to denote a unit functional that attains its norm on a non-zero vector u of a Banach space X , i.e., $\langle u^*, u \rangle = \|u\| \|u^*\| = \|u\|$. Clearly, a set $\{x \in X \mid \langle u^*, x \rangle = 1\}$ is a supporting hyperplane to the unit ball of X at $u/\|u\|$. For simplicity, we assume that $u^* = 0$ for $u = 0$. Let H be a supporting hyperplane at a unit vector u of the unit ball in a Euclidean space E and x be such that $\langle u^*, x \rangle \leq 0$. Then the norm of $\|u + x\|$

is at most $\sqrt{1 + \|x\|^2}$ ($\approx 1 + \rho_E(\|x\|)$ for sufficiently small $\|x\|$) with equality only for x parallel to H . That is, we see that vectors from the supporting hyperplane spoil the sum most badly and we measure the deviation of the unit sphere from a supporting hyperplane to bound the norm of the sum on each step. Similar statements can be proven in a uniformly smooth Banach space. However, it is not necessarily true that for an arbitrary unit vector u of a Banach space the maximum of norm $\|u + v\|$, where $\|v\|$ is fixed and $\langle u^*, v \rangle \leq 0$, is attained on the supporting hyperplane to the unit ball at u . But one can measure this deviation using the modulus of smoothness, which we do in the following simple lemma.

Lemma 3.1 *Let u be a unit vector of Banach space X and $x \in X$ such that $\langle u^*, x \rangle \leq 0$. Then*

$$\|u + x\| \leq 2\rho_X(\|x\|) + 1.$$

Proof Since $\langle u^*, x \rangle \leq 0$ and $H_u = \{q \in X \mid \langle u^*, q \rangle = 1\}$ is a supporting hyperplane for the unit ball of X , we have that $\|u - x\| \geq 1$. Therefore, by the definition of the modulus of smoothness, we obtain

$$\|u + x\| + \|u - x\| \leq 2\rho_X(\|x\|) + 2,$$

or, equivalently,

$$\|u + x\| \leq 2\rho_X(\|x\|) + 2 - \|u - x\| \leq 2\rho_X(\|x\|) + 1. \quad \square$$

Our proof follows the same line as in the above-mentioned Euclidean analogue with the use of Lemma 3.1 instead of the law of cosines.

Remark 3.2 The deviation of the unit sphere from the supporting hyperplane in a Banach space was studied by the author in [9]. In particular, it was shown that the inequality from Lemma 3.1 is asymptotically tight as $\|x\|$ tends to zero. On the other hand, it was proven in [11] that for any τ in an arbitrary Banach space, there is a unit vector u and a vector x parallel to a supporting hyperplane to the unit ball at u such that $\|u + x\| = \sqrt{1 + \|x\|^2} = \sqrt{1 + \tau^2}$. That is, the smallest upper bound is attained in the Euclidean case.

4 Proof of the Main Result

By setting $S_i = S$ for all $i \in \mathbb{N}$, we see that Corollary 1.2 implies Theorem 1.1. We give the proof of the corollary, which coincides with the proof of the theorem up to above-mentioned renaming of sets.

Proof To simplify the proof, we translate S_i to $S_i - a$ and then scale them in such a way that $D = \sup_i \text{diam } S_i = 1$. Then inequality (1.2) transforms into

$$\|a_k\| \leq \frac{2e^2}{k\rho_X^{-1}(1/k)}. \tag{4.1}$$

Let us construct a sequence $\{x_i\}_{i=1}^\infty$, where $x_i \in S_i$, that satisfies the latter inequality. We use the following **algorithm**:

- (1) x_1 is an arbitrary point from S_1 ; x_2 is an arbitrary point from S_2 .
- (2) For the constructed sequence $\{x_1, \dots, x_k\}$, $k \geq 2$, we choose $x_{k+1} \in S_{k+1}$ such that $\langle x_{k+1}, a_k^* \rangle \leq 0$ (that is, x_{k+1} is an arbitrary point of S_{k+1} if $a_k = 0$).

Firstly, the sequence $\{x_i\}_{i=1}^\infty$ is well defined. Indeed, there exists $q \in S_{k+1}$ such that $\langle q, p \rangle \leq 0$ for an arbitrary functional $p \in X^*$, since $0 \in \text{conv } S_{k+1}$. In the algorithm we choose a functional $p = a_k^*$ such that $\|a_k^*\| = 1$ and for $a_k = \frac{1}{k} \sum_{i \in [k]} x_i$, $a_k \neq 0$, the equality $\langle a_k^*, a_k \rangle = \|a_k\|$ is valid.

Secondly, let us show that $\{x_i\}_{i=1}^\infty$ satisfies inequality (4.1). Fix k . By inequality (2.1), we assume that $k \geq 3$. By definition put $\eta_k = 1/\rho_X^{-1}(1/k)$ and $u_k = ka_k$. If inequality $\|u_k\| \leq \eta_k$ holds, it implies the required inequality. Assume that $\|u_k\| > \eta_k$. Then let $m - 1$ be the biggest number in $[k - 1]$ such that $\|u_{m-1}\| \leq \eta_k$ and $\|u_m\| > \eta_k$. Such m exists since, by inequality (2.2), we have that $\|u_1\| = \|x_1\| \leq \eta_k$. Since $\eta_k \geq 1$ for $k \geq 3$, we have

$$\|u_m\| \leq \eta_k + 1 \leq 2\eta_k. \tag{4.2}$$

By the choice of m , we have that $\|u_{m+\ell-1}\| > \eta_k \geq 1$ for all $\ell \in [k - m]$. Hence, by the definition of η_k and the construction of $\{x_i\}_{i=1}^\infty$, we get

$$\begin{aligned} \|u_{m+\ell}\| &= \|u_{m+\ell-1} + x_{m+\ell}\| \\ &= \|u_{m+\ell-1}\| \left\| \frac{u_{m+\ell-1}}{\|u_{m+\ell-1}\|} + \frac{x_{m+\ell}}{\|u_{m+\ell-1}\|} \right\| \\ &\stackrel{\text{Lemma 3.1}}{\leq} \|u_{m+\ell-1}\| \left(1 + 2\rho_X \left(\frac{\|x_{m+\ell}\|}{\|u_{m+\ell-1}\|} \right) \right) \\ &\leq \|u_{m+\ell-1}\| \left(1 + 2\rho_X \left(\frac{1}{\eta_k} \right) \right) \\ &= \|u_{m+\ell-1}\| \left(1 + \frac{2}{k} \right). \end{aligned}$$

Combining these inequalities for all $\ell \in [k - m]$ and inequality (4.2), we obtain

$$\|u_k\| \leq \|u_m\| \left(1 + \frac{2}{k} \right)^{k-m} \leq 2e^2\eta_k.$$

Therefore, $a_k = u_k/k$ satisfies inequality (4.1). The theorem is proven. □

We note here that the authors used a similar idea in [10] to prove the convexity of a special type limit object in a uniformly smooth Banach space.

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