

# Covering a Ball by Smaller Balls

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**Abstract** We prove that, for any covering of a unit  $d$ -dimensional Euclidean ball by smaller balls, the sum of radii of the balls from the covering is greater than  $d$ . We also investigate the problem of finding lower and upper bounds for the sum of powers of radii of the balls covering a unit ball.

**Keywords** Euclidean ball · Sphere covering · Covering by homothets

**Mathematics Subject Classification** 05B40 · 52C17

## 1 Introduction

Let  $B \subset \mathbb{R}^d$  be a convex closed body. We say that the family of homothets  $\mathcal{F} = \{\lambda_1 B, \lambda_2 B, \dots\}$ ,  $\lambda_i \in (0, 1)$ , forms a translative covering of  $B$  if  $B \subseteq \bigcup_i (\lambda_i B + x_i)$ , where  $x_i$  are translation vectors in  $\mathbb{R}^d$ . A general question is to find necessary conditions on coefficients  $\lambda_i$  for existence of a translative covering. In 1990, Soltan formulated the following conjecture which was also stated in the book of Brass et al. (see [2, Sect. 3.2, Conj. 2]).

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**Conjecture 1.1** (Soltan) *For any covering of a convex body  $B \subset \mathbb{R}^d$  by its translative homothets with coefficients  $\lambda_1, \dots, \lambda_k$ ,*

$$\sum_{i=1}^k \lambda_i \geq d.$$

Following [5], we define

$$g(B) := \inf \left\{ \sum_{i=1}^k \lambda_i : B \subseteq \bigcup_{i=1}^k (\lambda_i B + x_i), \lambda_i \in (0, 1), x_i \in \mathbb{R}^d \right\}$$

and

$$g(d) = \inf \{g(B) : B \subset \mathbb{R}^d, B \text{ is a convex body}\}.$$

Then Conjecture 1.1 may be reformulated simply as  $g(d) \geq d$ .

In [8], Conjecture 1.1 was proven for the case  $d = 2$  and all convex bodies, for which there exists a covering with the sum of coefficients equal to 2, were found. In [5], the asymptotic version of the conjecture was proven, namely, it was shown that  $\lim_{d \rightarrow \infty} g(d)/d = 1$ .

In this paper, we prove Conjecture 1.1 for the case of a  $d$ -dimensional ball  $\mathbb{B}^d$  in Euclidean spaces of all dimensions  $d$ .

The celebrated result of Coxeter et al. [3] gives the lower bound  $O(d)$  on the density of coverings with sufficiently small spherical caps. Several papers such as [1, 4, 6, 7, 9] give upper bounds, typically,  $O(d \log d)$ , on the density of coverings or on the necessary number of caps given certain restrictions on caps' radii. From this point of view, the result of the paper is a rare exact lower bound for coverings of a sphere by spherical caps.

**Theorem 1.2** *In any covering of the unit Euclidean sphere in  $\mathbb{R}^d$ ,  $d \geq 2$ , by closed spherical caps smaller than a half-sphere, the sum of Euclidean radii of the caps is greater than  $d$ .*

Conjecture 1.1 for balls follows from Theorem 1.2 immediately. Constructing certain coverings of  $\mathbb{B}^d$  we can find the value of  $g(\mathbb{B}^d)$  precisely.

**Corollary 1.3** *For  $d \geq 2$ ,  $g(\mathbb{B}^d) = d$ .*

The paper is organized as follows. In Sect. 2, we will show how to prove Theorem 1.2 and Corollary 1.3. In Sect. 3, we will discuss similar problems concerning the sum of powers of radii in a covering of a ball by smaller balls.

## 2 Proof of the Main Theorem

*Proof of Theorem 1.2* Throughout the proof, unless it is stated otherwise, by the radius of a spherical cap we always mean its Euclidean radius and by the center of a cap we

mean its Euclidean center (Euclidean center of the cap’s boundary). From the very beginning, we can assume that none of the caps of a covering belongs to the union of all other caps. We also note that the number of caps in a covering is always at least  $d + 1$ . This may be proven, for instance, by induction: for any cap, a unit subsphere of codimension 1 not intersecting it must be covered by at least  $d$  other caps.

We will prove the theorem by induction for  $d \geq 2$ . In the case  $d = 2$ , any cap of radius  $r$  (a chord with length  $2r$ ) corresponds to a circular arc of length less than  $\pi r$ . Since the sum of lengths of such arcs is at least  $2\pi$ , the sum of radii is greater than 2.

For the induction step, we assume that the statement is true for  $d - 1$ ,  $d \geq 3$ , and aim to prove it for  $d$ .

If there are two non-intersecting spherical caps with the sum of radii at least 1, we consider a unit subsphere of codimension 1 separating them. By the induction hypothesis, the sum of radii of spherical caps covering it must be greater than  $d - 1$  and, in total, the sum of radii of all caps is greater than  $d$ . From this moment on we assume there are no pairs of caps like this.

If, on the other hand, there are two intersecting spherical caps with the sum of radii less than 1, we can substitute them by a bigger spherical cap covering both of them with a radius less than the sum of their radii. If the statement of the theorem is true after the substitution, then it was true before the substitution as well. By making substitutions of this kind, we can guarantee there are no pairs of caps like this either.

Consider any maximal spherical cap  $C_{\max}$  of a given covering  $\mathcal{C}$  of a unit sphere  $\mathbb{S}^{d-1}$ . Denote the boundary of this spherical cap by  $S$  and denote its radius by  $R$ .

By  $S'$  we denote the sphere centrally symmetric to  $S$  with respect to the center of  $\mathbb{S}^{d-1}$ . We claim that it is sufficient to consider only the situation when all other spherical caps of the covering intersect both  $S$  and  $S'$ . Consider an arbitrary cap  $C$  from  $\mathcal{C}$ . Since  $C$  intersects some other cap, the sum of the radii of  $C$  and this cap must be at least 1. Hence the sum of the radii of  $C$  and  $C_{\max}$  must be at least 1 and they must intersect. If  $C$  does not intersect  $S'$ , then, analogously to the case shown above, we can substitute

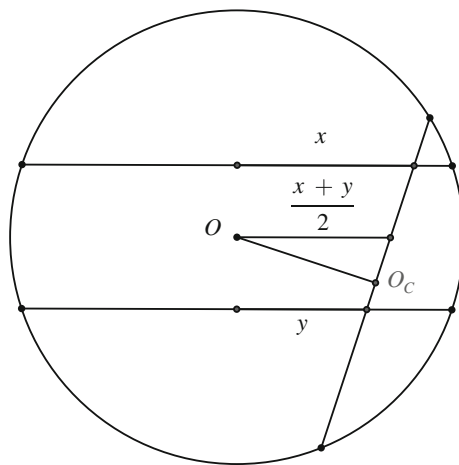


Fig. 1  $\frac{x+y}{2} \geq OO_C$

$C_{\max}$  and  $C$  by a cap covering both of them. Due to the triangle inequality, the radius of the substituting cap will be smaller than the sum of the radii of  $C_{\max}$  and  $C$ . By making such substitutions we can obtain a covering where the required condition holds.

Now assume that the radius of the cap  $C$  intersecting  $S$  and  $S'$  is  $r$ . Denote the distance from the center of  $S$  to the center of  $S \cap C$  by  $x$  and from the center of  $S'$  to the center of  $S' \cap C$  by  $y$ . Then the distance from the center of  $\mathbb{S}^{d-1}$  to the center of  $C$  is not greater than  $(x + y)/2$  (see Fig. 1 with the orthogonal projection along  $S \cap C$ ). Hence  $((x + y)/2)^2 \geq 1 - r^2$ . From this inequality and Jensen’s inequality for the concave function  $f(t) = \sqrt{R^2 - t^2}$ , we get

$$\sqrt{R^2 - x^2} + \sqrt{R^2 - y^2} \leq 2\sqrt{R^2 - \left(\frac{x + y}{2}\right)^2} \leq 2\sqrt{R^2 + r^2 - 1}. \tag{1}$$

We note that the left hand side of this inequality contains the sum of radii of  $S \cap C$  and  $S' \cap C$ .

Assume we have  $k$  caps  $C_1, \dots, C_k$  in the covering, not including  $C_{\max}$ , and define  $r_i, x_i, y_i$  for them just as above. Summing up the inequalities (1) for all these caps, we get

$$\sum_{i=1}^k \left(\sqrt{R^2 - x_i^2} + \sqrt{R^2 - y_i^2}\right) \leq 2 \sum_{i=1}^k \sqrt{R^2 + r_i^2 - 1}.$$

The left hand side of this inequality is the sum of radii of the coverings of  $S$  and  $S'$  and, by the induction hypothesis, it must be greater than  $2(d - 1)R$ . Therefore,

$$(d - 1)R < \sum_{i=1}^k \sqrt{R^2 + r_i^2 - 1}. \tag{2}$$

Using Jensen’s inequality for the concave function  $g(t) = \sqrt{t^2 - (1 - R^2)}$  and the fact that  $k \geq d$ ,

$$\begin{aligned} \sum_{i=1}^k \sqrt{R^2 + r_i^2 - 1} &\leq k\sqrt{\left(\frac{\sum_{i=1}^k r_i}{k}\right)^2 + R^2 - 1} = \sqrt{\left(\sum_{i=1}^k r_i\right)^2 - k^2(1 - R^2)} \\ &\leq \sqrt{\left(\sum_{i=1}^k r_i\right)^2 - d^2(1 - R^2)}. \end{aligned} \tag{3}$$

Combining inequalities (2) and (3), we get

$$\sqrt{\left(\sum_{i=1}^k r_i\right)^2 - d^2(1 - R^2)} > (d - 1)R, \text{ so } \sum_{i=1}^k r_i > \sqrt{(d - 1)^2 R^2 + d^2(1 - R^2)}.$$

Therefore, the sum of radii of all caps in the covering satisfies the inequality

$$R + \sum_{i=1}^k r_i > R + \sqrt{(d - 1)^2 R^2 + d^2(1 - R^2)},$$

which is at least  $d$  for any  $R \in [0, 1]$ . □

*Proof of Corollary 1.3* Theorem 1.2 implies that  $g(\mathbb{B}^d) \geq d$ . In order to prove the equality, it is sufficient to show that, for any given  $\varepsilon > 0$ , there is a set of balls covering the unit ball with the sum of radii less than  $d + \varepsilon$ .

Fix a positive  $\delta < 1/d$ . In a  $(d - 1)$ -dimensional subspace (all points with the last coordinate 0) consider the sphere  $S_\delta$  with the center at the origin and radius  $\delta$ . We choose points  $v_1, \dots, v_d$  on  $S_\delta$  so that they form a regular  $(d - 1)$ -dimensional simplex. We also take points  $v_+ = (0, \dots, 0, \sqrt{1 - (d - 1)^2\delta^2})$  and  $v_- = -v_+$ . We claim that the set consisting of  $d$  balls with centers at  $v_1, \dots, v_d$  and radius  $1 - \delta^2/2$  and two balls with centers at  $v_+, v_-$  and radius  $d\delta$  covers the  $d$ -dimensional unit ball with the center at the origin.

Consider an arbitrary point  $u$  such that one of the angles  $\angle(uv_i0)$  is obtuse. Then in the case  $u$  does not belong to a ball with the center  $v_i$ , we have  $u0^2 > uv_i^2 + v_i0^2 > (1 - \delta^2/2)^2 + \delta^2 > 1$ .

For each  $i$ ,  $\angle(uv_i0)$  is not obtuse if  $u$  belongs to the half-space formed by the hyperplane through  $v_i$  and perpendicular to  $0v_i$ . The intersection of these half-spaces is an infinite cylinder. The last coordinate axis is the axis of this cylinder. The base of the cylinder is formed by the regular simplex dual to the simplex formed by all  $v_i$  with respect to  $S_\delta$ . The distance from the vertices of the base to the origin is  $(d - 1)\delta$ . Hence the intersections of the cylinder at the base vertices with the unit sphere will be distant from the axis of the last coordinate by  $(d - 1)\delta$  and will have the last coordinate  $\pm\sqrt{1 - (d - 1)^2\delta^2}$ .

Now we can split the infinite cylinder into three parts: the part with the last coordinate at least  $\sqrt{1 - (d - 1)^2\delta^2}$ , the part with the last coordinate at most  $-\sqrt{1 - (d - 1)^2\delta^2}$  and the triangular prism between the hyperplanes defined by the last coordinate  $\pm\sqrt{1 - (d - 1)^2\delta^2}$ .

The first and the second part will be covered by balls with the centers  $v_+$  and  $v_-$ , respectively. These balls actually cover the spherical caps formed by the hyperplanes defined by the last coordinate  $\pm\sqrt{1 - (d - 1)^2\delta^2}$ .

In order to show that the prism is covered completely we divide it into  $d$  subprisms formed by the origin and each facet of the dual  $(d - 1)$ -dimensional simplex described above. The center of such facet is a point  $v_i$  for some  $i$ . For  $d > 2$ , the square of the distance from  $v_i$  to any vertex of the corresponding subprism is not greater than

$$((d - 1)^2\delta^2 - \delta^2) + (1 - (d - 1)^2\delta^2) = 1 - \delta^2 < \left(1 - \frac{1}{2}\delta^2\right)^2.$$

Therefore, the subprism is covered by the corresponding ball with the center  $v_i$ .

For the case  $d = 2$ , the same construction works but one needs to consider trapezoids formed by points  $(0, \pm\sqrt{1 - \delta^2} \mp 2\delta)$  instead of rectangles (subprisms).

The sum of radii of the balls in the family is  $d(1 - \delta^2/2) + 2d\delta$ , which is less than  $d + \varepsilon$  for a sufficiently small  $\delta$ . □

### 3 Discussion

Similarly to  $g(B)$  and  $g(d)$ , we can define  $g_\alpha(B)$  and  $g_\alpha(d)$  for the sums of powers of the homothety coefficients:

$$g_\alpha(B) := \inf \left\{ \sum_{i=1}^k \lambda_i^\alpha : B \subseteq \bigcup_{i=1}^k (\lambda_i B + x_i), \lambda_i \in (0, 1), x_i \in \mathbb{R}^d \right\},$$

and

$$g_\alpha(d) = \inf \{ g_\alpha(B) : B \subset \mathbb{R}^d, B \text{ is a convex body} \}.$$

The result from [5] may be formulated in these terms as  $\lim_{d \rightarrow \infty} g_\alpha(d)/d = 1$  for any fixed  $\alpha \geq 1$ .

Results somewhat similar to Theorem 1.2 and Corollary 1.3 hold for certain other values of  $\alpha$  as well. It is fairly easy to show that (1) for any  $\alpha > d$  and any natural  $d$ ,  $g_\alpha(\mathbb{B}^d) = 0$ , (2) for any  $\alpha \in (d - 1, d]$  and any natural  $d$ ,  $g_\alpha(\mathbb{B}^d) = 1$ , (3) for any  $\alpha \in (d - 2, d - 1]$  and any natural  $d \geq 2$ ,  $g_\alpha(\mathbb{B}^d) = 2$ .

Interestingly, for all these cases  $g_\alpha(\mathbb{R}^d) = d + 1 - \lceil \alpha \rceil$  (unless  $\alpha > d + 1$ , when the value of  $g$  is 0). This motivates us to formulate the following conjecture.

**Conjecture 3.1** *For all natural  $d$  and all  $\alpha$  such that  $0 \leq \alpha \leq d + 1$ ,  $g_\alpha(\mathbb{B}^d) = d + 1 - \lceil \alpha \rceil$ .*

Combining our results, we conclude that Conjecture 3.1 is true when  $\alpha \in [0, 1] \cap (d - 2, d + 1]$  for any  $d \geq 2$ .

We do not know how to prove that  $d + 1 + \lceil \alpha \rceil$  is a lower bound for  $g_\alpha(\mathbb{R})$ , still we are able to confirm it as an upper bound.

**Theorem 3.2** *For all natural  $d$  and all  $\alpha$  such that  $0 \leq \alpha \leq d + 1$ ,  $g_\alpha(\mathbb{B}^d) \leq d + 1 - \lceil \alpha \rceil$ .*

*Proof* Assume  $\alpha \in (n, n + 1]$ , where  $n \in \mathbb{N}$ . We will use a construction generalizing the construction from Corollary 1.3. We take a  $(d - n - 1)$ -dimensional space  $\Pi$  containing the center 0 of the unit ball  $\mathbb{B}^d$  and consider a sphere  $S_\delta$  in this space with center at 0 and radius  $\delta > 0$ . We select  $d - n$  points from  $S_\delta$  forming a regular simplex. Now for our covering we choose equal balls with centers at these points and radius  $1 - \delta^2/2$ . By  $S_\perp$  we denote the intersection of the orthogonal complement  $\Pi^\perp$  and the boundary of  $\mathbb{B}^d$ . The set of points not covered by the balls we have already chosen is an  $O(\delta)$ -neighborhood of  $S^\perp$ . Since  $S^\perp$  belongs to the  $n$ -dimensional space  $\Pi^\perp$ , we can choose its covering with sufficiently small spheres with density  $O(n \log n)$  and obtain a sufficiently small contribution of this covering into the sum of powers of radii. Overall, we will get the sum of powers of radii as close to  $d - n$  as we wish.  $\square$

Concluding the paper, we would like to make a general conjecture that the same lower bounds as in Conjecture 3.1 hold for all convex bodies.

**Conjecture 3.3** *For all natural  $d$  and all  $\alpha$  such that  $0 \leq \alpha \leq d + 1$ ,  $g_\alpha(d) = d + 1 - \lceil \alpha \rceil$ .*

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