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Collapsibility to a Subcomplex of a Given Dimension is NP-Complete

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Abstract In this paper we extend the works of Tancer, Malgouyres and Francés, showing that (d, k)-COLLAPSIBILITY is NP-complete for $d \geq k + 2$ except $(2, 0)$. By (d, k)-COLLAPSIBILITY we mean the following problem: determine whether a given d-dimensional simplicial complex can be collapsed to some k-dimensional subcomplex. The question of establishing the complexity status of (d, k)-COLLAPSIBILITY was asked by Tancer, who proved NP-completeness of $(d, 0)$ and $(d, 1)$-COLLAPSIBILITY (for $d \geq 3$). Our extended result, together with the known polynomial-time algorithms for $(2, 0)$ and $d = k + 1$, answers the question completely.

Keywords Simplicial complexes · Collapsibility · Discrete Morse theory · NP-hardness

Mathematics Subject Classification 05E45 · 68Q17

1 Introduction

Discrete Morse theory is a powerful combinatorial tool which allows to explicitly simplify cell complexes while preserving their homotopy type [1,4,6,9]. This is obtained through a sequence of "elementary collapses" of pairs of cells. Such a process might decrease the dimension of the starting complex, or sometimes even leave a single point (in which case we say that the starting complex was collapsible).

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The problem of algorithmically recognising collapsibility, or finding "good" sequences of elementary collapses, has been studied extensively [2,3,5,8,10,11]. Such problems proved to be computationally hard even for low-dimensional simplicial complexes. For 2-dimensional complexes there exists a polynomial-time algorithm to check collapsibility [8,10], but finding the minimum number of "critical" triangles (without which the remaining complex would be collapsible) is already NP-hard [5]. In dimension ≥ 3, collapsibility to some 1-dimensional subcomplex [10] or even to a single point [11] were proved to be NP-complete.

In [11], Tancer also introduced the general (d, k)-Collapsibility problem: determine whether a d-dimensional simplicial complex can be collapsed to some k-dimensional subcomplex. He showed that (d, k)-Collapsibility is NP-complete for $k \in \{0, 1\}$ and $d \geq 3$, extending the result of Malgouyres and Francés about NP-completeness of $(3, 1)$-Collapsibility [10]. Tancer also pointed out that the codimension 1 case ($d = k + 1$) is polynomial-time solvable as is the $(2, 0)$ case. He left open the question of determining the complexity status of (d, k)-Collapsibility in general.

In this short paper we extend Tancer's work, and prove that (d, k)-Collapsibility is NP-complete in all the remaining cases.

Theorem 1.1 *The* (d, k)-Collapsibility *problem is NP-complete for* $d \geq k + 2$*, except for the case* $(2, 0)$.

To do so, we prove that (d, k)-Collapsibility admits a polynomial-time reduction to $(d + 1, k + 1)$-Collapsibility (Theorem 3.1). Then the main result follows by induction. The base cases of the induction are given by NP-completeness of $(3, 1)$-Collapsibility (for codimension 2) and of $(d, 0)$-Collapsibility (for codimension $d \geq 3$).

2 Collapsibility and Discrete Morse Theory

We refer to [7] for the definition and the basic properties of simplicial complexes, and to [9] for the definition of elementary collapses. The simplicial complexes we consider do not contain the empty simplex, unless otherwise stated.

Our focus is the following decision problem.

Problem 1: (d, k)-Collapsibility**.**

Parameters:	Non-negative integers $d > k$.
Instance:	A finite d-dimensional simplicial complex X.
Question:	Can X be collapsed to some k-dimensional subcomplex?

We are now going to recall a few definitions of discrete Morse theory [4,6,9], so that we can restate the (d, k)-Collapsibility problem in terms of acyclic matchings.

Given a simplicial complex X, its *Hasse diagram* $H(X)$ is a directed graph in which the set of nodes is the set of simplexes of X, and an arc goes from σ to τ if and only if τ is a face of σ and $\dim(\sigma) = \dim(\tau) + 1$. We denote such an arc by $\sigma \to \tau$. A *matching* $\mathcal{M}$ on X is a set of arcs of $H(X)$ such that every node of $H(X)$ (i.e. every simplex of X) is contained in at most one arc in $\mathcal{M}$. Given a matching $\mathcal{M}$ on X, we

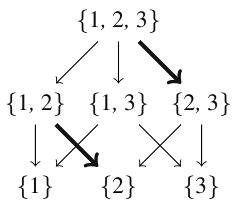

Fig. 1 An acyclic matching on the full simplicial complex on three vertices, with critical simplices {1, 3}, {1}, {3}

say that a simplex $\sigma \in X$ is *critical* if it does not belong to any arc in $\mathcal{M}$. Finally we say that a matching $\mathcal{M}$ on X is *acyclic* if the graph $H(X)^{\mathcal{M}}$, obtained from $H(X)$ by reversing the direction of each arc in $\mathcal{M}$, does not contain directed cycles.

Notice that the empty set is always a valid acyclic matching, for which all simplices are critical. See Fig. 1 for an example of a non-trivial acyclic matching on the full triangle.

By standard facts of discrete Morse theory (see for instance [9], Sect. 11.2), "collapsibility to some k-dimensional subcomplex" is equivalent to "existence of an acyclic matching such that the critical cells form a k-dimensional subcomplex". Notice that, given an acyclic matching $\mathcal{M}$ with no critical simplices of dimension $> k$, one can always remove from $\mathcal{M}$ the arcs between simplices of dimension $\leq k$ and obtain an acyclic matching where the critical simplices form a k-dimensional subcomplex.

Therefore the collapsibility problem can be restated as follows.

Problem 2: (d, k)-COLLAPSIBILITY (equivalent form)**.**

Parameters:	Non-negative integers $d > k$.
Instance:	A finite d-dimensional simplicial complex X.
Question:	Does X admit an acyclic matching such that all critical simplices have dimension $\leq k$?

To simplify the proof of Theorem 3.1 we quote the following useful result from [9], adapting it to our notation.

Theorem 2.1 (Patchwork theorem [9, Theorem 11.10]) *Let P be a poset. Let $\varphi\colon X \to P$ be an order-preserving map (where X is ordered by inclusion), and assume to have acyclic matchings on subposets $\varphi^{-1}(p)$ for all $p \in P$. Then the union of these matchings is itself an acyclic matching on X.*

Notice that the subposets $\varphi^{-1}(p)$ are not subcomplexes of X in general, but they still have a well-defined Hasse diagram (the induced subgraph of $H(X)$). Thus all the previous definitions (matching, critical simplex, acyclic matching) apply also to each subposet.

3 Main Result

Theorem 3.1 *Let $d > k \geq 0$. Then there is a polynomial-time reduction from (d, k)-*COLLAPSIBILITY *to $(d + 1, k + 1)$-*COLLAPSIBILITY.

Proof Let X be an instance of (d, k)-COLLAPSIBILITY, i.e. a d-dimensional simplicial complex. Let $V = \{v_1, \ldots, v_r\}$ be the vertex set of X. Construct an instance X' of $(d+1, k+1)$-COLLAPSIBILITY, i.e. a $(d+1)$-dimensional complex, as follows. Let $n \geq 1$ be the number of simplices in X. Introduce new vertices $w_1, \ldots, w_{n+1}$, and define X' as the simplicial complex on the vertex set $V' = \{v_1, \ldots, v_r, w_1, \ldots, w_{n+1}\}$ given by

$$X' = X \cup \Big\{\sigma \cup \{w_i\} \Big| \sigma \in X,\ i = 1, \ldots, n+1\Big\}.$$

Then X' has $n(n+2)$ simplices. Roughly speaking, X' is obtained from X by attaching $n+1$ cones of X to X. We are going to prove that X is a yes-instance of (d, k)-COLLAPSIBILITY if and only if X' is a yes-instance of $(d+1, k+1)$-COLLAPSIBILITY.

Suppose that X is a yes-instance of (d, k)-COLLAPSIBILITY. Then there exists an acyclic matching $\mathcal{M}$ on X such that all critical simplices have dimension $\leq k$. Construct a matching $\mathcal{M}'$ on X' as follows:

$$\begin{aligned}\mathcal{M}' = &\Big\{\sigma \cup \{w_1\} \to \sigma \Big| \sigma \in X\Big\} \cup \\ &\Big\{\sigma \cup \{w_i\} \to \tau \cup \{w_i\} \Big| (\sigma \to \tau) \in \mathcal{M},\ i = 2, \ldots, n+1\Big\}.\end{aligned}$$

This matching corresponds to collapsing the first cone together with X, and every other “base-less” cone by itself (as a copy of X). Notice that the critical simplices of $\mathcal{M}'$ do not form a subcomplex of X', even when the critical simplices of $\mathcal{M}$ form a subcomplex of X.

To prove that $\mathcal{M}'$ is acyclic, consider the set $P = \{w_1, \ldots, w_{n+1}\}$ with the partial order

$$w_i < w_j \text{ if and only if } i = 1 \text{ and } j > 1.$$

Let $\varphi\colon X' \to P$ be the order-preserving map given by

$$\varphi(\sigma) = \begin{cases} w_j & \text{if } \sigma \text{ contains } w_j \text{ for some } j \geq 2; \\ w_1 & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}'$ is a union of matchings $\mathcal{M}'_j$ on each fiber $\varphi^{-1}(w_j)$. The matching $\mathcal{M}'_1$ is acyclic on $\varphi^{-1}(w_1)$, since the arcs of $\mathcal{M}'_1$ define a cut of the Hasse diagram of $\varphi^{-1}(w_1)$. The Hasse diagram of each $\varphi^{-1}(w_j)$ for $j \geq 2$ is isomorphic to $H(X \cup \{\varnothing\})$ via the map $\sigma \cup \{w_j\} \mapsto \sigma$, and the matching $\mathcal{M}_j$ maps to $\mathcal{M}$. Since $\mathcal{M}$ is acyclic on $H(X)$, each $\mathcal{M}_j$ is also acyclic on $\varphi^{-1}(w_j)$. By the Patchwork Theorem (Theorem 2.1), $\mathcal{M}'$ is acyclic on X'.

The set of critical simplices of $\mathcal{M}'$ is

$$\mathrm{Cr}(X', \mathcal{M}') = \{w_1\} \cup \Big\{\sigma \cup \{w_i\} \Big| \sigma \in \mathrm{Cr}(X, \mathcal{M}) \cup \{\varnothing\},\ i = 2, \ldots, n+1\Big\}.$$

In particular, all critical simplices have dimension $\leq k+1$. Therefore X' is a yes-instance of $(d+1, k+1)$-COLLAPSIBILITY.

Conversely, suppose now that X' is a yes-instance of $(d+1, k+1)$-COLLAPSIBILITY. Let $\mathcal{M}'$ be an acyclic matching on X' such that all critical simplices have dimension $\leq k+1$. Since X contains n simplices, and there are $n+1$ cones, there must exist an index $j \in \{1, \ldots, n+1\}$ such that

$$\left(\sigma \cup \{w_j\} \to \sigma\right) \notin \mathcal{M}' \quad \forall\, \sigma \in X.$$

In other words, simplices containing w_j are only matched with simplices containing w_j. Then we can construct a matching $\mathcal{M}$ on X as follows:

$$\mathcal{M} = \left\{\sigma \to \tau \,\middle|\, \sigma, \tau \in X \text{ satisfying } \left(\sigma \cup \{w_j\} \to \tau \cup \{w_j\}\right) \in \mathcal{M}'\right\}.$$

Notice that if there is some 0-dimensional simplex $\sigma = \{v\} \in X$ such that $(\{v, w_j\} \to \{w_j\}) \in \mathcal{M}'$, then $\{v\}$ is critical with respect to $\mathcal{M}$ (it would be matched with $\tau = \varnothing$, which does not exist in X). The Hasse diagram of X injects into the Hasse diagram of the j-th cone via the map

$$\iota \colon \sigma \mapsto \sigma \cup \{w_j\},$$

and by construction $\mathcal{M}$ maps to $\mathcal{M}'$. Since $\mathcal{M}'$ is acyclic, $\mathcal{M}$ is also acyclic. The set of critical simplices of $\mathcal{M}$ is

$$\mathrm{Cr}(X, \mathcal{M}) = \left\{\sigma \in X \,\middle|\, \sigma \cup \{w_j\} \in \mathrm{Cr}(X', \mathcal{M}') \text{ or } \left(\sigma \cup \{w_j\} \to \{w_j\}\right) \in \mathcal{M}'\right\}.$$

In the first case $\sigma \cup \{w_j\}$ has dimension $\leq k+1$, and in the second case σ is 0-dimensional. In particular, all critical simplices have dimension $\leq k$. Therefore X is a yes-instance of (d, k)-COLLAPSIBILITY. □

The (d, k)-COLLAPSIBILITY problem admits a polynomial-time solution when $d = k+1$ and also for the case $(2, 0)$ [8,10,11]. Malgouyres and Francés [10] proved that $(3, 1)$-COLLAPSIBILITY is NP-complete, and Tancer [11] extended this result to (d, k)-COLLAPSIBILITY for $k \in \{0, 1\}$ and $d \geq 3$. Using this as the base step and Theorem 3.1 as the induction step, we obtain the following result.

Theorem 3.2 *The* (d, k)-COLLAPSIBILITY *problem is NP-complete for* $d \geq k+2$*, except for the case* $(2, 0)$.

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