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Proof of a Conjecture of Bárány, Katchalski and Pach

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Abstract Bárány, Katchalski and Pach (Proc Am Math Soc 86(1):109–114, 1982) (see also Bárány et al., Am Math Mon 91(6):362–365, 1984) proved the following quantitative form of Helly's theorem. If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. In Bárány et al. (Am Math Mon 91(6):362–365, 1984), the bound $v(d) \leq d^{2d^2}$ was proved and $v(d) \leq d^{cd}$ was conjectured. We confirm it.

Keywords Helly's theorem · Quantitative Helly theorem · Intersection of convex sets · Dvoretzky–Rogers lemma · John's ellipsoid · Volume

Mathematics Subject Classification 52A35

1 Introduction and Preliminaries

Theorem 1.1 *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that the volume of its intersection is* $\mathrm{vol}\,(\cap\mathcal{F}) > 0$. *Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with $|\mathcal{G}| \leq 2d$ and* $\mathrm{vol}\,(\cap\mathcal{G}) \leq e^{d+1} d^{2d+\frac{1}{2}}\, \mathrm{vol}\,(\cap\mathcal{F})$.

We recall the note from [2] (see also [3]) that the number $2d$ is optimal, as shown by the $2d$ half-spaces supporting the facets of the cube.

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The order of magnitude d^{cd} in the Theorem (and in the conjecture in [2]) is sharp as shown in Sect. 3.

Recently, other quantitative Helly type results have been obtained by De Loera et al. [5].

We introduce notations and tools that we will use in the proof. We denote the closed unit ball centered at the origin o in the d-dimensional Euclidean space $\mathbb{R}^d$ by $\mathbf{B}$. For the scalar product of $u, v \in \mathbb{R}^d$, we use $\langle u, v\rangle$, and the length of u is $|u| = \sqrt{\langle u, u\rangle}$. The tensor product $u \otimes u$ is the rank one linear operator that maps any $x \in \mathbb{R}^d$ to the vector $(u \otimes u)x = \langle u, x\rangle u \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we denote its polar by $A^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } x \in A\}$. The volume of a set is denoted by $\mathrm{vol}\,(\cdot)$.

Definition 1.2 We say that a set of vectors $w_1, \ldots, w_m \in \mathbb{R}^d$ with weights $c_1, \ldots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I, \tag{1}$$

where I is the identity operator on $\mathbb{R}^d$.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with non-empty interior. We recall John's theorem [8] (see also [1]).

Lemma 1.3 (John's theorem) *For any convex body K in $\mathbb{R}^d$, there is a unique ellipsoid of maximal volume in K. Furthermore, this ellipsoid is $\mathbf{B}$ if, and only if, there are points $w_1, \ldots, w_m \in \mathrm{bd}\,\mathbf{B} \cap \mathrm{bd}\,K$ (called* contact points*) and corresponding weights $c_1, \ldots, c_m > 0$ that form a John's decomposition of the identity.*

It is not difficult to see that if $w_1, \ldots, w_m \in \mathrm{bd}\,\mathbf{B}$ and corresponding weights $c_1, \ldots, c_m > 0$ form a John's decomposition of the identity, then $\{w_1, \ldots, w_m\}^* \subset d\mathbf{B}$, cf. [1] or [7, Thm. 5.1]. By polarity, we also obtain that $\frac{1}{d}\mathbf{B} \subset \mathrm{conv}(\{w_1, \ldots, w_m\})$.

One can verify that if Δ is a regular simplex in $\mathbb{R}^d$ such that the ball $\mathbf{B}$ is the largest volume ellipsoid in Δ, then

$$\mathrm{vol}\,(\Delta) = \frac{d^{d/2}(d+1)^{(d+1)/2}}{d!}. \tag{2}$$

We will use the following form of the Dvoretzky–Rogers lemma [6].

Lemma 1.4 (Dvoretzky–Rogers lemma) *Assume that $w_1, \ldots, w_m \in \mathrm{bd}\,\mathbf{B}$ and $c_1, \ldots, c_m > 0$ form a John's decomposition of the identity. Then there is an orthonormal basis $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \ldots, v_d\}$ of $\{w_1, \ldots, w_m\}$ such that*

$$v_i \in \mathrm{span}\{z_1, \ldots, z_i\} \quad \textit{and} \quad \sqrt{\frac{d-i+1}{d}} \leq \langle v_i, z_i\rangle \leq 1 \quad \textit{for all } i = 1, \ldots, d. \tag{3}$$

This lemma is usually stated in the setting of John's theorem, that is, when the vectors are contact points of a convex body K with its maximal volume ellipsoid, which is $\mathbf{B}$.

And often, it is assumed in the statement that K is symmetric about the origin, see for example [4]. Since we make no such assumption (in fact, we make no reference to K in the statement of Lemma 1.4), we give a proof in Sect. 4.

2 Proof of Theorem 1.1

Without loss of generality, we may assume that $\mathcal{F}$ consists of closed half-spaces, and also that $\operatorname{vol}(\cap\mathcal{F}) < \infty$, that is, $\cap\mathcal{F}$ is a convex body in $\mathbb{R}^d$. As shown in [3], by continuity, we may also assume that $\mathcal{F}$ is a finite family, that is $P = \cap\mathcal{F}$ is a d-dimensional polyhedron.

The problem is clearly affine invariant, so we may assume that $\mathbf{B} \subset P$ is the ellipsoid of maximal volume in P.

By Lemma 1.3, there are contact points $w_1, \ldots, w_m \in \operatorname{bd}\mathbf{B} \cap \operatorname{bd} P$ (and weights $c_1, \ldots, c_m > 0$) that form a John's decomposition of the identity. We denote their convex hull by $Q = \operatorname{conv}\{w_1. \ldots, w_m\}$. Lemma 1.4 yields that there is an orthonormal basis $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \ldots, v_d\}$ of the contact points $\{w_1, \ldots, w_m\}$ such that (3) holds.

Let $S_1 = \operatorname{conv}\{o, v_1, v_2, \ldots, v_d\}$ be the simplex spanned by these contact points, and let E_1 be the largest volume ellipsoid contained in S_1. We denote the center of E_1 by u. Let ℓ be the ray emanating from the origin in the direction of the vector $-u$. Clearly, the origin is in the interior of Q. In fact, by the remark following Lemma 1.3, $\frac{1}{d}\mathbf{B} \subset Q$. Let w be the point of intersection of the ray ℓ with $\operatorname{bd} Q$. Then $|w| \geq 1/d$. Let S_2 denote the simplex $S_2 = \operatorname{conv}\{w, v_1, v_2, \ldots, v_d\}$. See Fig. 1.

We apply a contraction with center w and ratio $\lambda = \frac{|w|}{|w-u|}$ on E_1 to obtain the ellipsoid E_2. Clearly, E_2 is centered at the origin and is contained in S_2. Furthermore,

$$\lambda = \frac{|w|}{|u| + |w|} \geq \frac{|w|}{1 + |w|} \geq \frac{1}{d+1}. \tag{4}$$

Since w is on $\operatorname{bd} Q$, by Caratheodory's theorem, w is in the convex hull of some set of at most d vertices of Q. By re-indexing the vertices, we may assume that $w \in \operatorname{conv}\{w_1, \ldots, w_k\}$ with $k \leq d$. Now,

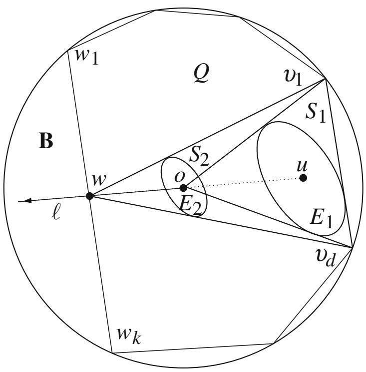

Fig. 1 Finding the ellipsoid E_2

$$E_2 \subset S_2 \subset \operatorname{conv}\{w_1, \ldots, w_k, v_1, \ldots, v_d\}. \tag{5}$$

Let $X = \{w_1, \ldots, w_k, v_1, \ldots, v_d\}$ be the set of these unit vectors, and let $\mathcal{G}$ denote the family of those half-spaces which support $\mathbf{B}$ at the points of X. Clearly, $|\mathcal{G}| \leq 2d$. Since the points of X are contact points of P and $\mathbf{B}$, we have that $\mathcal{G} \subseteq \mathcal{F}$. By (5),

$$\cap\, \mathcal{G} = X^* \subset E_2^*. \tag{6}$$

By (3),

$$\operatorname{vol}(S_1) \geq \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}} = \frac{1}{\sqrt{d!}\, d^{d/2}}. \tag{7}$$

Since $\mathbf{B} \subset \cap\mathcal{F}$, by (6) and (4), (2), (7) we have

$$\begin{aligned}\frac{\operatorname{vol}(\cap\mathcal{G})}{\operatorname{vol}(\cap\mathcal{F})} &\leq \frac{\operatorname{vol}\left(E_2^*\right)}{\operatorname{vol}(\mathbf{B})} = \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_2)} \leq (d+1)^d \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_1)} = (d+1)^d \frac{\operatorname{vol}(\Delta)}{\operatorname{vol}(S_1)} \\ &= \frac{d^{d/2}(d+1)^{(3d+1)/2}}{d!\operatorname{vol}(S_1)} = \frac{d^d d^{3d/2} e^{3/2}(d+1)^{1/2}}{(d!)^{1/2}} \leq e^{d+1} d^{2d+\frac{1}{2}},\end{aligned} \tag{8}$$

where Δ is as defined above (2). This completes the proof of Theorem 1.1.

Remark 2.1 In the proof, in place of the Dvoretzky–Rogers lemma, we could select the d vectors $v_1, \ldots, v_d$ from the contact points randomly: picking w_i with probability c_i/d for $i = 1, \ldots, m$, and repeating this picking independently d times. Pivovarov proved (cf. [9, Lem. 3]) that the expected volume of the random simplex S_1 obtained this way is the same as the right hand side in (7).

3 A Simple Lower Bound for $v(d)$

We outline a simple proof that one cannot hope a better bound in Theorem 1.1 than $d^{d/2}$ in place of $d^{2d+1/2}$. Indeed, consider the Euclidean ball $\mathbf{B}$, and a family $\mathcal{F}$ of (very many) supporting closed half space of $\mathbf{B}$ whose intersection is very close to $\mathbf{B}$. Suppose that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ of $2d$ members. Denote by σ the Haar probability measure on the sphere $R\mathbb{S}^{d-1}$, where $R = (d/(2\ln d))^{\frac{1}{2}}$. Let $H \in \mathcal{G}$ be one of the half spaces. Then

$$\sigma(R\mathbb{S}^{d-1} \setminus H) \leq \exp\left(\frac{-d}{2R^2}\right) \leq 1/(4d).$$

It follows that

$$\operatorname{vol}(\cap\mathcal{G}) \geq R^d \operatorname{vol}(\mathbf{B})\, \sigma(R\mathbb{S}^{d-1} \setminus (\cup\mathcal{G})) \geq \frac{1}{2} R^d \operatorname{vol}(\mathbf{B}) \geq d^{\frac{d}{2}-\varepsilon} \operatorname{vol}(\cap\mathcal{F})$$

for any $\varepsilon > 0$ if d is large enough.

4 Proof of Lemma 1.4

We follow the proof in [4].

Claim 4.1 *Assume that* $w_1, \dots, w_m \in \operatorname{bd} \mathbf{B}$ *and* $c_1, \dots, c_m > 0$ *form a John's decomposition of the identity. Then for any linear map* $T : \mathbb{R}^d \to \mathbb{R}^d$ *there is an* $\ell \in \{1, \dots, m\}$ *such that*

$$\langle w_\ell, T w_\ell \rangle \geq \frac{\operatorname{tr} T}{d}, \tag{9}$$

where tr T *denotes the trace of* T.

For matrices $A, B \in \mathbb{R}^{d\times d}$ we use $\langle A, B\rangle = \operatorname{tr}\left(AB^T\right)$ to denote their Frobenius product.

To prove the claim, we observe that

$$\frac{\operatorname{tr} T}{d} = \frac{1}{d}\langle T, I\rangle = \frac{1}{d}\sum_{i=1}^m c_i \langle T, w_i \otimes w_i\rangle = \frac{1}{d}\sum_{i=1}^m c_i \langle T w_i, w_i\rangle .$$

Since $\sum_{i=1}^m c_i = d$, the right hand side is a weighted average of the values $\langle T w_i, w_i\rangle$. Clearly, some value is at least the average, yielding Claim 4.1.

We define z_i and v_i inductively. First, let $z_1 = v_1 = w_1$. Assume that, for some $k < d$, we have found z_i and v_i for all $i = 1, \dots, k$. Let $F = \operatorname{span}\{z_1, \dots, z_k\}$, and let T be the orthogonal projection onto the orthogonal complement $F^\perp$ of F. Clearly, $\operatorname{tr} T = \dim F^\perp = d - k$. By Claim 4.1, for some $\ell \in \{1, \dots, m\}$ we have

$$|T w_\ell|^2 = \langle T w_\ell, w_\ell\rangle \geq \frac{d-k}{d}.$$

Let $v_{k+1} = w_\ell$ and $z_{k+1} = \frac{T w_\ell}{|T w_\ell|}$. Clearly, $v_{k+1} \in \operatorname{span}\{z_1, \dots, z_{k+1}\}$. Moreover,

$$\langle v_{k+1}, z_{k+1}\rangle = \frac{\langle T w_\ell, w_\ell\rangle}{|T w_\ell|} = \frac{|T w_\ell|^2}{|T w_\ell|} = |T w_\ell| \geq \sqrt{\frac{d-k}{d}},$$

finishing the proof of Lemma 1.4.

Note that in this proof, we did not use the fact that, in a John's decomposition of the identity, the vectors are balanced, that is $\sum_{i=1}^m c_i w_i = o$.

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