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A t_k Inequality for Arrangements of Pseudolines

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Abstract Consider arrangements of n pseudolines in the real projective plane. Let t_k denote the number of intersection points where exactly k pseudolines are incident. We present a new combinatorial inequality:

$$t_2 + 1.5t_3 \geq 8 + \sum_{k\geq 4}(2k - 7.5)t_k,$$

which holds if no more than $n - 3$ pseudolines intersect at one point. It looks similar but is unrelated to the Hirzebruch inequality for arrangements of complex lines in the complex projective plane. Based on this linear inequality, we construct lower bounds for the number of regions via n and the maximal number of (pseudo)lines passing through one point.

Keywords Pseudoline arrangement · t_k inequalities for arrangements of lines · Partitions of projective plane

1 Relations for t_k

By an arrangement of pseudolines we mean a finite collection of $n \geq 3$ smooth closed curves in the real projective plane $\mathbb{RP}^2$ such that

Curves do not self-intersect;
Curves intersect transversally at exactly one point;
There is no point where all pseudolines of the arrangement are incident.

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Let t_k, $2 \le k \le n-1$ denote the number of intersection points where exactly k pseudolines of the arrangement are incident. Each pseudoline of the arrangement is homotopically nontrivial and does not bound a disk in the plane $\mathbb{RP}^2$. Also, $t_n = 0$.

Some known relations for values of t_k are:

- $\sum_{k\ge 2} k(k-1)t_k = n(n-1)$, couple counting;
- $t_2 \ge 3 + \sum_{k\ge 4}(k-3)t_k$, Melchior [10];
- $\max\{t_2, t_3\} \ge n-1$ for $n \ge 25$, Erdős and Purdy [4];
- If $t_2 < n-1$, then $t_3 > cn^2$ for some positive c, Erdős and Purdy [4];
- $t_2 \ge \frac{6}{13}n$ for $n \ge 8$, Csima and Sawyer [1,2];
- $t_2 + 0.75t_3 \ge n + \sum_{k\ge 5}(2k-9)t_k$, if $t_{n-1} = t_{n-2} = 0$, Hirzebruch [8];
- $t_2 \ge \frac{n}{2}$ and $t_2 \ge 3[\frac{n}{4}]$ for sufficiently large, even and odd n, respectively, Green and Tao [6].

The Hirzebruch inequality holds for arrangements of complex lines in the complex projective plane; consequently, it also holds for arrangements of lines in the real projective plane. It is tight for several arrangements, e.g., for the real arrangements in Fig. 1 (in the right figure, three intersection points are at infinity). Tight examples of complex arrangements are presented in [8].

Some of these results were inspired and motivated by Sylvester's conjecture ($t_2 \ge 1$) [14], the Dirac–Motzkin conjecture ($t_2 \ge [\frac{n}{2}]$) [3], and the orchard problem ($t_3 \le [n(n-3)/6] + 1$, posed in [13]). One can find more problems related to values of t_k in the reviews by Erdős and Purdy [5] and Nilakantan [11], and Grünbaum's book [7].

2 Formulation of Main Results

2.1 Inequalities for t_k

Let us denote by p_j the number of regions bounded by exactly j arcs of pseudolines.

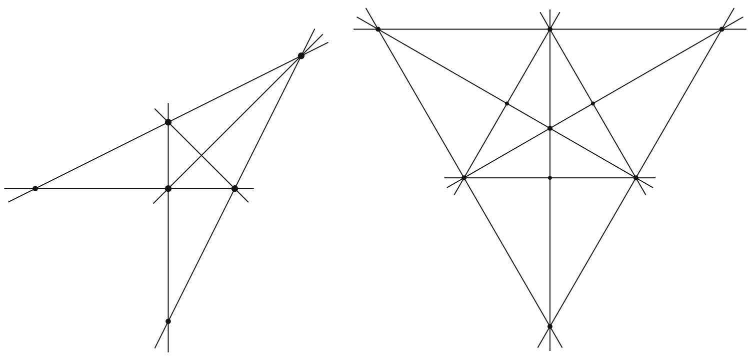

Fig. 1 $n = 6,\ t_2 = 3,\ t_3 = 4;\quad n = 9,\ t_2 = 6,\ t_3 = 4, t_4 = 3$

Lemma 1 (*Melchior* [10]).

$$\sum_{k\geq 2}(3-k)t_k = 3 + \sum_{j\geq 3}(j-3)p_j.$$

One may prove this using the Euler characteristic of the projective plane (see the next section for details). Since $p_j \geq 0$, we see that Melchior's inequality follows from this.

Theorem 1 *Suppose that $t_{n-1} = t_{n-2} = 0$ for arrangements of n pseudolines. Then, we have*

$$\text{(a)} \quad 2t_2 \leq 1 + 3p_4 + \sum_{j\geq 5} jp_j + \sum_{k\geq 3}(k-1.5)t_k;$$

$$\text{(b)} \quad t_2 + 1.5t_3 \geq 8 + \sum_{k\geq 4}(2k-7.5)t_k.$$

For convenience of reading, the proofs of Theorem 1(a), Lemma 1, and other lemmas are shifted to the next section. Now, we deduce Theorem 1(b) from Theorem 1(a) and Lemma 1. So, we have

$$\sum_{k\geq 2}(9-3k)t_k = 9 + 3p_4 + \sum_{j\geq 5}(3j-9)p_j,$$

$$3p_4 + \sum_{j\geq 5} jp_j \geq 2t_2 - \sum_{k\geq 3}(k-1.5)t_k - 1,$$

by Lemma 1 and Theorem 1(a), respectively. The inequalities

$$\sum_{k\geq 2}(9-3k)t_k \geq 9 + 2t_2 - \sum_{k\geq 3}(k-1.5)t_k - 1$$

$$\Longleftrightarrow t_2 + 1.5t_3 \geq 8 + \sum_{k\geq 4}(2k-7.5)t_k$$

follow from $3j - 9 \geq j$ for $j \geq 5$ and $p_j \geq 0$.

Remark 1 The inequalities of Theorem 1 are tight for and only for (up to combinatorial equivalence) the following arrangement of $n = 7$ pseudolines [this will become clear after Lemma 4 and the proof of Theorem 1(a)]. Let each of the distinct points A and B have four incident pseudolines, with one of these pseudolines passing through both points. Then, $t_4 = 2$, $t_2 = 9$, and $t_3 = t_k = 0$ for $k \geq 5$. For other arrangements of pseudolines with $t_{n-1} = t_{n-2} = 0$, we have

$$2t_2 \leq 3p_4 + \sum_{j\geq 5} jp_j + \sum_{k\geq 3}(k-1.5)t_k,$$

$$t_2 + 1.5t_3 \geq 9 + \sum_{k\geq 4}(2k-7.5)t_k.$$

Remark 2 The inequality in Theorem 1(b) looks similar to Hirzebruch's inequality. However, neither of them follows from the other in the case of arrangements of real lines, when both inequalities hold.

2.2 Bounds for Number of Regions

Consider an arrangement of n pseudolines and denote by m, $m \leq n-1$ the maximal number of pseudolines intersecting at one point. Let us associate with an arrangement a graph, drawn in the plane $\mathbb{RP}^2$. Vertices are intersection points, edges are arcs of pseudolines, and regions are connected components of the complement in the plane $\mathbb{RP}^2$ to the union of pseudolines. Let v, e, and f be the number of vertices, edges, and regions, respectively. The graph has no loops, because $t_n = 0$. Now, we construct lower bounds for f, which depend on the numbers n and m. Since the characteristic of a real projective plane is 1, we obtain

$$v - e + f = 1$$

by Euler's formula. As

$$v = \sum_{k \geq 2} t_k, \quad e = \sum_{k \geq 2} k t_k$$

and $t_k = 0$ for $k > m$, then

$$f - 1 = \sum_{k=2}^{m} (k-1) t_k.$$

The number of pairs of pseudolines is $\frac{n(n-1)}{2}$. Each pair of pseudolines intersects at exactly one point, and a point where k pseudolines are incident gives $\frac{k(k-1)}{2}$ such pairs. Thus, we have

$$n(n-1) = \sum_{k=2}^{m} k(k-1) t_k.$$

Let us consider the general linear inequality

$$\sum_{k \geq 2} \alpha_k t_k \geq \alpha_0 \tag{1}$$

for some real numbers $\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_n$, possibly depending on n; For example, we may take Melchior's or Hirzebruch's inequality, or Theorem 1(b). Suppose that there exist coefficients $c_1 = c_1(m, n) > 0$ and $c_2 = c_2(m, n) > 0$ such that

$$c_1 k(k-1) + c_2 \alpha_k \leq k - 1 \tag{2}$$

for all $2 \leq k \leq m$. Let us multiply both sides of (2) by t_k and sum up for $k = 2, \ldots, m$. Since $t_k \geq 0$, then

$$c_1 \sum_{k=2}^{m} k(k-1)t_k + c_2 \sum_{k=2}^{m} \alpha_k t_k \le \sum_{k=2}^{m} (k-1)t_k$$
$$\Longleftrightarrow c_1 n(n-1) + c_2 \sum_{k=2}^{m} \alpha_k t_k \le f - 1.$$

From the last inequality, inequality (1), and $c_2 > 0$, it follows that

$$f \ge c_1 n(n-1) + c_2\alpha_0 + 1 \qquad (3)$$

for positive c_1, c_2, satisfying (2).

The lower bounds for f in the form (3) were firstly obtained by the author in [12]. This bound was applied to Melchior's inequality to obtain a new proof of Martinov's theorem [9], which determines all possible pairs (n, f). In contrast to [12], here we apply the prescribed construction to the inequalities of Hirzebruch and Theorem 1(b). Thus, we obtain the following bounds, which are stronger than in [12]:

Theorem 2 (a) *Suppose that* $5 \le m < n-2$ *for arrangements of lines in the real projective plane. Then, we have*

$$f \ge \frac{(3m-10)n^2 + (m^2-6m+12)n}{m^2+3m-18} + 1.$$

(b) *Suppose that* $12 \le m < n-2$ *for arrangements of pseudolines in the real projective plane. Then, we have*

$$f \ge \frac{\big(3m-8.5\big)(n^2-n) + 9m^2 - 21m + 1}{m^2+3m-15}.$$

3 Proofs of Theorems 1(a) and 2, and Auxiliary Lemmas

Let us recall that $t_n = 0$ for arrangements of pseudolines. It follows that each arc of a pseudoline is incident to two different regions. Every pseudoline has at least two intersection points with other pseudolines of the arrangement. A region is bounded by arcs belonging to different pseudolines. Let us consider the graph associated with the arrangement. We call a vertex of the graph *ordinary* if it belongs to exactly two pseudolines (and, therefore, has degree 4 in the graph). An edge is called *double ordinary* if both of its endpoints are ordinary. We denote by e_0 the number of double ordinary edges. Let e_1 be the number of edges whose endpoints are both not ordinary. A region is called *triangular* if it is bounded by three edges; otherwise it is called *nontriangular*. Since $t_n = 0$, we see that every nontriangular region is bounded by at least four edges. A region is called *good* if it is bounded by at least four edges and its boundary contains at least one ordinary vertex. Let γ be an arbitrary good region, $e_0(\gamma)$ denote the number of double ordinary edges in the boundary of γ, $s(\gamma)$ denote the number of nonordinary vertices in the boundary of γ, and

$$\delta(\gamma) = \begin{cases} 0 & \text{if} \quad s(\gamma) \geq 1, \\ 1 & \text{if} \quad s(\gamma) = 0. \end{cases}$$

Let

$$s = \sum_{\text{good } \gamma} s(\gamma).$$

We shall briefly say “a vertex belongs to the region” if the vertex belongs to the boundary of the region. A double ordinary edge is called *perfect* if it is incident to two good regions. In other words, both regions whose boundaries contain this edge are good. Let e_p denote the number of perfect edges. A region is called *excellent* if it is bounded by four edges and all its vertices are ordinary (so, excellent regions are also good). We denote by f_e the number of excellent regions. Let φ denote the number of pairs (γ, κ) where γ is an excellent region, κ is a perfect edge, and κ is incident to γ.

Let us denote by G the graph associated with the arrangement. For a nonordinary vertex V, let us delete all edges lying on pseudolines passing through V. The degrees of the vertices will change, and some vertices could disappear (which belong to at most one pseudoline, not passing through V). Let us denote by G_V the obtained graph. Let us denote by γ_V the region of the graph G_V such that the vertex V belongs to the interior of γ_V.

For a nonordinary vertex $V \in G$, let us consider edges connecting V with nonordinary vertices of the graph G and good regions whose boundaries contain V; let $q(V)$ denote the sum of the number of these edges and double the number of these regions. Suppose that V is connected with nonordinary vertices by q_1 edges and belongs to q_2 good regions, then $q(V) = q_1 + 2q_2$. Let q denote the sum of the numbers $q(V)$ for all nonordinary vertices V of the graph G.

Proof of Lemma 1 The numbers of vertices v, edges e, and regions f can be found via t_k and p_j as follows:

$$v = \sum_{k \geq 2} t_k, \qquad e = \sum_{k \geq 2} k t_k = 0.5 \sum_{j \geq 3} j p_j,$$
$$f = 1 + \sum_{k \geq 2} (k-1) t_k = \sum_{j \geq 3} p_j.$$

Let us substitute these equations into the Euler formula:

$$3 = 3f - 2e - e + 3v = 3 \sum_{j \geq 3} p_j - \sum_{j \geq 3} j p_j - \sum_{k \geq 2} k t_k + 3 \sum_{k \geq 2} t_k$$
$$= \sum_{j \geq 3} (3-j) p_j + \sum_{k \geq 2} (3-k) t_k.$$

□

Lemma 2 (a) *Suppose that $t_{n-1} = 0$, then every double ordinary edge is incident to at least one good region. Also,*

$$\sum_{good\,\gamma} e_0(\gamma) = e_0 + e_p.$$

(b) *Suppose that $t_{n-2} = t_{n-1} = 0$, then every excellent region is incident to at least two perfect edges.*

Proof (a) Let us assume the contrary, i.e., that there exists a double ordinary edge with endpoints A and B such that both regions incident to it are triangular. Let l_1 and l_2 be the pseudolines passing through one of the points A, B and not containing the edge AB. Let l_1 intersect l_2 at the point C. Then, both triangular regions which are incident to the edge AB contain C. Hence, l_1 could not have intersection points except A and C. As A is an ordinary point, it follows that C belongs to $n-1$ pseudolines, which contradicts $t_{n-1} = 0$. So, every double ordinary edge is incident to at least one good region. Hence, there are $e_0 - e_p$ double ordinary edges which are incident to exactly one good region, and e_p double ordinary edges which are incident to two good regions.

(b) Let us prove that each pair of opposite edges of an excellent region contains at least one perfect edge. Assume the contrary, i.e., that both edges AB and CD of an excellent region $ABCD$ are not perfect, where A, B, C, D are vertices of G. Then, edges AB and CD are incident to triangular regions ABH and CDG. We denote by l_1 the pseudoline passing through points B and C. We denote by l_2 the pseudoline passing through points A and D. Then, the intersection point of l_1 and l_2 coincides with both points H and G. So $G = H$ and G belongs to $n-2$ pseudolines, all but two of which pass through points A, B and C, D. This is in contradiction with $t_{n-2} = 0$. □

Lemma 3

$$e_0 = 2t_2 + e_1 - \sum_{k\geq 3} kt_k.$$

Proof The graph contains $\sum_{k\geq 2} kt_k$ edges. There are $\sum_{k\geq 2} kt_k - e_0 - e_1$ edges with one ordinary endpoint. Every ordinary vertex is an endpoint of four edges that have at least one ordinary endpoint. Thus, the total number of ordinary endpoints over all edges is

$$4t_2 = 2e_0 + \sum_{k\geq 2} kt_k - e_0 - e_1.$$

□

Lemma 4 *Suppose that $t_{n-1} = t_{n-2} = 0$ and that there exist two points A, B such that each of the pseudolines in the arrangement contains at least one of them. Then, the statement of Theorem 1(a) holds:*

$$2t_2 \leq 1 + 3p_4 + \sum_{j \geq 5} jp_j + \sum_{k \geq 3} (k - 1.5)t_k.$$

Proof Let us denote by a and b the number of pseudolines passing through points A and B, respectively. We consider two cases.

In the first case, the arrangement does not contain a pseudoline passing through both points A and B. The inequalities $a \geq 3$ and $b \geq 3$ follow from $a+b = n$ and $t_{n-2} = 0$. So, we have

$$t_2 = ab, \qquad \sum_{k \geq 3} (k - 1.5)t_k = a + b - 3,$$
$$p_4 = ab - a - b + 3, \qquad p_j = 0 \quad \text{for} \quad j \geq 5.$$

In the second case, the arrangement contains a pseudoline passing through both of the points A and B. The inequalities $a \geq 4$ and $b \geq 4$ follow from $a + b = n + 1$ and $t_{n-2} = 0$. So, we have

$$t_2 = ab - a - b + 1, \quad \sum_{k \geq 3} (k - 1.5)t_k = a + b - 3,$$
$$p_4 = ab - 2a - 2b + 4, \qquad p_j = 0 \quad \text{for} \quad j \geq 5.$$

Now, it is easy to check that the required inequality holds in the first and second cases, for $a \geq 3$, $b \geq 3$ and $a \geq 4$, $b \geq 4$, respectively. □

Lemma 5 *Suppose that a good region γ is bounded by j edges. Then,*

$$s(\gamma) \leq j - 1 - e_0(\gamma) + \delta(\gamma), \tag{4}$$
$$s \leq \sum_{j \geq 4} (j - 1)p_j + \sum_{\text{good } \gamma} (\delta(\gamma) - e_0(\gamma)). \tag{5}$$

Proof Let us consider three cases.

(i) $e_0(\gamma) = 0$. Then $s(\gamma) \leq j - 1$, because the boundary of γ contains an ordinary point.

(ii) $e_0(\gamma) = j$. Then $s(\gamma) = 0$ and $\delta(\gamma) = 1$.

(iii) $0 < e_0(\gamma) < j$. Let us consider the boundary of γ consisting of j edges. Let the double ordinary edges in the boundary of γ determine $z(\gamma)$ connected components in the boundary of γ, where the boundary is considered a separate topological space, homeomorphic to a circle. So, each connected component is a union of consecutive double ordinary edges. From $0 < e_0(\gamma) < j$ it follows that $z(\gamma) \geq 1$ and that each connected component is homeomorphic to a segment. So, in each connected component, the number of vertices exceeds the number of edges by one. Hence, the boundary of γ contains at least $e_0(\gamma) + z(\gamma)$ ordinary vertices. Since $z(\gamma) \geq 1$, then $s(\gamma) \leq j - 1 - e_0(\gamma)$. Summing up (4) for all good regions γ, we get (5).

□

Lemma 6 *Suppose that* $t_n = t_{n-1} = 0$. *Then,*

$$s \le 3p_4 - e_0 + \sum_{j \ge 5} jp_j.$$

Proof There are $\sum_{\text{good } \gamma} \delta(\gamma)$ good regions, all vertices of which are ordinary. So,

$$\sum_{\text{good } \gamma} \delta(\gamma) \le f_e + \sum_{j \ge 5} p_j. \tag{6}$$

By Lemma 2(b), every excellent region is incident to at least two perfect edges, so $\varphi \ge 2f_e$. Every perfect edge is incident to at most two excellent regions, so $2e_p \ge \varphi$. Hence, $e_p \ge f_e$. The required inequality follows from Lemma 2(a) and inequalities (5), (6), and $e_p \ge f_e$. □

Lemma 7 *Suppose that there are no two points such that every pseudoline of the arrangement passes through at least one of them. Then, for every nonordinary vertex* V, *at least one of the following statements holds:*

(a) *There are at least three edges of the graph* G *connecting* V *with the vertices of* γ_V.
(b) *There are two edges of the graph* G *connecting* V *with the vertices of* γ_V, *and* V *is the vertex of a good region of the graph* G.
(c) V *is the vertex of at least two good regions of the graph* G.

Proof The graph G_V has at least two vertices for every vertex V. Each region of the graph G_V is bounded by at least three edges of G_V for every vertex V. Suppose that the vertex V is not connected by an edge of the graph G with some vertex U of the region γ_V. Then, U and V are vertices of some region $\gamma_{U,V} \subset \gamma_V$ of the graph G. The region $\gamma_{U,V}$ is bounded by at least four edges of the graph G. If $\gamma_{U,V}$ is not good, then all its vertices except V are vertices of the region γ_V, and so there are at least two edges of the graph G connecting V with the vertices of γ_V. If statement (a) is false for a vertex V, then one of the following cases holds:

(1) V is connected by edges of the graph G with exactly two vertices of the region γ_V.
(2) V is connected by edges of the graph G with at most one vertex of the region γ_V.

Case 1 Let these two vertices be U_1 and U_2.

Subcase 1.1 The points U_1, U_2, V do not belong to one pseudoline of the arrangement. Let W_1 and W_2 be the intersection points of the boundary of γ_V and the pseudolines passing through V, U_1 and V, U_2, respectively (so W_1 and W_2 are ordinary vertices of G). Then, on the part $U_1W_2W_1$ of the boundary of γ_V, there is at least one vertex U of γ_V [otherwise we would have that the points U_1, W_2, W_1 belong to a pseudoline of the arrangement, intersecting the pseudoline (U_1, V, W_1) at two points]. Then, U and V are vertices of some good region $\gamma_{U,V} \subset \gamma_V$ of the graph G, and we obtain statement (b).

Subcase 1.2 The points U_1, U_2, V belong to one pseudoline of the arrangement. Points U_1 and U_2 divide the boundary of γ_V into two open parts. Each part contains at least one vertex of γ_V (disjoint from U_1 and U_2) and two ordinary vertices of graph G. So, in the graph G there are at least two good regions with vertex V and we obtain statements (b) and (c).

Case 2

Subcase 2.1 Suppose that the vertex V is connected by an edge of the graph G with a vertex U in the boundary of region γ_V. The pseudoline passing through points V and U divides the region γ_V into two parts. Each of these parts contains at least one vertex of the region γ_V, disjoint from U. It follows that each part contains at least one good region for the graph G, so that V is a vertex of this good region.

Subcase 2.2 Suppose that V is not connected by an edge of the graph G with vertices in the boundary of region γ_V. Then, we may take a pseudoline of the arrangement passing through V and do the same as we did for a pseudoline passing through points V and U. So, in both subcases of case (2), we obtain statement (c). □

Proof of Theorem 1, part (a). Suppose there are two points such that each pseudoline of the arrangement passes through at least one of them, then we are done by Lemma 4. Thus, we may assume that there are no such points. By Lemma 7,

$$q(V) \geq 3 \quad \text{and} \quad q \geq 3 \sum_{k \geq 3} t_k.$$

Let us count q via edges and regions, then we have $q = 2e_1 + 2s$. Hence,

$$e_1 + s \geq 1.5 \sum_{k \geq 3} t_k. \tag{7}$$

From Lemmas 3 and 6 it follows that

$$3p_4 + \sum_{j \geq 5} j p_j - s \geq e_0 = 2t_2 + e_1 - \sum_{k \geq 3} k t_k$$
$$\Longrightarrow 3p_4 + \sum_{j \geq 5} j p_j + \sum_{k \geq 3} k t_k - 2t_2 \geq e_1 + s.$$

From the last inequality and inequality (7) it follows that

$$3p_4 + \sum_{j \geq 5} j p_j + \sum_{k \geq 3} k t_k - 2t_2 \geq 1.5 \sum_{k \geq 3} t_k$$
$$\Longrightarrow 3p_4 + \sum_{j \geq 5} j p_j + \sum_{k \geq 3} \big(k - 1.5\big) t_k \geq 2t_2.$$

□

Proof of Theorem 2 (*a*) We use Hirzebruch's inequality [8] as described in subsection "Bounds for Number of Regions" above. Hence, $m < n - 2$ and we may present the inequality

$$t_2 + 0.75t_3 \geq n + \sum_{k\geq 5}(2k-9)t_k$$

in the form (1) with

$$\alpha_0 = n, \quad \alpha_2 = 1, \quad \alpha_3 = 0.75, \quad \alpha_4 = 0, \quad \alpha_i = 9 - 2i \quad \text{for} \quad i \geq 5.$$

Let us take positive (for $m \geq 5$) numbers

$$c_1 = \frac{3m-10}{m^2+3m-18}, \qquad c_2 = \frac{m^2-3m+2}{m^2+3m-18}.$$

The system (2) takes on the form

$$1 \geq 2c_1 + c_2, \quad 2 \geq 6c_1 + 0.75c_2, \quad 3 \geq 12c_1,$$
$$i - 1 \geq c_1 i(i-1) - c_2(2i-9) \quad \text{for} \quad 5 \leq i \leq m.$$

Let us check these inequalities for $m \geq 5$ and for given c_1, c_2. The first three are obvious; to verify the last one for $5 \leq i \leq m$, let us consider the quadratic polynomial

$$c_1 i(i-1) - c_2(2i-9) - (i-1) = \frac{(i-m)(3mi - 10(m+i) + 24)}{(m-3)(m+6)} \leq 0,$$

because $3mi - 10(m+i) + 24 > 0$ for $m \geq 6$ and $i \geq 5$. So, we obtain (3) for given c_1, c_2, and hence the inequality of Theorem 2(a).

(*b*) Hence, $m < n - 2$, then the inequality of Theorem 1(b) is valid and can be presented in the form (1) with

$$\alpha_0 = 8, \quad \alpha_2 = 1, \quad \alpha_3 = 1.5, \quad \alpha_k = 7.5 - 2k \quad \text{for} \quad k \geq 4.$$

Let us take positive (for $m \geq 12$) numbers

$$c_1 = \frac{3m-8.5}{m^2+3m-15}, \qquad c_2 = \frac{m^2-3m+2}{m^2+3m-15}.$$

The system (2) takes on the form

$$1 \geq 2c_1 + c_2, \quad 2 \geq 6c_1 + 1.5c_2,$$
$$i - 1 \geq c_1 i(i-1) - c_2(2i-7.5) \quad \text{for} \quad 4 \leq i \leq m.$$

Let us check these inequalities for $m \geq 12$ and for given c_1, c_2. The first two are obvious; to verify the last one for $4 \leq i \leq m$, consider the quadratic polynomial

$$c_1 i(i-1) - c_2\big(2i - 7.5\big) - (i-1) \\ = \frac{(i-m)(3mi - 8.5(m+i) + 19.5)}{m^2 + 3m - 15} \leq 0,$$

because $3mi - 8.5(m+i) + 19.5 > 0$ for $m \geq 12$ and $i \geq 4$. So, we obtain (3) for given c_1, c_2, and hence the inequality of Theorem 2(b). □

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