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# On the Lower Bound in the Lattice Point Remainder Problem for a Parallelepiped

**Mordechay B. Levin**[1]

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**Abstract** Let $\Gamma \subset \mathbb{R}^s$ be a lattice, obtained from a module in a totally real algebraic number field. Let $G$ be an axis parallel parallelepiped, and let $|G|$ be a volume of $G$. In this paper we prove that

$$\limsup_{|G|\to\infty} \frac{|\det \Gamma \#(\Gamma \cap G) - |G||}{\ln^{s-1}|G|} > 0.$$

Thus the known estimate $\det \Gamma \#(\Gamma \cap G) = |G| + O(\ln^{s-1}|G|)$ is exact. We obtain also a similar result for the low discrepancy sequence corresponding to $\Gamma$.

**Keywords** Lattice point problem · Low discrepancy sequences · Totally real algebraic number field

**Mathematics Subject Classification** Primary 11P21, 11K38, 11R80

## 1 Introduction

### 1.1 Lattice Points

Let $\Gamma \subset \mathbb{R}^s$ be a lattice, i.e., a discrete subgroup of $\mathbb{R}^s$ with a compact fundamental set $\mathbb{R}^s/\Gamma$, $\det \Gamma = \mathrm{vol}(\mathbb{R}^s/\Gamma)$. Let $N_1, \dots, N_s > 0$ be reals, $\mathbf{N} =$

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Mordechay B. Levin
mlevin@math.biu.ac.il

1 Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, Israel

$(N_1, \dots, N_s)$, $B_{\mathbf{N}} = [0, N_1) \times \cdots \times [0, N_s)$, $\mathrm{vol}(B_{\mathbf{N}})$ the volume of $B_{\mathbf{N}}$, $tB_{\mathbf{N}}$ the dilatation of $B_{\mathbf{N}}$ by a factor $t > 0$, $tB_{\mathbf{N}} + \mathbf{x}$ the translation of $tB_{\mathbf{N}}$ by a vector $\mathbf{x} \in \mathbb{R}^s$, $(x_1, \dots, x_s) \cdot (y_1, \dots, y_s) = (x_1 y_1, \dots, x_s y_s)$, and let $(x_1, \dots, x_s) \cdot B_{\mathbf{N}} = \{(x_1, \dots, x_s) \cdot (y_1, \dots, y_s) \mid (y_1, \dots, y_s) \in B_{\mathbf{N}}\}$. Let

$$\mathcal{N}(B_{\mathbf{N}} + \mathbf{x}, \Gamma) = \#(B_{\mathbf{N}} + \mathbf{x} \cap \Gamma) = \sum_{\boldsymbol{\gamma} \in \Gamma} \mathbb{1}_{B_{\mathbf{N}}+\mathbf{x}}(\boldsymbol{\gamma}) \tag{1.1}$$

be the number of points of the lattice $\Gamma$ lying inside the parallelepiped $B_{\mathbf{N}}$, where we denote by $\mathbb{1}_{B_{\mathbf{N}}+\mathbf{x}}(\boldsymbol{\gamma})$ the indicator function of $B_{\mathbf{N}}+\mathbf{x}$. We define the error $\mathcal{R}(B_{\mathbf{N}}+\mathbf{x}, \Gamma)$ by setting

$$\mathcal{N}(B_{\mathbf{N}} + \mathbf{x}, \Gamma) = (\det \Gamma)^{-1} \mathrm{vol}(B_{\mathbf{N}}) \ + \ \mathcal{R}(B_{\mathbf{N}} + \mathbf{x}, \Gamma). \tag{1.2}$$

Let $\mathrm{Nm}(\mathbf{x}) = x_1 x_2 \dots x_s$ for $\mathbf{x} = (x_1, \dots, x_s)$. The lattice $\Gamma \subset \mathbb{R}^s$ is *admissible* if

$$\mathrm{Nm}\ \Gamma = \inf_{\boldsymbol{\gamma} \in \Gamma \setminus \{0\}} |\mathrm{Nm}(\boldsymbol{\gamma})| > 0.$$

Let $\Gamma$ be an admissible lattice. In 1994, Skriganov [27] proved the following theorem:

**Theorem A** *Let* $\mathbf{t} = (t_1, \dots, t_s)$. *Then*

$$|\mathcal{R}(\mathbf{t} \cdot [-1/2, 1/2)^s + \mathbf{x}, \Gamma)| < c_0(\Gamma) \log_2^{s-1}(2 + |\mathrm{Nm}(\mathbf{t})|), \tag{1.3}$$

*where the constant* $c_0(\Gamma)$ *depends upon the lattice* $\Gamma$ *only by means of the invariants* $\det \Gamma$ *and* $\mathrm{Nm}\ \Gamma$.

In [27, p. 205], Skriganov conjectured that the bound (1.3) is the best possible. In this paper we prove this conjecture.

Let $\mathcal{K}$ be a totally real algebraic number field of degree $s \geq 2$, and let $\sigma$ be the canonical embedding of $\mathcal{K}$ in the Euclidean space $\mathbb{R}^s$, $\sigma : \mathcal{K} \ni \xi \to \sigma(\xi) = (\sigma_1(\xi), \dots, \sigma_s(\xi)) \in \mathbb{R}^s$, where $\{\sigma_j\}_{j=1}^s$ are $s$ distinct embeddings of $\mathcal{K}$ in the field $\mathbb{R}$ of real numbers. Let $N_{\mathcal{K}/\mathbb{Q}}(\xi)$ be the norm of $\xi \in \mathcal{K}$. By [6, p. 404],

$$N_{\mathcal{K}/\mathbb{Q}}(\xi) = \sigma_1(\xi) \cdots \sigma_s(\xi) \quad \text{and} \quad |N_{\mathcal{K}/\mathbb{Q}}(\alpha)| \geq 1$$

for all algebraic integers $\alpha \in \mathcal{K} \setminus \{0\}$. We see that $|\mathrm{Nm}(\sigma(\xi))| = |N_{\mathcal{K}/\mathbb{Q}}(\xi)|$. Let $\mathcal{M}$ be a full $\mathbb{Z}$ module in $\mathcal{K}$ and let $\Gamma_{\mathcal{M}}$ be the lattice corresponding to $\mathcal{M}$ under the embedding $\sigma$. Let $(c_{\mathcal{M}})^{-1} > 0$ be an integer such that $(c_{\mathcal{M}})^{-1}\gamma$ are algebraic integers for all $\gamma \in \mathcal{M}$. Hence

$$\mathrm{Nm}\ \Gamma_{\mathcal{M}} \geq c_{\mathcal{M}}^s.$$

Therefore, $\Gamma_{\mathcal{M}}$ is an admissible lattice. In the following, we will use notations $\Gamma = \Gamma_{\mathcal{M}}$, and $N = N_1 N_2 \cdots N_s \geq 2$. In Sect. 2 we will prove the following theorem:

**Theorem 1** *With the above notations, there exist* $c_1(\mathcal{M}) > 0$ *such that*

$$\sup_{\boldsymbol{\theta}\in[0,1]^s} |\mathcal{R}(B_{\boldsymbol{\theta}.\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}})| \geq c_1(\mathcal{M}) \log_2^{s-1} N \tag{1.4}$$

*for all* $\mathbf{x} \in \mathbb{R}^s$.

In [15, Chap. 5], Lang considered the lattice point problem in the adelic setting. In [15,25], the upper bound for the lattice point remainder problem in parallelotopes was found. In a forthcoming paper, we will prove that the lower bound (1.4) can be extended to the adelic case (see [18]). Namely, we will prove that the upper bound in [25] is exact for the case of totally real algebraic number fields.

### 1.2 Low Discrepancy Sequences

Let $(\beta_{k,N})_{k=0}^{N-1}$ be a $N$-point set in an $s$-dimensional unit cube $[0, 1)^s$, $B_{\mathbf{y}} = [0, y_1) \times \cdots \times [0, y_s)$,

$$\Delta(B_{\mathbf{y}}, (\beta_{k,N})_{k=0}^{N-1}) = \#\{0 \leq k < N \mid \beta_{k,N} \in B_{\mathbf{y}}\} - N y_1 \ldots y_s. \tag{1.5}$$

We define the star *discrepancy* of a $N$-point set $(\beta_{k,N})_{k=0}^{N-1}$ as

$$D^*(N) = D^*((\beta_{k,N})_{k=0}^{N-1}) = \sup_{0<y_1,\ldots,y_s\leq 1} \Big|\frac{1}{N}\Delta(B_{\mathbf{y}}, (\beta_{k,N})_{k=0}^{N-1})\Big|. \tag{1.6}$$

In 1954, Roth proved that there exists a constant $\dot{c}_1 > 0$, such that

$$N D^*\big((\beta_{k,N})_{k=0}^{N-1}\big) > \dot{c}_1 (\ln N)^{\frac{s-1}{2}},$$

for all $N$-point sets $(\beta_{k,N})_{k=0}^{N-1}$.

**Definition 1** A sequence of point sets $((\beta_{k,N})_{k=0}^{N-1})_{N=1}^{\infty}$ is of *low discrepancy* (abbreviated l.d.p.s.) if $D^*((\beta_{k,N})_{k=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$ for $N \to \infty$.

For examples of l.d.p.s. see e.g. in [3,10,27]. Consider a lower bound for l.d.p.s. According to the well-known conjecture (see, e.g., [3, p. 283]), there exists a constant $\dot{c}_2 > 0$ such that

$$N D^*\big((\beta_{k,N})_{k=0}^{N-1}\big) > \dot{c}_2 (\ln N)^{s-1} \tag{1.7}$$

for all $N$-point sets $(\beta_{k,N})_{k=0}^{N-1}$. In 1972, W. Schmidt proved this conjecture for $s = 2$. In 1989, Beck [1] proved that $N D^*(N) \geq \dot{c} \ln N (\ln \ln N)^{1/8-\epsilon}$ for $s = 3$ and some $\dot{c} > 0$. In 2008, Bilyk et al. (see [4, p. 147], [5, p. 2]) proved in all dimensions $s \geq 3$ that there exists some $\dot{c}(s)$, $\eta > 0$ for which the following estimate holds for all $N$-point sets: $N D^*(N) > \dot{c}(s)(\ln N)^{\frac{s-1}{2}+\eta}$.

There exists another conjecture on the lower bound for the discrepancy function: there exists a constant $\dot{c}_3 > 0$ such that

$$N D^*\big((\beta_{k,N})_{k=0}^{N-1}\big) > \dot{c}_3 (\ln N)^{s/2} \tag{1.8}$$

for all $N$-point sets $(\beta_{k,N})_{k=0}^{N-1}$ (see [4, p. 147], [5, p. 3] and [8, p. 153]).

Let $\mathcal{W} = (\Gamma_{\mathcal{M}} + \mathbf{x}) \cap [0,1)^{s-1} \times [0,\infty)$. We enumerate $\mathcal{W}$ by the sequence $(z_{1,k}(\mathbf{x}), z_{2,k}(\mathbf{x}))$ with $z_{1,k}(\mathbf{x}) \in [0,1)^{s-1}$, $z_{2,k}(\mathbf{x}) \in [0,\infty)$, and $z_{2,i}(\mathbf{x}) < z_{2,j}(\mathbf{x})$ for $i < j$. In [27], Skriganov proved that the point set $((\beta_{k,N}(\mathbf{x}))_{k=0}^{N-1})$ with $\beta_{k,N}(\mathbf{x}) = (z_{1,k}(\mathbf{x}), z_{2,k}(\mathbf{x})/z_{2,N}(\mathbf{x}))$ is of low discrepancy (see also [17]). In Sect. 2.10 we will prove

**Theorem 2** *With the notations as above, there exist $c_2(\mathcal{M})$ such that*

$$N D^*\big((\beta_{k,N}(\mathbf{x}))_{k=0}^{N-1}\big) \geq c_2(\mathcal{M}) \log_2^{s-1} N \tag{1.9}$$

*for all $\mathbf{x} \in \mathbb{R}^s$.*

This result supports conjecture (1.7). In [19,20], we proved that (1.9) is also true for the Halton sequence, and $(t, s)$-sequences.

We note that the constant $c_2$ depends on the chosen module $\mathcal{M}$. Hence we get a lower bound for translations of one concrete lattice. We do not understand if $c_2(\mathcal{M})$ is uniformly bounded from below for all module $\mathcal{M}$. However, it seems that conjecture (1.7) is more likely than conjecture (1.8), because the following result of Beck [2]:

Consider a Kronecker's lattice $\{(n, n\alpha_1 + m_1, \ldots, n\alpha_{s-1} + m_{s-1}) | (n, m_1, \ldots, m_{s-1}) \in \mathbb{Z}^s\}$ and the corresponding Kronecker's sequence $\mathcal{P}_N = \{(\{n\alpha_1\}, \ldots, \{n\alpha_{s-1}\}, n/N)\}_{n=0}^{N-1}$, where $\alpha = (\alpha_1, \ldots, \alpha_{s-1}) \in \mathbb{R}^{s-1}$. Then that for almost all $\alpha \in \mathbb{R}^{s-1}$, we have that $D(\mathcal{P}_N) > c(s)(\log N)^{s-1} \log\log N$, with a uniform constant $c(s)$ depending only on the dimension $s$.

## 2 Proof of Theorems

In this paper we consider a fundamental units of the field $\mathcal{K}$ and the appropriate toral automorphisms $A_1, \ldots, A_{s-1}$. Applying the profound Chevalley's result [9], we construct a Hecke character, corresponding to $A_1, \ldots, A_{s-1}$.

The main idea of this paper is to express the essential part of the normalized discrepancy function as a truncated $L$function with the above Hecke character. Using the non-vanishing property of an $L$-function, we obtain the assertion of Theorem 1.

Let us describe the main steps of the proof of Theorem 1:

In Sect. 2.1, we use the Poisson summation formula and the standard trick of 'smoothing'. This allows to express the discrepancy function $\mathcal{R}_\theta$ in terms of absolutely convergent Fourier's series. Next we decompose the domain of the summation in three parts, and we obtain that $\mathcal{R}_\theta = \mathcal{A}_\theta + \mathcal{B}_\theta + \mathcal{C}_\theta$. Using the expectation function $E$, we get $\sup_\theta |R_\theta| \geq |E(\mathcal{A}_\theta)| - |E(\mathcal{B}_\theta)| - |E(\mathcal{C}_\theta)|$. Hence, to obtain the assertion of Theorem 1, it is sufficient to find the lower bound of $|E(\mathcal{A}_\theta)|$ and the upper bounds of $|E(\mathcal{B}_\theta)|$ and $|E(\mathcal{C}_\theta)|$.

In Sect. 2.2, we consider the fundamental domain of the field $\mathcal{K}$. We apply [30] to estimate the error term in the lattice point problem in a compact convex body. We use these results to compute the difference between an $L$-function and the corresponding truncated $L$-function, and also to estimate the value of the domain of the summation in the Fourier's series of $\mathcal{A}_\theta$.

In Sect. 2.3, we use the Chevalley theorem [9] to construct a special Hecke character.

In Sect. 2.4, we consider the truncated $L$-function $\vartheta$, with the above Hecke character. Using the estimates of Sect. 2.2 and the non-vanishing property of $L$-function, we obtain the lower bound of $\vartheta$.

In Sect. 2.5, we find the lower bound of $|E(\mathcal{A}_\theta)|$. First, we decompose the domain of the summation in seven parts, and we get that $\mathcal{A}_\theta = \mathcal{A}_0+\mathcal{A}_1+\cdots+\mathcal{A}_6$. Using results of Sect. 2.2, we compute $|E(\mathcal{A}_1)|+\cdots+|E(\mathcal{A}_6)|$. In addition, we decompose $\mathcal{A}_0$ in several parts and we select the main part $\mathcal{A}_7(\Gamma^\perp+\mathbf{x})$. Lemma 12 is the main result of this subsection. Let $\Gamma^\perp = A\mathbb{Z}^s$, $\dot{Z}_p = \{(a_1,\dots,a_s)^\top | a_i \in \{0,1,\dots,p-1\}, i = 1,\dots,s\}$, and $\Lambda_p = A\dot{Z}_p^s$, where $p$ is obtained from the Chevalley theorem (see Theorem C). In Lemma 12, we prove that $p^{-s}\sum_{\mathbf{b}\in\Lambda_p} |A_7(\Gamma^\perp + \mathbf{b}/p)|^2$ may be estimated from below as a part of the corresponding $L$-function. Next, using results of Sect. 2.4, we get the lower bound of $|E(\mathcal{A}_\theta)|$.

In Sect. 2.6, we cite some inequalities from [27].

In Sect. 2.7, we use the dyadic decomposition method (see, e.g., [27]) to obtain the convenient expressions for $E(\mathcal{B}_\theta)$ and $E(\mathcal{C}_\theta)$.

In Sect. 2.8, we apply inequalities from Sect. 2.6 to obtain the upper bound estimate for $|E(\mathcal{B}_\theta)|$.

In Sect. 2.9, we apply the Koksma–Hlawka inequality and Theorem A to obtain the upper bound estimate for $|E(\mathcal{C}_\theta)|$.

### 2.1 Poisson Summation Formula

It is known that the set $\mathcal{M}^\perp$ of all $\beta \in \mathcal{K}$, for which $\mathrm{Tr}_{\mathcal{K}/\mathbb{Q}}(\alpha\beta) \in \mathbb{Z}$ for all $\alpha \in \mathcal{M}$, is also a full $\mathbb{Z}$ module (*the dual of the module* $\mathcal{M}$) of the field $K$ (see [6, p. 94]). Recall that the dual lattice $\Gamma_{\mathcal{M}}^\perp$ consists of all vectors $\boldsymbol{\gamma}^\perp \in \mathbb{R}^s$ such that the inner product $\langle\boldsymbol{\gamma}^\perp, \boldsymbol{\gamma}\rangle$ belongs to $\mathbb{Z}$ for each $\boldsymbol{\gamma} \in \Gamma$. Hence $\Gamma_{\mathcal{M}^\perp} = \Gamma_{\mathcal{M}}^\perp$. Let $\mathcal{O}$ be the ring of integers of the field $\mathcal{K}$, and let $a\mathcal{M}^\perp \subseteq \mathcal{O}$ for some $a \in \mathbb{Z}\setminus 0$. By (1.1), we have $\mathcal{N}(B_{\mathbf{N}}+\mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{N}(a^{-1}B_{\mathbf{N}}+a^{-1}\mathbf{x}, \Gamma_{a^{-1}\mathcal{M}})$. Therefore, to prove Theorem 1 it suffices consider only the case $\mathcal{M}^\perp \subseteq \mathcal{O}$. We set

$$p_1 = \min\{b \in \mathbb{Z} \mid b\mathcal{O} \subseteq \mathcal{M}^\perp \subseteq \mathcal{O},\ \ b > 0\}. \tag{2.1}$$

We will use the same notations for elements of $\mathcal{O}$ and $\Gamma_{\mathcal{O}}$. Let $\mathcal{D}_{\mathcal{M}}$ be the ring of coefficients of the full module $\mathcal{M}$, $\mathcal{U}_{\mathcal{M}}$ be the group of units of $\mathcal{D}_M$, and let $\eta_1,\dots,\eta_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{M}}$. According to the Dirichlet theorem (see e.g., [6, p. 112]), every unit $\varepsilon \in \mathcal{U}_{\mathcal{M}}$ has a unique representation in the form

$$\varepsilon = (-1)^a \eta_1^{a_1}\cdots\eta_{s-1}^{a_{s-1}}, \tag{2.2}$$

where $a_1,\dots,a_{s-1}$ are rational integers and $a \in \{0,1\}$. It is easy to proof (see e.g. [19, Lemma 1]) that there exists a constant $c_3 > 1$ such that for all $\mathbf{N}$ there exists $\eta(\mathbf{N}) \in \mathfrak{U}_{\mathcal{M}}$ with $|N_i^{'} N^{-1/s}| \in [1/c_3, c_3]$, where $N_i^{'} = N_i|\sigma_i(\eta(\mathbf{N}))|$, $i = 1,\dots,s$, and $N = N_1\cdots N_s$. Let $\sigma(\eta(\mathbf{N})) = (\sigma_1(\eta(\mathbf{N})),\dots,\sigma_s(\eta(\mathbf{N})))$. We see that $\sigma(\eta(\mathbf{N}))\cdot(\boldsymbol{\theta}\cdot B_{\mathbf{N}}+\mathbf{x}) = \boldsymbol{\theta}\cdot B_{\mathbf{N}'}+\mathbf{x}_1$ and

$$\boldsymbol{\gamma} \in \Gamma_{\mathcal{M}} \cap (\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}) \Leftrightarrow \boldsymbol{\gamma} \cdot \sigma(\eta(\mathbf{N})) \in \Gamma_{\mathcal{M}} \cap (\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1)),$$

with $\mathbf{x}_1 = \sigma(\eta(\mathbf{N}) \cdot \mathbf{x} + \sigma(\eta(\mathbf{N})) \cdot \mathbf{N}/2 - \mathbf{N}'/2$. Hence

$$\mathcal{N}(\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{N}(\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1, \Gamma_{\mathcal{M}}).$$

By (1.2), we have

$$\mathcal{R}(\boldsymbol{\theta} \cdot B_{\mathbf{N}} + \mathbf{x}, \Gamma_{\mathcal{M}}) = \mathcal{R}(\boldsymbol{\theta} \cdot B_{\mathbf{N}'} + \mathbf{x}_1, \Gamma_{\mathcal{M}}).$$

Therefore, without loss of generality, we can assume that

$$N_i N^{-1/s} \in [1/c_3, c_3], \quad i = 1, \dots, s. \tag{2.3}$$

Note that in this paper $O$-constants and constants $c_1, c_2, \dots$ depend only on $\mathcal{M}$.

We shall need the Poisson summation formula:

$$\det \Gamma \sum_{\boldsymbol{\gamma} \in \Gamma} f(\boldsymbol{\gamma} - X) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp}} \widehat{f}(\boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle), \tag{2.4}$$

where

$$\widehat{f}(Y) = \int_{\mathbb{R}^s} f(X) e(\langle \mathbf{y}, \mathbf{x} \rangle) \mathrm{d}\mathbf{x}$$

is the Fourier transform of $f(X)$, and $e(x) = \exp(2\pi\sqrt{-1}x)$, $\langle \mathbf{y}, \mathbf{x} \rangle = y_1 x_1 + \cdots + y_s x_s$. Formula (2.4) holds for functions $f(\mathbf{x})$ with period lattice $\Gamma$ if one of the functions $f$ or $\widehat{f}$ is integrable and belongs to the class $C^{\infty}$ (see e.g. [28, p. 251]).

Let $\widehat{\mathbb{1}}_{B_{\mathbf{N}}}(\boldsymbol{\gamma})$ be the Fourier transform of the indicator function $\mathbb{1}_{B_{\mathbf{N}}}(\boldsymbol{\gamma})$. It is easy to prove that $\widehat{\mathbb{1}}_{B_{\mathbf{N}}}(\mathbf{0}) = N_1 \cdots N_s$ and

$$\widehat{\mathbb{1}}_{B_{\mathbf{N}}}(\boldsymbol{\gamma}) = \prod_{i=1}^{s} \frac{e(N_i \gamma_i) - 1}{2\pi\sqrt{-1}\gamma_i} = \prod_{i=1}^{s} \frac{\sin(\pi N_i \gamma_i)}{\pi \gamma_i} e\Big(\sum_{i=1}^{s} N_i \gamma_i / 2\Big) \text{ for } \mathrm{Nm}(\boldsymbol{\gamma} \neq 0). \tag{2.5}$$

We fix a nonnegative even function $\omega(x)$, $x \in \mathbb{R}$, of the class $C^{\infty}$, with a support inside the segment $[-1/2, 1/2]$, and satisfying the condition $\int_{\mathbb{R}} \omega(x)\mathrm{d}x = 1$. We set $\Omega(\mathbf{x}) = \omega(x_1) \cdots \omega(x_s)$, $\Omega_{\tau}(\mathbf{x}) = \tau^{-s} \Omega(\tau^{-1} x_1, \dots, \tau^{-1} x_s)$, $\tau > 0$, and

$$\widehat{\Omega}(\mathbf{y}) = \int_{\mathbb{R}^s} e(\langle \mathbf{y}, \mathbf{x} \rangle) \Omega(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{2.6}$$

Notice that the Fourier transform $\widehat{\Omega}_{\tau}(\mathbf{y}) = \widehat{\Omega}(\tau \mathbf{y})$ of the function $\Omega_{\tau}(\mathbf{y})$ satisfies the bound

$$|\widehat{\Omega}(\tau \mathbf{y})| < \dot{c}(s, \omega)(1 + \tau |\mathbf{y}|)^{-2s}. \tag{2.7}$$

It is easy to see that

$$\widehat{\Omega}(\mathbf{y}) = \widehat{\Omega}(\mathbf{0}) + O(|\mathbf{y}|) = 1 + O(|\mathbf{y}|) \quad \text{for} \quad |\mathbf{y}| \to 0. \tag{2.8}$$

**Lemma 1** *There exists a constant $c > 0$ such that we have for $N > c$*

$$|\mathcal{R}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)| \le 2^s,$$

*where*

$$\ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = (\det \Gamma)^{-1} \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp}\setminus\{0\}} \widehat{\mathbb{1}}_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau\boldsymbol{\gamma}) e(\langle \boldsymbol{\gamma}, \mathbf{x}\rangle), \quad \tau = N^{-2}. \tag{2.9}$$

*Proof* Let $B^{\pm\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} = [0, \max(0, \theta_1 N_1 \pm \tau)) \times \cdots \times [0, \max(0, \theta_s N_s \pm \tau))$, and let $\mathbb{1}_B(x)$ be the indicator function of $B$. We consider the convolutions of the functions $\mathbb{1}_{B^{\pm\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\boldsymbol{\gamma})$ and $\Omega_\tau(\mathbf{y})$:

$$\Omega_\tau * \mathbb{1}_{B^{\pm\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}) = \int_{\mathbb{R}^s} \Omega_\tau(\mathbf{x} - \mathbf{y}) \mathbb{1}_{B^{\pm\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{y}) d\mathbf{y}. \tag{2.10}$$

It is obvious that the nonnegative functions (2.10) are of class $C^\infty$ and are compactly supported in $\tau$-neighborhoods of the bodies $B^{\pm\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}$, respectively. We obtain

$$\mathbb{1}_{B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}) \le \mathbb{1}_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}) \le \mathbb{1}_{B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}), \quad \mathbb{1}_{B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}) \le \Omega_\tau * \mathbb{1}_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}) \le \mathbb{1}_{B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\mathbf{x}). \tag{2.11}$$

Replacing $\mathbf{x}$ by $\boldsymbol{\gamma} - \mathbf{x}$ in (2.11) and summing these inequalities over $\boldsymbol{\gamma} \in \Gamma = \Gamma_{\mathcal{M}}$, we find from (1.1) that

$$\mathcal{N}(B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \le \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \le \mathcal{N}(B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma),$$

and

$$\mathcal{N}(B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \le \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \le \mathcal{N}(B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma),$$

where

$$\dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \sum_{\boldsymbol{\gamma} \in \Gamma} \Omega_\tau * \mathbb{1}_{B_{\boldsymbol{\theta}\cdot\mathbf{N}}}(\boldsymbol{\gamma} - \mathbf{x}). \tag{2.12}$$

Hence

$$\begin{aligned} &-\mathcal{N}(B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) + \mathcal{N}(B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \\ &\quad \le \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) \le \mathcal{N}(B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \mathcal{N}(B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma). \end{aligned}$$

Thus

$$|\mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)| \le \mathcal{N}(B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \mathcal{N}(B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \upsilon\Gamma). \tag{2.13}$$

Consider the right side of this inequality. We have that $B^{+\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}} \setminus B^{-\tau}_{\boldsymbol{\theta}\cdot\mathbf{N}}$ is the union of boxes $B^{(i)},\ i = 1, \dots, 2^s - 1$, where

$$\mathrm{vol}(B^{(i)}) \le \mathrm{vol}(B^{+\tau}_{\mathbf{N}}) - \mathrm{vol}(B^{-\tau}_{\mathbf{N}}) \le \prod_{i=1}^{s}(N_i+\tau) - \prod_{i=1}^{s}(N_i-\tau)$$
$$\le N\Big(\prod_{i=1}^{s}(1+\tau) - \prod_{i=1}^{s}(1-\tau)\Big) < \ddot{c}_s N\tau = \ddot{c}_s/N, \quad \tau = N^{-2},$$

with some $\ddot{c}_s > 0$. From (2.1), we get $\mathcal{M} \supseteq p_1^{-1}\mathcal{O}$. Hence $|\mathrm{Nm}(\boldsymbol{\gamma})| \ge p_1^{-s}$ for $\boldsymbol{\gamma} \in \Gamma_{\mathcal{M}} \setminus \mathbf{0}$. We see that $|\mathrm{Nm}(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)| \le \mathrm{vol}(B^{(i)} + \mathbf{x}) < p_1^{-s}$ for $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in B^{(i)} + \mathbf{x}$ and $N > \ddot{c}_s p_1^s$. Therefore, the box $B^{(i)} + \mathbf{x}$ contains at most one point of $\Gamma_{\mathcal{M}}$ for $N > \ddot{c} p_1^s$. By (2.13), we have

$$|\dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \mathcal{N}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)| \le 2^s - 1 \quad \text{for} \quad N > \ddot{c} p_1^s. \tag{2.14}$$

Let

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) - \frac{\mathrm{vol}(B_{\boldsymbol{\theta}\cdot\mathbf{N}})}{\det \Gamma}. \tag{2.15}$$

By (2.12), we obtain that $\dot{\mathcal{N}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)$ is a periodic function of $\mathbf{x} \in \mathbb{R}^n$ with the period lattice $\Gamma$. Applying the Poisson summation formula to the series (2.12), and bearing in mind that $\widehat{\Omega}_\tau(\mathbf{y}) = \widehat{\Omega}(\tau\mathbf{y})$, we get from (2.9)

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = \ddot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma).$$

Note that (2.7) ensure the absolute convergence of the series (2.9) over $\boldsymbol{\gamma} \in \Gamma^\perp \setminus \{0\}$. Using (1.2), (2.14) and (2.15) , we obtain the assertion of Lemma 1. □

Let $\eta(t) = \eta(|t|)$, $t \in \mathbb{R}^1$ be an even function of the class $C^\infty$; moreover, let $\eta(t) = 0$ for $|t| \le 1$, $0 \le \eta(t) \le 1$ for $|t| \le 2$ and $\eta(t) = 1$ for $|t| \ge 2$. Let $n = s^{-1}\log_2 N$, $M = [\sqrt{n}]$ , and

$$\eta_M(\boldsymbol{\gamma}) = 1 - \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})|/M). \tag{2.16}$$

By (2.5) and (2.9), we have

$$\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma) = (\pi^s \det \Gamma)^{-1}(\mathcal{A}(\mathbf{x}, M) + \mathcal{B}(\mathbf{x}, M)), \tag{2.17}$$

where

$$\mathcal{A}(\mathbf{x}, M) = \sum_{\boldsymbol{\gamma}\in\Gamma^\perp\setminus\mathbf{0}} \prod_{i=1}^{s} \sin(\pi\theta_i N_i \gamma_i) \frac{\eta_M(\boldsymbol{\gamma})\widehat{\Omega}(\tau\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle + \dot{x})}{\mathrm{Nm}(\boldsymbol{\gamma})},$$
$$\mathcal{B}(\mathbf{x}, M) = \sum_{\boldsymbol{\gamma}\in\Gamma^\perp\setminus\mathbf{0}} \prod_{i=1}^{s} \sin(\pi\theta_i N_i \gamma_i) \frac{(1-\eta_M(\boldsymbol{\gamma}))\widehat{\Omega}(\tau\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{x}\rangle + \dot{x})}{\mathrm{Nm}(\boldsymbol{\gamma})},$$

with $\dot{x} = \sum_{1\le i\le s} \theta_i N_i \gamma_i/2$. Let

$$\mathbf{E}(f) = \int_{[0,1]^s} f(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}.$$

By the triangle inequality, we get

$$\pi^s \det\Gamma \sup_{\boldsymbol{\theta}\in[0,1]^s} |\dot{\mathcal{R}}(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{x}, \Gamma)| \ge |\mathbf{E}(\mathcal{A}(\mathbf{x}, M))| - |\mathbf{E}(\mathcal{B}(\mathbf{x}, M))|. \tag{2.18}$$

In Sect. 2.5 we will find the lower bound of $|\mathbf{E}(\mathcal{A}(\mathbf{x}, M))|$ and in Sect. 2.9 we will find the upper bound of $|\mathbf{E}(\mathcal{B}(\mathbf{x}, M))|$.

### 2.2 The Logarithmic Space and the Fundamental Domain

We consider Dirichlet's Unit Theorem (2.2) applied to the ring of integers $\mathcal{O}$. Let $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_{s-1}$ be the set of fundamental units of $\mathcal{U}_{\mathcal{O}}$. We set $l_i(\mathbf{x}) = \ln|x_i|$, $i = 1, \dots, s$, $\mathbf{l}(\mathbf{x}) = (l_1(\mathbf{x}), \dots, l_s(\mathbf{x}))$, $\mathbf{1} = (1, \dots, 1)$, where $\mathbf{x} \in \mathbb{R}^s$ and $\mathrm{Nm}(\mathbf{x}) \neq \mathbf{0}$. By [6, p. 311], the set of vectors $\mathbf{1}, \mathbf{l}(\varepsilon_1), \dots, \mathbf{l}(\varepsilon_{s-1}))$ is a basis for $\mathbb{R}^s$. Any vector $\mathbf{l}(\mathbf{x}) \in \mathbb{R}^s$ ($\mathbf{x} \in \mathbb{R}^s$, $\mathrm{Nm}(\mathbf{x}) \neq \mathbf{0}$) can be represented in the form

$$\mathbf{l}(\mathbf{x}) = \xi\mathbf{1} + \xi_1\mathbf{l}(\boldsymbol{\varepsilon}_1) + \dots + \xi_{s-1}\mathbf{l}(\boldsymbol{\varepsilon}_{s-1}), \tag{2.19}$$

where $\xi, \xi_1, \dots, \xi_{s-1}$ are real numbers. In the following we will need the next definition.

**Definition 2** [6, p. 312] A subset $\mathcal{F}$ of the space $\mathbb{R}^s$ is called a *fundamental domain* for the field $\mathcal{K}$ if it consists of all points $\mathbf{x}$ which satisfy the following conditions: $\mathrm{Nm}(\mathbf{x}) \neq \mathbf{0}$, in the representation (2.19) the coefficients $\xi_i$ ($i = 1, \dots, s-1$) satisfy the inequality $0 \le \xi_i < 1$, $x_1 > 0$.

**Theorem B** [6, p. 312] *In every class of associate numbers* ($\neq 0$) *of the field* $\mathcal{K}$*, there is one and only one number whose geometric representation in the space* $\mathbb{R}^s$ *lies in the fundamental domain* $\mathcal{F}$.

**Lemma A** [30, p. 59, Thm. 2, Ref. 3] *Let* $\dot{\Gamma} \subset \mathbb{R}^k$ *be a lattice,* $\det\dot{\Gamma} = 1$*,* $\mathcal{Q} \subset \mathbb{R}^k$ *a compact convex body and r the radius of its greatest sphere in the interior. Then*

$$\mathrm{vol}(\mathcal{Q})\big(1 - \frac{\sqrt{k}}{2r}\big) \le \#\dot{\Gamma} \cap \mathcal{Q} \le \mathrm{vol}(\mathcal{Q})\big(1 + \frac{\sqrt{k}}{2r}\big),$$

*provided* $r > \sqrt{k}/2$.

Let $\dot{\Gamma} \subset \mathbb{R}^k$ be an arbitrary lattice. We derive from Lemma A

$$\sup_{\mathbf{x}\in\mathbb{R}^s} |\#\dot{\Gamma} \cap (t\mathcal{Q} + \mathbf{x}) - t^k\mathrm{vol}(\mathcal{Q})/\det\dot{\Gamma}| = O(t^{k-1}) \quad \text{for} \quad t \to \infty. \tag{2.20}$$

See also [11, pp. 141, 142].

**Lemma 2** *Let* $\boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} = \max_{1\le i\le s} |(\boldsymbol{\varepsilon}^{\mathbf{k}})_i|$ *and* $\boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} = \min_{1\le i\le s} |(\boldsymbol{\varepsilon}^{\mathbf{k}})_i|$. *There exists a constant* $c_4, c_5 > 0$, *such that*

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \le e^t\} = c_4 t^{s-1} + O(t^{s-2}) \tag{2.21}$$

*and*

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} \ge e^{-t}\} = c_5 t^{s-1} + O(t^{s-2}). \tag{2.22}$$

*Proof* By (2.19), we have that the left hand sides of (2.21) and (2.22) are equal to

$$\#\Big\{\mathbf{k} \in \mathbb{Z}^{s-1}\Big| \sum_{i=1}^{s-1} k_i l_j(\boldsymbol{\varepsilon}_i) \le t,\ \ j = 1, \dots, s\Big\},$$

and

$$\#\Big\{\mathbf{k} \in \mathbb{Z}^{s-1}\Big| \sum_{i=1}^{s-1} k_i l_j(\boldsymbol{\varepsilon}_i) \ge -t,\ \ j = 1, \dots, s\Big\},$$

respectively. Let

$$\mathcal{Q}_1 = \{\mathbf{x} \in \mathbb{R}^{s-1} | \dot{x_j} \le 1,\ \ j \in [1, s]\} \ \text{ and } \ \mathcal{Q}_2 = \{\mathbf{x} \in \mathbb{R}^{s-1} | \dot{x_j} \ge -1,\ \ j \in [1, s]\},$$

with $\dot{x_j} = x_1 l_j(\boldsymbol{\varepsilon}_1) + \cdots + x_{s-1} l_j(\boldsymbol{\varepsilon}_{s-1})$. We see $\dot{x_1} + \cdots + \dot{x_s} = 0$. Hence $\dot{x_j} \ge -s+1$ for $\mathbf{x} \in \mathcal{Q}_1$ and $\dot{x_j} \le s-1$ for $\mathbf{x} \in \mathcal{Q}_2$ ($j = 1, \dots, s$). By [6, p. 115], we get $\det(l_i(|\boldsymbol{\varepsilon}_j|)_{i,j=1,\dots,s-1}) \ne 0$. Hence, $\mathcal{Q}_i$ is the compact convex set in $\mathbb{R}^{s-1}$, $i = 1, 2$. Applying (2.20) with $k = s-1$, and $\dot{\Gamma} = \mathbb{Z}^{s-1}$, we obtain the assertion of Lemma 2. □

Let $cl(\mathcal{K})$ be the ideal class group of $\mathcal{K}$, $h = \#cl(\mathcal{K})$, and $cl(\mathcal{K}) = \{C_1, \dots, C_h\}$. In the ideal class $C_i$, we choose an integral ideal $\mathfrak{a}_i$, $i = 1, \dots, h$. Let $\mathfrak{N}(\mathfrak{a})$ be the absolute norm of ideal $\mathfrak{a}$. If $h = 1$, then we set $p_2 = 1$ and $\Gamma_1 = \Gamma_{\mathcal{O}}$. Let $h > 1$, $i \in [1, h]$,

$$\mathcal{M}_i = \{u \in \mathcal{O} \mid u \equiv 0 \bmod \mathfrak{a}_i\}, \quad \Gamma_i = \sigma(\mathcal{M}_i), \quad \text{and} \quad p_2 = \prod_{i=1}^{h} \mathfrak{N}(\mathfrak{a}_i). \tag{2.23}$$

**Lemma 3** *Let* $w \ge 1$, $i \in [1, h]$, $\mathbb{F}_{M_1}(\boldsymbol{\varsigma}) = \{\mathbf{y} \in \mathcal{F} \mid |\mathrm{Nm}(\mathbf{y})| < M_1, \mathrm{sgn}(y_i) = \varsigma_i, i = 1, \dots, s\}$, *where* $\mathrm{sgn}(y) = y/|y|$ *for* $y \ne 0$ *and* $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_s) \in \{-1, 1\}^s$. *Then there exists* $c_{6,i} > 0$ *such that*

$$\sup_{\mathbf{x}\in\mathbb{R}^s} \Big| \sum_{\gamma\in(w\Gamma_i+\mathbf{x})\cap\mathbb{F}_{M_1}(\boldsymbol{\varsigma})} 1 - c_{6,i} M_1/w^s \Big| = O(M_1^{1-1/s}) \quad \text{for} \quad M_1 \to \infty.$$

*Proof* It is easy to see that $\mathbb{F}_{M_1}(\boldsymbol{\varsigma}) = {M_1}^{1/s}\mathbb{F}_1(\boldsymbol{\varsigma})$. By [6, p. 312], the fundamental domain $\mathcal{F}$ is a cone in $\mathbb{R}^s$. Let $\dot{\mathbb{F}} = \{\mathbf{y} \in \mathcal{F} |\ |y_i| \le y_0, \mathrm{sgn}(y_i) = \varsigma_i, i = 1, \dots, s\}$ and let $\ddot{\mathbb{F}} = \{\mathbf{y} \in \dot{\mathbb{F}} \mid |\mathrm{Nm}(\mathbf{y})| \ge 1\}$, where $y_0 = \sup_{\mathbf{y}\in\mathbb{F}_1(\boldsymbol{\varsigma}), i=1,\dots,s} |y_i|$. We see that

$\mathbb{F}_1(\varsigma) = \dot{\mathbb{F}} \setminus \ddot{\mathbb{F}}$ and $\dot{\mathbb{F}}, \ddot{\mathbb{F}}$ are compact convex sets. Using (2.20) with $k = s$, $\dot{\Gamma} = w\Gamma_i$, and $t = {M_1}^{1/s}$, we obtain the assertion of Lemma 3. □

### 2.3 Construction of a Hecke Character by Using Chevalley's Theorem

Let $\mathfrak{m}$ be an integral ideal of the number field $\mathcal{K}$, and let $\mathcal{J}^{\mathfrak{m}}$ be the group of all ideals of $\mathcal{K}$ which are relatively prime to $\mathfrak{m}$. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

**Definition 3** [24, p. 470] A *Hecke character* mod $\mathfrak{m}$ is a character $\chi : \mathcal{J}^{\mathfrak{m}} \to S^1$ for which there exists a pair of characters

$$\chi_f : (\mathcal{O}/\mathfrak{m})^* \to S^1, \qquad \chi_\infty : (\mathbb{R}^*)^s \to S^1,$$

such that

$$\chi((a)) = \chi_f(a)\chi_\infty(a)$$

for every algebraic integer $a \in \mathcal{O}$ relatively prime to $\mathfrak{m}$.

The character taking the value one for all group elements will be called the trivial character.

**Definition 4** Let $A_1, \dots, A_d$ be invertible $s \times s$ commuting matrices with integer entries. A sequence of matrices $A_1, \dots, A_d$ is said to be *partially hyperbolic* if for all $(n_1, \dots, n_d) \in \mathbb{Z}^d \setminus \{0\}$ none of the eigenvalues of $A_1^{n_1} \dots A_d^{n_d}$ are roots of unity.

We need the following variant of Chevalley's theorem ([9], see also [29]):

Theorem C [13, p. 282, Th. 6.2.6] Let $A_1, \dots, A_d \in GL(s, \mathbb{Z})$ be commuting partially hyperbolic matrices with determinants $w_1, \dots, w_d$, $p^{(k)}$ the product of the first $k$ primes numbers relatively prime to $w_1, \dots, w_d$. If $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{Z}^s$ and there are $d$ sequences $\{j_i^{(k)}, 1 \le i \le d\}$ of integers such that

$$A_1^{j_1^{(n)}} \cdots A_d^{j_d^{(k)}} \tilde{\mathbf{z}} \equiv \mathbf{z} \pmod{p^{(k)}}, \qquad k = 1, 2, \dots,$$

then there exists a vector $(j_1^{(0)}, \dots, j_d^{(0)}) \in \mathbb{Z}^s$ such that

$$A_1^{j_1^{(0)}} \cdots A_d^{j_d^{(0)}} \tilde{\mathbf{z}} = \mathbf{z}. \tag{2.24}$$

Let

$$\mu = \begin{cases} 1 & \text{if } s \text{ is odd,} \\ 2 & \text{if } s \text{ is even and } \nexists \boldsymbol{\varepsilon} \text{ with } N_{\mathcal{K}/Q}(\boldsymbol{\varepsilon}) = -1, \\ 3 & \text{if } s \text{ is even and } \exists \boldsymbol{\varepsilon} \text{ with } N_{\mathcal{K}/Q}(\boldsymbol{\varepsilon}) = -1. \end{cases} \tag{2.25}$$

Let $\mu \in \{1, 2\}$. By [6, p. 117], we see that there exist units $\boldsymbol{\varepsilon}_i \in \mathcal{U}_{\mathcal{O}}$ with $N_{\mathcal{K}/Q}(\boldsymbol{\varepsilon}_i) = 1$, $i = 1, \dots, s-1$, such that every $\boldsymbol{\varepsilon} \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows:

$$\boldsymbol{\varepsilon} = (-1)^a \boldsymbol{\varepsilon}_1^{k_1} \cdots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} \quad \text{with} \quad (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1}, \ a \in \{0, 1\}. \tag{2.26}$$

Let $\mu = 3$. By [6, p. 117], there exist units $\boldsymbol{\varepsilon}_i \in \mathcal{U}_{\mathcal{O}}$ with $N_{\mathcal{K}/Q}(\boldsymbol{\varepsilon}_i) = 1$, $i = 1, \dots, s-1$ and $N_{\mathcal{K}/Q}(\boldsymbol{\varepsilon}_0) = -1$, such that every $\boldsymbol{\varepsilon} \in \mathcal{U}_{\mathcal{O}}$ can be uniquely represented as follows:

$$\boldsymbol{\varepsilon} = (-1)^{a_1}\boldsymbol{\varepsilon}_0^{a_2}\boldsymbol{\varepsilon}_1^{k_1}\cdots\boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} \quad \text{with} \quad (k_1,\dots,k_{s-1}) \in \mathbb{Z}^{s-1},\ a_1, a_2 \in \{0,1\}. \tag{2.27}$$

Consider the case $\mu = 1$. Let $I_i = \operatorname{diag}((\sigma_j(\boldsymbol{\varepsilon}_i))_{1\le j\le s})$, $i = 1, \dots, s-1$, $\Gamma_{\mathcal{O}} = \sigma(\mathcal{O})$, $\mathbf{f}_1, \dots, \mathbf{f}_s$ be a basis of $\Gamma_{\mathcal{O}}$, $\mathbf{e}_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^s$, $i = 1, \dots, s$ a basis of $\mathbb{Z}^s$. Let $Y$ be the $s \times s$ matrix with $\mathbf{e}_i Y = \mathbf{f}_i$, $i = 1, \dots, s$. We have $\mathbb{Z}^s Y = \Gamma_{\mathcal{O}}$. Let $A_i = Y I_i Y^{-1}$, $i = 1, \dots, s-1$. We see $\mathbb{Z}^s A_i = \mathbb{Z}^s$ $(i = 1, \dots, s-1)$. Hence, $A_i$ is the integer matrix with $\det A_i = \det I_i = 1$ $(i = 1, \dots, s-1)$.

Let $\tilde{\mathbf{z}} = (1, \dots, 1)$ and $\mathbf{z} = -\tilde{\mathbf{z}}$. Let $h > 1$, and let $A_s = p_2 I$, where $I$ is the identity matrix. Taking into account that $(\boldsymbol{\varepsilon}_1^{k_1} \dots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} p_2^{k_s})_j = 1$ for some $j \in [1, s]$ if and only if $k_1 = \cdots = k_s = 0$, we get that $A_1, \dots, A_s$ are commuting partially hyperbolic matrices. By Definition 4, $-1$ is not the eigenvalue of $A_1^{k_1} \dots A_s^{k_s}$, and $\tilde{\mathbf{z}} A_1^{k_1} \dots A_s^{k_s} \neq \mathbf{z}$ for all $(k_1, \dots, k_s) \in \mathbb{Z}^s$. Applying Theorem D with $d = s$, we have that there exists an integer $i \ge 1$ such that $(p_2, p^{(i)}) = 1$,

$$\tilde{\mathbf{z}} A_1^{k_1} \dots A_{s-1}^{k_{s-1}} \not\equiv \mathbf{z} \pmod{p^{(i)}} \quad \text{for all} \quad (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1},$$

and

$$(\boldsymbol{\varepsilon}_1^{k_1} \dots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}})_j \not\equiv -1 \pmod{p^{(i)}} \quad \text{for all} \quad (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1}, \quad j \in [1, s]. \tag{2.28}$$

We denote this $p^{(i)}$ by $p_3$. We have $(p_2, p_3) = 1$. If $h = 1$, then we apply Theorem D with $d = s - 1$.

Let $\mathfrak{p}_3 = p_3 \mathcal{O}$ and $\mathbb{P} = \mathcal{O}/\mathfrak{p}_3$. Denote the projection map $\mathcal{O} \to \mathbb{P}$ by $\pi_1$. Let $\mathcal{O}^*$ be the set of all integers of $\mathcal{O}$ which are relatively prime to $\mathfrak{p}_3$, $\mathbb{P}^* = \pi_1(\mathcal{O}^*)$,

$$\mathcal{E}_j = \{v \in \mathbb{P}^* \mid \exists\, (k_1, \dots, k_{s-1}) \in \mathbb{Z}^{s-1} \text{ with } v \equiv (-1)^j \boldsymbol{\varepsilon}_1^{k_1} \dots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} \pmod{\mathfrak{p}_3}\},$$

where $j = 0, 1$, and $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. By (2.28), $\mathcal{E}_0 \cap \mathcal{E}_1 = \emptyset$. Let

$$\chi_{1,p_3}(v) = (-1)^j \quad \text{for} \quad v \in \mathcal{E}_j,\ j = 0, 1. \tag{2.29}$$

We see that $\chi_{1,p_3}$ is the character on group $\mathcal{E}$. We need the following known assertion (see e.g. [12, p. 63], [14, p. 446, Chap. 8, Sect. 2, Ex. 4]) :

**Lemma B** *Let $\dot{G}$ be a finite abelian group, $\dot{H}$ is a subgroup of $\dot{G}$, and $\chi_{\dot{H}}$ is a character of $\dot{H}$. Then there exists a character $\chi_{\dot{G}}$ of $\dot{G}$ such that $\chi_{\dot{H}}(h) = \chi_{\dot{G}}(h)$ for all $h \in \dot{H}$.*

Applying Lemma B, we can extend the character $\chi_{1,p_3}$ to a character $\chi_{2,p_3}$ of group $\mathbb{P}^*$. Now we extend $\chi_{2,p_3}$ to a character $\chi_{3,p_3}$ of group $\mathcal{O}^*$ by setting

$$\chi_{3,p_3}(v) = \chi_{2,p_3}(\pi_1(v)) \quad \text{for} \quad v \in \mathcal{O}^*. \tag{2.30}$$

Let

$$\chi_{4,p_3}(v) = \chi_{3,p_3}(v)\chi_\infty(v) \quad \text{with} \quad \chi_\infty(v) = \mathrm{Nm}(v)/|\mathrm{Nm}(v)|,$$

for $v \in \mathcal{O}^*$, and let

$$\chi_{5,p_3}((v)) = \chi_{4,p_3}(v). \tag{2.31}$$

We need to prove that the right hand side of (2.31) does not depend on units $\boldsymbol{\varepsilon} \in \mathcal{U}_\mathcal{O}$. Let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1^{k_1} \ldots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}}$. By (2.26), (2.29), and (2.30), we have $\chi_{3,p_3}(\boldsymbol{\varepsilon}) = 1, \mathrm{Nm}(\boldsymbol{\varepsilon}) = 1$, and $\chi_\infty(\boldsymbol{\varepsilon}) = 1$. Therefore

$$\chi_{4,p_3}(v\boldsymbol{\varepsilon}) = \chi_{3,p_3}(v\boldsymbol{\varepsilon})\chi_\infty(v\boldsymbol{\varepsilon}) = \chi_{3,p_3}(v)\chi_{3,p_3}(\boldsymbol{\varepsilon})\chi_\infty(v)\chi_\infty(\boldsymbol{\varepsilon}) = \chi_{3,p_3}(v)\chi_\infty(v).$$

Now let $\boldsymbol{\varepsilon} = -1$. Bearing in mind that $\chi_{3,p_3}(-1) = -1$, $\mathrm{Nm}(-1) = -1$, and $\chi_\infty(-1) = -1$, we obtain $\chi_{4,p_3}(-1) = 1$. Hence, definition (2.31) is correct. Let $\mathcal{I}^{\mathfrak{p}_3}$ be the group of all principal ideals of $\mathcal{K}$ which are relatively prime to $\mathfrak{p}_3$. Let

$$\chi_{6,p_3}((v_1/v_2)) = \chi_{5,p_3}((v_1))/\chi_{5,p_3}((v_2)) \quad \text{for} \quad v_1, v_2 \in \mathcal{O}^*.$$

Let $\mathcal{P}^{\mathfrak{p}_3}$ is the group of fractional principal ideals $(a)$ such that $a \equiv 1 \bmod \mathfrak{p}_3$ and $\sigma_i(a) > 0,\ i = 1, \ldots, s$. Let $\pi_2 : \mathcal{I}^{\mathfrak{p}_3} \to \mathcal{I}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$ be the projection map. Bearing in mind that $\chi_{6,p_3}(\mathfrak{a}) = 1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_3}$, we define

$$\chi_{7,p_3}(\pi_2(\mathfrak{a})) = \chi_{6,p_3}(\mathfrak{a}) \quad \text{for} \quad \mathfrak{a} \in \mathcal{I}^{\mathfrak{p}_3}.$$

By [23, p. 94, Lemma. 3.3], $\mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$ is the finite abelian group. Applying Lemma B, we extend the character $\chi_{7,p_3}$ to a character $\chi_{8,p_3}$ of group $\mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$. We have $\chi_{8,p_3}(\mathfrak{a}) = 1$ for $\mathfrak{a} \in \mathcal{P}^{\mathfrak{p}_3}$, and we set $\chi_{9,p_3}(\mathfrak{a}) = \chi_{8,p_3}(\pi_3(\mathfrak{a}))$, where $\pi_3$ is the proection map $\mathcal{J}^{\mathfrak{p}_3} \to \mathcal{J}^{\mathfrak{p}_3}/\mathcal{P}^{\mathfrak{p}_3}$. It is easy to verify

$$\begin{aligned}\chi_{9,p_3}((v)) &= \chi_{8,p_3}(\pi_3((v))) = \chi_{7,p_3}(\pi_3((v))) = \chi_{7,p_3}(\pi_2((v))) \\ &= \chi_{6,p_3}((v)) = \chi_{4,p_3}(v) = \chi_{3,p_3}(v)\chi_\infty(v)\end{aligned}$$

for $\mathfrak{a} \in \mathcal{I}^{\mathfrak{p}_3}$. Thus we have constructed a nontrivial Hecke character.

Case $\mu = 2$. We repeat the construction of the case $\mu = 1$, taking $p_3 = 1$ and $\chi_{4,p_3}((v)) = \mathrm{Nm}(v)/|\mathrm{Nm}(v)|$.

Case $\mu = 3$. Similarly to the case $\mu = 1$, we have that there exists $i > 0$ with

$$\boldsymbol{\varepsilon}_1^{k_1} \ldots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} \not\equiv \boldsymbol{\varepsilon}_0 (\mathrm{mod}\ p^{(i)}) \quad \text{for all} \quad (k_1, \ldots, k_{s-1}) \in \mathbb{Z}^{s-1}. \tag{2.32}$$

We denote this $p^{(i)}$ by $p_3$. Let

$$\mathcal{E}_j = \{v \in \mathcal{P}^* \mid \exists\, (k_1, \ldots, k_{s-1}) \in \mathbb{Z}^{s-1} \text{ with } v \equiv \boldsymbol{\varepsilon}_0^j \boldsymbol{\varepsilon}_1^{k_1} \ldots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}} (\mathrm{mod}\ p_3\mathcal{O})\},$$

where $j = 0, 1$, and $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. By (2.32), $\mathcal{E}_0 \cap \mathcal{E}_1 = \emptyset$. Let

$$\chi_{2,p_3}(v) = (-1)^j \quad \text{for} \quad v \in \mathcal{E}_j,\ j = 0, 1.$$

Next, we repeat the construction of the case $\mu = 1$, and we verify the correction of definition (2.31). Thus, we have proved the following lemma:

**Lemma 4** *Let* $\mu \in \{1, 2, 3\}$. *There exists* $p_3 = p_3(\mu) \geq 1$, $(p_2, p_3) = 1$, *a nontrivial Hecke character* $\dot{\chi}_{p_3}$, *and a character* $\ddot{\chi}_{p_3}$ *on group* $(\mathcal{O}/p_3\mathcal{O})^*$ *such that*

$$\dot{\chi}_{p_3}((v)) = \tilde{\chi}_{p_3}(v), \quad \text{with} \quad \tilde{\chi}_{p_3}(v) = \ddot{\chi}_{p_3}(v)\mathrm{Nm}(v)/|\mathrm{Nm}(v)|,$$

*for* $v \in \mathcal{O}^*$, *and* $\ddot{\chi}_{p_3}(v) = 0$ *for* $(v, p_3\mathcal{O}) \neq 1$.

### 2.4 Non-vanishing of $L$-functions

With every Hecke character $\chi$ mod $\mathfrak{m}$, we associate its $L$-function

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ varies over the integral ideals of $\mathcal{K}$, and we put $\chi(\mathfrak{a}) = 0$ whenever $(\mathfrak{a}, \mathfrak{m}) \neq 1$.

**Theorem C** [15, p. 313, Thm. 2] *Let* $\chi$ *be a nontrivial Hecke character. Then*

$$L(1, \chi) \neq 0.$$

**Theorem D** [21, p. 128, Thm. 10.1.4] *Let* $(a_k)_{k\geq 1}$ *be a sequence of complex numbers, and let* $\sum_{k<x} a_k = O(x^\delta)$, *for some* $\delta > 0$. *Then*

$$\sum_{n\geq 1} a_n/n^s \tag{2.33}$$

*converges for* $\Re(s) > \delta$.

**Theorem E** [23, p. 464, Prop. I] *If the series* (2.33) *converges at a point* $s_0$, *then it converges also in the open half-plane* $\Re s > \Re s_0$, *the convergence being uniform in every angle* $\arg(s - s0) < c < \pi/2$. *Thus* (2.33) *defines a function regular in* $\Re s > \Re s_0$.

*Let* $\mathbf{f}_1, \dots, \mathbf{f}_s$ *be a basis of* $\Gamma_{\mathcal{O}}$, *and let* $\mathbf{f}_1^{\perp}, \dots, \mathbf{f}_s^{\perp}$ *be a dual basis (i.e.* $\langle \mathbf{f}_i, \mathbf{f}_i^{\perp}\rangle = 1$, $\langle \mathbf{f}_i, \mathbf{f}_j^{\perp}\rangle = 0,\ 1 \leq i, j \leq s,\ i \neq j$). *Let*

$$\Lambda_w = \{a_1\mathbf{f}_1^{\perp} + \cdots + a_s\mathbf{f}_s^{\perp} \mid 0 \leq a_i \leq w - 1,\ i = 1, \dots, s\}, \tag{2.34}$$

*and* $\Lambda_w^* = \{\mathbf{b} \in \Lambda_w \mid (w, \mathbf{b}) = 1\}$.

**Lemma 5** *With notations as above,*

$$\rho(M, j) := \sum_{\boldsymbol{\gamma}\in\Gamma_j\cap\mathcal{F},\ |\mathrm{Nm}(\boldsymbol{\gamma})|<M/2} \tilde{\chi}_{p_3}(\boldsymbol{\gamma}) = O(M^{1-1/s}), \qquad j \in [1, h], \tag{2.35}$$

*and*

$$\sum_{\mathfrak{N}(\mathfrak{a})<M/2} \dot{\chi}_{p_3}(\mathfrak{a}) = O(M^{1-1/s}) \tag{2.36}$$

*for $M \to \infty$, where $\mathfrak{a}$ varies over the integral ideals of $\mathcal{K}$.*

*Proof* By Lemma 4, we have

$$\rho(M, j) = \sum_{\mathbf{a}\in\Lambda^*_{p_3}} \ddot{\chi}_{p_3}(\mathbf{a}) \sum_{\varsigma_i\in\{-1,+1\},\ i=1,\dots,s} \varsigma_1\cdots\varsigma_s \dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j),$$

where

$$\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j) = \sum_{\substack{\boldsymbol{\gamma}\in\Gamma_j\cap\mathcal{F},\ \boldsymbol{\gamma}\equiv\mathbf{a} \bmod p_3,\\ |\mathrm{Nm}(\boldsymbol{\gamma})|<M/2,\ \mathrm{sgn}(\gamma_i)=\varsigma_i,\ i=1,\dots,s}} 1.$$

Using Lemma 3 with $M_1 = M/2$ and $w = p_3$ , we get

$$\dot{\rho}(\mathbf{a}, \boldsymbol{\varsigma}, j) = \sum_{\substack{\boldsymbol{\gamma}\in(p_3\Gamma_j+\mathbf{a})\cap\mathcal{F},\ |\mathrm{Nm}(\boldsymbol{\gamma})|<M/2\\ \mathrm{sgn}(\gamma_i)=\varsigma_i,\ i=1,\dots,s}} 1 = c_{6,j}M/p_3^s + O(M^{1-1/s}).$$

Therefore

$$\begin{aligned}\rho(M, j) &= \sum_{\mathbf{a}\in\Lambda^*_{p_3}} \ddot{\chi}_{p_3}(\mathbf{a}) \sum_{\varsigma_i\in\{-1,+1\},\ i=1,\dots,s} \varsigma_1\cdots\varsigma_s (c_{6,j}M/p_3^s + O(M^{1-1/s}))\\ &= O(M^{1-1/s}).\end{aligned}$$

Hence, the assertion (2.35) is proved. The assertion (2.36) can be proved similarly (see also [7, p. 210, Thm. 1], [22, p. 142, and p.144, Thm 11.1.5]). □

**Lemma 6** *There exists $M_0 > 0$, $i_0 \in [1, h]$, and $c_7 > 0$, such that*

$$|\rho_0(M, i_0)| \geq c_7 \quad \text{for} \quad M > M_0 \quad \text{with} \quad \rho_0(M, i) = \sum_{\boldsymbol{\gamma}\in\Gamma_i\cap\mathcal{F},\ |\mathrm{Nm}(\boldsymbol{\gamma})|<M/2} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma})}{|\mathrm{Nm}(\boldsymbol{\gamma})|}.$$

*Proof* Let $cl(\mathcal{K}) = \{C_1, \dots, C_h\}$, $\mathfrak{a}_i \in C_i$ be an integral ideal, $i = 1, \dots, s$, and let $C_1$ be the class of principal ideals. Consider the inverse ideal class $C_i^{-1}$. We set $\dot{\mathfrak{a}}_i = \{\mathfrak{a}_1, \dots, \mathfrak{a}_h\} \cap C_i^{-1}$. Then for any $\mathfrak{a} \in C_i$ the product $\mathfrak{a}\dot{\mathfrak{a}}_i$ will be a principal ideal: $\mathfrak{a}\dot{\mathfrak{a}}_i = (\alpha)$, $(\alpha \in \mathcal{K})$. By [6, p. 310], we have that the mapping $\mathfrak{a} \to (\alpha)$ establishes a one to one correspondence between integral ideal $\mathfrak{a}$ of the class $C_i$ and principal ideals divisible by $\dot{\mathfrak{a}}_i$. Let

$$\rho_1(M) = \sum_{\mathfrak{N}(\mathfrak{a})<M/2} \dot{\chi}_{p_3}(\mathfrak{a})/\mathfrak{N}(\mathfrak{a}).$$

Similarly to [6, p. 311], we get

$$\rho_1(M) = \sum_{1\le i\le h} \sum_{\mathfrak{a}\in C_i, \mathfrak{N}(\mathfrak{a})<M/2} \frac{\dot{\chi}_{p_3}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})} = \sum_{1\le i\le h} \sum_{\substack{\mathfrak{a}\in C_1, \mathfrak{N}(\mathfrak{a}/\dot{\mathfrak{a}}_i))<M/2 \\ \mathfrak{a}\equiv 0 \bmod \dot{\mathfrak{a}}_i}} \frac{\dot{\chi}_{p_3}(\mathfrak{a}/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(\mathfrak{a}/\dot{\mathfrak{a}}_i))}.$$

Let

$$\rho_2(M,i) = \sum_{\substack{\mathfrak{a}\in C_1,\ \mathfrak{N}(\mathfrak{a})<M/2 \\ \mathfrak{a}\equiv 0 \bmod \dot{\mathfrak{a}}_i}} \dot{\chi}_{p_3}(\mathfrak{a})/\mathfrak{N}(\mathfrak{a}).$$

We see

$$\rho_1(M) = \sum_{1\le i\le h} \frac{\dot{\chi}_{p_3}(1/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(1/\dot{\mathfrak{a}}_i)} \rho_2(M\mathfrak{N}(\dot{\mathfrak{a}}_i), i). \tag{2.37}$$

By Lemma 4, we obtain $\tilde{\chi}_{p_3}(\boldsymbol{\gamma})/|\mathrm{Nm}(\boldsymbol{\gamma})| = \dot{\chi}_{p_3}((\boldsymbol{\gamma}))/\mathfrak{N}((\boldsymbol{\gamma}))$. Using Theorem B, we get $\rho_0(M,i) = \rho_2(M,i)$. From (2.36), Theorem C, Theorem D, and Theorem E, we derive $\rho_1(M) \overset{M\to\infty}{\longrightarrow} L(1,\dot{\chi}_{p_3}) \neq 0$. By (2.35) and Theorem D, we obtain that there exists a complex number $\rho_i$ such that $\rho_0(M,i) \overset{M\to\infty}{\longrightarrow} \rho_i$, $i=1,\dots,h$. Hence, there exists $M_0 > 0$ such that

$$|L(1,\dot{\chi}_{p_3})|/2 \le |\rho_1(M)| \quad \text{and} \quad |\rho_i - \rho_2(M,i)| \le |L(1,\dot{\chi}_{p_3})|(8\beta)^{-1}, \tag{2.38}$$

with $\beta = \sum_{1\le i\le h} \mathfrak{N}(\dot{\mathfrak{a}}_i)$ for $M \ge M_0$. Let $\rho = \max_{1\le i\le h} |\rho_i| = |\rho_{i_0}|$.

Using (2.37), we have

$$\begin{aligned}|L(1,\dot{\chi}_{p_3})|/2 \le |\rho_1(M)| &\le \rho\beta + \Big|\sum_{1\le i\le h} \frac{\dot{\chi}_{p_3}(1/\dot{\mathfrak{a}}_i)}{\mathfrak{N}(1/\dot{\mathfrak{a}}_i)}(\rho_i - \rho_2(M\mathfrak{N}(\dot{\mathfrak{a}}_i), i))\Big| \\ &\le \rho\beta + |L(1,\dot{\chi}_{p_3})|/8 \quad \text{for} \quad M > M_0.\end{aligned}$$

By (2.38), we get for $M > M_0$

$$\rho \ge |L(1,\dot{\chi}_{p_3})|(4\beta)^{-1} \quad \text{and} \quad |\rho_0(M,i_0)| = |\rho_2(M,i_0)| \ge |L(1,\dot{\chi}_{p_3})|(8\beta)^{-1}.$$

Therefore, Lemma 6 is proved. □

**Lemma 7** *There exists $M_2 > 0$ such that*

$$|\vartheta| \ge c_7/2 \quad \text{for} \quad M > M_2, \quad \text{where} \quad \vartheta = \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F}} \frac{\ddot{\chi}_{p_3}(\boldsymbol{\gamma})\eta_M(\boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})}.$$

*Proof* Let $\dot{\eta}_M(k) = 1 - \eta(2|k|/M)$,

$$\vartheta_1 = \sum_{\substack{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F} \\ |\mathrm{Nm}(\boldsymbol{\gamma})|<M/2}} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma})}{|\mathrm{Nm}(\boldsymbol{\gamma})|} \quad \text{and} \quad \vartheta_2 = \sum_{\substack{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F} \\ M/2\le|\mathrm{Nm}(\boldsymbol{\gamma})|\le M}} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma})\dot{\eta}_M(\mathrm{Nm}(\boldsymbol{\gamma}))}{|\mathrm{Nm}(\boldsymbol{\gamma})|}.$$

From (2.16), we get $\eta_M(\boldsymbol{\gamma}) = \dot{\eta}_M(\mathrm{Nm}(\boldsymbol{\gamma}))$, $\eta_M(\boldsymbol{\gamma}) = 1$ for $|\mathrm{Nm}(\boldsymbol{\gamma})| \leq M/2$, and $\eta_M(\boldsymbol{\gamma}) = 0$ for $|\mathrm{Nm}(\boldsymbol{\gamma})| \geq M$. Using Lemma 4, we derive

$$\vartheta = \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F},\ |\mathrm{Nm}(\boldsymbol{\gamma})|\leq M} \frac{\tilde{\chi}_{p_3}(\boldsymbol{\gamma})\dot{\eta}_M(\mathrm{Nm}(\boldsymbol{\gamma}))}{|\mathrm{Nm}(\boldsymbol{\gamma})|} \quad \text{and} \quad \vartheta = \vartheta_1 + \vartheta_2. \tag{2.39}$$

Bearing in mind that $\mathrm{Nm}(\boldsymbol{\gamma}) \in \mathbb{Z}$ and $\mathrm{Nm}(\boldsymbol{\gamma}) \neq 0$, we have

$$\vartheta_2 = \sum_{M/2\leq\dot{n}\leq M} \frac{a_{\dot{n}}\dot{\eta}_M(k)}{k} \quad \text{with} \quad a_{\dot{n}} = \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F},\ |\mathrm{Nm}(\boldsymbol{\gamma})|=\dot{n}} \tilde{\chi}_{p_3}(\boldsymbol{\gamma}).$$

Applying Abel' transformation

$$\sum_{m<k\leq\dot{n}} g_k f_k = g_{\dot{n}} F_{\dot{n}} - \sum_{m<k\leq\dot{n}-1} (g_{k+1} - g_k) F_k, \quad \text{where} \quad F_k = \sum_{m<i\leq k} f_i,$$

with $f_k = a_k,\ g_k = \dot{\eta}_M(k)/k$ and $F_k = \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F},\ M/2-0.1<|\mathrm{Nm}(\boldsymbol{\gamma})|\leq k} \tilde{\chi}_{p_3}(\boldsymbol{\gamma})$, we obtain

$$\vartheta_2 = \dot{\eta}_M(M) F_M/M - \sum_{M/2-0.1<k\leq M-1} (\dot{\eta}_M(k+1)/(k+1) - \dot{\eta}_M(k)/k) F_k. \tag{2.40}$$

Bearing in mind that $0 \leq \dot{\eta}_M(x) \leq 1$ and $\eta^{'}(x) = O(1)$, for $|x| \leq 2$, we get

$$\begin{aligned} |\dot{\eta}_M(k+1)/(k+1) - \dot{\eta}_M(k)/k| &\leq |\dot{\eta}_M(k+1)/(k+1) - \dot{\eta}_M(k+1)/k| \\ &\quad + |(\dot{\eta}_M(k+1) - \dot{\eta}_M(k))/k| \\ &\leq 1/k^2 + 2(kM)^{-1} \sup_{x\in[0,2]} |\eta^{'}(x)| = O(k^{-2}). \end{aligned}$$

Taking into account that $F_k = O(M^{1-1/s})$ (see (2.35)), we have from (2.40) that $\vartheta_2 = O(M^{-1/s})$. Using Lemma 6 and (2.39), we obtain the assertion of Lemma 7. □

### 2.5 The Lower Bound Estimate for $\mathbf{E}(\mathcal{A}(\mathbf{x}, M))$

Let $n = s^{-1}\log_2 N$ with $N = N_1 \cdots N_s$, $\tau = N^{-2}$, $M = [\sqrt{n}]$, and

$$\begin{aligned} &G_0 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| > M\}, \\ &G_1 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M,\ \max_i |\gamma_i| \geq 1/\tau^2\}, \\ &G_2 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M,\ 1/\tau^2 > \max_i |\gamma_i| \geq n/\tau\}, \\ &G_3 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M,\ n/\tau > \max_i |\gamma_i| \geq n^{-s}/\tau\}, \\ &G_4 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M,\ \max_i |\gamma_i| < n^{-s}\tau^{-1},\ \ n^{-s} > N^{1/s} \min_i |\gamma_i|\}, \end{aligned}$$

$$G_5 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} | \, |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M, \ \max_i |\gamma_i| < n^{-s}\tau^{-1}, \ N^{1/s} \min_i |\gamma_i| \in [n^{-s}, n^s]\},$$
$$G_6 = \{\boldsymbol{\gamma} \in \Gamma^{\perp} \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \leq M, \quad \max_i |\gamma_i| < n^{-s}\tau^{-1}, \qquad N^{1/s} \min_i |\gamma_i| > n^s\}. \tag{2.41}$$

We see that

$$\Gamma^{\perp} \setminus \mathbf{0} = G_0 \cup \cdots \cup G_6 \quad \text{and} \quad G_i \cap G_j = \emptyset, \ \text{for } i \neq j.$$

Let $p = p_1 p_2 p_3$, $\mathbf{b} \in \Delta_p$. By (2.16) and (2.17), we have

$$\mathcal{A}(\mathbf{b}/p, M) = \sum_{0 \leq i \leq 6} \mathcal{A}_i(\mathbf{b}/p, M) \quad \text{and} \quad \mathcal{A}_0(\mathbf{b}/p, M) = 0, \tag{2.42}$$

where

$$\mathcal{A}_i(\mathbf{b}/p, M) = \sum_{\boldsymbol{\gamma} \in G_i} \prod_{i=1}^{s} \sin(\pi\theta_i N_i \gamma_i) \frac{\eta_M(\boldsymbol{\gamma})\widehat{\Omega}(\tau\boldsymbol{\gamma})e(\langle \boldsymbol{\gamma}, \mathbf{b}/p\rangle + \dot{x})}{\mathrm{Nm}(\boldsymbol{\gamma})}, \tag{2.43}$$

with $\dot{x} = \sum_{1 \leq i \leq s} \theta_i N_i \gamma_i / 2$.

We will use the following simple decomposition (see notations from Sect. 2.2 and (2.25)–(2.27)):

$$G_i = \bigcup_{1 \leq j \leq M} \ \bigcup_{\boldsymbol{\gamma}_0 \in \Gamma^{\perp} \cap \mathcal{F}, |\mathrm{Nm}(\boldsymbol{\gamma}_0)| \in (j-1, j]} \times \bigcup_{a_1, a_2 = 0,1} \left\{ \boldsymbol{\gamma} \in G_i \ \left| \boldsymbol{\gamma} = \boldsymbol{\gamma}_0 (-1)^{a_1} \boldsymbol{\varepsilon}_0^{a_2} \boldsymbol{\varepsilon}^{\mathbf{k}}, \ \mathbf{k} \in \mathbb{Z}^{s-1} \right.\right\}, \quad i \in [1, 6], \tag{2.44}$$

where $\mathbf{k} = (k_1, \ldots, k_{s-1})$, $\boldsymbol{\varepsilon}^{\mathbf{k}} = \boldsymbol{\varepsilon}_1^{k_1} \cdots \boldsymbol{\varepsilon}_{s-1}^{k_{s-1}}$, and $\boldsymbol{\varepsilon}_0 = 1$ for $\mu = 1, 2$.

**Lemma 8** *With notations as above*

$$\mathcal{A}_i(\mathbf{b}/p, M) = O(n^{s-3/2} \ln n), \quad \text{where} \quad M = [\sqrt{n}] \quad \text{and} \quad i \in [1, 5].$$

*Proof* By (2.43), we have

$$|\mathcal{A}_i(\mathbf{b}/p, M)| \leq \sum_{\boldsymbol{\gamma} \in G_i} \prod_{1 \leq j \leq s} \left| \sin(\pi\theta_j N_j \gamma_j) \widehat{\Omega}(\tau\boldsymbol{\gamma}) / \mathrm{Nm}(\boldsymbol{\gamma}) \right|. \tag{2.45}$$

*Case* $i = 1$. Applying (2.20), we obtain $\#\{\boldsymbol{\gamma} \in \Gamma^{\perp} \ : \ j \leq |\boldsymbol{\gamma}| \leq j+1\} = O(j^{s-1})$. By (2.7) we get $\widehat{\Omega}(\tau\boldsymbol{\gamma}) = O((\tau|\boldsymbol{\gamma}|)^{-2s})$ for $\boldsymbol{\gamma} \in G_1$. From (2.45) and (2.41), we have

$$\mathcal{A}_1(\mathbf{b}/p, M) = O\Big( \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp}, \max_{i \in [1,s]} |\gamma_i| \geq 1/\tau^2} \tau^{-2s} (\max_{i \in [1,s]} |\gamma_i|)^{-2s} \Big)$$

$$= O\Big( \sum_{j \geq \tau^{-2}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma^{\perp} \\ \max_i |\gamma_i| \in [j, j+1)}} \tau^{-2s} \big( \max_{i \in [1,s]} |\gamma_i| \big)^{-2s} \Big)$$

$$= O\Big( \sum_{j \geq \tau^{-2}} \frac{\tau^{-2s}}{j^{s+1}} \Big) = O(1).$$

*Case* $i = 2$. By (2.7) we obtain $\widehat{\Omega}(\tau \boldsymbol{\gamma}) = O(n^{-2s})$ for $\boldsymbol{\gamma} \in G_2$. By [6, pp. 312, 322], the points of $\Gamma_{\mathcal{O}} \cap \mathcal{F}$ can be arranged in a sequence $\dot{\boldsymbol{\gamma}}^{(k)}$ so that

$$|\mathrm{Nm}(\dot{\boldsymbol{\gamma}}^{(1)})| \leq |\mathrm{Nm}(\dot{\boldsymbol{\gamma}}^{(2)})| \leq \cdots \quad \text{and } c^{(1)} k \leq |\mathrm{Nm}(\dot{\boldsymbol{\gamma}}^{(k)})| \leq c^{(2)} k, \tag{2.46}$$

$k = 1, 2, \ldots$ for some $c^{(2)} > c^{(1)} > 0$. Let $\boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} = \max_{1 \leq i \leq s} |(\boldsymbol{\varepsilon}^{\mathbf{k}})_i|$ and $\boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} = \min_{1 \leq i \leq s} |(\boldsymbol{\varepsilon}^{\mathbf{k}})_i|$. Using Lemma 2, we get

$$\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \leq \tau^{-4}\} = O(n^{s-1}), \quad \text{where} \quad \tau = N^{-2} = e^{-2sn}. \tag{2.47}$$

Applying (2.44)–(2.47), we have

$$\mathcal{A}_2(\mathbf{b}/p, M) = O\Big( \sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1},\ \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \leq \tau^{-2}} n^{-2s} \Big) = O(M n^{-2s+s-1}) = O(1).$$

*Case* $i = 3$. Using Lemma 2, we obtain

$$\begin{aligned}
&\#\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \in [n^{-s-1}/\tau, n^{s+1}/\tau]\} \\
&\quad = c_4 (\ln^{s-1}(n^{s+1}/\tau) - \ln^{s-1}(n^{-s-1}/\tau)) + O(n^{s-2}) \\
&\quad = O\Big( |\ln^{s-1} \tau| \Big( \Big(1 + \frac{(s+1)\ln n}{|\ln \tau|}\Big)^{s-1} - \Big(1 - \frac{(s+1)\ln n}{|\ln \tau|}\Big)^{s-1} \Big) \Big) \\
&\quad = O(n^{s-2} \ln n).
\end{aligned} \tag{2.48}$$

Applying (2.44)–(2.47), we get

$$\mathcal{A}_3(\mathbf{b}/p, M) = O\Big( \sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1}, \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \in [n^{-s-1}/\tau, n^{s+1}/\tau]} 1 \Big) = O(M n^{s-2} \ln n).$$

*Case* $i = 4$. We see $\min_{1 \leq i \leq s} |\sin(\pi N_i \gamma_i)| = O(n^{-s})$ for $\boldsymbol{\gamma} \in G_4$. Applying (2.44)–(2.47), we have

$$|\mathcal{A}_4(\mathbf{b}/p, M)| = O\Big( \sum_{1 \leq j \leq M} \sum_{\mathbf{k} \in \mathbb{Z}^{s-1},\ \boldsymbol{\varepsilon}^{\mathbf{k}}_{\max} \leq \tau^{-4}} n^{-s} \Big) = O(M n^{-2}).$$

*Case* $i = 5$. Similarly to (2.48), we obtain from Lemma 2 that

$$\{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} \in [n^{-s-1} N^{-1/s}, n^{s+1} N^{-1/s}]\} = O(n^{s-2} \ln n).$$

Therefore

$$\mathcal{A}_3(\mathbf{b}/p, M) = O\Big(\sum_{1\le j\le M} \sum_{\mathbf{k}\in\mathbb{Z}^{s-1}, \boldsymbol{\varepsilon}^{\mathbf{k}}_{\min}\in[n^{-s-1}N^{-1/s}, n^{s+1}N^{-1/s}]} 1\Big) = O(Mn^{s-2}\ln n).$$

Hence, Lemma 8 is proved. □

Let $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_s)$, $\mathbf{1} = (1, 1, \dots, 1)$, and

$$\breve{\mathcal{A}}_6(\mathbf{b}/p, M, \boldsymbol{\varsigma}) = \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \sum_{\boldsymbol{\gamma}\in G_6} \frac{\widehat{\Omega}(\tau\boldsymbol{\gamma})\eta_M(\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle)}{\mathrm{Nm}(\boldsymbol{\gamma})} \tag{2.49}$$

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) = (\dot{\theta}_1(\boldsymbol{\varsigma}), \dots, \dot{\theta}_s(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1 + \varsigma_i)\theta_i N_i/4,\ i = 1, \dots, s.$

By (2.43), we see

$$\mathcal{A}_6(\mathbf{b}/p, M) = \sum_{\boldsymbol{\varsigma}\in\{1,-1\}^s} \breve{\mathcal{A}}_6(\mathbf{b}/p, M, \boldsymbol{\varsigma}). \tag{2.50}$$

**Lemma 9** *With notations as above*

$$\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) = \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) + O(1),$$

*where*

$$\dot{\mathcal{A}}_i(\mathbf{b}/p, M, -\mathbf{1}) = (-2\sqrt{-1})^{-s} \sum_{\boldsymbol{\gamma}\in G_i} \frac{\eta_M(\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{b}/p\rangle)}{\mathrm{Nm}(\boldsymbol{\gamma})}, \quad i = 1, 2, \dots \tag{2.51}$$

*Proof* By (2.49) and (2.50), we have

$$\begin{aligned}&|\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \breve{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1})|\\ &\quad = O\Big(\sum_{\substack{\boldsymbol{\varsigma}\in\{1,-1\}^s\\ \boldsymbol{\varsigma}\neq-\mathbf{1}}} \sum_{\boldsymbol{\gamma}\in G_6} \sum_{1\le i\le s} \frac{|\mathbf{E}(e(\varsigma_i\theta_i N_i\gamma_i/4))|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big).\end{aligned}$$

Bearing in mind that

$$\mathbf{E}(e(\theta_i z)) = \frac{e(z) - 1}{2\pi\sqrt{-1}z} \tag{2.52}$$

and that $|N_i\gamma_i| \ge n^s/c_3$ for $\boldsymbol{\gamma} \in G_6$ (see (2.3), and (2.41)), we get

$$|\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \breve{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1})| = O\Big(\sum_{\boldsymbol{\gamma}\in G_6} n^{-s}|\mathrm{Nm}(\boldsymbol{\gamma})|^{-1}\Big).$$

By (2.49) and (2.51), we obtain

$$|\breve{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) - \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1})| = O\Big(\sum_{\boldsymbol{\gamma}\in G_6} \frac{|\widehat{\Omega}(\tau\boldsymbol{\gamma}) - 1|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big).$$

By (2.8) and (2.41), we see $\widehat{\Omega}(\tau\boldsymbol{\gamma}) = 1 + O(n^{-s})$ for $\boldsymbol{\gamma} \in G_6$. From (2.41), (2.44) and (2.47), we have $\# G_6 = O(Mn^{s-1})$. Hence

$$\mathbf{E}(\mathcal{A}_6(\mathbf{b}/p, M)) - \dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) = O\Big(\sum_{\boldsymbol{\gamma}\in G_6} n^{-s}|\mathrm{Nm}(\boldsymbol{\gamma})|^{-1}\Big) = O(1).$$

Therefore, Lemma 9 is proved. □

Let

$$G_7 = \bigcup_{\boldsymbol{\gamma}_0\in\Gamma^\perp\cap\mathcal{F}, |\mathrm{Nm}(\boldsymbol{\gamma}_0)|\le M} \ \bigcup_{a_1,a_2=0,1} \ \bigcup_{\mathbf{k}\in\mathcal{Y}_N} T_{\boldsymbol{\gamma}_0,a_1,a_2,\mathbf{k}}, \tag{2.53}$$

with

$$\mathcal{Y}_N = \{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} \ge N^{-1/s}\}, \tag{2.54}$$

and

$$T_{\boldsymbol{\gamma}_0,a_1,a_2,\mathbf{k}} = \{\boldsymbol{\gamma} \in \Gamma^\perp \mid \boldsymbol{\gamma} = \boldsymbol{\gamma}_0(-1)^{a_1}\boldsymbol{\varepsilon}_0^{a_2}\boldsymbol{\varepsilon}^{\mathbf{k}}\}.$$

We note that $\# T_{\boldsymbol{\gamma}_0,a_1,a_2,\mathbf{k}} \le 1$ (may be $\boldsymbol{\gamma}_0(-1)^{a_1}\boldsymbol{\varepsilon}_0^{a_2}\boldsymbol{\varepsilon}^{\mathbf{k}} \notin \Gamma^\perp$).

**Lemma 10** *With notations as above*

$$\mathbf{E}(\mathcal{A}(\mathbf{b}/p, M)) = \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1}) + O(n^{s-3/2}\ln n), \quad \text{where} \quad M = [\sqrt{n}]. \tag{2.55}$$

*Proof* By (2.51), we have

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1})| = O(\#(G_7 \setminus G_6) + \#(G_6 \setminus G_7)).$$

Consider $\boldsymbol{\gamma} \in G_6$ (see (2.41)). Bearing in mind that $\min_{1\le i\le s}|\gamma_i| \ge n^s N^{-1/s}$, we get

$$|\gamma_i| = |\mathrm{Nm}(\boldsymbol{\gamma})| \prod_{[1,s]\ni j\ne i} |\gamma_j|^{-1} \le n^{-s(s-1)}N^{1+(s-1)/s} < n^{-s}/\tau, \quad \text{with} \quad \tau = N^{-2}.$$

Thus

$$G_6 = \{\boldsymbol{\gamma} \in \Gamma^\perp \mid |\mathrm{Nm}(\boldsymbol{\gamma})| \le M, \ N^{1/s}\min_i|\gamma_i| > n^s\}.$$

From (2.53), we obtain $G_7 \supseteq G_6$. Bearing in mind that $|\mathrm{Nm}(\boldsymbol{\gamma})| \ge 1$ for $\boldsymbol{\gamma} \in \Gamma^\perp \setminus \mathbf{0}$, we have that $G_6 \supseteq G_5$, where

$$G_5 = \bigcup_{\boldsymbol{\gamma}_0\in\Gamma^\perp\cap\mathcal{F}, \ |\mathrm{Nm}(\boldsymbol{\gamma}_0)|\le M} \ \bigcup_{a_1,a_2=0,1} \ \bigcup_{\mathbf{k}\in\dot{\mathcal{Y}}_N} T_{\boldsymbol{\gamma}_0,a_1,a_2,\mathbf{k}},$$

with

$$\dot{\mathcal{Y}}_N = \{\mathbf{k} \in \mathbb{Z}^{s-1} \mid N^{1/s}\boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} \ge n^{2s}\}. \tag{2.56}$$

By Lemma 3, we get $\#\{\boldsymbol{\gamma}_0 \in \Gamma^\perp \cap \mathcal{F}, |\mathrm{Nm}(\boldsymbol{\gamma}_0)| \le M\} = O(M)$. Therefore

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1})| = O(M\#(\mathcal{Y}_N \setminus \dot{\mathcal{Y}}_N)).$$

Using Lemma 2, we obtain

$$\begin{aligned}\#(\mathcal{Y}_N \setminus \dot{\mathcal{Y}}_N) &= \{\mathbf{k} \in \mathbb{Z}^{s-1} \mid \boldsymbol{\varepsilon}^{\mathbf{k}}_{\min} \in [N^{-1/s}, n^{2s} N^{-1/s}]\} \\ &= c_5\big(\ln^{s-1}(N^{1/s}) - \ln^{s-1}(n^{-2s} N^{1/s})\big) + O(n^{s-2}) \\ &= O\Big(\ln^{s-1} N\Big(\Big(1 - \Big(1 - \frac{2s^2 \log_2 n}{\ln N}\Big)^{s-1}\Big)\Big)\Big) \\ &= O(n^{s-2} \ln n), \quad n = s^{-1} \log_2 N.\end{aligned}$$

Hence

$$|\dot{\mathcal{A}}_6(\mathbf{b}/p, M, -\mathbf{1}) - \dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1})| = O(M n^{s-2} \ln n).$$

Applying Lemmas 8 and 9, we get the assertion of Lemma 10. □

Let

$$\delta_w(\boldsymbol{\gamma}) = \begin{cases} 1 & \text{if } \boldsymbol{\gamma} \in w\mathcal{O}, \\ 0 & \text{otherwise.}\end{cases}$$

**Lemma 11** *Let* $\boldsymbol{\gamma} \in \mathcal{O}$, *then*

$$\frac{1}{w^s} \sum_{\mathbf{y} \in \Lambda_w} e(\langle \boldsymbol{\gamma}, \mathbf{y}\rangle / w) = \delta_w(\gamma).$$

*Proof* It easy to verify that

$$\frac{1}{v} \sum_{0 \le k < w} e(kb/w) = \dot{\delta}_w(b), \quad \text{where} \quad \dot{\delta}_w(b) = \begin{cases} 1 & \text{if} \quad b \equiv 0 \bmod w, \\ 0 & \text{otherwise.}\end{cases} \tag{2.57}$$

Let $\boldsymbol{\gamma} = d_1 \mathbf{f}_1 + \cdots + d_s \mathbf{f}_s$, and $\mathbf{y} = a_1 \mathbf{f}_1^{\perp} + \cdots + a_s \mathbf{f}_s^{\perp}$ (see (2.34)). We have $\langle \boldsymbol{\gamma}, \mathbf{y}\rangle = a_1 d_1 + \cdots + a_s d_s$. Bearing in mind that $\boldsymbol{\gamma} \in w\mathcal{O}$ if and only if $d_i \equiv 0 \bmod w$ $(i = 1, \ldots, s)$, we obtain from (2.57) the assertion of Lemma 11. □

**Lemma 12** *There exist* $\mathbf{b} \in \Lambda_p$, $c_8 > 0$ *and* $N_0 > 0$ *such that*

$$|\mathbf{E}(\mathcal{A}(\mathbf{b}/p, M))| > c_8 n^{s-1} \quad \text{for} \quad N > N_0.$$

*Proof* We consider the case $\mu = 1$. The proof for the cases $\mu = 2, 3$ is similar.

By (2.51) and Lemma 11, we have

$$\begin{aligned}\varrho &:= \frac{2^{2s}}{p^s} \sum_{\mathbf{b} \in \Lambda_p} |\dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1})|^2 = \sum_{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in G_7} \frac{\eta_M(\boldsymbol{\gamma}_1)\eta_M(\boldsymbol{\gamma}_2)\delta_p(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)}{\mathrm{Nm}(\boldsymbol{\gamma}_1)\mathrm{Nm}(\boldsymbol{\gamma}_2)} \\ &= \sum_{\mathbf{b} \in \Lambda_p} \Big| \sum_{\boldsymbol{\gamma} \in G_7,\ \boldsymbol{\gamma} \equiv \mathbf{b} \bmod p} \frac{\eta_M(\boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})}\Big|^2. \end{aligned} \tag{2.58}$$

Bearing in mind that $\eta_M(\boldsymbol{\gamma}) = 0$ for $|\mathrm{Nm}(\boldsymbol{\gamma})| \geq M$ (see (2.16)), we get from (2.53) that

$$\varrho = \sum_{\mathbf{b} \in \Lambda_p} \Big| \sum_{\varsigma=-1,1} \sum_{\mathbf{k} \in \mathcal{Y}_N} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma^{\perp} \cap \mathcal{F},\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \in \Gamma^{\perp} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b} \bmod p}} \frac{\eta_M(\varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma})}{\mathrm{Nm}(\varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma})} \Big|^2.$$

We consider only $\mathbf{b} = p_1 \mathbf{b}_0 \in \Lambda_p$, where $\mathbf{b}_0 \in \Lambda_{p_2 p_3}$ and $p = p_1 p_2 p_3$. By (2.1), we obtain $\Gamma_{p_1 \mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$ and $\Gamma_{p_1 \mathcal{O}} = \{\boldsymbol{\gamma} \in \Gamma^{\perp} | \gamma \equiv \mathbf{0} \bmod p_1\}$. Hence, we can take $\Gamma_{p_1 \mathcal{O}}$ instead of $\Gamma^{\perp}$. We see $\varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \in \Gamma_{\mathcal{O}}$ for all $\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}}$, $\mathbf{k} \in \mathbb{Z}^{s-1}$ and $\varsigma \in \{-1, 1\}$. Thus

$$\varrho \geq \sum_{\mathbf{b} \in \Lambda_{p_2 p_3}} \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b} \bmod p_2 p_3}} \frac{\eta_M(p_1 \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma})}{\mathrm{Nm}(p_1 \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma})} \Big|^2.$$

By Lemma 4, $(p_2, p_3) = 1$. Hence, there exists $w_2, w_3 \in \mathbb{Z}$ such that $p_2 w_2 \equiv 1 \bmod p_3$ and $p_3 w_3 \equiv 1 \bmod p_2$. It is easy to verify that if $\dot{\mathbf{b}}_2, \ddot{\mathbf{b}}_2 \in \Lambda_{p_2}$ (see (2.34)), $\dot{\mathbf{b}}_3, \ddot{\mathbf{b}}_3 \in \Lambda_{p_3}$, and $(\dot{\mathbf{b}}_2, \dot{\mathbf{b}}_3) \neq (\ddot{\mathbf{b}}_2, \ddot{\mathbf{b}}_3)$, then

$$\dot{\mathbf{b}}_2 p_3 w_3 + \dot{\mathbf{b}}_3 p_2 w_2 \not\equiv \ddot{\mathbf{b}}_2 p_3 w_3 + \ddot{\mathbf{b}}_3 p_2 w_2 \bmod p_2 p_3.$$

Therefore

$$\Lambda_{p_2 p_3} = \{\mathbf{b} \in \Lambda_{p_2 p_3} \mid \ \exists\, \mathbf{b}_2 \in \Lambda_{p_2},\ \mathbf{b}_3 \in \Lambda_{p_3} \ \text{ with } \ \mathbf{b} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3\}.$$

Thus

$$\begin{aligned} \varrho &\geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3}} \frac{\eta_M(p_1 \boldsymbol{\gamma})}{\mathrm{Nm}(p_1 \varsigma \boldsymbol{\gamma})} \Big|^2 \\ &\geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \Big| \ddot{\chi}_{p_3}(\mathbf{b}_3) \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3}} \frac{\eta_M(p_1 \boldsymbol{\gamma})}{\mathrm{Nm}(p_1 \varsigma \boldsymbol{\gamma})} \Big|^2 \\ &= \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3}} \frac{\ddot{\chi}_{p_3}(\varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_M(p_1 \boldsymbol{\gamma})}{\mathrm{Nm}(p_1 \varsigma \boldsymbol{\gamma})} \Big|^2. \end{aligned}$$

Using the Cauchy–Schwartz inequality, we have

$$p_3^s \varrho \geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \Big| \sum_{\mathbf{b}_3 \in \Lambda_{p_3}} \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k} \in \mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma} \in \Gamma_{\mathcal{O}} \cap \mathcal{F} \\ \varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3}} \frac{\ddot{\chi}_{p_3}(\varsigma \boldsymbol{\varepsilon}^{\mathbf{k}} \boldsymbol{\gamma}) \eta_M(p_1 \boldsymbol{\gamma})}{p_1^s \mathrm{Nm}(\varsigma \boldsymbol{\gamma})} \Big|^2.$$

We see that $\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 \equiv \mathbf{b}_2 \bmod p_2$ if and only if there exists $\mathbf{b}_3 \in \Lambda_{p_3}$ such that $\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma} \equiv \mathbf{b}_2 p_3 w_3 + \mathbf{b}_3 p_2 w_2 \bmod p_2 p_3$. Hence

$$p_1^{2s} p_3^s \varrho \geq \sum_{\mathbf{b}_2 \in \Lambda_{p_2}} \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k}\in\mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma}\in\Gamma_{\mathcal{O}}\cap\mathcal{F} \\ \varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma}\equiv\mathbf{b}_2 \bmod p_2}} \frac{\ddot{\chi}_{p_3}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})\eta_M(p_1\boldsymbol{\gamma})}{\mathrm{Nm}(\varsigma\boldsymbol{\gamma})}\Big|^2. \tag{2.59}$$

By (2.23), we get $\Gamma_{i_0} = \varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\Gamma_{i_0}$ for all $\mathbf{k} \in \mathbb{Z}^{s-1}$, $\varsigma \in \{-1, 1\}$, and there exists $\Phi_{i_0} \subseteq \Lambda_{p_2}$ with

$$\Gamma_{i_0} = \bigcup_{\mathbf{b}\in\Phi_{i_0}} (p_2\Gamma_{\mathcal{O}} + \mathbf{b}), \quad \text{where} \quad (p_2\Gamma_{\mathcal{O}} + \mathbf{b}_1) \cap (p_2\Gamma_{\mathcal{O}} + \mathbf{b}_2) = \emptyset, \text{ for } \mathbf{b}_1 \neq \mathbf{b}_2.$$

We consider in (2.59) only $\mathbf{b}_2 \in \Phi_{i_0}$. Applying the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} p_1^{2s} p_2^s p_3^s \varrho &\geq \Big| \sum_{\mathbf{b}_2\in\Phi_{i_0}} \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k}\in\mathcal{Y}_N}} \sum_{\substack{\boldsymbol{\gamma}\in\Gamma_{\mathcal{O}}\cap\mathcal{F} \\ \varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma}\equiv\mathbf{b}_2 \bmod p_2}} \frac{\ddot{\chi}_{p_3}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})\eta_M(p_1\boldsymbol{\gamma})}{\mathrm{Nm}(\varsigma\boldsymbol{\gamma})}\Big|^2 \\ &= \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k}\in\mathcal{Y}_N}} \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F}} \frac{\ddot{\chi}_{p_3}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})\eta_M(p_1\boldsymbol{\gamma})}{\mathrm{Nm}(\varsigma\boldsymbol{\gamma})}\Big|^2. \end{aligned}$$

Using Lemma 4, we get

$$\begin{aligned} \ddot{\chi}_{p_3}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})\frac{|\mathrm{Nm}(\boldsymbol{\gamma})|}{\mathrm{Nm}(\varsigma\boldsymbol{\gamma})} &= \ddot{\chi}_{p_3}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})\frac{\mathrm{Nm}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})}{|\mathrm{Nm}(\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})|} \\ &= \dot{\chi}_{p_3}((\varsigma\boldsymbol{\varepsilon}^{\mathbf{k}}\boldsymbol{\gamma})) = \dot{\chi}_{p_3}((\boldsymbol{\gamma})) = \ddot{\chi}_{p_3}(\boldsymbol{\gamma})\frac{|\mathrm{Nm}(\boldsymbol{\gamma})|}{\mathrm{Nm}(\boldsymbol{\gamma})}. \end{aligned}$$

Hence

$$p_1^{2s} p_2^s p_3^s \varrho \geq \Big| \sum_{\substack{\varsigma=-1,1 \\ \mathbf{k}\in\mathcal{Y}_N}} \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F}} \frac{\ddot{\chi}_{p_3}(\boldsymbol{\gamma})\eta_M(p_1\boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})}\Big|^2.$$

Bearing in mind that $\eta_M(p_1\boldsymbol{\gamma}) = \eta_{M/p_1^s}(\boldsymbol{\gamma})$ (see (2.16)), we obtain

$$p_1^{2s} p_2^s p_3^s \varrho \geq 4\#\mathcal{Y}_N^2 \Big| \sum_{\boldsymbol{\gamma}\in\Gamma_{i_0}\cap\mathcal{F}} \frac{\ddot{\chi}_{p_3}(\boldsymbol{\gamma})\eta_{M/p_1^s}(\boldsymbol{\gamma})}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big|^2.$$

Applying Lemma 2, we have from (2.54) that $\#\mathcal{Y}_N \geq 0.5c_5(n/s)^{s-1}$ for $N \geq \dot{N}_0$ with some $\dot{N}_0 > 1$, and $n = s^{-1}\log_2 N$. By Lemma 7 and (2.58), we obtain

$$\sup_{\mathbf{b}\in\Lambda_p} |\dot{\mathcal{A}}_7(\mathbf{b}/p, M, -\mathbf{1})| \geq 2^{-s}\varrho^{1/2}$$
$$\geq c_7(2p_1^2p_2p_3)^{-s}\#\mathcal{Y}_N \geq 0.5c_5c_7(2p_1^2p_2p_3s)^{-s}n^{s-1},$$

with $M=[\sqrt{n}]=[\sqrt{\log_2 N}]\geq M_2+\log_2\dot{N}_0$. Using Lemma 10, we get the assertion of Lemma 12. □

### 2.6 Auxiliary Lemmas

We need the following notations and results from [27]:

**Lemma C** [27, Lemma 3.2] *Let* $\dot{\Gamma}\subset\mathbb{R}^s$ *be an admissible lattice. Then*

$$\sup_{\mathbf{x}\in\mathbb{R}^s}\sum_{\boldsymbol{\gamma}\in\dot{\Gamma}}\prod_{1\leq i\leq s}(1+|\gamma_i-x_i|)^{-2s}\leq H_{\dot{\Gamma}}$$

*where the constant* $H_{\dot{\Gamma}}$ *depends upon the lattice* $\dot{\Gamma}$ *only by means of the invariants* $\det\dot{\Gamma}$ *and* $\mathrm{Nm}\ \dot{\Gamma}$.

Let $f(t),\ t\in\mathbb{R}$, be a function of the class $C^\infty$; moreover let $f(t)$ and all derivatives $f^{(k)}$ belong to $L^1(\mathbb{R})$. We consider the following integrals for $\dot{\tau}>0$:

$$I(\dot{\tau},\xi)=\int_{-\infty}^{\infty}\frac{\eta(t)\widehat{\omega}(\dot{\tau}t)e(-\xi t)}{t}\mathrm{d}t,\ \ J_f(\dot{\tau},\xi)=\int_{-\infty}^{\infty}f(t)\widehat{\omega}(\dot{\tau}t)e(-\xi t)\mathrm{d}t. \quad (2.60)$$

**Lemma D** [27, Lemma 4.2] *For all* $\alpha>0$ *and* $\beta>0$*, there exists a constant* $\breve{c}_{(\alpha,\beta)}>0$ *such that*

$$\max(|I(\dot{\tau},\xi)|,|J_f(\dot{\tau},\xi)|)<\breve{c}_{(\alpha,\beta)}(1+\dot{\tau})^{-\alpha}(1+|\xi|)^{-\beta}.$$

Let $m(t),\ t\in\mathbb{R}$, be an even non negative function of the class $C^\infty$; moreover $m(t)=0$ for $|t|\leq 1$, $m(t)=0$ for $|t|\geq 4$, and

$$\sum_{q=-\infty}^{+\infty}m(2^{-q}t)=1. \quad (2.61)$$

For examples of such functions see e.g. [27, Ref. 5.16]. Let $\dot{\mathbf{p}}=(\dot{p}_1,\dots,\dot{p}_s)$, $\dot{p}_i>0,\ i=1,\dots,s$, $a>0$, $x_0=\gamma_0=1$,

$$\widehat{W}_{a,i}(\dot{\mathbf{p}},\mathbf{x})=\frac{\widehat{\omega}(\dot{p}_1x_1)\eta(ax_1)}{x_1}\prod_{j=2}^{s}\frac{\widehat{\omega}(\dot{p}_jx_j)m(x_j)}{x_j}\frac{1}{x_i}\quad\text{for}\quad \mathrm{Nm}\ \mathbf{x}\neq 0, \quad (2.62)$$

and $\widehat{W}_{a,i}(\dot{\mathbf{p}},\mathbf{x})=0$ for $\mathrm{Nm}(\mathbf{x})=0,\ \ i=0,1,\dots,s$. Let

$$\breve{W}_{a,i}(\dot{\Gamma},\dot{\mathbf{p}},\mathbf{x})=\sum_{\boldsymbol{\gamma}\in\dot{\Gamma}^{\perp}\setminus\mathbf{0}}\widehat{W}_{a,i}(\dot{\mathbf{p}},\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma},\mathbf{x}\rangle). \quad (2.63)$$

By (2.6) and (2.7), we see that the series (2.63) converge absolutely, and $\widehat{W}_{a,i}(\dot{\mathbf{p}}, \mathbf{x})$ belongs to the class $C^{\infty}$. Therefore, we can use Poisson's summation formula (2.4):

$$\breve{W}_{a,i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x}) = \det \dot{\Gamma} \sum_{\boldsymbol{\gamma} \in \dot{\Gamma}} W_{a,i}(\dot{\mathbf{p}}, \gamma - \mathbf{x}), \tag{2.64}$$

where $\widehat{W}_{a,i}(\dot{\mathbf{p}}, \mathbf{x})$ and $W_{a,i}(\dot{\mathbf{p}}, \mathbf{x})$ are related by the Fourier transform. Using (2.62), we derive

$$W_{a,i}(\dot{\mathbf{p}}, \mathbf{x}) = \prod_{j \in \{1,\dots,s\} \setminus \{i\}} w_1^{(1)}(\dot{p}_j, x_j) \prod_{j \in \{1,\dots,s\} \cap \{i\}} w_j^{(2)}(\dot{p}_j, x_j),$$

where co-factors can be described as follows (see also [27, Ref. 6.14–6.17]):

If $j = 1$ and $i \neq 1$, then

$$w_1^{(1)}(\tau, \xi) = \int_{-\infty}^{\infty} \frac{1}{t} \eta(at) \widehat{\omega}(\tau t) e(-\xi t) \mathrm{d}t = I(a^{-1}\tau, a^{-1}\xi). \tag{2.65}$$

Note that here we used formula (2.60). If $j = 1$ and $i = 1$, then

$$w_1^{(2)}(\tau, \xi) = \int_{-\infty}^{\infty} \frac{1}{t^2} \eta(at) \widehat{\omega}(\tau t) e(-\xi t) \mathrm{d}t = a J_{f_1}(a^{-1}\tau, a^{-1}\xi).$$

Note that here we used formula (2.60) with $f_1(t) = \eta(t)/t^2$. If $j \geq 2$, then

$$w_j^{(l)}(\tau, \xi) = \int_{-\infty}^{\infty} \frac{1}{t^l} m(t) \widehat{\omega}(\tau t) e(-\xi t) \mathrm{d}t = J_{f_2}(\tau, \xi). \tag{2.66}$$

Here we used formula (2.60) with $f_2(t) = m(t)/t^l \quad j = 2, \dots, s, \;\; l = 1, 2$.

Applying Lemma D, we obtain for $0 < a \leq 1$ that

$$|w_1^{(l)}(\tau, \xi)| < \breve{c}_{(2s,2s)}(1 + a^{-1}|\xi|)^{-2s} \text{ and } |w_j^{(l)}(\tau, \xi)| < \breve{c}_{(2s,2s)}(1 + |\xi|)^{-2s}, \tag{2.67}$$

with $j = 2, \dots, s$, and $l = 1, 2$. Now, using (2.64) and Lemma C, we get (see also [27, Ref. 6.18, 6.19, 3.7, 3.10, 3,13]):

**Lemma E** *Let* $\dot{\Gamma} \subset \mathbb{R}^s$ *be an admissible lattice, and* $0 < a \leq 1$ . *Then*

$$\sup_{\mathbf{x} \in \mathbb{R}^s} |\breve{W}_{a,i}(\dot{\Gamma}, \dot{\mathbf{p}}, \mathbf{x})| \leq \breve{c}_{(2s,2s)} \det \dot{\Gamma} \, H_{\dot{\Gamma}}.$$

### 2.7 Dyadic Decomposition of $\mathcal{B}(\mathbf{b}/p, M)$

Using the definition of the function $m(x)$ (see (2.61)), we set

$$\mathbb{M}(\mathbf{x}) = \prod_{j=2}^{s} m(x_j). \tag{2.68}$$

Let $2^{\mathbf{q}} = (2^{q_1}, \ldots, 2^{q_s})$, and

$$\begin{aligned}\psi_{\mathbf{q}}(\boldsymbol{\gamma}) &= \mathbb{M}(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma})\widehat{\Omega}(\tau\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}),\\ \mathcal{B}_{\mathbf{q}}(M) &= \mathcal{B}_{\mathbf{q}}(\mathbf{b}/p, M)\\ &= \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \prod_{i=1}^{s} \sin(\pi\theta_i N_i\gamma_i)(1-\eta_M(\boldsymbol{\gamma}))\psi_{\mathbf{q}}(\boldsymbol{\gamma}))e(\langle\boldsymbol{\gamma}, \mathbf{b}/p\rangle + \dot{x}),\end{aligned} \tag{2.69}$$

with $\dot{x} = \sum_{1\le i\le s} \theta_i N_i\gamma_i/2$.

By (2.17) and (2.61), we have

$$\mathcal{B}(\mathbf{b}/p, M) = \sum_{Q\in L} \mathcal{B}_{\mathbf{q}}(M), \tag{2.70}$$

with $\mathcal{L} = \{\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{Z}^s \mid q_1 + \cdots + q_s = 0\}$.

Let

$$\widetilde{\mathcal{B}}_{\mathbf{q}}(M) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \prod_{i=1}^{s} \sin(\pi\theta_i N_i\gamma_i)\eta(\gamma_1 2^{-q_1}/M)\psi_{\mathbf{q}}(\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{b}/p\rangle + \dot{x}), \tag{2.71}$$

and

$$\begin{aligned}\mathcal{C}_{\mathbf{q}}(M) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \prod_{i=1}^{s} &\sin(\pi\theta_i N_i\gamma_i)(1-\eta_M(\boldsymbol{\gamma}))\\ &\times(1-\eta(\gamma_1 2^{-q_1}/M))\psi_{\mathbf{q}}(\boldsymbol{\gamma})e(\langle\boldsymbol{\gamma}, \mathbf{b}/p\rangle + \dot{x}).\end{aligned}$$

According to (2.16), we get $\eta_M(\boldsymbol{\gamma}) = 1 - \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})|/M)$, $\eta(x) = 0$ for $|x| \le 1$, $\eta(x) = \eta(-x)$ and $\eta(x) = 1$ for $|x| \ge 2$. Let $\eta(\gamma_1 2^{-q_1}/M)m(\gamma_2 2^{-q_2})\cdots m(\gamma_s 2^{-q_s}) \neq 0$, then $|\mathrm{Nm}(\boldsymbol{\gamma})| \ge M$ (see (2.61)), and

$$(1-\eta_M(\boldsymbol{\gamma}))\eta(\gamma_1 2^{-q_1}/M) = \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})|/M)\eta(\gamma_1 2^{-q_1}/M) = \eta(\gamma_1 2^{-q_1}/M).$$

Hence

$$\mathcal{B}_{\mathbf{q}}(M) = \widetilde{\mathcal{B}}_{\mathbf{q}}(M) + \mathcal{C}_{\mathbf{q}}(M). \tag{2.72}$$

Let $n = s^{-1}\log_2 N,\ \tau = N^{-2}$ and

$$\begin{aligned}
\mathcal{G}_1 &= \{\mathbf{q} \in \mathcal{L} \mid \max_{i=1,\ldots,s} q_i \geq -\log_2 \tau + \log_2 n\},\\
\mathcal{G}_2 &= \{\mathbf{q} \in \mathcal{L} \setminus \mathcal{G}_1 \mid \min_{i=2,\ldots,s} q_i \leq -n - 1/2\log_2 n\},\\
\mathcal{G}_3 &= \{\mathbf{q} \in \mathcal{L} \mid -n - 1/2\log_2 n < \min_{i=2,\ldots,s} q_i,\ \max_{i=1,\ldots,s} q_i < -\log_2 \tau + \log_2 n\},\\
\mathcal{G}_4 &= \{\mathbf{q} \in \mathcal{G}_3 \mid q_1 \geq -n + s\log_2 n\},\\
\mathcal{G}_5 &= \{\mathbf{q} \in \mathcal{G}_3 \mid -n - s\log_2 n \leq q_1 < -n + s\log_2 n\},\\
\mathcal{G}_6 &= \{\mathbf{q} \in \mathcal{G}_3 \mid q_1 < -n - s\log_2 n\}.
\end{aligned} \tag{2.73}$$

We see

$$\mathcal{L} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3, \quad \mathcal{G}_3 = \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \quad \text{and} \quad \mathcal{G}_i \cap \mathcal{G}_j = \emptyset,\ \text{for } i \neq j \tag{2.74}$$

and $i, j \in [1, 3]$ or $i, j \in [4, 6]$. Let

$$\mathcal{B}_i(M) = \sum_{\mathbf{q}\in\mathcal{G}_i} \mathcal{B}_{\mathbf{q}}(M). \tag{2.75}$$

By (2.70), we obtain

$$\mathcal{B}(\mathbf{b}/p, M) = \mathcal{B}_1(M) + \mathcal{B}_2(M) + \mathcal{B}_3(M). \tag{2.76}$$

Let

$$\widetilde{\mathcal{B}}_3(M) = \sum_{\mathbf{q}\in\mathcal{G}_3} \widetilde{\mathcal{B}}_{\mathbf{q}}(M), \quad \widetilde{\mathcal{C}}_3(M) = \sum_{\mathbf{q}\in\mathcal{G}_3} \mathcal{C}_{\mathbf{q}}(M). \tag{2.77}$$

Applying (2.72) and (2.75), we get

$$\mathcal{B}_3(M) = \widetilde{\mathcal{B}}_3(M) + \widetilde{\mathcal{C}}_3(M). \tag{2.78}$$

By (2.7), we obtain the absolute convergence of the following series

$$\sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} |\widehat{\Omega}(\tau\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma})|.$$

Hence, the series (2.71), (2.75) and (2.77) converges absolutely.

Let

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \eta(\gamma_1 2^{-q_1}/M)\psi_{\mathbf{q}}(\boldsymbol{\gamma}) e(\langle\boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle) \tag{2.79}$$

with $\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) = (\dot{\theta}_1(\boldsymbol{\varsigma}), \dots, \dot{\theta}_s(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1+\varsigma_i)\theta_i N_i/4, \quad i = 1, \dots, s$. By (2.71), we have

$$\widetilde{\mathcal{B}}_{\mathbf{q}}(M) = \sum_{\boldsymbol{\varsigma} \in \{1,-1\}^s} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}). \tag{2.80}$$

Let $\boldsymbol{\varsigma}_2 = -\mathbf{1} = -(1, 1, \dots, 1)$, $\boldsymbol{\varsigma}_3 = \dot{\mathbf{1}} = (1, -1, \dots, -1)$, and let

$$\widetilde{\mathcal{B}}_{3,1}(M) = \sum_{\mathbf{q} \in \mathcal{G}_3} \sum_{\substack{\boldsymbol{\varsigma} \in \{1,-1\}^s \\ \boldsymbol{\varsigma} \neq \boldsymbol{\varsigma}_2, \boldsymbol{\varsigma}_3}} \varsigma_1 \cdots \varsigma_s (2\sqrt{-1})^{-s} \breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}), \tag{2.81}$$

$$\widetilde{\mathcal{B}}_{i,j}(M) = (-1)^{s+j} (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_i} \breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}_j), \quad i = 3, 4, 5, 6, \quad j = 2, 3. \tag{2.82}$$

Using (2.77) and (2.80), we derive

$$\widetilde{\mathcal{B}}_3(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \widetilde{\mathcal{B}}_{3,2}(M) + \widetilde{\mathcal{B}}_{3,3}(M).$$

Bearing in mind (2.74), we obtain

$$\widetilde{\mathcal{B}}_3(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \sum_{i=4,5,6} \sum_{j=2,3} \widetilde{\mathcal{B}}_{i,j}(M). \tag{2.83}$$

Let

$$\widetilde{\mathcal{B}}_{6,j,k}(M) = (-1)^{s+j} (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_6} \breve{\mathcal{B}}^{(k)}_{\mathbf{q}}(M, \boldsymbol{\varsigma}_j), \quad j = 2, 3, \quad k = 1, 2, \tag{2.84}$$

where

$$\breve{\mathcal{B}}^{(1)}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1 2^{-q_1}/M) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) \eta(2^{n+\log_2 n}\gamma_1) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle)$$

and

$$\breve{\mathcal{B}}^{(2)}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1 2^{-q_1}/M) \psi_{\mathbf{q}}(\boldsymbol{\gamma}) (1 - \eta(2^{n+\log_2 n}\gamma_1)) e(\langle \boldsymbol{\gamma}, \mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}) \rangle).$$

From (2.79), (2.82) and (2.84) , we get

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = \breve{\mathcal{B}}^{(1)}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) + \breve{\mathcal{B}}^{(2)}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) \quad \text{and} \quad \widetilde{\mathcal{B}}_{6,j}(M) = \widetilde{\mathcal{B}}_{6,j,1}(M) + \widetilde{\mathcal{B}}_{6,j,2}(M).$$

So, we proved the following lemma:

**Lemma 13** *With notations as above, we get from* (2.76), (2.78) *and* (2.83)

$$\mathcal{B}(\mathbf{b}/p, M) = \bar{\mathcal{B}}(M) + \widetilde{\mathcal{C}}_3(M), \tag{2.85}$$

*where*

$$\bar{\mathcal{B}}(M) = \mathcal{B}_1(M) + \mathcal{B}_2(M) + \widetilde{\mathcal{B}}_3(M) \tag{2.86}$$

*and*

$$\widetilde{\mathcal{B}}_3(M) = \widetilde{\mathcal{B}}_{3,1}(M) + \sum_{j=2,3} (\widetilde{\mathcal{B}}_{4,j}(M) + \widetilde{\mathcal{B}}_{5,j}(M) + \widetilde{\mathcal{B}}_{6,j,1}(M) + \widetilde{\mathcal{B}}_{6,j,2}(M)). \tag{2.87}$$

### 2.8 The Upper Bound Estimate for $\mathbf{E}(\bar{\mathcal{B}}(M))$

**Lemma 14** *With notations as above*

$$\mathcal{B}_1(M) = O(1).$$

*Proof* Let $\mathbf{q} \in \mathcal{G}_1$, and let $j = q_{i_0} = \max_{1 \le i \le s} q_i$, $i_0 \in [1, \dots, s]$. By (2.73), we have $j \ge -\log_2 \tau + \log_2 n$. Using (2.69), we obtain

$$|\mathcal{B}_{\mathbf{q}}(M)| \le \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} \Big| \prod_{i=1}^{s} \sin(\pi \theta_i N_i \gamma_i) \frac{\mathbb{M}(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma}) \widehat{\Omega}(\tau \boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})} \Big|. \tag{2.88}$$

From (2.68) and (2.61), we get

$$|\mathcal{B}_{\mathbf{q}}(M)| \le \rho_1 + \rho_2 \quad \text{with} \quad \rho_i = \sum_{\boldsymbol{\gamma} \in \mathcal{X}_i} \frac{|\mathbb{M}(\boldsymbol{\gamma}) \widehat{\Omega}(\tau 2^{\mathbf{q}} \cdot \boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}, \tag{2.89}$$

where

$$\mathcal{X}_1 = \{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0} \mid |\gamma_1| \le 2^{4sj},\ |\gamma_i| \in [1, 4],\ i = 2, \dots, s\},$$

and

$$\mathcal{X}_2 = \{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0} \mid |\gamma_1| > 2^{4sj},\ |\gamma_i| \in [1, 4],\ i = 2, \dots, s\}.$$

We consider the admissible lattice $2^{-\mathbf{q}} \cdot \Gamma^{\perp}$, where $\mathrm{Nm}(\Gamma^{\perp}) \ge 1$. Using Theorem A, we obtain that there exists a constant $c_9 = c_9(\Gamma^{\perp})$ such that

$$\#\{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \mid |\gamma_i| \le 4, i = 2, \dots, s,\ 2^{4(s-1)}|\gamma_1| \in [k, 2k]\} \le c_9 k, \tag{2.90}$$

where $k = 1, 2, \dots$.

Let $i_0 = 1$. We see that $\tau 2^{q_1} = \tau 2^j \geq 2^{\log_2 n} = n$. By (2.7), (2.88) and (2.90), we get

$$\mathcal{B}_{\mathbf{q}}(M) = O\Big(\sum_{k\geq 0} \sum_{\substack{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0},\ 1\leq|\gamma_i|\leq 4,\ i\geq 2\\ 2^{4(s-1)}|\gamma_1|\in[2^k,2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_1}\gamma_1)|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big) = O\Big(\sum_{k\geq 0}(1+\tau 2^{q_1+k})^{-2s}\Big).$$

Hence

$$\mathcal{B}_{\mathbf{q}}(M) = O((\tau 2^j)^{-2s}). \tag{2.91}$$

Let $i_0 \geq 2$. Bearing in mind (2.7) and (2.90), we have

$$\begin{aligned}\rho_1 &= O\Big(\sum_{0\leq k\leq 4s(j+1)} \sum_{\substack{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0},\ 1\leq|\gamma_i|\leq 4,\ i\geq 2\\ 2^{4(s-1)}|\gamma_1|\in[2^k,2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_{i_0}}\gamma_{q_{i_0}})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big)\\ &= O\Big(\sum_{0\leq k\leq 4s(j+1)} (1+\tau 2^{q_{i_0}})^{-2s}\Big).\end{aligned}$$

Hence

$$\rho_1 = O(j(1+\tau 2^j)^{-2s}). \tag{2.92}$$

Taking into account that $q_1 = -(q_2 + \cdots + q_s) \geq -(s-1)j$ and $\tau 2^j \geq n$, we obtain

$$\begin{aligned}\rho_2 &= O\Big(\sum_{k\geq 4sj} \sum_{\substack{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0},\ 1\leq|\gamma_i|\leq 4,\ i\geq 2\\ 2^{4(s-1)}|\gamma_1|\in[2^k,2^{k+1}]}} \frac{|\widehat{\omega}(\tau 2^{q_1}\gamma_{q_1})\widehat{\omega}(\tau 2^{q_{i_0}}\gamma_{q_{i_0}})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big)\\ &= O\Big(\sum_{k\geq 4sj}(1+\tau 2^{q_1+k})^{-2s}(1+\tau 2^{q_{i_0}})^{-2s}\Big) = O\big((1+\tau 2^{q_{i_0}})^{-2s}\big).\end{aligned}$$

Therefore

$$\rho_2 = O((1+\tau 2^j)^{-2s}). \tag{2.93}$$

Thus

$$\mathcal{B}_{\mathbf{q}}(M) = O(j(\tau 2^j)^{-2s}). \tag{2.94}$$

From (2.20), we have

$$\sum_{\mathbf{q}\in\mathbb{Z}^s,\ q_1+\cdots+q_s=0,\ \max_i q_i = j} 1 = O(j^{s-2}). \tag{2.95}$$

By (2.73), (2.75), (2.94) and (2.91), we get

$$\begin{aligned}\mathcal{B}_1(M) = \sum_{\mathbf{q}\in\mathcal{G}_1}\mathcal{B}_{\mathbf{q}}(M) &= O\Big(\sum_{j\geq -\log_2\tau+\log_2 n} \sum_{\mathbf{q}\in\mathcal{L},\max_i q_i = j} j(\tau 2^j)^{-2s}\Big)\\ &= O\Big(\sum_{j\geq -\log_2\tau+\log_2 n} j^s(\tau 2^j)^{-2s}\Big) = O(n^s(n)^{-2s}) = O(1).\end{aligned}$$

Hence, Lemma 14 is proved. □

**Lemma 15** *With notations as above*

$$|\mathcal{B}_2(M)| + |\widetilde{\mathcal{B}}_{6,2,2}(M) + \widetilde{\mathcal{B}}_{6,3,2}(M)| = O(n^{s-3/2}).$$

*Proof* We consider $\mathcal{B}_2(M)$ (see (2.69), (2.73) and (2.75)). Let $\mathbf{q} \in \mathcal{G}_2$, and let $j = -q_{i_0} = \min_{2\le i\le s} q_i, i_0 \in [2,\dots,s]$. We see $j \ge n+1/2\log_2 n$ and $|\sin(\pi N_{i_0}\gamma_{i_0})| \le \pi N_{i_0} 2^{-j+2}$ for $m(2^{-q_{i_0}}\gamma_{i_0}) \ne 0$. By (2.88) and (2.89), we obtain

$$\mathcal{B}_{\mathbf{q}}(M) = O(\rho_1+\rho_2) \quad \text{with} \quad \rho_i = \sum_{\mathbf{q}\in\mathcal{X}_i} \frac{|N^{1/s}2^{-j}\mathbb{M}(\boldsymbol{\gamma})\widehat{\Omega}(\tau 2^{\mathbf{q}}\cdot\boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}.$$

Similarly to (2.92), (2.93), we get

$$\rho_1 = O\Big(\sum_{0\le k\le 4s(j+1)} \sum_{\substack{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0},\ 1\le|\gamma_i|\le 4,\ i\ge 2\\ 2^{4(s-1)}|\gamma_1|\in[2^k,2^{k+1}]}} \frac{N^{1/s}2^{-j}}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big)$$
$$= O\Big(\sum_{0\le k\le 4s(j+1)} N^{1/s}2^{-j}\Big) = O(jN^{1/s}2^{-j}).$$

We see

$$\rho_2 = O\Big(\sum_{k\ge 4sj} \sum_{\substack{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0},\ 1\le|\gamma_i|\le 4,\ i\ge 2\\ 2^{4(s-1)}|\gamma_1|\in[2^k,2^{k+1}]}} \frac{N^{1/s}2^{-j}|\widehat{\omega}(\tau 2^{q_1}\gamma_{q_1})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big).$$

We have $\max_{1\le i\le s} q_i \le -\log_2\tau + \log_2 n$ for $\mathbf{q}\in\mathcal{G}_2$. Hence $q_1 = -(q_2+\dots+q_s) \ge (s-1)(\log_2\tau - \log_2 n)$ and $\tau 2^{q_1} \ge \tau^s n^{-s+1} = 2^{-2ns}n^{-s+1} > 2^{-2sj}$. Thus

$$\rho_2 = O\Big(N^{1/s}2^{-j}\sum_{k\ge 4sj}(1+\tau 2^{q_1+k})^{-2s}\Big)$$
$$= O\Big(N^{1/s}2^{-j}\sum_{k\ge 4sj}2^{-2s(k-2sj)}\Big) = O(N^{1/s}2^{-j}).$$

Bearing in mind (2.95), we derive

$$\mathcal{B}_2(M) = \sum_{\mathbf{q}\in\mathcal{G}_2}\mathcal{B}_{\mathbf{q}}(M) = O\Big(\sum_{j\ge n+1/2\log_2 n}\ \sum_{\mathbf{q}\in\mathcal{L},\min_{2\le i\le s}q_i=-j} jN^{1/s}2^{-j}\Big)$$
$$= O\Big(\sum_{j\ge n+1/2\log_2 n} j^{s-1}N^{1/s}2^{-j}\Big) = O(n^{s-3/2}).$$

Consider $\rho := \breve{\mathcal{B}}^{(2)}_{\mathbf{q}}(M, \dot{\mathbf{1}}) + \breve{\mathcal{B}}^{(2)}_{\mathbf{q}}(M, -\mathbf{1})$. By (2.69) and (2.84), we have

$$\rho = O\Big( \sum_{\boldsymbol{\gamma} \in \Gamma^{\perp} \setminus \mathbf{0}} | \sin(\pi\theta_1 N_1 \gamma_1)\eta(\gamma_1 2^{-q_1}/M) M(2^{-\mathbf{q}} \cdot \boldsymbol{\gamma})\widehat{\Omega}(\tau\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}) \times (1 - \eta(2^{n+\log_2 n}\gamma_1))e(\langle \boldsymbol{\gamma}, \mathbf{b}/p\rangle)|\Big)$$
$$= O\Big( \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\dot{\Gamma}^{\perp} \setminus \mathbf{0}} | \sin(\pi\theta_1 N_1 2^{q_1}\gamma_1)(1 - \eta(2^{q_1+n+\log_2 n}\gamma_1))\mathbb{M}(\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma})|\Big).$$

Applying (2.16), (2.68) and (2.90), we obtain

$$\rho = O\Big( \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\Gamma^{\perp} \setminus \mathbf{0},\ |\gamma_1| \le 2^{-q_1-n-\log_2 n+4}} |N_1 2^{q_1}\gamma_1 \mathbb{M}(\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma})|\Big) = O(1/n).$$
$$= O\Big( \sum_{\substack{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\cdot\Gamma^{\perp} \setminus \mathbf{0},\ 1 \le |\gamma_i| \le 4,\ i \ge 2 \\ |\gamma_1| \le 2^{-q_1-n-\log_2 n+4}}} N_1 2^{q_1}\Big) = O(N_1 2^{q_1} 2^{-q_1-n-\log_2 n+4}) = O(1/n).$$

We get from (2.73) that

$$\#\mathcal{G}_3 = O(n^{s-1}). \tag{2.96}$$

By (2.73) and (2.84), we get $\widetilde{\mathcal{B}}_{6,2,2}(M) + \widetilde{\mathcal{B}}_{6,3,2}(M) = (n^{s-2})$.

Hence, Lemma 15 is proved. □

**Lemma 16** *With notations as above*

$$|\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M))| + |\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M))| + |\widetilde{\mathcal{B}}_{5,2}(M)| + |\widetilde{\mathcal{B}}_{5,3}(M)| = O(n^{s-3/2}).$$

*Proof* By (2.69) and (2.79), we have

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\cdot\Gamma^{\perp} \setminus \mathbf{0}} \eta(\gamma_1/M)\psi_{\mathbf{q}}(2^{\mathbf{q}} \cdot \boldsymbol{\gamma})e(\langle \boldsymbol{\gamma}, \mathbf{x}\rangle)$$
$$= \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}\cdot\Gamma^{\perp} \setminus \mathbf{0}} \frac{\widehat{\omega}(2^{q_1}\tau\gamma_1)\eta(\gamma_1/M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j}\tau\gamma_j)m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, \mathbf{x}\rangle), \tag{2.97}$$

with $\mathbf{x} = 2^{\mathbf{q}} \cdot (\mathbf{b}/p + \dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma}))$ and $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1 + \varsigma_i)\theta_i N_i/4,\ i = 1, \dots, s$.

Applying (2.64) and Lemma E with $\dot{\Gamma} = 2^{-\mathbf{q}}\Gamma$, $i = 0$, and $\dot{\mathbf{p}} = \tau 2^{\mathbf{q}}$, we get

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}) = O(1).$$

Using (2.73), we obtain $\#\mathcal{G}_5 = O(n^{s-2}\log_2 n)$.

By (2.82) , we get

$$\widetilde{\mathcal{B}}_{5,i}(M) = O\Big( \sum_{\mathbf{q} \in \mathcal{G}_5} |\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})|\Big) = O(n^{s-2}\log_2 n), \quad i = 2, 3. \tag{2.98}$$

Consider $\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M))$ and $\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M))$. Let

$$\mathbf{E}_i(f) = \int_0^1 f(\boldsymbol{\theta}) \mathrm{d}\theta_i.$$

Let $\boldsymbol{\varsigma} \neq -\mathbf{1}$. Then there exists $i_0 = i_0(\boldsymbol{\varsigma}) \in [1, s]$ with $\varsigma_{i_0} = 1$.
By (2.52) and (2.97), we have

$$\mathbf{E}_{i_0}(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \frac{e(N_{i_0} 2^{q_{i_0}} \gamma_{i_0}/2) - 1}{\pi \sqrt{-1} N_{i_0} 2^{q_{i_0}} \gamma_{i_0}} \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1/M)}{\gamma_1} \times \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle),$$

with some $\mathbf{x} \in \mathbb{R}^s$. Hence

$$\mathbf{E}_{i_0}(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})) = O\Big(N_{i_0}^{-1} 2^{-q_{i_0}} \sup_{\mathbf{x} \in \mathbb{R}^s} \Big| \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \widehat{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\gamma}, i_0) e(\langle \boldsymbol{\gamma}, \mathbf{x} \rangle) \Big| \Big),$$

where

$$\widehat{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\gamma}, i_0) = \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1/M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} \frac{1}{\gamma_{i_0}}.$$

Applying (2.64) and Lemma E with $\dot{\Gamma} = 2^{-\mathbf{q}} \Gamma$, and $\dot{\mathbf{p}} = \tau 2^{\mathbf{q}}$, we obtain

$$\mathbf{E}(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma})) = \mathbf{E}(\mathbf{E}_{i_0}(\breve{\mathcal{B}}_{\mathbf{q}}(M, \boldsymbol{\varsigma}))) = O(N_{i_0}^{-1} 2^{-q_{i_0}}). \tag{2.99}$$

By (2.81), we have $i_0(\boldsymbol{\varsigma}) \geq 2$ and

$$\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M)) = O\Big( \sum_{\substack{\boldsymbol{\varsigma} \in \{1,-1\}^s \\ \boldsymbol{\varsigma} \neq -\mathbf{1}, \dot{\mathbf{1}}}} \sum_{\mathbf{q} \in \mathcal{G}_3} N_{i_0(\boldsymbol{\varsigma})}^{-1} 2^{-q_{i_0(\varsigma)}} \Big).$$

Using (2.73), we get $\#\{\mathbf{q} \in \mathcal{G}_3 \mid q_{i_0} = j\} = O(n^{s-2})$ and $j \geq -n - 1/2 \log_2 n$. Hence

$$\mathbf{E}(\widetilde{\mathcal{B}}_{3,1}(M)) = O\Big(n^{s-2} \sum_{j \geq -n - 1/2 \log_2 n} N^{-1/s} 2^{-j}\Big) = O(n^{s-3/2}). \tag{2.100}$$

From (2.73), we get $q_1 \geq -n + s \log_2 n$ for $\mathbf{q} \in \mathcal{G}_4$. Applying (2.82), (2.96) and (2.99) with $i_0(\boldsymbol{\varsigma}) = 1$, we obtain

$$\mathbf{E}(\widetilde{\mathcal{B}}_{4,3}(M)) = O\Big( \sum_{\mathbf{q} \in \mathcal{G}_4} N_1^{-1} 2^{-q_1} \Big) = O\Big(n^{s-1} \sum_{q_1 \geq -n + s \log_2 n} N^{-1/s} 2^{-q_1}\Big) = O(1).$$

By (2.98) and (2.100), Lemma 16 is proved. □

**Lemma 17** *With notations as above*

$$\widetilde{\mathcal{B}}_{4,2}(M) = O(n^{s-3/2}).$$

*Proof* By (2.97), we have

$$\breve{\mathcal{B}}_{\mathbf{q}}(M, -\mathbf{1}) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}} \cdot \Gamma^{\perp} \setminus \mathbf{0}} \frac{\widehat{\omega}(2^{q_1} \tau \gamma_1) \eta(\gamma_1/M)}{\gamma_1} \prod_{j=2}^{s} \frac{\widehat{\omega}(2^{q_j} \tau \gamma_j) m(\gamma_j)}{\gamma_j} e(\langle \boldsymbol{\gamma}, 2^{\mathbf{q}} \cdot \mathbf{b}/p \rangle).$$

From (2.65), we derive that $I(d, v) = 0$ for $v = 0$. Hence $w_1^{(1)}(\tau, 0) = 0$. Now applying (2.64)–(2.67) with $\dot{\Gamma}^{\perp} = 2^{-\mathbf{q}} \cdot \Gamma^{\perp}$, $i = 0$ and $a = M^{-1}$, we get

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\mathbf{1})| \leq \breve{c}_{(2s,2s)} \det \Gamma \sum_{\boldsymbol{\gamma} \in 2^{\mathbf{q}} \cdot \Gamma, \ \gamma_1 \neq (\mathbf{b}/p)_1} (1 + M |\gamma_1 - 2^{q_1} (\mathbf{b}/p)_1|)^{-2s}$$
$$\times \prod_{i=2}^{s} (1 + |\gamma_i - 2^{q_i} (\mathbf{b}/p)_i|)^{-2s}.$$

Bearing in mind (2.1), we get $p_1 \Gamma_{\mathcal{O}} \subseteq \Gamma^{\perp} \subseteq \Gamma_{\mathcal{O}}$. Taking into account that $p = p_1 p_2 p_3$ and $\mathbf{b} \in \Gamma_{\mathcal{O}}$, we obtain

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\mathbf{1})| \leq \breve{c}_{(2s,2s)} \det \Gamma p^{2s^2} \sum_{\boldsymbol{\gamma} \in p 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}} (1 + M |\gamma_1|)^{-2s} \prod_{i=2}^{s} (1 + |\gamma_i|)^{-2s}. \quad (2.101)$$

We have

$$|\breve{\mathcal{B}}_{\mathbf{q}}(M, -\mathbf{1})| \leq \breve{c}_{(2s,2s)} \det \Gamma p^{2s^2} (a_1 + a_2), \quad (2.102)$$

where

$$a_1 = \sum_{\boldsymbol{\gamma} \in p 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max |\gamma_i| \leq M^{1/s}} (1 + M |\gamma_1|)^{-2s} \prod_{i=2}^{s} (1 + |\gamma_i|)^{-2s},$$

and

$$a_2 = \sum_{\boldsymbol{\gamma} \in p 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max |\gamma_i| > M^{1/s}} (1 + M |\gamma_1|)^{-2s} \prod_{i=2}^{s} (1 + |\gamma_i|)^{-2s}.$$

We see that $|\gamma_1| \geq M^{-(s-1)/s}$ for $\max_{1 \leq i \leq s} |\gamma_i| \leq M^{1/s}$. Applying Theorem A, we have

$$a_1 \leq M^{-2} \sum_{\boldsymbol{\gamma} \in p 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0}, \max |\gamma_i| \leq M^{1/s}} 1 = O(M^{-1}),$$

and

$$a_2 \leq \sum_{j \geq M^{1/s}} \sum_{\substack{\boldsymbol{\gamma} \in p 2^{\mathbf{q}} \cdot \Gamma \setminus \mathbf{0} \\ \max |\gamma_i| \in [j, j+1)}} j^{-2s} = O\Big( \sum_{j \geq M^{1/s}} j^{-s} \Big) = O(M^{-(s-1)/s}). \quad (2.103)$$

Taking into account that $\#\mathcal{G}_3 = O(n^{s-1})$ (see (2.96)), we get from (2.102) and (2.82) that

$$\widetilde{\mathcal{B}}_{4,2}(M) = O\Big(\sum_{\mathbf{q}\in\mathcal{G}_4} \breve{\mathcal{B}}_{\mathbf{q}}(M,-\mathbf{1})\Big) = O\Big(\sum_{\mathbf{q}\in\mathcal{G}_3} M^{-1/2}\Big) = O(M^{-1/2}n^{s-1}).$$

Hence, Lemma 17 is proved. □

**Lemma 18** *With notations as above*

$$\widetilde{\mathcal{B}}_{6,2,1}(M) + \widetilde{\mathcal{B}}_{6,3,1}(M) = O(n^{s-3/2}), \quad M = [\sqrt{n}].$$

*Proof* Let $M_1 = 2^{-q_1-n-\log_2 n}$. By (2.73), we get $M_1 \geq n \geq 2M$ for $\mathbf{q} \in \mathcal{G}_6$ and $n \geq 4$. From (2.16), we have $\eta(\gamma_1/M)\eta(\gamma_1/M_1) = \eta(\gamma_1/M_1)$. Using (2.69), (2.79) and (2.84), we derive similarly to (2.97) that

$$\breve{\mathcal{B}}^{(1)}_{\mathbf{q}}(M,\boldsymbol{\varsigma}_j) = \sum_{\boldsymbol{\gamma}\in 2^{\mathbf{q}}\cdot\Gamma\setminus\mathbf{0}} \frac{\widehat{\omega}(2^{q_1}\tau\gamma_1)\eta(\gamma_1/M_1)}{\gamma_1} \times \prod_{i=2}^{s} \frac{\widehat{\omega}(2^{q_j}\tau\gamma_j)m(\gamma_j)}{\gamma_j} e(\langle\boldsymbol{\gamma}, 2^{\mathbf{q}}\cdot(\mathbf{b}/p+(j-2)\theta_1 N_1(1,0,\dots,0))\rangle)$$

with $j = 2,3$, $\boldsymbol{\varsigma}_2 = -\mathbf{1}$ and $\boldsymbol{\varsigma}_3 = \dot{\mathbf{1}}$.

By (2.66), we obtain that, $J_{f_2}(\tau,v) = 0$ with $f_2(t) = m(t)/t$ for $v = 0$ . Hence $w_2^{(1)}(\tau,0) = 0$. Now applying (2.64)–(2.67) with $\dot{\Gamma}^{\perp} = 2^{-\mathbf{q}}\cdot\Gamma^{\perp}$, $i = 0$ and $a = M_1^{-1} = 2^{q_1+n+\log_2 n}$, we get analogously to (2.101)

$$|\breve{\mathcal{B}}^{(1)}_{\mathbf{q}}(M,\boldsymbol{\varsigma}_j)| \leq \breve{c}_{(2s,2s)} \det\Gamma p^{2s^2} \sum_{\boldsymbol{\gamma}\in p2^{\mathbf{q}}\cdot\Gamma\setminus\mathbf{0}} (1+M_1|\gamma_1 - x(j)|)^{-2s} \prod_{i=2}^{s}(1+|\gamma_i|)^{-2s},$$

with $x(j) = (j-2)p\theta_1 2^{q_1} N_1$. We have

$$|\breve{\mathcal{B}}^{(1)}_{\mathbf{q}}(M,\boldsymbol{\varsigma}_j)| \leq \breve{c}_{(2s,2s)} \det\Gamma p^{2s^2}(a_3+a_4), \tag{2.104}$$

where

$$a_3 = \sum_{\boldsymbol{\gamma}\in p2^{\mathbf{q}}\cdot\Gamma\setminus\mathbf{0},\ \max|\gamma_i|\leq M^{1/s}} (1+M_1|\gamma_1 - x(j)|)^{-2s} \prod_{i=2}^{s}(1+|\gamma_i|)^{-2s},$$

and

$$a_4 = \sum_{\boldsymbol{\gamma}\in p2^{\mathbf{q}}\cdot\Gamma,\max|\gamma_i|> M^{1/s}} (1+M_1|\gamma_1 - x(j)|)^{-2s} \prod_{i=2}^{s}(1+|\gamma_i|)^{-2s}.$$

We see that $|\gamma_1| \geq M^{-(s-1)/s}$ for $\max_{1\leq i\leq s}|\gamma_i| \leq M^{1/s}$. Bearing in mind that $|x(j)| \leq c_3 pn^{-s}$ for $\mathbf{q} \in \mathcal{G}_6$, we obtain $|\gamma_1| \geq 2|x(j)|$ for $M = [\sqrt{n}]$ and $N > 8psc_3$. Applying Theorem A, we get

$$a_3 \leq 2^{2s} M_1^{-2s} M^{2(s-1)} \sum_{\boldsymbol{\gamma}\in p2^{\mathbf{q}}\cdot\Gamma,\ \max|\gamma_i|\leq M^{1/s}} 1 = O(M^{-1}).$$

Similarly to (2.103), we have

$$a_4 \leq \sum_{j\geq M^{1/s}} \sum_{\substack{\boldsymbol{\gamma}\in p2^{\mathbf{q}}\cdot\Gamma\setminus\mathbf{0}\\ \max|\gamma_i|\in[j,j+1)}} j^{-2s} = O\Big(\sum_{j\geq M^{1/s}} j^{-s}\Big) = O(M^{-(s-1)/s}).$$

By (2.73) and (2.96), we obtain $\#\mathcal{G}_6 \leq \#\mathcal{G}_3 = O(n^{s-1})$. We get from (2.84) and (2.104) that

$$\widetilde{\mathcal{B}}_{6,2,1}(M) + \widetilde{\mathcal{B}}_{6,3,1}(M) = O\Big(\sum_{\mathbf{q}\in\mathcal{G}_6,\ j=2,3} \breve{\mathcal{B}}_{\mathbf{q}}^{(1)}(M, \varsigma_j)\Big) = O(M^{-1/2}n^{s-1}).$$

Hence, Lemma 18 is proved. □

Using (2.87), (2.86) and Lemmas 14–18, we obtain

**Corollary 1** *With notations as above*

$$\mathbf{E}(\bar{\mathcal{B}}(M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

### 2.9 The Upper Bound Estimate for $\mathbf{E}(\widetilde{\mathcal{C}}_3(M))$ and Koksma–Hlawka Inequality

Let

$$\begin{aligned} \mathcal{G}_7 &= \{\mathbf{q} \in \mathcal{G}_3 \mid -\log_2\tau - s\log_2 n \leq \max_{i=1,\dots,s} q_i < -\log_2\tau + \log_2 n\}. \\ \mathcal{G}_8 &= \{\mathbf{q} \in \mathcal{G}_3 \setminus \mathcal{G}_7 \mid q_1 < -n - 1/2\log_2 n\}, \\ \mathcal{G}_9 &= \{\mathbf{q} \in \mathcal{G}_3 \setminus \mathcal{G}_7 \mid q_1 \geq -n - 1/2\log_2 n\}, \end{aligned} \tag{2.105}$$

and let

$$\widetilde{\mathcal{C}}_i(M) = \sum_{\mathbf{q}\in\mathcal{G}_i} \mathcal{C}_{\mathbf{q}}(M), \quad i = 7, 8, 9.$$

It is easy to see that

$$\mathcal{G}_3 = \mathcal{G}_7 \cup \mathcal{G}_8 \cup \mathcal{G}_9, \quad \text{and} \quad \mathcal{G}_i \cap \mathcal{G}_j = \emptyset, \quad \text{for } i \neq j.$$

Hence

$$\widetilde{\mathcal{C}}_3(M) = \widetilde{\mathcal{C}}_7(M) + \widetilde{\mathcal{C}}_8(M) + \widetilde{\mathcal{C}}_9(M). \tag{2.106}$$

From (2.71), we have similarly to (2.79) that

$$\mathcal{C}_{\mathbf{q}}(M) = \sum_{\boldsymbol{\varsigma}\in\{1,-1\}^s} \varsigma_1\cdots\varsigma_s(2\sqrt{-1})^{-s}\breve{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma}), \tag{2.107}$$

where

$$\breve{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma}) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \psi_{\mathbf{q}}(\boldsymbol{\gamma})(1-\eta_M(\boldsymbol{\gamma}))(1-\eta(\gamma_1 2^{-q_1}/M))e(\langle\boldsymbol{\gamma},\mathbf{b}/p+\dot{\boldsymbol{\theta}}(\boldsymbol{\varsigma})\rangle),$$

with $\dot{\theta}_i(\boldsymbol{\varsigma}) = (1+\varsigma_i)\theta_i N_i/4,\ i=1,\dots,s.$

By (2.107) and (2.105), we get

$$\widetilde{\mathcal{C}}_9(M) = \widetilde{\mathcal{C}}_{10}(M) + \widetilde{\mathcal{C}}_{11}(M), \tag{2.108}$$

where

$$\widetilde{\mathcal{C}}_{10}(M) = \sum_{\mathbf{q}\in\mathcal{G}_9}\sum_{\substack{\boldsymbol{\varsigma}\in\{1,-1\}^s\\ \boldsymbol{\varsigma}\neq-\mathbf{1}}} \varsigma_1\cdots\varsigma_s(2\sqrt{-1})^{-s}\breve{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma}), \tag{2.109}$$

and

$$\widetilde{\mathcal{C}}_{11}(M) = (-1)^s(2\sqrt{-1})^{-s}\sum_{\mathbf{q}\in\mathcal{G}_9}\breve{\mathcal{C}}_{\mathbf{q}}(M,-\mathbf{1}). \tag{2.110}$$

**Lemma 19** *With notations as above*

$$\mathbf{E}(\widetilde{\mathcal{C}}_i(M)) = O(n^{s-3/2}),\quad i=7,8,10,\quad M=[\sqrt{n}].$$

*Proof* Let $\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0}$. By (2.16), (2.61) and (2.68), we have $(1-\eta_M(\boldsymbol{\gamma}))(1-\eta(\gamma_1/M))\mathbb{M}(\boldsymbol{\gamma})\neq 0$ only if $2^{-2s+3}M\le|\gamma_1|\le 2M,\ |\gamma_i|\in[1,4],\ i=2,\dots,s.$ From (2.71), we derive

$$\mathcal{C}_{\mathbf{q}}(M) = O\Big(\sum_{\boldsymbol{\gamma}\in\mathcal{X}}\Big|\prod_{i=1}^{s}\sin(\pi\theta_i N_i 2^{q_i}\gamma_i)\frac{\mathbb{M}(\boldsymbol{\gamma})\widehat{\Omega}(\tau 2^{\mathbf{q}}\cdot\boldsymbol{\gamma})}{\mathrm{Nm}(\boldsymbol{\gamma})}\Big|\Big) \tag{2.111}$$

where

$$\mathcal{X} = \{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0}\mid 2^{-2s+3}M\le|\gamma_1|\le 2M,\ |\gamma_i|\in[1,4],\ i=2,\dots,s\}.$$

Bearing in mind (2.90), we get $\mathcal{C}_{\mathbf{q}}(M)=O(1)$.

Using (2.20), (2.73) and (2.105), we obtain $\#\mathcal{G}_7 = O(n^{s-2}\log_2 n)$. Applying(2.105), we get

$$\widetilde{\mathcal{C}}_7(M) = \sum_{\mathbf{q}\in\mathcal{G}_7}\mathcal{C}_{\mathbf{q}}(M) = O(n^{s-2}\log_2 n). \tag{2.112}$$

Consider $\widetilde{\mathcal{C}}_8(M)$. Let $\boldsymbol{\gamma} \in \mathcal{X}$. Then $|\sin(\pi\theta_1 N_1 2^{q_1}\gamma_1)| \leq \pi M N_1 2^{1+q_1}$.

By (2.111), we have

$$\mathcal{C}_{\mathbf{q}}(M) = O\Big(\sum_{\boldsymbol{\gamma}\in\mathcal{X}} \frac{|MN^{1/s}2^{q_1}\widehat{\Omega}(\tau 2^{\mathbf{q}}\cdot\boldsymbol{\gamma})|}{|\mathrm{Nm}(\boldsymbol{\gamma})|}\Big) = O(MN^{1/s}2^{q_1}).$$

Using (2.20) and (2.105), we derive $\#\{\mathbf{q} \in \mathcal{G}_8 | q_1 = d\} = O(n^{s-2})$. Hence

$$\begin{aligned}\widetilde{\mathcal{C}}_8(M) &= \sum_{\mathbf{q}\in\mathcal{G}_8} \mathcal{C}_{\mathbf{q}}(M) = O\Big(\sum_{j\geq n+0.5\log_2 n}\ \sum_{\mathbf{q}\in\mathcal{G}_8,\ q_1=-j} MN^{1/s}2^{-j}\Big)\\ &= O\Big(n^{s-2}M \sum_{j\geq n+0.5\log_2 n} 2^{n-j}\Big) = O(n^{s-2}).\end{aligned} \tag{2.113}$$

Consider $\widetilde{\mathcal{C}}_{10}(M)$. From (2.109), we get that there exists $i_0 = i_0(\boldsymbol{\varsigma}) \in [1, s]$ with $\varsigma_{i_0} = 1$. By (2.52), (2.69) and (2.107), we have

$$\mathbf{E}_{i_0}(\check{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma})) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} \dot{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\gamma}) \frac{e(N_i\gamma_{i_0}/2)-1}{\pi\sqrt{-1}N_{i_0}\gamma_{i_0}} e(\langle\boldsymbol{\gamma},\mathbf{x}\rangle)$$

with some $\mathbf{x} \in \mathbb{R}^s$, where

$$\dot{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\gamma}) = (1-\eta_M(\boldsymbol{\gamma}))(1-\eta(\gamma_1 2^{-q_1}/M))\widehat{\Omega}(\tau\cdot\boldsymbol{\gamma})\mathbb{M}(2^{-\mathbf{q}}\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

Hence

$$\mathbf{E}_{i_0}(\check{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma})) = O\Big(N_{i_0}^{-1}2^{-q_{i_0}} \sum_{\boldsymbol{\gamma}\in 2^{-\mathbf{q}}\cdot\Gamma^{\perp}\setminus\mathbf{0}} |\ddot{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\gamma},i_0)|\Big),$$

with

$$\ddot{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\gamma},i_0) = \frac{(1-\eta_M(\boldsymbol{\gamma}))(1-\eta(\gamma_1/M))}{\gamma_1} \prod_{j=2}^{s} \frac{m(\gamma_j)}{\gamma_j}\frac{1}{\gamma_{i_0}}.$$

Applying (2.111), we obtain $\max_{\gamma\in\mathcal{X},i\in[1,s]} |1/\gamma_i| = O(1)$.

By (2.16) and (2.90), we have

$$\begin{aligned}\mathbf{E}(\check{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma})) &= \mathbf{E}(\mathbf{E}_{i_0}(\check{\mathcal{C}}_{\mathbf{q}}(M,\boldsymbol{\varsigma})))\\ &= O\Big(N_{i_0}^{-1}2^{-q_{i_0}} \sum_{\boldsymbol{\gamma}\in\mathcal{X}} 1/|\mathrm{Nm}(\boldsymbol{\gamma})|\Big)\\ &= O(N_{i_0}^{-1}2^{-q_{i_0}}).\end{aligned}$$

Similarly to (2.99)–(2.100), we get from (2.105) and (2.73), that

$$\mathbf{E}(\widetilde{\mathcal{C}}_{10}(M)) = O\Big( \sum_{\substack{\boldsymbol{\varsigma}\in\{1,-1\}^s \\ \boldsymbol{\varsigma}\neq -\mathbf{1}}} \sum_{\mathbf{q}\in\mathcal{G}_9} N_{i_0(\boldsymbol{\varsigma})}^{-1} 2^{-q_{i_0(\boldsymbol{\varsigma})}}\Big)$$
$$= O\Big( \sum_{1\leq i\leq s} \sum_{j\leq n+0.5\log_2 n} \sum_{\mathbf{q}\in\mathcal{G}_9, q_i=-j} 2^{-n+j}\Big)$$
$$= O\Big(n^{s-2} \sum_{j\leq 1/2\log_2 n} 2^j\Big) = O(n^{s-3/2}).$$

Using (2.112) and (2.113), we obtain the assertion of Lemma 19. □

**Lemma 20** *With notations as above*

$$\mathbf{E}(\widetilde{\mathcal{C}}_3(M)) = \widetilde{\mathcal{C}}_{12}(M) + O(n^{s-3/2}), \quad M = [\sqrt{n}],$$

*where*

$$\widetilde{\mathcal{C}}_{12}(M) = (-1)^s(2\sqrt{-1})^{-s} \sum_{\mathbf{q}\in\mathcal{G}_9} \sum_{\boldsymbol{\gamma}_0\in\Delta_p} e(\langle\boldsymbol{\gamma}_0, \mathbf{b}/p\rangle)\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0), \tag{2.114}$$

*with*

$$\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = M^{-1} \sum_{\boldsymbol{\gamma}\in\Gamma_{M,\mathbf{q}}(\boldsymbol{\gamma}_0)} g(\boldsymbol{\gamma}), \quad g(\mathbf{x}) = \eta(2\mathrm{Nm}(\mathbf{x}))(1-\eta(x_1))\mathbb{M}(\mathbf{x})/\mathrm{Nm}(\mathbf{x}),$$

*and*

$$\Gamma_{M,\mathbf{q}}(\boldsymbol{\gamma}_0) = (p2^{-\mathbf{q}}\cdot\Gamma^{\perp} + \boldsymbol{\gamma}_0)\cdot(1/M, 1, 1, \dots, 1).$$

*Proof* By (2.106), (2.108) and Lemma 19, it is enough to prove that

$$\widetilde{\mathcal{C}}_{11}(M) = \widetilde{\mathcal{C}}_{12}(M) + O(n^{s-3/2}).$$

Consider $\check{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1})$. Let

$$\bar{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1}) = \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} (1-\eta_M(\boldsymbol{\gamma}))e(\langle\boldsymbol{\gamma}, \mathbf{b}/p\rangle))$$
$$\times\eta(2^{-q_1}\gamma_1/M)\mathbb{M}(2^{-\mathbf{q}}\cdot\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

By (2.107), we have

$$|\check{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1}) - \bar{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1})| \leq \sum_{\boldsymbol{\gamma}\in\Gamma^{\perp}\setminus\mathbf{0}} |(1-\eta_M(\boldsymbol{\gamma}))\eta(2^{-q_1}\gamma_1/M)\mathbb{M}(2^{-\mathbf{q}}\cdot\boldsymbol{\gamma})|$$
$$\times|(\widehat{\Omega}(\tau\boldsymbol{\gamma}) - 1)/\mathrm{Nm}(\boldsymbol{\gamma})|.$$

We examine the case $(1-\eta(\gamma_1 2^{-q_1}/M))\mathbb{M}(2^{-\mathbf{q}}\boldsymbol{\gamma}) \neq 0$. By (2.16) and (2.61), we get $|\gamma_1| \leq M 2^{q_1+1}$ and $|\gamma_i| \leq 2^{q_i+2},\ i \geq 2$.

Hence, we obtain from (2.73) and (2.105), that $|\tau\gamma_i| \leq 4n^{-s+1/2},\ i \geq 1$ for $\mathbf{q} \in \mathcal{G}_9$.

Applying (2.8), we get $\widehat{\Omega}(\tau\boldsymbol{\gamma}) = 1 + O(n^{-s+1/2})$ for $\mathbf{q} \in G_9$. Bearing in mind (2.90), we have

$$\breve{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1}) = \bar{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1}) + O(n^{-1}). \tag{2.115}$$

Taking into account that $\eta(0) = 0$ (see (2.16)), we get

$$\bar{\mathcal{C}}_{\mathbf{q}}(M, -\mathbf{1}) = \sum_{\boldsymbol{\gamma}_0 \in \Delta_p} e(\langle \boldsymbol{\gamma}_0, \mathbf{b}/p \rangle) \acute{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0),$$

with

$$\acute{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = \sum_{\boldsymbol{\gamma} \in 2^{-\mathbf{q}}(p\Gamma^{\perp} + \boldsymbol{\gamma}_0)} \eta(2|\mathrm{Nm}(\boldsymbol{\gamma})|/M)(1-\eta(\gamma_1/M))\mathbb{M}(\boldsymbol{\gamma})/\mathrm{Nm}(\boldsymbol{\gamma}).$$

It is easy to verify that $\acute{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = \breve{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0)$. By (2.110) and (2.114), we obtain

$$\begin{aligned}\widetilde{\mathcal{C}}_{11}(M) &= (-1)^s (2\sqrt{-1})^{-s} \sum_{\mathbf{q} \in \mathcal{G}_9} \Big( \sum_{\boldsymbol{\gamma}_0 \in \Delta_p} e(\langle \boldsymbol{\gamma}_0, \mathbf{b}/p \rangle) \breve{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) + O(n^{-1}) \Big) \\ &= \widetilde{\mathcal{C}}_{12}(M) + O(n^{s-2}).\end{aligned}$$

Hence, Lemma 20 is proved. □

We consider Koksma–Hlawka inequality (see e.g. [10, pp. 10, 11]):

**Definition 5** Let a function $f \ : \ [0,1]^s \to \mathbb{R}$ have continuous partial derivative $\partial^l f^{(F_l)}/\partial x_{i_1} \cdots \partial x_{i_l}$ on on the $s-l$ dimensional face $F_l$, defined by $x_{i_1} = \cdots = x_{i_l} = 1$, and let

$$V^{(s-l)}(f^{F_l}) = \int_{F_l} \Big| \frac{\partial^l f^{(F_l)}}{\partial x_{i_1} \cdots \partial x_{i_1}} \Big| dx_{i_1} \cdots dx_{i_l}.$$

Then the number

$$V(f) = \sum_{0 \leq l < s} \sum_{F_l} V^{(s-l)}(f^{F_l})$$

is called a Hardy and Krause variation.

**Theorem F** (Koksma–Hlawka) *Let $f$ be of bounded variation on $[0,1]^s$ in the sense of Hardy and Krause. Let $((\beta_{k,K})_{k=0}^{K-1})$ be a $K$-point set in an $s$-dimensional unit cube $[0,1)^s$. Then we have*

$$\Big| \frac{1}{K} \sum_{0 \leq k \leq K-1} f(\beta_{k,K}) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \Big| \leq V(f) D((\beta_{k,K})_{k=0}^{K-1}).$$

**Lemma 21** *With notations as above*

$$\mathbf{E}(\widetilde{\mathcal{C}}_3(M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

*Proof* By (2.114) $g(\mathbf{x}) = \eta(2\mathrm{Nm}(\mathbf{x}))(1 - \eta(x_1)))\mathbb{M}(\mathbf{x})/\mathrm{Nm}(\mathbf{x})$. We have that $g$ is the odd function, with respect to each coordinate, and $g(\mathbf{x}) = 0$ for $\mathbf{x} \notin [-2, 2] \times [-4, 4]^{s-1}$. Hence

$$\int_{[-2,2]\times[-4,4]^{s-1}} g(\mathbf{x})\mathrm{d}\mathbf{x} = 0.$$

Let $f(\mathbf{x}) = g((4x_1 - 2, 8x_2 - 4, \dots, 8x_s - 4))$. It is easy to verify that $f(\mathbf{x}) = 0$ for $\mathbf{x} \notin [0, 1]^s$, and

$$\int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{[-2,2]\times[-4,4]^{s-1}} g(\mathbf{x})\mathrm{d}\mathbf{x} = 0.$$

We see that $f$ is of bounded variation on $[0, 1]^s$ in the sense of Hardy and Krause. Let $\ddot{\Gamma}(\boldsymbol{\gamma}_0) = \{((\gamma_1 + 2)/4, (\gamma_2 + 4)/8, \dots, (\gamma_s + 4)/8) \mid \boldsymbol{\gamma} \in \Gamma_{M,\mathbf{q}}(\boldsymbol{\gamma}_0)\}$.

Using (2.114), we obtain

$$\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = M^{-1} \sum_{\boldsymbol{\gamma}\in\ddot{\Gamma}(\boldsymbol{\gamma}_0)} f(\boldsymbol{\gamma}).$$

Let $H = \ddot{\Gamma}(\boldsymbol{\gamma}_0) \cap [0, 1)^s$, and $K = \#H$. Applying Theorem A, we get $K \in [c_1 M, c_2 M]$ for some $c_1, c_2 > 0$. We enumerate the set $H$ by a sequence $((\beta_{k,K})_{k=0}^{K-1})$.

By Theorem A, we have $D((\beta_{k,K})_{k=0}^{K-1}) = O(M^{-1} \ln^{s-1} M)$.

Using Theorem F, we obtain $\check{\mathcal{C}}_{\mathbf{q}}(\boldsymbol{\gamma}_0) = O(M^{-1} \ln^{s-1} M)$.

Bearing in mind that $\#\mathcal{G}_3 = O(n^{s-1})$ (see (2.96)), we derive from (2.114) that $\widetilde{\mathcal{C}}_{12}(M) = O(n^{s-1} M^{-1} \ln^{s-1} M)$.

Applying Lemma 20, we obtain the assertion of the Lemma 21. □

Now using (2.85), Corollary 1 and Lemma 21, we get

**Corollary 2** *With notations as above*

$$\mathbf{E}(\mathcal{B}(\mathbf{b}/p, M)) = O(n^{s-5/4}), \quad M = [\sqrt{n}].$$

Let $\mathbf{N} = (N_1, \dots, N_s)$, $N = N_1 \cdots N_s$, $n = s^{-1} \log_2 N$, $c_9 = 0.25(\pi^s \det \Gamma)^{-1} c_8$ and $M = [\sqrt{n}]$. From Lemma 12, Corollary 2 and (2.18), we obtain that there exist $N_0 > 0$, and $\mathbf{b} \in \Delta_p$ such that

$$\sup_{\boldsymbol{\theta}\in[0,1]^s} |\mathbf{E}(\mathcal{R})(B_{\boldsymbol{\theta}\cdot\mathbf{N}} + \mathbf{b}/p, \Gamma)| \geq c_9 n^{s-1} \quad \text{for} \quad N > N_0. \tag{2.116}$$

### 2.10 End of Proof

*End of the proof of Theorem 1.*

We set $\widetilde{\mathcal{R}}(\mathbf{z}, \mathbf{y}) = \mathcal{R}(B_{\mathbf{y}-\mathbf{z}} + \mathbf{z}, \Gamma)$, where $y_i \geq z_i$ $(i = 1, \dots, s)$ (see (1.2)). Let us introduce the difference operator $\dot{\Delta}_{a_i,h_i} : \mathbb{R}^s \to \mathbb{R}$, defined by the formula

$$\dot{\Delta}_{a_i,h_i}\tilde{\mathcal{R}}(\mathbf{z}, \mathbf{y}) = \tilde{\mathcal{R}}(\mathbf{z}, (y_1, \dots, y_{i-1}, h_i, y_{i+1}, \dots, y_s)) \\ -\tilde{\mathcal{R}}(\mathbf{z}, (y_1, \dots, y_{i-1}, a_i, y_{i+1}, \dots, y_s)).$$

Similarly to [26, p. 160, Ref. 7], we derive

$$\dot{\Delta}_{a_1,h_1} \cdots \dot{\Delta}_{a_s,h_s}\widetilde{\mathcal{R}}(\mathbf{z}, \mathbf{y}) = \widetilde{\mathcal{R}}(\mathbf{a}, \mathbf{h}), \tag{2.117}$$

where $h_i \geq a_i \geq z_i$ $(i = 1, \dots, s)$. Let $\mathbf{f}_1, \dots, \mathbf{f}_s$ be a basis of $\Gamma$. We have that $F = \{\rho_1\mathbf{f}_1 + \dots + \rho_s\mathbf{f}_s \mid (\rho_1, \dots, \rho_s) \in [0, 1)^s\}$ is the fundamental set of $\Gamma$. It is easy to see that $\mathcal{R}(B_\mathbf{N}+\mathbf{x}, \Gamma) = \mathcal{R}(B_\mathbf{N}+\mathbf{x}+\boldsymbol{\gamma}, \Gamma)$ for all $\boldsymbol{\gamma} \in \Gamma$. Hence, we can assume in Theorem 1 that $\mathbf{x} \in F$. Similarly, we can assume in Corollary 2 that $\mathbf{b}/p \in F$. We get that there exists $\boldsymbol{\gamma}_0 \in \Gamma$ with $|\boldsymbol{\gamma}_0| \leq 4 \max_i |\mathbf{f}_i|$ and $x_i < (\mathbf{b}/p)_i + \gamma_{0,i}$, $i = 1, \dots, s$. Let $\mathbf{b}_1 = \mathbf{b} + p\boldsymbol{\gamma}_0$. By (2.116), we have that there exists $\boldsymbol{\theta} \in [0, 1]^s$ and $\mathbf{b} \in \Delta_p$ such that

$$|\widetilde{\mathcal{R}}(\mathbf{b}_1/p, \mathbf{b}_1/p + \boldsymbol{\theta} \cdot \mathbf{N})| \geq c_9 n^{s-1}. \tag{2.118}$$

Let $\mathcal{S} = \{\mathbf{y} \mid y_i = (\mathbf{b}_{/}p)_i, (\mathbf{b}_{/}p)_i + \theta_i N_i, \ i = 1, \dots, s\}$. We see $\#\mathcal{S} = 2^s$. From (2.117), we obtain that $\widetilde{\mathcal{R}}(\mathbf{b}_1/p, \mathbf{b}_1/p + \boldsymbol{\theta} \cdot \mathbf{N})$ is the sum of $2^s$ numbers $\pm\widetilde{\mathcal{R}}(\mathbf{x}, \mathbf{y}^j)$, where $\mathbf{y}^j \in \mathcal{S}$. By (2.118), we get

$$|\mathcal{R}(B_{\mathbf{y}-\mathbf{x}} + \mathbf{x}, \Gamma)| = |\widetilde{\mathcal{R}}(\mathbf{x}, \mathbf{y})| \geq 2^{-s} c_9 n^{s-1} \quad \text{for some} \quad \mathbf{y} \in \mathcal{S}.$$

Therefore, Theorem 1 is proved. □

*Proof of Theorem 2* We follow [17, p. 86] and [19, p. 1]. Let $n \geq 1$, $N \in [2^n, 2^{n+1})$, $\mathbf{y} = (y_1, \dots, y_s)$ and $\Gamma = \Gamma_\mathcal{M}$. By (1.2) and (1.5), we have

$$N\Delta(B_\mathbf{y}, (\beta_{k,N}(\mathbf{x}))_{k=0}^{N-1}) = \varphi_1 - y_1 \cdots y_s\varphi_2, \tag{2.119}$$

where

$$\varphi_1 = \mathcal{N}(B_{(y_1,\dots,y_{s-1},y_s z_{2,N}(\mathbf{x}))} + \mathbf{x}, \Gamma) \ \text{ and } \ \varphi_2 = N = \mathcal{N}(B_{(1,\dots,1,z_{2,N}(\mathbf{x}))} + \mathbf{x}, \Gamma).$$

Let

$$\alpha_1 = \mathcal{N}(B_{(y_1,\dots,y_{s-1},y_s N \det\Gamma)} + \mathbf{x}, \Gamma) \ \text{ and } \ \alpha_2 = \mathcal{N}(B_{(1,\dots,1,N \det\Gamma)} + \mathbf{x}, \Gamma).$$

Applying Theorem A, we get

$$z_{2,N}(\mathbf{x})(\det\Gamma)^{-1}-N = O(n^{s-1}), \quad \varphi_2-\alpha_2 = z_{2,N}(\mathbf{x})(\det\Gamma)^{-1}-N+O(\log_2^{s-1} n),$$

and

$$\varphi_1 - \alpha_1 = y_1...y_s(z_{2,N}(\mathbf{x})(\det\Gamma)^{-1} - N) + O(\log_2^{s-1} n).$$

From (2.119), we derive

$$N\Delta(B_{\mathbf{y}}, (\beta_{k,N}(\mathbf{x}))_{k=0}^{N-1}) = \alpha_1 - y_1 \cdots y_{s-1}\alpha_2 + O(\log_2^{s-1} n) \tag{2.120}$$

By (1.2), we obtain

$$\alpha_1 - y_1 \cdots y_{s-1}\alpha_2 = \beta_1 - y_1 \cdots y_{s-1}\beta_2 \tag{2.121}$$

with

$$\beta_1 = \mathcal{R}(B_{(y_1,\dots,y_{s-1},y_s N \det\Gamma)} + \mathbf{x}, \Gamma) \ \text{ and } \ \beta_2 = \mathcal{R}(B_{(1,\dots,1,N\det\Gamma)} + \mathbf{x}, \Gamma).$$

Let $y_0 = 0.125\min(1, 1/\det\Gamma, (c_1(\mathcal{M})/c_0(\Gamma))^{1/(s-1)}), \boldsymbol{\theta} = (\theta_1, \dots, \theta_s), y_i = y_0\theta_i$, $i = 1, \dots, s-1$, and $y_s = \theta_s$. Using Theorem A, we get

$$\begin{aligned} |y_1 \cdots y_s \mathcal{R}(B_{(1,\dots,1,N\det\Gamma)} + \mathbf{x}, \Gamma)| &\le y_0^{s-1} c_0(\Gamma)\log_2^{s-1}(2 + N\det\Gamma) \\ &\le (2y_0)^{s-1} c_0(\Gamma)\log_2^{s-1} N \\ &\le 0.25 c_1(\mathcal{M}) n^{s-1} \quad \text{for} \quad N > \det\Gamma + 2. \end{aligned} \tag{2.122}$$

Applying Theorem 1, we have

$$\begin{aligned} &\sup_{\boldsymbol{\theta}\in[0,1)^s} |\mathcal{R}(B_{(\theta_1 y_0,\dots,\theta_{s-1}y_0,\theta_s N\det\Gamma)} + \mathbf{x}, \Gamma)| \\ &\ge c_1(\mathcal{M})\log_2^{s-1}(y_0^{s-1}\det\Gamma N) \\ &\ge c_1(\mathcal{M}) n^{s-1}(1 + n^{-1}(s-1)\log_2(y_0^{s-1}\det\Gamma)) \ge 0.5 c_1(\mathcal{M}) n^{s-1} \end{aligned}$$

for $n > 10(s-1)|\log_2(y_0^{s-1}\det\Gamma)|$. Using (1.6), (2.120), (2.121) and (2.122), we get the assertion of Theorem 2. □

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