

The Maximum Number of Faces of the Minkowski Sum of Two Convex Polytopes

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Abstract We derive tight bounds for the maximum number of k -faces, $0 \leq k \leq d - 1$, of the Minkowski sum, $P_1 + P_2$, of two d -dimensional convex polytopes P_1 and P_2 , as a function of the number of vertices of the polytopes. For even dimensions $d \geq 2$, the maximum values are attained when P_1 and P_2 are cyclic d -polytopes with disjoint vertex sets. For odd dimensions $d \geq 3$, the maximum values are attained when P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$ -neighborly d -polytopes, whose vertex sets are chosen appropriately from two distinct d -dimensional moment-like curves.

Keywords High-dimensional geometry · Discrete geometry · Combinatorial geometry · Combinatorial complexity · Minkowski sum · Convex polytopes

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1 Introduction

Let P, Q be two d -dimensional polytopes, or simply d -polytopes, in the Euclidean d -dimensional space \mathbb{E}^d . Their Minkowski sum, $P + Q$, which is again a d -polytope in \mathbb{E}^d , is defined as the set $\{p + q \mid p \in P, q \in Q\}$. Minkowski sums are fundamental structures in both Mathematics and Computer Science. They appear in a variety of different subjects including Combinatorial Geometry, Computational Geometry, Computer Algebra, Computer-Aided Design & Solid Modeling, Motion Planning, Assembly Planning, Robotics (see [6,25] and the references therein), and, more recently, Game Theory [22], Computational Biology [21], and Operations Research [27].

Despite their apparent importance, little is known about the worst-case combinatorial complexity, i.e., the total number of faces, of Minkowski sums in dimensions four and higher. In two dimensions, the worst-case complexity of Minkowski sums is well understood. Given two convex polygons P and Q with n and m vertices, respectively, the maximum number of vertices and edges of $P + Q$ is $n + m$ [4]. This result can be immediately generalized (e.g., by induction) to any number of summands. If P is convex and Q is non-convex (or vice versa), the worst-case complexity of $P + Q$ is $\Theta(nm)$, while if both P and Q are non-convex the complexity of their Minkowski sum can be as high as $\Theta(n^2m^2)$ [4]. When P and Q are 3-polytopes (embedded in the 3-dimensional Euclidean space), the worst-case complexity of $P + Q$ is $\Theta(nm)$, if both P and Q are convex, and $\Theta(n^3m^3)$, if both P and Q are non-convex (e.g., see [5]). For the intermediate cases, i.e., if only one of P and Q is convex, see [24].

Given two convex d -polytopes P_1 and P_2 in \mathbb{E}^d , $d \geq 2$, with n_1 and n_2 vertices, respectively, we can easily get a straightforward upper bound of $O((n_1 + n_2)^{\lfloor \frac{d+1}{2} \rfloor})$ on the complexity of $P_1 + P_2$ by means of the following reduction: embed P_1 and P_2 in the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$ of \mathbb{E}^{d+1} , respectively; then the weighted Minkowski sum $(1 - \lambda)P_1 + \lambda P_2 = \{(1 - \lambda)p_1 + \lambda p_2 \mid p_1 \in P_1, p_2 \in P_2\}$, $\lambda \in (0, 1)$, of P_1 and P_2 is the intersection of the convex hull P of P_1 and P_2 with the hyperplane $\{x_1 = \lambda\}$. The embedding and reduction described above are essentially what are known as the *Cayley embedding* and *Cayley trick*, respectively, whereas the $(d + 1)$ -polytope P is called the *Cayley polytope* of P_1 and P_2 [11]. From this reduction it is obvious that the worst-case complexity of $(1 - \lambda)P_1 + \lambda P_2$ is bounded from above by the complexity of the Cayley polytope, which is in $O((n_1 + n_2)^{\lfloor \frac{d+1}{2} \rfloor})$. Furthermore, the complexity of the weighted Minkowski sum of P_1 and P_2 is independent of λ , in the sense that, for any value of $\lambda \in (0, 1)$, the polytopes we get by intersecting P with $\{x_1 = \lambda\}$ are combinatorially equivalent. In fact, since $P_1 + P_2$ is nothing but $\frac{1}{2}P_1 + \frac{1}{2}P_2$ scaled by a factor of 2, the complexity of the weighted Minkowski sum of two convex polytopes is the same as the complexity of their unweighted Minkowski sum. The “obvious” upper bound for the complexity of the Minkowski sum stemming from the complexity of the Cayley polytope is tight in even dimensions. In odd dimensions, $d \geq 3$, however, this upper bound may not be tight; in this case, the worst-case complexity of $P_1 + P_2$ is in $\Theta(n_1 n_2^{\lfloor \frac{d}{2} \rfloor} + n_2 n_1^{\lfloor \frac{d}{2} \rfloor})$ (cf. [13]), which is a refinement over of the “obvious” upper bound when n_1 and n_2 asymptotically differ. In terms of exact bounds on the number of faces of the Minkowski sum of two polytopes, results are known only when the two summands are convex. Besides the trivial bound for convex

polygons (2-polytopes), mentioned in the previous paragraph, the first result of this nature was shown by Gritzmann and Sturmfels [10]: given r polytopes P_1, P_2, \dots, P_r in \mathbb{E}^d , with a total of n non-parallel edges, the number of l -faces $f_l(P_1 + P_2 + \dots + P_r)$ of $P_1 + P_2 + \dots + P_r$ is bounded from above by $2\binom{n}{l} \sum_{j=0}^{d-1-l} \binom{n-l-1}{j}$. This bound is attained when the polytopes P_i are *zonotopes*, and their generating edges are in general position.

Regarding bounds as a function of the number of vertices or facets of the summands, Fukuda and Weibel [7] have shown that, given two 3-polytopes P_1 and P_2 in \mathbb{E}^3 , the number of k -faces of $P_1 + P_2$, $0 \leq k \leq 2$, is bounded from above as follows:

$$\begin{aligned} f_0(P_1 + P_2) &\leq n_1 n_2, \\ f_1(P_1 + P_2) &\leq 2n_1 n_2 + n_1 + n_2 - 8, \\ f_2(P_1 + P_2) &\leq n_1 n_2 + n_1 + n_2 - 6, \end{aligned} \tag{1}$$

where n_j is the number of vertices of P_j , $j = 1, 2$. Weibel [25] has also derived similar expressions in terms of the number of facets m_j of P_j , $j = 1, 2$, namely

$$\begin{aligned} f_0(P_1 + P_2) &\leq 4m_1 m_2 - 8m_1 - 8m_2 + 16, \\ f_1(P_1 + P_2) &\leq 8m_1 m_2 - 17m_1 - 17m_2 + 40, \\ f_2(P_1 + P_2) &\leq 4m_1 m_2 - 9m_1 - 9m_2 + 26. \end{aligned}$$

All these bounds are tight. Fogel et al. [5] have further generalized some of these bounds in the case of r summands. More precisely, they have shown that given r 3-polytopes P_1, P_2, \dots, P_r in \mathbb{E}^3 , where P_j has $m_j \geq d + 1$ facets, the number of facets of the Minkowski sum $P_1 + P_2 + \dots + P_r$ is bounded from above by

$$\sum_{1 \leq i < j \leq r} (2m_i - 5)(2m_j - 5) + \sum_{i=1}^r m_i + \binom{r}{2},$$

and this bound is tight.

In dimensions four and higher, there are no results that relate the worst-case number of k -faces of the Minkowski sum of two or more convex polytopes with the number of facets of the summands. There are, however, bounds on the number of k -faces of the Minkowski sum of convex polytopes, as a function of the number of vertices of the summands. Fukuda and Weibel [7] have shown that the number of vertices of the Minkowski sum of r d -polytopes P_1, \dots, P_r , where $r \leq d - 1$ and $d \geq 2$, is bounded from above by $\prod_{i=1}^r n_i$, where n_i is the number of vertices of P_i , and this bound is tight. On the other hand, for $r \geq d$, this bound cannot be attained: Sanyal [23] has shown that for $r \geq d$, $f_0(P_1 + \dots + P_r)$ is bounded from above by $(1 - \frac{1}{(d+1)^d}) \prod_{i=1}^r n_i$, which is, clearly, strictly smaller than $\prod_{i=1}^r n_i$. For higher-dimensional faces, i.e., for $k \geq 1$, Fukuda and Weibel [7] have proven what they call the *trivial upper bound*, namely that the number of k -faces of the Minkowski sum of r d -polytopes is bounded as follows:

$$f_k(P_1 + P_2 + \dots + P_r) \leq \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \dots + s_r = k+r}} \prod_{i=1}^r \binom{n_i}{s_i}, \quad 0 \leq k \leq d - 1, \quad (2)$$

where n_i is the number of vertices of P_i , and the s_i 's take integral values. Furthermore, it is shown in [7] that these bounds are tight for $d \geq 4$, $r \leq \lfloor \frac{d}{2} \rfloor$, and for all k with $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$, i.e., for the cases where both the number of summands and the dimension of the faces considered are small. The above-mentioned ranges for the parameters d , r , and k for which the trivial upper bound is tight are not the best possible: Karavelas and Tzanaki [15] have shown that the trivial upper bound is tight for $d \geq 3$, $2 \leq r \leq d - 1$ and for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$, and these ranges are maximal. This result immediately implies a tight worst-case asymptotic bound on the complexity of the Minkowski sum of r n -vertex d -polytopes, namely $\Theta(n^{\lfloor \frac{d+r-1}{2} \rfloor})$.

We end our discussion of the previous work related to this paper by some results presented in Weibel [26]. In this paper, Weibel considers the case where the number of summands r is at least as big as the dimension of the polytopes. In this setting, he gives a relation between the number of k -faces of the Minkowski sum of r polytopes, $r \geq d \geq 2$, and the number of k -faces of the Minkowski sum of subsets of the original set of r polytopes that are of size at most $d - 1$. In more detail, if we have r d -polytopes P_1, P_2, \dots, P_r in \mathbb{E}^d , where $r \geq d$, that are in general position, then the following relation holds for any k with $0 \leq k \leq d - 1$:

$$\begin{aligned} f_k(P_1 + P_2 + \dots + P_r) &= \alpha + \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \sum_{S \in \mathcal{C}_j^r} (f_k(P_S) - \alpha) \\ &\leq \sum_{S \in \mathcal{C}_{d-1}^r} f_k(P_S), \end{aligned} \quad (3)$$

where \mathcal{C}_j^r is the family of subsets of $\{1, 2, \dots, r\}$ of cardinality j , P_S is the Minkowski sum of the polytopes in S , and, finally, $\alpha = 2$ if $k = 0$ and d is odd, $\alpha = 0$, otherwise. Weibel used this relation to derive tight upper bounds on the number of vertices of the Minkowski sum of r d -polytopes in \mathbb{E}^d , when $r \geq d$. In particular, the following tight upper bound holds (cf. [26, Thm. 3]):

$$f_0(P_1 + P_2 + \dots + P_r) \leq \alpha + \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \sum_{S \in \mathcal{C}_j^r} \left(\prod_{i \in S} f_0(P_i) - \alpha \right),$$

where \mathcal{C}_j^r and α are defined as for relation (3).

In this paper, we extend previous results on the exact maximum number of faces of the Minkowski sum of two convex d -polytopes.¹ More precisely, we show that given two d -polytopes P_1 and P_2 in \mathbb{E}^d with $n_1 \geq d + 1$ and $n_2 \geq d + 1$ vertices, respectively,

¹ In the rest of the paper, all polytopes are considered to be convex.

the maximum number of k -faces of the Minkowski sum $P_1 + P_2$ is bounded from above as follows:

$$f_{k-1}(P_1 + P_2) \leq f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $1 \leq k \leq d$, and $C_d(n)$ stands for the cyclic d -polytope with n vertices. The expressions above are shown to be tight for any $d \geq 2$ and for all $1 \leq k \leq d$, and match with the corresponding expressions for two and three dimensions (cf. rel. (1)), as well as the expressions in (2) for $r = 2$ and for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 2$.

To prove the upper bounds, we use the Cayley embedding already described above. Given the d -polytopes P_1 and P_2 in \mathbb{E}^d , we embed P_1 and P_2 in the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$ of \mathbb{E}^{d+1} . We consider the Cayley polytope $P = CH_{d+1}(P_1, P_2)$ of P_1 and P_2 and argue that, for the purposes of the worst-case upper bounds, it suffices to consider the case where P is simplicial, except possibly for its two facets P_1 and P_2 . We concentrate on the set \mathcal{F} of faces of P that are neither faces of P_1 nor faces of P_2 . The reason that we focus on \mathcal{F} is that there is a bijection between the k -faces of \mathcal{F} and the $(k-1)$ -faces of $P_1 + P_2$, $1 \leq k \leq d$, and, thus, deriving upper bounds of the number of $(k-1)$ -faces of $P_1 + P_2$ reduces to deriving upper bounds for the number of k -faces of \mathcal{F} . We then proceed in a manner analogous to that used by McMullen [20] to prove the Upper Bound Theorem for polytopes.

We consider the f -vector $\mathbf{f}(\mathcal{F})$ of \mathcal{F} , and from this we define the h -vector $\mathbf{h}(\mathcal{F})$ of \mathcal{F} and continue by

- (i) deriving Dehn–Sommerville-like equations for \mathcal{F} , expressed in terms of the elements of $\mathbf{h}(\mathcal{F})$ and the g -vectors of the boundary complexes of P_1 and P_2 , and
- (ii) establishing a recurrence relation for the elements of $\mathbf{h}(\mathcal{F})$.

From the latter, we inductively compute upper bounds on the elements of $\mathbf{h}(\mathcal{F})$, which we combine with the Dehn–Sommerville-like equations for \mathcal{F} , to get refined upper bounds for the “left-most half” of the elements of $\mathbf{h}(\mathcal{F})$, i.e., for the values $h_k(\mathcal{F})$ with $k > \lfloor \frac{d+1}{2} \rfloor$. We then establish our upper bounds by computing $\mathbf{f}(\mathcal{F})$ from $\mathbf{h}(\mathcal{F})$.

To prove the tightness of our upper bounds, we distinguish between even and odd dimensions. In even dimensions $d \geq 2$, we show that the k -faces of the Minkowski sum of any two cyclic d -polytopes with n_1 and n_2 vertices, respectively, whose vertex sets are distinct, attain the upper bounds we have proved. In odd dimensions $d \geq 3$, the construction that establishes the tightness of our bounds is more intricate. We consider the $(d-1)$ -dimensional moment curve $\boldsymbol{\gamma}(t) = (t, t^2, t^3, \dots, t^{d-1})$, $t > 0$, and define two vertex sets V_1 and V_2 with n_1 and n_2 vertices on $\boldsymbol{\gamma}(t)$, respectively. For $i = 1, 2$, we define P_i as the convex hull in \mathbb{E}^d of the vertices in V_i . We embed P_1, P_2 in \mathbb{E}^{d+1} so that they lie in two affinely independent hyperplanes of \mathbb{E}^{d+1} and then appropriately perturb their vertices so that they become combinatorially equivalent to d -dimensional cyclic polytopes. We next argue that, by the way we have chosen V_1 and V_2 , the number of k -faces of the Minkowski sum $P_1 + P_2$ attains its maximum possible value. At a very high/qualitative level, the appropriate choice we refer to

above amounts to choosing V_1 and V_2 so that the parameter values on $\gamma(t)$ of the vertices in V_1 and V_2 lie within two disjoint intervals of \mathbb{R} that are sufficiently away from each other.

The structure of the rest of the paper is as follows: In Sect. 2 we formally give various definitions and recall a version of the Upper Bound Theorem for polytopes that will be useful later in the paper. In Sect. 3 we define what we call *bineighborly* polytopal complexes and prove some properties associated with them. The reason why we introduce this new notion is the fact that the tightness of our upper bounds is shown to be equivalent to requiring that the Cayley polytope of P_1 and P_2 is bineighborly. In Sect. 4 we prove our upper bounds on the number of faces of the Minkowski sum of two polytopes. In Sect. 5 we describe our worst-case constructions and show that these constructions attain the upper bounds proved in Sect. 4. We conclude the paper with Sect. 6, where we summarize our results, and state open problems and directions for future work.

2 Definitions and Preliminaries

A *convex polytope*, or simply *polytope*, P in \mathbb{E}^d is the convex hull of a finite set of points V in \mathbb{E}^d . The minimal subset V' of V for which the convex hull of V' is P is called the *vertex set* of P . A polytope P can equivalently be described as the intersection of all the closed halfspaces containing V . A *face* of P is the intersection of P with a hyperplane H for which the polytope is contained in one of the two closed halfspaces delimited by H . Such a hyperplane is called a *supporting hyperplane* of P . The dimension of a face of P is the dimension of its affine hull. A k -face of P is a k -dimensional face of P . We consider the polytope itself as a trivial d -dimensional face; all the other faces are called *proper faces*. We use the term *d -polytope* to refer to a polytope the trivial face of which is d -dimensional. For a d -polytope P , the 0-faces of P are its *vertices*, the 1-faces of P are its *edges*, the $(d-2)$ -faces of P are called *ridges*, while the $(d-1)$ -faces are called *facets*. For $0 \leq k \leq d$ we denote by $f_k(P)$ the number of k -faces of P . Note that every k -face F of P is also a k -polytope whose faces are all the faces of P contained in F . A k -simplex in \mathbb{E}^d , $k \leq d$, is the convex hull of any $k+1$ affinely independent points in \mathbb{E}^d . A polytope is called *simplicial* if all its proper faces are simplices. Equivalently, P is simplicial if for every vertex v of P and every face $F \subset P$, v does not belong to the affine hull of the vertices in $F \setminus \{v\}$.

A *polytopal complex* \mathcal{C} is a finite collection of polytopes in \mathbb{E}^d such that (i) $\emptyset \in \mathcal{C}$; (ii) if $P \in \mathcal{C}$, then all the faces of P are also in \mathcal{C} ; and (iii) the intersection $P \cap Q$ for two polytopes P and Q in \mathcal{C} is a face of both P and Q . The dimension $\dim(\mathcal{C})$ of \mathcal{C} is the largest dimension of a polytope in \mathcal{C} . A polytopal complex is called *pure* if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the *facets* of \mathcal{C} . We use the term *d -complex* to refer to a polytopal complex whose maximal faces are d -dimensional (i.e., the dimension of \mathcal{C} is d). A polytopal complex is called *simplicial* if all its faces are simplices. Finally, a polytopal complex \mathcal{C}' is called a *subcomplex* of a polytopal complex \mathcal{C} if all faces of \mathcal{C}' are also faces of \mathcal{C} .

One important class of polytopal complexes arises from polytopes. More precisely, a d -polytope P , together with all its faces, including the empty face, forms a d -complex, denoted by $\mathcal{C}(P)$. The only maximal face of $\mathcal{C}(P)$, which is clearly the only facet of $\mathcal{C}(P)$, is the polytope P itself. Moreover, all proper faces of P form a pure $(d - 1)$ -complex, called the *boundary complex* $\mathcal{C}(\partial P)$, or simply ∂P of P . The facets of ∂P are just the facets of P , and the dimension of ∂P is clearly $\dim(\partial P) = \dim(P) - 1 = d - 1$.

Given a d -polytope P in \mathbb{E}^d , consider a facet F of P and call H the supporting hyperplane of F (with respect to P). For an arbitrary point p in \mathbb{E}^d , we say that p is *beyond* (resp., *beneath*) the facet F of P , if p lies in the open halfspace of H that does not contain P (resp., contains the interior of P). Furthermore, we say that an arbitrary point v' is *beyond* the vertex v of P if for every facet F of P containing v , v' is beyond F , while for every facet F of P not containing v , v' is beneath F . For a vertex v of P , the *star* of v , denoted by $\text{star}(v, P)$, is the polytopal complex of all faces of P that contain v , and their faces. The *link* of v , denoted by $\text{link}(v, P)$ or P/v , is the subcomplex of $\text{star}(v, P)$ consisting of all the faces of $\text{star}(v, P)$ that do not contain v .

Definition 1 ([28, Rem. 8.3]) Let \mathcal{C} be a pure simplicial polytopal d -complex. A shelling $\mathcal{S}(\mathcal{C})$ of \mathcal{C} is a linear ordering F_1, F_2, \dots, F_s of the facets of \mathcal{C} such that, for all $1 < j \leq s$, the intersection $F_j \cap (\bigcup_{i=1}^{j-1} F_i)$ of the facet F_j with the previous facets is non-empty and pure $(d - 1)$ -dimensional. In other words, for every $i < j$, there exists some $\ell < j$ such that the intersection $F_i \cap F_j$ is contained in $F_\ell \cap F_j$ and such that $F_\ell \cap F_j$ is a facet of F_j .

The shelling definition above is the specialization, to simplicial complexes, of the shelling definition for general polytopal complexes (cf. [28, Def. 8.1]). We refrained from stating the general definition, as in the sequel of the paper we will only consider shellings of simplicial polytopal complexes. A polytopal complex that has a shelling is called *shellable*, while not all polytopal complexes are shellable. It was a major result in polytopal theory that the boundary complex of a polytope is always shellable (cf. [2]).

Consider a pure shellable simplicial polytopal complex \mathcal{C} and let $\mathcal{S}(\mathcal{C}) = \{F_1, \dots, F_s\}$ be a shelling order of its facets. The *restriction* $R(F_j)$ of a facet F_j is the set of all vertices $v \in F_j$ such that $F_j \setminus \{v\}$ is contained in one of the earlier facets.² The main observation here is that when we construct \mathcal{C} according to the shelling $\mathcal{S}(\mathcal{C})$, the new faces at the j -th step of the shelling are exactly the vertex sets G with $R(F_j) \subseteq G \subseteq F_j$ (cf. [28, Sect. 8.3]). Moreover, notice that $R(F_1) = \emptyset$ and $R(F_i) \neq R(F_j)$ for all $i \neq j$.

The f -vector $\mathbf{f}(P) = (f_{-1}(P), f_0(P), \dots, f_{d-1}(P))$ of a d -polytope P (or its boundary complex ∂P) is defined as the $(d + 1)$ -dimensional vector consisting of the numbers $f_k(P)$ of k -faces of P , $-1 \leq k \leq d - 1$, where $f_{-1}(P) = 1$ refers to the empty set. The h -vector $\mathbf{h}(P) = (h_0(P), h_1(P), \dots, h_d(P))$ of a d -polytope P (or its boundary complex ∂P) is defined as the $(d + 1)$ -dimensional vector, where

² For simplicial faces, we identify the face with its defining vertex set.

$$h_k(P) := \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(P), \quad 0 \leq k \leq d. \tag{4}$$

It is easy to verify from the defining equations of the $h_k(P)$'s that the elements of $f(P)$ determine the elements of $h(P)$ and vice versa (see also [28, Sect. 8.3] and below). In particular, the elements of $f(P)$ can be written in terms of the elements of $h(P)$ as follows:

$$f_{k-1}(P) = \sum_{i=0}^k \binom{d-i}{k-i} h_i(P), \quad 0 \leq k \leq d. \tag{5}$$

For simplicial polytopes, the number $h_k(P)$ counts the number of facets of P in a shelling of ∂P , whose restriction has size k ; this number is independent of the particular shelling chosen (cf. [28, Thm. 8.19]). Moreover, the elements of $f(P)$ (or, equivalently, $h(P)$) are not linearly independent; they satisfy the so-called *Dehn–Sommerville equations*, which can be written in a very concise form as $h_k(P) = h_{d-k}(P)$, $0 \leq k \leq d$. An important implication of the existence of the Dehn–Sommerville equations is that if we know the face numbers $f_k(P)$ for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 1$, we can determine the remaining face numbers $f_k(P)$ for all $\lfloor \frac{d}{2} \rfloor \leq k \leq d - 1$ (cf. relations (4) and (5)).

Both the f -vector and h -vector of a simplicial d -polytope are related to the so-called g -vector. For a simplicial d -polytope P , its g -vector is the $(\lfloor \frac{d}{2} \rfloor + 1)$ -dimensional vector $g(P) = (g_0(P), g_1(P), \dots, g_{\lfloor \frac{d}{2} \rfloor}(P))$, where $g_0(P) = 1$ and $g_k(P) = h_k(P) - h_{k-1}(P)$, $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$ (see also [28, Sect. 8.6]). Using the convention that $h_{d+1}(P) = 0$, we can actually extend the definition of $g_k(P)$ for all $0 \leq k \leq d + 1$, while using the Dehn–Sommerville equations for P yields $g_{d+1-k}(P) = -g_k(P)$, $0 \leq k \leq d + 1$. We can then express $f(P)$ in terms of $g(P)$ as follows:

$$f_{k-1}(P) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} g_j(P) \left(\binom{d+1-j}{d+1-k} - \binom{j}{d+1-k} \right), \quad 0 \leq k \leq d + 1.$$

As a final note for this section, the Upper Bound Theorem for polytopes can be expressed in terms of their g -vector:

Corollary 2 [28, Cor. 8.38] *We consider simplicial d -polytopes P of fixed dimension d and fixed number of vertices $n = g_1(P) + d + 1$. $f(P)$ is component-wise maximal if and only if all the components of $g(P)$ are maximal, with*

$$g_k(P) = \binom{g_1(P) + k - 1}{k} = \binom{n - d - 2 + k}{k}, \tag{6}$$

for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Also, $f_{k-1}(P)$ is maximal if and only if $g_i(P)$ is maximal for all i with $i \leq \min\{k, \lfloor \frac{d}{2} \rfloor\}$.

The relation between the f -vector and h -vector of a polytopal complex is better manipulated using generating functions. For a pure simplicial $(d - 1)$ -complex \mathcal{C} , its f -polynomial and h -polynomial, respectively, are defined as

$$f(\mathcal{C}; t) = \sum_{i=0}^d f_{i-1}t^{d-i} = f_d + f_{d-1}t + \cdots + f_{-1}t^d, \tag{7}$$

$$h(\mathcal{C}; t) = \sum_{i=0}^d h_it^{d-i} = h_d + h_{d-1}t + \cdots + h_0t^d. \tag{8}$$

The relation between the f -vector and h -vector (cf. relations (4) and (5)) can then be expressed as

$$f(\mathcal{C}; t) = h(\mathcal{C}; t + 1) \tag{9}$$

or equivalently as

$$h(\mathcal{C}; t) = f(\mathcal{C}; t - 1). \tag{10}$$

If P is a simplicial d -polytope, then its boundary complex ∂P is a simplicial $(d - 1)$ -complex. We can therefore use relations (9) and (10) to define $f(\partial P; t)$ and $h(\partial P; t)$. Furthermore, we may define the g -polynomial $g(\partial P; t)$ of P as

$$g(\partial P; t) := \sum_{i=0}^d g_it^{d+1-i} = g_d t + g_{d-1}t^2 + \cdots + g_0t^{d+1}. \tag{11}$$

The relation between the g - and h -vector of ∂P can be expressed in terms of generating polynomials as

$$g(\partial P; t) = (t - 1)h(\partial P; t). \tag{12}$$

Furthermore, the Dehn–Sommerville equations for a simplicial d -polytope are captured in the following reciprocal relation:

$$t^d h(\partial P; \frac{1}{t}) = h(\partial P; t). \tag{13}$$

We end this section with a technical lemma. Its proof may be found in Appendix.

Lemma 3 *For any d -polytope P*

$$d h(\partial P; t) + (1 - t) h'(\partial P; t) = \sum_{v \in \text{vert}(P)} h(\partial P/v; t), \tag{14}$$

where $\text{vert}(P)$ is the vertex set of P (or ∂P), and $h'(\cdot; t)$ denotes the derivative of $h(\cdot; t)$ with respect to t .

3 Bineighborly Polytopal Complexes

Let \mathcal{C} be a d -complex with vertex set V . Let $\{V_1, V_2\}$ be a partition of V and define \mathcal{C}_1 (resp., \mathcal{C}_2) to be the subcomplex of \mathcal{C} consisting of all the faces of \mathcal{C} whose vertices are vertices in V_1 (resp., V_2). We start with a useful definition:

Definition 4 Let \mathcal{C} be a d -complex. We say that \mathcal{C} is (k, V_1) -bineighborly if we can partition the vertex set V of \mathcal{C} into two non-empty subsets V_1 and $V_2 = V \setminus V_1$ such that for every $\emptyset \subset S_j \subseteq V_j$, $j = 1, 2$, with $|S_1| + |S_2| \leq k$, the union $S_1 \cup S_2$ is the vertex set of a $(|S_1| + |S_2| - 1)$ -face of \mathcal{C} .

We introduce the notion of bineighborly polytopal complexes because they play an important role when considering the maximum complexity of the Minkowski sum of two d -polytopes P_1 and P_2 . As we will see in the upcoming section, the number of $(k - 1)$ -faces of $P_1 + P_2$ is maximal for all $1 \leq k \leq l$, $l \leq \lfloor \frac{d-1}{2} \rfloor$, if and only if the Cayley polytope P of P_1 and P_2 is $(l + 1, V_1)$ -bineighborly, where V_1 stands for the vertex set of P_1 . Even more interestingly, in any odd dimension $d \geq 3$, the number of k -faces of $P_1 + P_2$ is maximized for all $0 \leq k \leq d - 1$, if and only if P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly. In the rest of this section we highlight some properties of bineighborly polytopal complexes that will be useful in the upcoming sections.

A direct consequence of our definition is the following: suppose that \mathcal{C} is a (l, V_1) -bineighborly polytopal complex, and let F be a k -face of \mathcal{C} , $1 \leq k < l$, such that at least one vertex of F is in V_1 and at least one vertex of F is in V_2 ; then F is a k -simplex. Another immediate consequence of Definition 4 is that a k -neighborly d -complex is also (k, V') -bineighborly for every non-empty subset V' of its vertex set:

Corollary 5 Let \mathcal{C} be a k -neighborly d -complex, with vertex set V . Then, for every V' , with $\emptyset \subset V' \subset V$, \mathcal{C} is (k, V') -bineighborly.

It is easy to see that if a d -complex \mathcal{C} is (k, V_1) -bineighborly, then it is $(k - 1)$ -neighborly, as the following straightforward lemma suggests.

Lemma 6 Let \mathcal{C} be a (k, V_1) -bineighborly d -complex, $k \geq 2$. Then \mathcal{C} is $(k - 1)$ -neighborly.

Proof Let $S \subseteq V$ be of size $k - 1$. Choose $v \in V \setminus S$ such that $(S \cup \{v\}) \cap V_i \neq \emptyset$ for $i = 1, 2$, and let $S' = S \cup \{v\}$. Since \mathcal{C} is (k, V_1) -bineighborly, S' is the vertex set of a $(k - 1)$ -face of \mathcal{C} and, as argued above, a $(k - 1)$ -simplex. Clearly, S is the vertex set of a $(k - 2)$ -face of S' and hence an element of \mathcal{C} . \square

The following lemma is in some sense the converse of Lemma 6.

Lemma 7 Let \mathcal{C} be a (k, V_1) -bineighborly d -complex and let its two subcomplexes \mathcal{C}_1 and \mathcal{C}_2 be k -neighborly. Then \mathcal{C} is also k -neighborly.

Proof Let S be a non-empty subset of V of size k . Consider the following, mutually exclusive cases:

- (i) S consists of vertices of both V_1 and V_2 . Then, since \mathcal{C} is (k, V_1) -bineighborly, S defines a $(k - 1)$ -face of \mathcal{C} .

- (ii) S consists of vertices of V_j only, $j = 1, 2$. Since C_j is k -neighborly, S defines a $(k - 1)$ -face of C_j . However, C_j is a subcomplex of C , which further implies that S is also a face of C .

Hence, for every vertex subset S of V of size k , S defines a $(k - 1)$ -face of C , i.e., C is k -neighborly. □

Consider again a d -complex C with vertex set $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$. Let \mathcal{B} be the set of faces of C that are not faces of either C_1 or C_2 . The following lemma gives tight upper bounds for the number of faces in \mathcal{B} . In what follows, we denote by n_j the cardinality of V_j , $j = 1, 2$.

Lemma 8 *The number of $(k - 1)$ -faces of \mathcal{B} is bounded from above as follows:*

$$f_{k-1}(\mathcal{B}) \leq \sum_{j=1}^{k-1} \binom{n_1}{j} \binom{n_2}{k-j} = \binom{n_1 + n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k}, \quad 1 \leq k \leq d, \quad (15)$$

where equality holds if and only if C is (k, V_1) -bineighborly.

Proof For the purposes of proving an upper bound for elements of the f -vector of \mathcal{B} , it suffices to consider the case where C_1 and C_2 are in *general position* with respect to each other, in which case \mathcal{B} is simplicial.

Under the assumption that \mathcal{B} is simplicial, we can identify each face of \mathcal{B} with its uniquely defined vertex set. Consider a $(k - 1)$ -face F of \mathcal{B} . The face F is a $(k - 1)$ -simplex and, seen as a vertex set, it is neither a subset of V_1 nor V_2 . Clearly, the number $\#_k \mathcal{B}$ of vertex sets of cardinality k of \mathcal{B} , which are neither subsets of V_1 nor V_2 , is bounded by above by the number $\#_k(V_1, V_2)$ of k -element subsets of V not fully contained in either V_1 or V_2 ; this is precisely the right-hand side of (15). Finally, notice that $\#_k \mathcal{B}$ is equal to $\#_k(V_1, V_2)$ if and only if for every non-empty subset S_1 of V_1 and every non-empty subset S_2 of V_2 , where $|S_1| + |S_2| = k$, the set $S_1 \cup S_2$ is the vertex set of a $(k - 1)$ -face of C . In other words, equality in (15) can only hold if and only if C is (k, V_1) -bineighborly. □

4 Upper Bounds

Let P_1 and P_2 be two d -polytopes in \mathbb{E}^d , with n_1 and n_2 vertices, respectively. Let us consider the Cayley embedding of P_1 and P_2 , i.e., we embed P_1 (resp., P_2) in the hyperplane Π_1 (resp., Π_2) of \mathbb{E}^{d+1} with equation $\{x_1 = 0\}$ (resp., $\{x_1 = 1\}$). Then the Minkowski sum $P_1 + P_2$ (scaled by a factor of 2) is the d -polytope we get when intersecting the Cayley polytope $P = CH_{d+1}(P_1, P_2)$ of P_1 and P_2 with the hyperplane $\tilde{\Pi}$ with equation $\{x_1 = \frac{1}{2}\}$ (see Fig. 1). This immediately implies that the k -faces of the Minkowski sum $P_1 + P_2$ correspond bijectively to the $(k + 1)$ -faces of P not contained in either P_1 or P_2 .

Karavelas and Tzanaki [13, Lem. 2] have shown that the vertices of P_1 and P_2 can be perturbed in such a way that

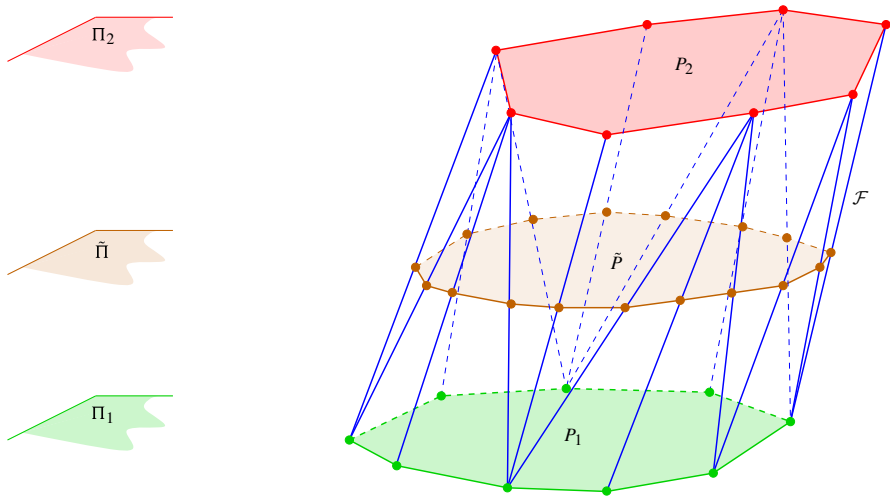


Fig. 1 The d -polytopes P_1 and P_2 are embedded in the hyperplanes $\Pi_1 = \{x_1 = 0\}$ and $\Pi_2 = \{x_1 = 1\}$ of \mathbb{E}^{d+1} . The polytope \tilde{P} is the intersection of the Cayley polytope of P_1 and P_2 with the hyperplane $\tilde{\Pi} = \{x_1 = \frac{1}{2}\}$

- (i) the vertices of P'_1 and P'_2 remain in Π_1 and Π_2 , respectively, and both P'_1 and P'_2 are simplicial;
- (ii) the Cayley polytope P' of P'_1 and P'_2 is also simplicial, except possibly the facets P'_1 and P'_2 ; and
- (iii) the number of vertices of P'_1 and P'_2 is the same as the number of vertices of P_1 and P_2 , respectively, whereas $f_k(P) \leq f_k(P')$ for all $k \geq 1$,

where P'_1 and P'_2 are the polytopes in Π_1 and Π_2 we get after perturbing the vertices of P_1 and P_2 , respectively. In view of this result, it suffices to consider the case where both P_1, P_2 , as well as their Cayley polytope P , are simplicial complexes, except possibly the facets P_1 and P_2 of P .

In the rest of this section, we consider that this is the case: P is considered simplicial, with the possible exception of its two facets P_1 and P_2 . Let \mathcal{F} be the set of proper faces of P having non-empty intersection with $\tilde{\Pi}$. Note that $\tilde{P} = P \cap \tilde{\Pi}$ is a d -polytope, which is, in general, non-simplicial, and whose proper non-trivial faces are intersections of the form $F \cap \tilde{\Pi}$ where $F \in \mathcal{F}$. As we have already observed above, \tilde{P} is combinatorially equivalent to the Minkowski sum $P_1 + P_2$. Furthermore,

$$f_{k-1}(P_1 + P_2) = f_{k-1}(\tilde{P}) = f_k(\mathcal{F}), \quad 1 \leq k \leq d. \tag{16}$$

The remainder of this section is devoted to deriving upper bounds for $f_k(\mathcal{F})$, which, by relation (16), become upper bounds for $f_{k-1}(P_1 + P_2)$.

Let \mathcal{K} be the polytopal complex whose faces are all the faces of \mathcal{F} , as well as the faces of P that are subfaces of faces in \mathcal{F} . Clearly, the d -faces of \mathcal{K} are exactly the d -faces of \mathcal{F} , and thus, \mathcal{K} is a pure simplicial d -complex, with the d -faces of \mathcal{F} being the facets of \mathcal{K} . Moreover, the set of k -faces of \mathcal{K} is the disjoint union of the sets of k -faces of $\mathcal{F}, \partial P_1$ and ∂P_2 . This implies

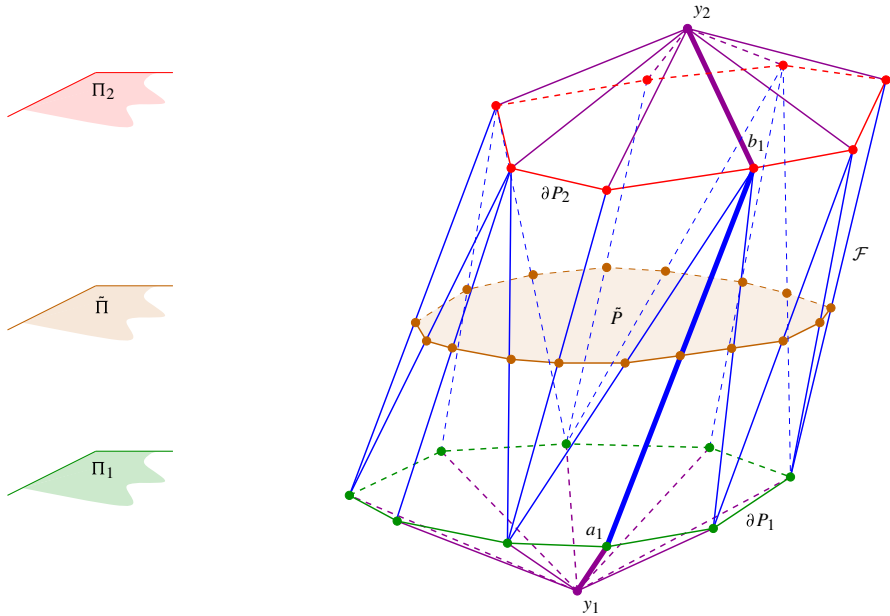


Fig. 2 The polytope Q is created by adding two vertices y_1 and y_2 . The vertex y_1 (resp., y_2) is below P_1 (resp., above P_2) and is visible by the vertices of P_1 (resp., P_2) only

$$f_k(\mathcal{K}) = f_k(\mathcal{F}) + f_k(\partial P_1) + f_k(\partial P_2), \quad -1 \leq k \leq d, \quad (17)$$

where $f_d(\partial P_j) = 0, j = 1, 2$. Since $\mathcal{K}, \partial P_1$, and ∂P_2 are complexes, we have, by definition, that $f_{-1}(\mathcal{K}) = f_{-1}(\partial P_1) = f_{-1}(\partial P_2) = 1$. In order for relation (17) to be valid for $k = -1$, we conventionally set $f_{-1}(\mathcal{F}) = -1$. We next express (17) in terms of generating functions:

$$\begin{aligned} \mathfrak{f}(\mathcal{K}; t) &= \sum_{i=0}^{d+1} f_{i-1}(\mathcal{K})t^{d+1-i} \\ &= \sum_{i=0}^{d+1} f_{i-1}(\mathcal{F})t^{d+1-i} + \sum_{i=0}^{d+1} f_{i-1}(\partial P_1)t^{d+1-i} + \sum_{i=0}^{d+1} f_{i-1}(\partial P_2)t^{d+1-i} \\ &= \mathfrak{f}(\mathcal{F}; t) + t \sum_{i=0}^d f_{i-1}(\partial P_1)t^{d-i} + t \sum_{i=0}^d f_{i-1}(\partial P_2)t^{d-i} \\ &= \mathfrak{f}(\mathcal{F}; t) + t \mathfrak{f}(\partial P_1; t) + t \mathfrak{f}(\partial P_2; t), \end{aligned} \quad (18)$$

where we used that $\dim(\mathcal{K}) = \dim(\mathcal{F}) = d, \dim(\partial P_i) = d - 1$, and $f_d(\partial P_1) = f_d(\partial P_2) = 0$.

For $i = 1, 2$, let y_i be a point beyond the facet P_i of P , and beneath every other facet of P (see Fig. 2). We call Q the $(d + 1)$ -polytope we get by taking the *stellar subdivisions* of P with y_1 and y_2 , respectively (cf. [28, Problem 3.0]). Notice that since

the facets P_1 and P_2 are disjoint, the order in which we perform the stellar subdivisions does not matter. The boundary complex ∂Q is then the disjoint union of the set of

- (i) faces in the star \mathcal{S}_1 of y_1 , faces in the star \mathcal{S}_2 of y_2 , and
- (ii) faces in \mathcal{F} .

This implies that

$$f_k(\partial Q) = f_k(\mathcal{F}) + f_k(\mathcal{S}_1) + f_k(\mathcal{S}_2), \quad 0 \leq k \leq d, \tag{19}$$

where $f_0(\mathcal{F}) = 0$. Since $\mathcal{S}_i = \text{star}(y_i, \partial P_i)$, we can further write

$$f_k(\mathcal{S}_j) = f_k(\partial P_j) + f_{k-1}(\partial P_j), \quad 0 \leq k \leq d, \quad i = 1, 2, \tag{20}$$

where $f_{-1}(\partial P_j) = 1$ and $f_d(\partial P_j) = 0$. Combining relations (19) and (20), we get

$$f_k(\partial Q) = f_k(\mathcal{F}) + f_k(\partial P_1) + f_{k-1}(\partial P_1) + f_k(\partial P_2) + f_{k-1}(\partial P_2), \quad 0 \leq k \leq d. \tag{21}$$

As for (17), we now express (21) in terms of generating functions (recall that $\dim(\mathcal{F}) = \dim(\partial Q) = d$ and $\dim(\partial P_i) = d - 1$):

$$\mathfrak{f}(\partial Q; t) = \mathfrak{f}(\mathcal{F}; t) + (t + 1)\mathfrak{f}(\partial P_1; t) + (t + 1)\mathfrak{f}(\partial P_2; t). \tag{22}$$

We call \mathcal{K}_j , $j = 1, 2$, the subcomplex of ∂Q consisting of faces of \mathcal{K} or faces of \mathcal{S}_j . \mathcal{K}_j is a pure simplicial d -complex, the facets of which are either facets of \mathcal{S}_j or \mathcal{K} . Furthermore, \mathcal{K}_j is shellable. To see this, first notice that ∂Q is shellable (Q is a polytope). Consider a line shelling (cf. [28, Sect. 8.2]) F_1, F_2, \dots, F_s of ∂Q that shells $\text{star}(y_2, \partial Q)$ last, and let $F_{\lambda+1}, F_{\lambda+2}, \dots, F_s$ be the facets of ∂Q that correspond to \mathcal{S}_2 . Trivially, the subcomplex of ∂Q , the facets of which are $F_1, F_2, \dots, F_\lambda$, is shellable; however, this subcomplex is nothing but \mathcal{K}_1 . The argument for \mathcal{K}_2 is analogous.

The k -faces of \mathcal{K}_j , $j = 1, 2$, are either k -faces of \mathcal{K} or k -faces of the star \mathcal{S}_j of y_j that contain y_j . The latter faces are in one-to-one correspondence with the $(k - 1)$ -faces of ∂P_j , i.e., we get

$$f_k(\mathcal{K}_j) = f_k(\mathcal{K}) + f_{k-1}(\partial P_j). \tag{23}$$

Expressing (23) in terms of generating functions, we get

$$\mathfrak{f}(\mathcal{K}_j; t) = \mathfrak{f}(\mathcal{K}; t) + \mathfrak{f}(\partial P_j; t). \tag{24}$$

Notice that Q is a simplicial $(d + 1)$ -polytope, while \mathcal{K} , \mathcal{K}_1 , and \mathcal{K}_2 are simplicial d -complexes; hence their h -vectors are well defined. We define the f -vector of \mathcal{F} to be the $(d + 2)$ -vector $\mathbf{f}(\mathcal{F}) = (f_{-1}(\mathcal{F}), f_0(\mathcal{F}), \dots, f_d(\mathcal{F}))$, recalling that $f_{-1}(\mathcal{F}) = -1$. From $\mathbf{f}(\mathcal{F})$ we can also define the $(d + 2)$ -vector $\mathbf{h}(\mathcal{F}) = (h_0(\mathcal{F}), h_1(\mathcal{F}), \dots, h_{d+1}(\mathcal{F}))$, where

$$h_k(\mathcal{F}) = \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}), \quad 0 \leq k \leq d + 1. \tag{25}$$

We naturally call this vector the h -vector of \mathcal{F} (see also relation (4)). As for polytopal complexes and polytopes, the f -vector of \mathcal{F} defines the h -vector of \mathcal{F} and vice versa (cf. (4) and (5)).

The next lemma associates the elements of $\mathbf{h}(\partial Q)$, $\mathbf{h}(\mathcal{K})$, $\mathbf{h}(\mathcal{K}_1)$, $\mathbf{h}(\mathcal{K}_2)$, $\mathbf{h}(\mathcal{F})$, $\mathbf{h}(\partial P_1)$, and $\mathbf{h}(\partial P_2)$ via generating functions. The last among the relations in the lemma can be thought of as the analogue of the Dehn–Sommerville equations for \mathcal{F} .

Lemma 9

$$\mathbf{h}(Q; t) = \mathbf{h}(\mathcal{F}; t) + t\mathbf{h}(P_1; t) + t\mathbf{h}(P_2; t), \tag{26}$$

$$\mathbf{h}(\mathcal{K}; t) = \mathbf{h}(\mathcal{F}; t) + \mathfrak{g}(P_1; t) + \mathfrak{g}(P_2; t), \tag{27}$$

$$\mathbf{h}(\mathcal{K}_j; t) = \mathbf{h}(\mathcal{K}; t) + \mathbf{h}(P_j; t), \quad j = 1, 2, \tag{28}$$

$$t^{d+1}\mathbf{h}(\mathcal{F}; \frac{1}{t}) = \mathbf{h}(\mathcal{F}; t) + \mathfrak{g}(\partial P_1; t) + \mathfrak{g}(\partial P_2; t). \tag{29}$$

Proof Using (22) we have

$$\begin{aligned} \mathbf{h}(\partial Q; t) &= \mathfrak{f}(\partial Q; t - 1) \\ &= \mathfrak{f}(\mathcal{F}; t - 1) + t\mathfrak{f}(\partial P_1; t - 1) + t\mathfrak{f}(\partial P_2; t - 1) \\ &= \mathbf{h}(\mathcal{F}; t) + t\mathbf{h}(\partial P_1; t) + t\mathbf{h}(\partial P_2; t). \end{aligned}$$

Using (12) and (18), we arrive at the following:

$$\begin{aligned} \mathbf{h}(\mathcal{K}; t) &= \mathfrak{f}(\mathcal{K}; t - 1) \\ &= \mathfrak{f}(\mathcal{F}; t - 1) + (t - 1)\mathfrak{f}(\partial P_1; t - 1) + (t - 1)\mathfrak{f}(\partial P_2; t - 1) \\ &= \mathbf{h}(\mathcal{F}; t) + (t - 1)\mathbf{h}(\partial P_1; t) + (t - 1)\mathbf{h}(\partial P_2; t) \\ &= \mathbf{h}(\mathcal{F}; t) + \mathfrak{g}(\partial P_1; t) + \mathfrak{g}(\partial P_2; t). \end{aligned}$$

Similarly, from (24) we get

$$\mathbf{h}(\mathcal{K}_j; t) = \mathfrak{f}(\mathcal{K}_j; t - 1) = \mathfrak{f}(\mathcal{K}; t - 1) + \mathfrak{f}(\partial P_j; t - 1) = \mathbf{h}(\mathcal{K}; t) + \mathbf{h}(\partial P_j; t).$$

To prove (29) recall the Dehn–Sommerville equations (13) for the $(d + 1)$ -polytope ∂Q and the d -polytopes $\partial P_i, i = 1, 2$:

$$t^{d+1}\mathbf{h}(\partial Q; \frac{1}{t}) = \mathbf{h}(\partial Q; t) \tag{30}$$

and

$$t^d\mathbf{h}(\partial P_i; \frac{1}{t}) = \mathbf{h}(\partial P_i; t).$$

Substituting $\frac{1}{t}$ for t in (26) and multiplying both sides with t^{d+1} , we get

$$\begin{aligned} t^{d+1}h(\partial Q; \frac{1}{t}) &= t^{d+1}h(\mathcal{F}; \frac{1}{t}) + \frac{t^{d+1}}{t}h(\partial P_1; \frac{1}{t}) + \frac{t^{d+1}}{t}h(\partial P_2; \frac{1}{t}) \\ &= t^{d+1}h(\mathcal{F}; \frac{1}{t}) + t^d h(\partial P_1; \frac{1}{t}) + t^d h(\partial P_2; \frac{1}{t}) \\ &= t^{d+1}h(\mathcal{F}; \frac{1}{t}) + h(\partial P_1; t) + h(\partial P_2; t). \end{aligned} \tag{31}$$

Using (26) and (30), relation (31) becomes

$$t^{d+1}h(\mathcal{F}; \frac{1}{t}) + h(\partial P_1; t) + h(\partial P_2; t) = h(\mathcal{F}; t) + t h(\partial P_1; t) + t h(\partial P_2; t),$$

or equivalently

$$\begin{aligned} t^{d+1}h(\mathcal{F}; \frac{1}{t}) &= h(\mathcal{F}; t) + (t - 1)h(\partial P_1; t) + (t - 1)h(\partial P_2; t) \\ &= h(\mathcal{F}; t) + g(\partial P_1; t) + g(\partial P_2; t), \end{aligned}$$

which completes our proof. □

Definition 10 Let $p(t) = \sum_{i=0}^d p_i t^i$ and $q(t) = \sum_{i=0}^d q_i t^i$ be two polynomial functions of degree at most d . We will write $p(t) \preceq q(t)$ if and only if $p(t)$ is coefficient-wise lesser than or equal to $q(t)$, i.e., $p_i \leq q_i, 0 \leq i \leq d$.

Lemma 11 For $j = 1, 2$ and all $v \in V_j$, we have

$$t h(\mathcal{K}/v; t) - t g(\partial P_j/v; t) \preceq h(\mathcal{K}; t) - g(\partial P_j; t). \tag{32}$$

Proof We are going to prove our claim for $j = 1$; the case $j = 2$ is entirely analogous. To prove relation (32), we rewrite it in terms of the elements of the h - and g -vectors involved. More precisely, it suffices to show that, for all $0 \leq k \leq d + 1$,

$$h_k(\mathcal{K}/v) - g_k(\partial P_1/v) \leq h_k(\mathcal{K}) - g_k(\partial P_1). \tag{33}$$

First, notice that for $k = 0$ the statement of lemma is trivial since $\mathcal{K}/v, \partial P_1/v, \mathcal{K}$ and ∂P_1 are simplicial complexes and, thus, $h_0(\mathcal{K}/v) = g_0(\partial P_1/v) = h_0(\mathcal{K}) = g_0(\partial P_1) = 1$. Thus, in what follows we will assume that $k > 0$.

Fix a vertex $v \in V_1$. Call \mathcal{X}_1 the set of faces of \mathcal{K} that are not faces of ∂P_1 , i.e., $\mathcal{X}_1 = \mathcal{K} \setminus \partial P_1$. It is easy to verify that \mathcal{X}_1 is also the set of faces of \mathcal{K}_1 that are not faces of \mathcal{S}_1 , i.e., $\mathcal{X}_1 = \mathcal{K}_1 \setminus \mathcal{S}_1$. Similarly, the faces in \mathcal{X}_1/v is the set of faces \mathcal{K}/v which are not faces in $\partial P_1/v$, or, equivalently, the set of faces of \mathcal{K}_1/v that are not faces of \mathcal{S}_1/v , i.e., $\mathcal{X}_1/v = (\mathcal{K}_1/v) \setminus (\mathcal{S}_1/v)$ (see also the left two subfigures in Fig. 3). Hence we have

$$f_k(\mathcal{X}_1) = f_k(\mathcal{K}) - f_k(\partial P_1) = f_k(\mathcal{K}_1) - f_k(\mathcal{S}_1), \quad -1 \leq k \leq d - 1,$$

and

$$f_k(\mathcal{X}_1/v) = f_k(\mathcal{K}/v) - f_k(\partial P_1/v) = f_k(\mathcal{K}_1/v) - f_k(\mathcal{S}_1/v), \quad -1 \leq k \leq d - 2.$$

Rewriting these equations in terms of generating functions, we obtain

$$f(\mathcal{X}_1; t) = f(\mathcal{K}; t) - t f(\partial P_1; t) = f(\mathcal{K}_1; t) - f(\mathcal{S}_1; t)$$

and

$$f(\mathcal{X}_1/v; t) = f(\mathcal{K}/v; t) - t f(\partial P_1/v; t) = f(\mathcal{K}_1/v; t) - f(\mathcal{S}_1/v; t).$$

Using (10), we further deduce that

$$\begin{aligned} h(\mathcal{X}_1; t) &= h(\mathcal{K}; t) - (t - 1)h(\partial P_1; t) = h(\mathcal{K}; t) - g(\partial P_1; t) \\ &= h(\mathcal{K}_1; t) - h(\mathcal{S}_1; t) \end{aligned} \tag{34}$$

and

$$\begin{aligned} h(\mathcal{X}_1/v; t) &= h(\mathcal{K}/v; t) - (t - 1)h(\partial P_1/v; t) = h(\mathcal{K}/v; t) - g(\partial P_1/v; t) \\ &= h(\mathcal{K}_1/v; t) - h(\mathcal{S}_1/v; t). \end{aligned}$$

Consider now a shelling $\mathbb{S}(\partial Q)$ of ∂Q that shells the star of y_1 first and the star of y_2 last. Such a shelling does exist since y_1 and y_2 are not visible from each other: y_j is beyond the facet P_j of \mathcal{C} , $j = 1, 2$, while P_1 and P_2 are parallel to each other (the purple numbers in the top-left subfigure of Fig. 3 indicate such a shelling). The shelling $\mathbb{S}(\partial Q)$ gives a shelling $\mathbb{S}(\mathcal{K}_1)$ for \mathcal{K}_1 (we just have to discard the facets at the end of $\mathbb{S}(\partial Q)$ that are facets in \mathcal{S}_2), which shells the star of y_1 in ∂Q first. In turn, $\mathbb{S}(\mathcal{K}_1)$ induces a shelling $\mathbb{S}(\mathcal{K}_1/v)$ for \mathcal{K}_1/v , which shells the star of y_1 in \mathcal{K}_1/v first (this induced shelling is shown for the star of v in ∂Q in the top-right subfigure of Fig. 3).³

Let us now consider the dual graph $G^\Delta(\partial Q)$ of ∂Q , oriented⁴ according to the shelling $\mathbb{S}(\partial Q)$, as well as the dual graph $G^\Delta(\partial Q/v)$ of $\partial Q/v$, also oriented according to the shelling $\mathbb{S}(\partial Q/v)$ (these graphs are shown in red in the top two subfigures of Fig. 3). We will denote by $V^\Delta(\mathcal{Y})$ the vertices of $G^\Delta(\partial Q)$ that are the duals of the facets in ∂Q that belong to \mathcal{Y} , where \mathcal{Y} stands for a subset of the set of proper faces of ∂Q ; for simplicity we will abbreviate $V^\Delta(\partial Q)$ to V^Δ .

³ For these particular shellings, $h_k(\mathcal{X}_1)$ and $h_k(\mathcal{X}_1/v)$ have a geometric interpretation: $h_k(\mathcal{X}_1)$ (resp., $h_k(\mathcal{X}_1/v)$) counts the number of restrictions of size k of the facets of \mathcal{K}_1 (resp., \mathcal{K}_1/v) in \mathcal{X}_1 (resp., \mathcal{X}_1/v) in the shelling $\mathbb{S}(\mathcal{K}_1)$ (resp., $\mathbb{S}(\mathcal{K}_1/v)$) of \mathcal{K}_1 (resp., \mathcal{K}_1/v).

To see this, rewrite relation (34) in its element-wise form:

$$h_k(\mathcal{K}_1) = h_k(\mathcal{X}_1) + h_k(\mathcal{S}_1),$$

and recall that $\mathbb{S}(\mathcal{K}_1)$ shells \mathcal{S}_1 first. Since the shellings of \mathcal{K}_1 and \mathcal{S}_1 coincide as long as we shell \mathcal{S}_1 , we get a contribution of one to both $h_k(\mathcal{K}_1)$ and $h_k(\mathcal{S}_1)$ for every restriction of $\mathbb{S}(\mathcal{K}_1)$ of size k . After the shelling $\mathbb{S}(\mathcal{K}_1)$ has left \mathcal{S}_1 , a restriction of size k of the shelling contributes one to $h_k(\mathcal{K}_1)$ and, thus, necessarily, to $h_k(\mathcal{X}_1)$. But these restrictions are precisely the restrictions corresponding to the facets of \mathcal{X}_1 in the shelling $\mathbb{S}(\mathcal{K}_1)$. The argumentation for $h_k(\mathcal{X}_1/v)$ is entirely similar.

⁴ Given two facets F_i and F_j in the shelling $\mathbb{S}(\partial Q)$ (resp., $\mathbb{S}(\partial Q/v)$) of ∂Q (resp., $\partial Q/v$) that share a ridge, the edge connecting F_i and F_j in the dual graph $G^\Delta(\partial Q)$ (resp., $G^\Delta(\partial Q/v)$) is oriented from F_i to F_j if and only if $i < j$.

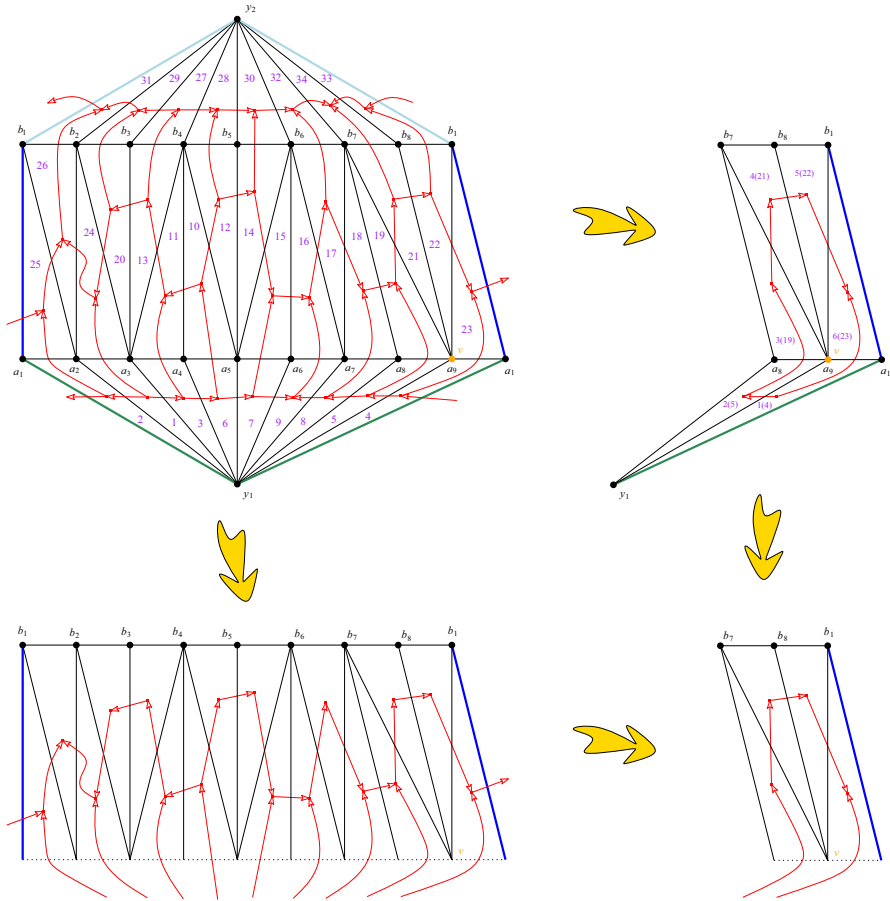


Fig. 3 *Top left* The complex ∂Q from Fig. 2, cut along the edges $y_1a_1-a_1b_1-b_1y_2$ and embedded in the plane. The colored edges are identified. The vertex v is shown in orange. The purple numbers refer to a shelling of ∂Q that shells $\text{star}(y_1, \partial Q)$ first and $\text{star}(y_2, \partial Q)$ last. The directed graph in red is the dual graph $G^\Delta(\partial Q)$ of ∂Q whose edge orientations correspond to the shelling of ∂Q indicated in the figure. *Top right* The complex $\text{star}(v, \partial Q)$. The purple numbers refer to the shelling order of $\text{star}(v, \partial Q)$ induced by the shelling in the figure to the left (with the numbers in parenthesis being the order of the corresponding facets in the shelling of ∂Q). The directed graph in red is the dual graph $G^\Delta(\text{star}(v, \partial Q))$ of $\text{star}(v, \partial Q)$ that corresponds to the shelling of $\text{star}(v, \partial Q)$ shown in the figure; observe that $G^\Delta(\partial Q/v)$ is the subgraph of $G^\Delta(\partial Q)$ corresponding to the vertices of $G^\Delta(\text{star}(v, \partial Q))$ that are duals of facets in $\text{star}(v, \partial Q)$. *Bottom left* The set of faces \mathcal{X}_1 of ∂Q , along with the portion of $G^\Delta(\partial Q)$ that corresponds to the vertices that are duals of facets of \mathcal{X}_1 . *Bottom right* The set of faces $\text{star}(v, \mathcal{X}_1)$ of $\text{star}(v, \partial Q)$, along with the portion of $G^\Delta(\text{star}(v, \partial Q))$ that corresponds to the vertices that are duals of facets of $\text{star}(v, \mathcal{X}_1)$. For the particular example shown: $\mathbf{h}(\partial Q) = (1, 16, 16, 1)$, $\mathbf{h}(\partial Q/v) = (1, 4, 1)$, $\mathbf{h}(\mathcal{X}_1) = (0, 8, 9, 0)$, and $\mathbf{h}(\mathcal{X}_1/v) = (0, 3, 1)$

Since $\mathbb{S}(\partial Q/v)$ is induced from $\mathbb{S}(\partial Q)$, $G^\Delta(\partial Q/v)$ is isomorphic to the subgraph of $G^\Delta(\partial Q)$ defined over $V^\Delta(\text{star}(v, \partial Q))$. Moreover, $h_k(\partial Q)$ counts the number of vertices of V^Δ of in-degree equal to k [12], while $h_k(\mathcal{S}_1)$ counts the number of vertices of $V^\Delta(\mathcal{S}_1)$ of in-degree k in $G^\Delta(\partial Q)$ (for the particular shelling $\mathbb{S}(\partial Q)$ of ∂Q that

we have chosen). Consequently, $h_k(\mathcal{X}_1)$ counts the number of vertices of $V^\Delta(\mathcal{X}_1)$ of in-degree k in $G^\Delta(\partial Q)$; in an analogous manner, we can conclude that $h_k(\mathcal{X}_1/v)$ counts the number of vertices of $V^\Delta(\text{star}(v, \mathcal{X}_1))$ with in-degree k in $G^\Delta(\partial Q/v)$ (refer to the bottom two subfigures of Fig. 3 to verify these facts for the particular example shown in the figure). Since, however, $G^\Delta(\partial Q/v)$ is a subgraph of $G^\Delta(\partial Q)$, the number of vertices of $V^\Delta(\text{star}(v, \mathcal{X}_1))$ with in-degree k cannot exceed the number of vertices of $V^\Delta(\mathcal{X}_1)$ with in-degree k . Hence, $h_k(\mathcal{X}_1/v) \leq h_k(\mathcal{X}_1)$, which is precisely relation (33). □

Using Lemma 11, we can now derive the following generating function inequality, which is essential in our upper bound proof.

Lemma 12 *The following inequality holds:*

$$t(1-t)h'(\mathcal{F}; t) \preceq [n_1 + n_2 - t(d+1)]h(\mathcal{F}; t) + n_1g(\partial P_2; t) + n_2g(\partial P_1; t). \tag{35}$$

Proof Let us denote by V the vertex set of ∂Q and by V_j the vertex set of ∂P_j , $j = 1, 2$.

Applying Lemma 3 to Q , P_1 , and P_2 , we get the following relations:

$$(d+1)h(\partial Q; t) + (1-t)h'(\partial Q; t) = \sum_{v \in V} h(\partial Q/v; t), \tag{36}$$

$$dh(\partial P_j; t) + (1-t)h'(\partial P_j; t) = \sum_{v \in V_j} h(\partial P_j/v; t), \quad j = 1, 2. \tag{37}$$

Recall that the link of y_j in ∂Q is ∂P_j , $j = 1, 2$, and observe that the link of $v \in V_j$ in ∂Q coincides with \mathcal{K}_j/v . Expanding relation (36) by means of relation (26), we deduce

$$\begin{aligned} & (d+1)\left[h(\mathcal{F}; t) + t \sum_{j=1}^2 h(\partial P_j; t)\right] \\ & + (1-t)\left[h'(\mathcal{F}; t) + \sum_{j=1}^2 [h(\partial P_j; t) + th'(\partial P_j; t)]\right] \\ & = (d+1)h(\mathcal{F}; t) + (1-t)h'(\mathcal{F}; t) + t[dh(\partial P_1; t) + (1-t)h'(\partial P_1; t)] \\ & + t[dh(\partial P_2; t) + (1-t)h'(\partial P_2; t)] + h(\partial P_1; t) + h(\partial P_2; t) \\ & = \sum_{v \in V} h(\partial Q/v; t) = h(\partial Q/y_1; t) + h(\partial Q/y_2; t) + \sum_{v \in V_1 \cup V_2} h(\partial Q/v; t) \\ & = h(\partial P_1; t) + h(\partial P_2; t) + \sum_{v \in V_1} h(\mathcal{K}_1/v; t) + \sum_{v \in V_2} h(\mathcal{K}_2/v; t). \end{aligned}$$

Utilizing relations (37), the above equation is equivalent to

$$(d + 1)h(\mathcal{F}; t) + (1 - t)h'(\mathcal{F}; t) = \sum_{j=1}^2 \sum_{v \in V_j} [h(\mathcal{K}_j/v; t) - t h(\partial P_j/v; t)]. \tag{38}$$

Using an analogous argumentation to that used for deriving relation (24), we deduce that, for any $v \in V_j, j = 1, 2$, we have

$$f(\mathcal{K}_j/v; t) = f(\mathcal{K}/v; t) + f(\partial P_j/v; t).$$

Using relation (10), we readily get

$$h(\mathcal{K}_j/v; t) = h(\mathcal{K}/v; t) + h(\partial P_j/v; t).$$

Substituting in (38), we finally get

$$\begin{aligned} &(d + 1)h(\mathcal{F}; t) + (1 - t)h'(\mathcal{F}; t) \\ &= \sum_{j=1}^2 \sum_{v \in V_j} [h(\mathcal{K}/v; t) + h(\partial P_j/v; t) - t h(\partial P_j/v; t)] \\ &= \sum_{j=1}^2 \sum_{v \in V_j} [h(\mathcal{K}/v; t) - g(\partial P_j/v; t)]. \end{aligned} \tag{39}$$

Thus, by applying Lemma 11 and using relation (28), we get for every vertex $v \in V_1$

$$\begin{aligned} t \sum_{v \in V_1} [h(\mathcal{K}/v; t) - g(\partial P_1/v; t)] &\preccurlyeq \sum_{v \in V_1} [h(\mathcal{K}; t) - g(\partial P_1; t)] \\ &= n_1 [h(\mathcal{F}; t) + g(\partial P_2; t)], \end{aligned}$$

Similarly, by applying Lemma 11 and using relation (28), we get for every vertex $v \in V_2$

$$\begin{aligned} t \sum_{v \in V_2} [h(\mathcal{K}/v; t) - g(\partial P_2/v; t)] &\preccurlyeq \sum_{v \in V_2} [h(\mathcal{K}; t) - g(\partial P_2; t)] \\ &= n_2 [h(\mathcal{F}; t) + g(\partial P_1; t)]. \end{aligned}$$

We thus arrive at the following inequality, for $0 \leq k \leq d$,

$$\begin{aligned} &t(d + 1)h(\mathcal{F}; t) + t(1 - t)h'(\mathcal{F}; t) \\ &\preccurlyeq (n_1 + n_2) h(\mathcal{F}; t) + n_1 g(\partial P_2; t) + n_2 g(\partial P_1; t), \end{aligned} \tag{40}$$

which gives the inequality in the statement of the lemma. □

Corollary 13 For all $0 \leq k \leq d$,

$$h_{k+1}(\mathcal{F}) \leq \frac{n_1 + n_2 - d - 1 + k}{k + 1} h_k(\mathcal{F}) + \frac{n_1}{k + 1} g_k(\partial P_2) + \frac{n_2}{k + 1} g_k(\partial P_1). \tag{41}$$

Proof Expanding the generating functions in relation (40), we have

$$\begin{aligned} & t \sum_{k=0}^{d+1} (d + 1)h_k(\mathcal{F}) t^{d+1-k} + t(1 - t) \sum_{k=0}^{d+1} (d + 1 - k)h_k(\mathcal{F}) t^{d-k} \\ & \preceq (n_1 + n_2) \sum_{k=0}^{d+1} h_k(\mathcal{F}) t^{d+1-k} + n_1 \sum_{k=0}^{d+1} g_k(\partial P_2) t^{d+1-k} + n_2 \sum_{k=0}^{d+1} g_k(\partial P_1) t^{d+1-k}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \sum_{k=0}^{d+1} (d + 1 - k)h_k(\mathcal{F}) t^{d+1-k} + \sum_{k=0}^{d+1} kh_k(\mathcal{F}) t^{d+2-k} \\ & \preceq (n_1 + n_2) \sum_{k=0}^{d+1} h_k(\mathcal{F}) t^{d+1-k} + n_1 \sum_{k=0}^{d+1} g_k(\partial P_2) t^{d+1-k} + n_2 \sum_{k=0}^{d+1} g_k(\partial P_1) t^{d+1-k}. \end{aligned}$$

Setting $\lambda = k - 1$ in the second sum of the left-hand side, we get

$$\begin{aligned} \sum_{\lambda=-1}^d (\lambda + 1)h_{\lambda+1}(\mathcal{F}) t^{d+1-\lambda} & \preceq (n_1 + n_2 - d - 1 + k) \sum_{k=0}^{d+1} h_k(\mathcal{F}) t^{d+1-k} \\ & + n_1 \sum_{k=0}^{d+1} g_k(\partial P_2) t^{d+1-k} + n_2 \sum_{k=0}^{d+1} g_k(\partial P_1) t^{d+1-k}. \end{aligned}$$

Equating terms of equal power of t , we get the following relation for all $0 \leq k \leq d$:

$$(k + 1)h_{k+1}(\mathcal{F}) \leq (n_1 + n_2 - d - 1 + k)h_k(\mathcal{F}) + n_1 g_k(\partial P_2) + n_2 g_k(\partial P_1),$$

which gives relation (41). □

Using the recurrence relation from Corollary 13, we get the following bounds on the elements of $\mathbf{h}(\mathcal{F})$.

Lemma 14 For all $0 \leq k \leq d + 1$,

$$h_k(\mathcal{F}) \leq \binom{n_1 + n_2 - d - 2 + k}{k} - \binom{n_1 - d - 2 + k}{k} - \binom{n_2 - d - 2 + k}{k}. \tag{42}$$

Equality holds for all k with $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d+1}{2} \rfloor$ and P is (l, V_1) -bineighborly.

Proof We show the desired bound by induction on k . Clearly, the bound holds (as equality) for $k = 0$, since

$$\begin{aligned}
 h_0(\mathcal{F}) &= -1 = 1 - 1 - 1 \\
 &= \binom{n_1 + n_2 - d - 2 + 0}{0} - \binom{n_1 - d - 2 + 0}{0} - \binom{n_2 - d - 2 + 0}{0}.
 \end{aligned}
 \tag{43}$$

Suppose now that the bound holds for $h_k(\mathcal{F})$, where $k \geq 0$. Using the recurrence relation (41), in conjunction with the upper bounds for the elements of the g -vector of a polytope from Corollary 2, and since for $k \geq 0$, $n_1 + n_2 - d - 1 + k \geq d + 1 > 0$, we have

$$\begin{aligned}
 &h_{k+1}(\mathcal{F}) \\
 &\leq \frac{n_1 + n_2 - d - 1 + k}{k + 1} h_k(\mathcal{F}) + \frac{n_1}{k + 1} g_k(\partial P_2) + \frac{n_2}{k + 1} g_k(\partial P_1) \\
 &\leq \frac{n_1 + n_2 - d - 1 + k}{k + 1} \\
 &\quad \times \left[\binom{n_1 + n_2 - d - 2 + k}{k} - \binom{n_1 - d - 2 + k}{k} - \binom{n_2 - d - 2 + k}{k} \right] \\
 &\quad + \frac{n_1}{k + 1} \binom{n_2 - d - 2 + k}{k} + \frac{n_2}{k + 1} \binom{n_1 - d - 2 + k}{k} \\
 &= \frac{n_1 + n_2 - d - 1 + k}{k + 1} \binom{n_1 + n_2 - d - 2 + k}{k} \\
 &\quad - \frac{n_1 - d - 1 + k}{k + 1} \binom{n_1 - d - 2 + k}{k} - \frac{n_2 - d - 1 + k}{k + 1} \binom{n_2 - d - 2 + k}{k} \\
 &= \binom{n_1 + n_2 - d - 1 + k}{k + 1} - \binom{n_1 - d - 1 + k}{k + 1} - \binom{n_2 - d - 1 + k}{k + 1}.
 \end{aligned}
 \tag{44}$$

Let us now turn to our equality claim. The claim for $l = 0$ is obvious (cf. (43)), so we assume below that $l \geq 1$.

Suppose first that P is (l, V_1) -bineighborly. Then, we have

$$f_{i-1}(\mathcal{F}) = \binom{n_1 + n_2}{i} - \binom{n_1}{i} - \binom{n_2}{i}, \quad 0 \leq i \leq l.
 \tag{45}$$

Substituting $f_{i-1}(\mathcal{F})$ from (45) in the defining (25) for $h(\mathcal{F})$, we get, for all $0 \leq k \leq l$

$$\begin{aligned}
 h_k(\mathcal{F}) &= \sum_{i=0}^k (-1)^{k-i} \binom{d + 1 - i}{d + 1 - k} f_{i-1}(\mathcal{F}) \\
 &= \sum_{i=0}^k (-1)^{k-i} \binom{d + 1 - i}{d + 1 - k} \left(\binom{n_1 + n_2}{i} - \binom{n_1}{i} - \binom{n_2}{i} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{d+1-k} \binom{n_1+n_2}{i} - \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{d+1-k} \binom{n_1}{i} \\
 &\quad - \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{d+1-k} \binom{n_2}{i} \\
 &= \binom{n_1+n_2-d-2+k}{k} - \binom{n_2-d-2+k}{k} - \binom{n_2-d-2+k}{k},
 \end{aligned}$$

where for the last equality we used the fact that $\binom{d+1-i}{d+1-k} = 0$ for $i > k$, in conjunction with the following combinatorial identity (cf. [9, Eq. (5.25)], [28, Exer. 8.20]):

$$\sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{s}{k-n} (-1)^k = (-1)^{l+m} \binom{s-m-1}{l-m-n}.$$

In the equation above we set $k \leftarrow i, l \leftarrow d+1, m \leftarrow d+1-k, n \leftarrow 0$, while s stands for either n_1+n_2, n_1 or n_2 . We thus conclude that (42) holds as equality for all $0 \leq k \leq l$.

Suppose now that inequality (42) holds as equality for all $0 \leq k \leq l$. Solving (25) in terms of $f(\mathcal{F})$ (cf. also (5)) and substituting $h_i(\mathcal{F}), 0 \leq i \leq l$, from (42), we get

$$\begin{aligned}
 &f_{l-1}(\mathcal{F}) \\
 &= \sum_{i=0}^{d+1} \binom{d+1-i}{l-i} h_i(\mathcal{F}) \\
 &= \sum_{i=0}^{d+1} \binom{d+1-i}{l-i} \left(\binom{n_1+n_2-d-2+i}{i} - \binom{n_1-d-2+i}{i} - \binom{n_2-d-2+i}{i} \right) \\
 &= \sum_{i=0}^{d+1} \binom{d+1-i}{l-i} \binom{n_1+n_2-d-2+i}{i} - \sum_{i=0}^{d+1} \binom{d+1-i}{l-i} \binom{n_1-d-2+i}{i} \\
 &\quad - \sum_{i=0}^{d+1} \binom{d+1-i}{l-i} \binom{n_2-d-2+i}{i} \\
 &= \sum_{i=0}^{d+1} \binom{d+1-i}{d+1-l} \binom{n_1+n_2-d-2+i}{n_1+n_2-d-2} - \sum_{i=0}^{d+1} \binom{d+1-i}{d+1-l} \binom{n_1-d-2+i}{n_1-d-2} \\
 &\quad - \sum_{i=0}^{d+1} \binom{d+1-i}{d+1-l} \binom{n_2-d-2+i}{n_2-d-2} \tag{46}
 \end{aligned}$$

$$\begin{aligned}
 &= \binom{(d+1) + (n_1+n_2-d-2) + 1}{(d+1-l) + (n_1+n_2-d-2) + 1} \\
 &\quad - \binom{(d+1) + (n_1-d-2) + 1}{(d+1-l) + (n_1-d-2) + 1} - \binom{(d+1) + (n_2-d-2) + 1}{(d+1-l) + (n_2-d-2) + 1} \tag{47}
 \end{aligned}$$

$$\begin{aligned}
 &= \binom{n_1 + n_2}{n_1 + n_2 - l} - \binom{n_1}{n_1 - l} - \binom{n_2}{n_2 - l} \\
 &= \binom{n_1 + n_2}{l} - \binom{n_1}{l} - \binom{n_2}{l},
 \end{aligned}$$

where, in order to get from (46) to (47), we used the combinatorial identity (cf. [9, Eq. (5.26)]):

$$\sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1},$$

with $k \leftarrow i, l \leftarrow d + 1, m \leftarrow d + 1 - k, q \leftarrow s - d - 2, n \leftarrow s - d - 2$, and s stands for either $n_1 + n_2, n_1$ or n_2 . Hence, P is (l, V_1) -bineighborly. \square

Using the Dehn–Sommerville-like relations (49), in conjunction with the bounds from the previous lemma, we derive alternative bounds for $h_k(\mathcal{F})$, which are of interest since they refine the bounds for $h_k(\mathcal{F})$ from Lemma 14 for large values of k , namely for $k > \lfloor \frac{d+1}{2} \rfloor$. More precisely:

Lemma 15 For all $0 \leq k \leq d + 1$,

$$h_{d+1-k}(\mathcal{F}) \leq \binom{n_1 + n_2 - d - 2 + k}{k}. \tag{48}$$

Equality holds for all k with $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d}{2} \rfloor$ and P is l -neighborly.

Proof The Dehn–Sommerville-like relation (29) corresponds to the following equalities for the elements of $\mathbf{h}(\mathcal{F})$:

$$h_{d+1-k}(\mathcal{F}) = h_k(\mathcal{F}) + g_k(\partial P_1) + g_k(\partial P_2), \quad 0 \leq k \leq d + 1. \tag{49}$$

The upper bound claim in (48) is then a direct consequence of relation (49), the upper bounds from Lemma 14, and the Upper Bound Theorem for polytopes as stated in Corollary 2.

The rest of the proof deals with the equality claim. In view of the Dehn–Sommerville-like equations $h_{d+1-k}(\mathcal{F}) = h_k(\mathcal{F}) + g_k(\partial P_1) + g_k(\partial P_2)$, the inequality (48) holds as equality for all $0 \leq k \leq l$, where $l \leq \lfloor \frac{d}{2} \rfloor$, if and only if the following two conditions hold:

- (i) inequality (42) holds as equality for all $0 \leq k \leq l \leq \lfloor \frac{d}{2} \rfloor$,
- (ii) for $j = 1, 2$ and all $0 \leq k \leq l \leq \lfloor \frac{d}{2} \rfloor$, we have $g_k(\partial P_j) = \binom{n_j - d - 2 + k}{k}$.

The first condition holds true if and only if P is (l, V_1) -bineighborly, while the second condition holds true if and only if $P_j, j = 1, 2$, is l -neighborly. Therefore, inequality (48) holds as equality for all $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d}{2} \rfloor, P$ is (l, V_1) -bineighborly and both P_1, P_2 are l -neighborly. In view of Lemma 7, we conclude that equality in (48) holds for all $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d}{2} \rfloor$ and P is l -neighborly. \square

We are now ready to compute upper bounds for the face numbers of \mathcal{F} . Using relation (9), in conjunction with the bounds on the elements of $\mathbf{h}(\mathcal{F})$ from Lemmas 14 and 15, we get, for $0 \leq k \leq d + 1$:

$$\begin{aligned}
 f_{k-1}(\mathcal{F}) &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=\lfloor \frac{d+1}{2} \rfloor + 1}^{d+1} \binom{d+1-i}{k-i} h_i(\mathcal{F}) \\
 &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} h_{d+1-i}(\mathcal{F}) \\
 &\leq \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \left(\binom{n_1+n_2-d-2+i}{i} - \sum_{j=1}^2 \binom{n_j-d-2+i}{i} \right) \\
 &\quad + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \binom{n_1+n_2-d-2+i}{i} \\
 &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \binom{n_1+n_2-d-2+i}{i} \\
 &\quad + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \binom{n_1+n_2-d-2+i}{i} \\
 &\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^2 \binom{n_j-d-2+i}{i} \tag{50} \\
 &= \sum_{i=0}^{\frac{d+1}{2} *} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \binom{n_1+n_2-d-2+i}{i} \\
 &\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^2 \binom{n_j-d-2+i}{i} \tag{51} \\
 &= f_{k-1}(C_{d+1}(n_1+n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^2 \binom{n_j-d-2+i}{i},
 \end{aligned}$$

where $C_d(n)$ stands for the cyclic d -polytope with n vertices, and $\sum_{i=0}^{\frac{\delta}{2} *} T_i$ denotes the sum of the elements $T_0, T_1, \dots, T_{\lfloor \frac{\delta}{2} \rfloor}$ where the last term is halved if δ is even; in order to get from (50) to (51), we used an identity proved in Appendix. The following lemma summarizes our results.

Lemma 16 For all $0 \leq k \leq d + 1$

$$f_{k-1}(\mathcal{F}) \leq f_{k-1}(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $C_d(n)$ stands for the cyclic d -polytope with n vertices. Furthermore:

- (i) Equality holds for all $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d+1}{2} \rfloor$ and P is (l, V_1) -bineighborly.
- (ii) For $d \geq 2$ even, equality holds for all $0 \leq k \leq d + 1$ if and only if P is $\lfloor \frac{d}{2} \rfloor$ -neighborly.
- (iii) For $d \geq 3$ odd, equality holds for all $0 \leq k \leq d + 1$ if and only if P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly.

Since $f_{k-1}(P_1 + P_2) = f_k(\mathcal{F})$ for all $1 \leq k \leq d$, we arrive at the central theorem of this section, stating upper bounds for the face numbers of the Minkowski sum of two d -polytopes.

Theorem 17 Let P_1 and P_2 be two d -polytopes in \mathbb{E}^d , $d \geq 2$, with $n_1 \geq d + 1$ and $n_2 \geq d + 1$ vertices, respectively. Let also P be the Cayley polytope of P_1 and P_2 in \mathbb{E}^{d+1} . Then, for $1 \leq k \leq d$, we have

$$f_{k-1}(P_1 + P_2) \leq f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right).$$

Furthermore:

- (i) Equality holds for all $1 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d-1}{2} \rfloor$ and P is $(l + 1, V_1)$ -bineighborly.
- (ii) For $d \geq 2$ even, equality holds for all $1 \leq k \leq d$ if and only if P is $\lfloor \frac{d}{2} \rfloor$ -neighborly.
- (iii) For $d \geq 3$ odd, equality holds for all $1 \leq k \leq d$ if and only if P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly.

5 Tightness of the Upper Bounds

In the previous section we proved upper bounds on the face numbers of the Minkowski sum $P_1 + P_2$ of two polytopes P_1 and P_2 , and we provided necessary and sufficient conditions for these bounds to hold. However, there is one remaining important question: Are these bounds tight? In this section we give a positive answer to this question.

We recall, from the introductory section, the already known results, and discuss how they are related to the results in this paper. It is already known (e.g., cf. [4]) that the maximum number of vertices/edges of the Minkowski sum of two polygons

(i.e., 2-polytopes) is the sum of the vertices/edges of the summands. These match our expressions for $d = 2$ in Theorem 17. Fukuda and Weibel [7] have shown tight expressions for the number of k -faces, $0 \leq k \leq 2$, of the Minkowski sum of two 3-polytopes P_1 and P_2 , as a function of the number of vertices of P_1 and P_2 . These maximal values are given in relations (1) and match our expressions for $d = 3$ in Theorem 17. In the same paper, Fukuda and Weibel have shown that given r d -polytopes P_1, P_2, \dots, P_r , the number of k -faces of $P_1 + P_2 + \dots + P_r$ is bounded from above as per relation (2). These bounds have been shown to be tight for $d \geq 4, r \leq \lfloor \frac{d}{2} \rfloor$, and for all k with $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$. For $r = 2$, the upper bounds in (2) reduce to

$$f_k(P_1 + P_2) \leq \sum_{j=1}^{k+1} \binom{n_1}{j} \binom{n_2}{k+2-j}, \quad 0 \leq k \leq d-1, \tag{52}$$

and are tight for all k , with $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 2$. According to Fukuda and Weibel [7], these upper bounds are attained when considering two cyclic d -polytopes P_1 and P_2 , with n_1 and n_2 vertices, respectively, with disjoint vertex sets. As we show below, this construction gives, in fact, tight bounds on the number of k -faces of the Minkowski sum for all $0 \leq k \leq d-1$, when the dimension d is even.

Theorem 18 *Let $d \geq 2$ and d is even. Consider two cyclic d -polytopes P_1 and P_2 with disjoint vertex sets on the d -dimensional moment curve, and let n_j be the number of vertices of $P_j, j = 1, 2$. Then, for all $1 \leq k \leq d$,*

$$f_{k-1}(P_1 + P_2) = f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $C_d(n)$ stands for the cyclic d -polytope with n vertices.

Proof Let V_1 and V_2 be two disjoint sets of points on the d -dimensional moment curve of cardinality n_1 and n_2 , respectively. Let P_1 and P_2 be the corresponding cyclic d -polytopes, and call P their Cayley polytope in \mathbb{E}^{d+1} . As in the previous section, we define \mathcal{F} as the set of proper faces of P whose vertex set has non-empty intersection with both V_1 and V_2 . We then get

$$\begin{aligned} f_{\lfloor \frac{d}{2} \rfloor - 1}(\mathcal{F}) &= f_{\lfloor \frac{d}{2} \rfloor - 2}(P_1 + P_2) = \sum_{j=1}^{\lfloor \frac{d}{2} \rfloor - 1} \binom{n_1}{j} \binom{n_2}{\lfloor \frac{d}{2} \rfloor - j} \\ &= \binom{n_1 + n_2}{\lfloor \frac{d}{2} \rfloor} - \binom{n_1}{\lfloor \frac{d}{2} \rfloor} - \binom{n_2}{\lfloor \frac{d}{2} \rfloor}, \end{aligned}$$

which, by Lemma 8, implies that P is $(\lfloor \frac{d}{2} \rfloor, V_1)$ -bineighborly. Using Lemma 7, in conjunction with the fact that both P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$ -neighborly, we further conclude

that P is $\lfloor \frac{d}{2} \rfloor$ -neighborly. Hence, by Theorem 17, our upper bounds in Theorem 17 are attained for all face numbers of $P_1 + P_2$. \square

If $d \geq 5$ and d is odd, however, the construction in [7] gives tight bounds for $f_k(P_1 + P_2)$ for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 2$, which according to Theorem 17 are not sufficient to establish that the bounds are tight for the face numbers of all dimensions. To establish the tightness of the bounds in Theorem 17 for all the face numbers of all dimensions, we need to construct two d -polytopes P_1 and P_2 , with n_1 and n_2 vertices, respectively, such that

$$f_{\lfloor \frac{d}{2} \rfloor}(\mathcal{F}) = f_{\lfloor \frac{d}{2} \rfloor - 1}(P_1 + P_2) = \binom{n_1 + n_2}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_1}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_2}{\lfloor \frac{d+1}{2} \rfloor},$$

or, equivalently, construct two d -polytopes P_1 and P_2 such that P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly.

The rest of this section is devoted to this construction. Before getting into the technical details, we outline our approach. In what follows $d \geq 3$ and d is odd. We denote by $\boldsymbol{\gamma}(t)$, $t > 0$, the $(d - 1)$ -dimensional moment curve, i.e., $\boldsymbol{\gamma}(t) = (t, t^2, \dots, t^{d-1})$, and we define two additional curves $\boldsymbol{\gamma}_1(t; \zeta)$ and $\boldsymbol{\gamma}_2(t; \zeta)$ in \mathbb{E}^{d+1} , as follows:

$$\begin{aligned} \boldsymbol{\gamma}_1(t; \zeta) &= (0, t, \zeta t^d, t^2, t^3, \dots, t^{d-1}), \\ \boldsymbol{\gamma}_2(t; \zeta) &= (1, \zeta t^d, t, t^2, t^3, \dots, t^{d-1}), \end{aligned} \quad t > 0, \quad \zeta \geq 0. \tag{53}$$

Notice that $\boldsymbol{\gamma}_1(t; \zeta)$ and $\boldsymbol{\gamma}_2(t; \zeta)$, with $\zeta > 0$, are d -dimensional moment-like curves,⁵ embedded in the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$, respectively. Choose $n_1 + n_2$ real numbers $\alpha_i, i = 1, \dots, n_1$, and $\beta_i, i = 1, \dots, n_2$, such that $0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n_1}$ and $0 < \beta_1 < \beta_2 < \dots < \beta_{n_2}$. Let τ be a strictly positive parameter determined below, and let U_1 and U_2 be the $(d - 1)$ -dimensional point sets:

$$\begin{aligned} U_1 &= \{\boldsymbol{\gamma}_1(\alpha_1 \tau), \boldsymbol{\gamma}_1(\alpha_2 \tau), \dots, \boldsymbol{\gamma}_1(\alpha_{n_1} \tau)\}, \\ U_2 &= \{\boldsymbol{\gamma}_2(\beta_1), \boldsymbol{\gamma}_2(\beta_2), \dots, \boldsymbol{\gamma}_2(\beta_{n_2})\}, \end{aligned} \tag{54}$$

where $\boldsymbol{\gamma}_j(\cdot)$ is used to denote $\boldsymbol{\gamma}_j(\cdot; 0)$, for simplicity. Notice that U_1 and U_2 consist of points on the moment curve $\boldsymbol{\gamma}(t)$, embedded in the $(d - 1)$ -subspaces $\{x_1 = 0, x_3 = 0\}$ and $\{x_1 = 1, x_2 = 0\}$ of \mathbb{E}^{d+1} , respectively. Call Q_j the cyclic $(d - 1)$ -polytope defined as the convex hull of the points in $U_j, j = 1, 2$. We first show that, for sufficiently small τ , any subset U of $\lfloor \frac{d+1}{2} \rfloor$ vertices of $U_1 \cup U_2$ such that $U \cap U_j \neq \emptyset, j = 1, 2$, defines a $\lfloor \frac{d}{2} \rfloor$ -face of $Q = CH_{d+1}(\{Q_1, Q_2\})$; in other words, we show that, for sufficiently small τ , the $(d + 1)$ -polytope Q is $(\lfloor \frac{d+1}{2} \rfloor, U_1)$ -bineighborly. We then appropriately perturb U_1 and U_2 (by considering a positive value for ζ) so that they become d -dimensional. Let V_1, V_2 be the perturbed vertex sets, and P_1, P_2 be the resulting d -polytopes (V_j is the vertex set of P_j). The final step of our construction

⁵ They are images of moment curves under invertible linear transformations.

amounts to considering the Cayley polytope P of P_1 and P_2 , and arguing that, if the perturbation parameter ζ is sufficiently small, then P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly. In view of Theorem 17, this establishes the tightness of our bounds for all face numbers of $P_1 + P_2$.

We start off with a technical lemma. Its proof may be found in Appendix.

Lemma 19 *Fix two integers $k, \ell \geq 2$. Let $D_{k,\ell}(\tau)$ be the $(k+\ell) \times (k+\ell)$ determinant:*

$$D_{k,\ell}(\tau) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ x_1\tau & x_2\tau & \cdots & x_k\tau & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & y_1 & y_2 & \cdots & y_\ell \\ x_1^2\tau^2 & x_2^2\tau^2 & \cdots & x_k^2\tau^2 & y_1^2 & y_2^2 & \cdots & y_\ell^2 \\ x_1^3\tau^3 & x_2^3\tau^3 & \cdots & x_k^3\tau^3 & y_1^3 & y_2^3 & \cdots & y_\ell^3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_1^m\tau^m & x_2^m\tau^m & \cdots & x_k^m\tau^m & y_1^m & y_2^m & \cdots & y_\ell^m \end{vmatrix}, \quad m = k + \ell - 3,$$

where $0 < x_1 < x_2 < \cdots < x_k, 0 < y_1 < y_2 < \cdots < y_\ell$, and $\tau > 0$. Then, there exists some $\tau_0 > 0$ (that depends on the x_i 's, the y_i 's, k , and ℓ) such that for all $\tau \in (0, \tau_0)$, the determinant $D_{k,\ell}(\tau)$ is strictly positive.

We now formally proceed with our construction. As described above, consider the vertex sets U_1 and U_2 (cf. (54)) and call Q_j the cyclic $(d - 1)$ -polytope with vertex set $U_j, j = 1, 2$. As in the previous section, call $\tilde{\Pi}$ the hyperplane of \mathbb{E}^{d+1} with equation $\{x_1 = \lambda\}, \lambda \in (0, 1)$. Let $Q = CH_{d+1}(\{Q_1, Q_2\})$, and let \mathcal{F}_Q be the set of proper faces of Q with non-empty intersection with $\tilde{\Pi}$, i.e., \mathcal{F}_Q consists of all the proper faces of Q , the vertex set of which has non-empty intersection with both U_1 and U_2 . The following lemma establishes the first step toward our construction.

Lemma 20 *There exists a sufficiently small positive value τ^* for τ such that the $(d + 1)$ -polytope Q is $(\lfloor \frac{d+1}{2} \rfloor, U_1)$ -bineighborly.*

Proof Let $t_i = \alpha_i\tau, t_i^\varepsilon = (\alpha_i + \varepsilon)\tau, 1 \leq i \leq n_1$, and $s_i = \beta_i, s_i^\varepsilon = \beta_i + \varepsilon, 1 \leq i \leq n_2$, where $\varepsilon > 0$ is chosen such that $\alpha_i + \varepsilon < \alpha_{i+1}$, for all $1 \leq i < n_1$, and $\beta_i + \varepsilon < \beta_{i+1}$, for all $1 \leq i < n_2$.

Choose a subset U of $U_1 \cup U_2$ of size $\lfloor \frac{d+1}{2} \rfloor$ such that $U \cap U_j \neq \emptyset, j = 1, 2$. We denote by μ (resp., ν) the cardinality of $U \cap U_1$ (resp., $U \cap U_2$), and, clearly, $\mu + \nu = \lfloor \frac{d+1}{2} \rfloor$. Let $\gamma_1(t_{i_1}), \gamma_1(t_{i_2}), \dots, \gamma_1(t_{i_\mu})$ be the vertices in $U \cap U_1$, where $i_1 < i_2 < \dots < i_\mu$, and analogously, let $\gamma_2(s_{j_1}), \gamma_2(s_{j_2}), \dots, \gamma_2(s_{j_\nu})$ be the vertices in $U \cap U_2$, where $j_1 < j_2 < \dots < j_\nu$. Let $\mathbf{x} = (x_1, x_2, \dots, x_{d+1})$ and define the $(d + 2) \times (d + 2)$ determinant $H_U(\mathbf{x})$ as follows:

$$H_U(\mathbf{x}) = - \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \mathbf{x} & \gamma_1(t_{i_1}) & \gamma_1(t_{i_1}^\varepsilon) & \cdots & \gamma_1(t_{i_\mu}) & \gamma_1(t_{i_\mu}^\varepsilon) & \gamma_2(s_{j_1}) & \gamma_2(s_{j_1}^\varepsilon) & \cdots & \gamma_2(s_{j_\nu}) & \gamma_2(s_{j_\nu}^\varepsilon) \end{vmatrix}. \quad (55)$$

The equation $H_U(\mathbf{x}) = 0$ is the equation of a hyperplane in \mathbb{E}^{d+1} that passes through the points in U .

Consider first the case $\mathbf{u} \in U_1 \setminus U$. Then, $\mathbf{u} = \boldsymbol{\gamma}_1(t) = (0, t, 0, t^2, t^3, \dots, t^{d-1})$, $t = \alpha\tau$, for some $\alpha \notin \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_\mu}\}$, in which case $H_U(\mathbf{u})$ becomes

$$\begin{aligned}
 &H_U(\mathbf{u}) \\
 &= - \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ \boldsymbol{\gamma}_1(t) & \boldsymbol{\gamma}_1(t_{i_1}) & \boldsymbol{\gamma}_1(t_{i_1}^\varepsilon) & \dots & \boldsymbol{\gamma}_1(t_{i_\mu}) & \boldsymbol{\gamma}_1(t_{i_\mu}^\varepsilon) & \boldsymbol{\gamma}_2(s_{j_1}) & \boldsymbol{\gamma}_2(s_{j_1}^\varepsilon) & \dots & \boldsymbol{\gamma}_2(s_{j_\nu}) & \boldsymbol{\gamma}_2(s_{j_\nu}^\varepsilon) \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ t & t_{i_1} & t_{i_1}^\varepsilon & \dots & t_{i_\mu} & t_{i_\mu}^\varepsilon & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & s_{j_1} & s_{j_1}^\varepsilon & \dots & s_{j_\nu} & s_{j_\nu}^\varepsilon \\ t^2 & t_{i_1}^2 & (t_{i_1}^\varepsilon)^2 & \dots & t_{i_\mu}^2 & (t_{i_\mu}^\varepsilon)^2 & s_{j_1}^2 & (s_{j_1}^\varepsilon)^2 & \dots & s_{j_\nu}^2 & (s_{j_\nu}^\varepsilon)^2 \\ t^3 & t_{i_1}^3 & (t_{i_1}^\varepsilon)^3 & \dots & t_{i_\mu}^3 & (t_{i_\mu}^\varepsilon)^3 & s_{j_1}^3 & (s_{j_1}^\varepsilon)^3 & \dots & s_{j_\nu}^3 & (s_{j_\nu}^\varepsilon)^3 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ t^{d-1} & t_{i_1}^{d-1} & (t_{i_1}^\varepsilon)^{d-1} & \dots & t_{i_\mu}^{d-1} & (t_{i_\mu}^\varepsilon)^{d-1} & s_{j_1}^{d-1} & (s_{j_1}^\varepsilon)^{d-1} & \dots & s_{j_\nu}^{d-1} & (s_{j_\nu}^\varepsilon)^{d-1} \end{vmatrix}.
 \end{aligned}$$

Observe now that we can transform $H_U(\mathbf{u})$ in the form of the determinant $D_{k,\ell}(\tau)$ of Lemma 19, where $k = 2\mu + 1$ and $\ell = 2\nu$, by means of the following determinant transformations:

- (i) Subtract the third row of $H_U(\mathbf{u})$ from the first.
- (ii) Shift the first column of $H_U(\mathbf{u})$ to the right so that all columns of $H_U(\mathbf{u})$ are arranged in increasing order according to their parameter. Clearly, this can be done with an *even* number of column swaps.

The case $\mathbf{u} \in U_2 \setminus U$ is entirely analogous.

We thus conclude that, for any specific choice of U , and for any specific point $\mathbf{u} \in (U_1 \cup U_2) \setminus U$, there exists some $\tau_0 > 0$ (cf. Lemma 19) that depends on \mathbf{u} and U and such that for all $\tau \in (0, \tau_0)$, $H_U(\mathbf{u}) > 0$. Since the number of all such choices is finite, it suffices to consider a value τ^* for τ that is small enough so that all possible determinants $H_U(\mathbf{u})$ are strictly positive. For this specific choice of τ , every subset of U of $U_1 \cup U_2$, where $|U| = \lfloor \frac{d+1}{2} \rfloor$, $U \cap U_j \neq \emptyset$, $j = 1, 2$, defines a $\lfloor \frac{d}{2} \rfloor$ -face of Q , which means that Q is $(\lfloor \frac{d+1}{2} \rfloor, U_1)$ -bineighborly. □

We are now ready to perform the last step of our construction. In the remainder of this section we assume that τ is equal to τ^* so that the polytopes Q_1, Q_2 constructed above have all the properties mentioned in the proof of Lemma 20. We consider the vertex sets U_1, U_2 of the polytopes Q_1, Q_2 , respectively, and perturb them to get the vertex sets V_1 and V_2 . We do this by considering vertices on the curves $\boldsymbol{\gamma}_1(t; \zeta)$ and $\boldsymbol{\gamma}_2(t; \zeta)$, with $\zeta > 0$ instead of the curves $\boldsymbol{\gamma}_1(t)$ and $\boldsymbol{\gamma}_2(t)$ (cf. (53)). More precisely, define the sets V_1 and V_2 as

$$\begin{aligned}
 V_1 &= \{\boldsymbol{\gamma}_1(\alpha_1\tau^*; \zeta), \boldsymbol{\gamma}_1(\alpha_2\tau^*; \zeta), \dots, \boldsymbol{\gamma}_1(\alpha_{n_1}\tau^*; \zeta)\}, \quad \text{and} \\
 V_2 &= \{\boldsymbol{\gamma}_2(\beta_1; \zeta), \boldsymbol{\gamma}_2(\beta_2; \zeta), \dots, \boldsymbol{\gamma}_2(\beta_{n_2}; \zeta)\},
 \end{aligned}
 \tag{56}$$

where $\zeta > 0$. Let P_j be the convex hull of the vertices in V_j , $j = 1, 2$, and notice that P_j is a $\lfloor \frac{d}{2} \rfloor$ -neighborly d -polytope. Let P be the Cayley polytope of P_1 and P_2 , and let \mathcal{F}_P be the set of proper faces of P , the vertex set of which has non-empty intersection with both V_1 and V_2 . The following lemma establishes the final step of our construction. In view of Theorem 17, it also establishes the tightness of our bounds for all face numbers of $P_1 + P_2$.

Lemma 21 *There exists a sufficiently small positive value ζ^* for ζ such that the $(d + 1)$ -polytope P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly.*

Proof Similarly to what we have done in the proof of Lemma 20, let $t_i = \alpha_i \tau^*$, $t_i^\varepsilon = (\alpha_i + \varepsilon) \tau^*$, $1 \leq i \leq n_1$, and $s_i = \beta_i$, $s_i^\varepsilon = \beta_i + \varepsilon$, $1 \leq i \leq n_2$, where $\varepsilon > 0$ is chosen such that $\alpha_i + \varepsilon < \alpha_{i+1}$, for all $1 \leq i < n_1$, and $\beta_i + \varepsilon < \beta_{i+1}$, for all $1 \leq i < n_2$.

Choose V , a subset of $V_1 \cup V_2$ of size $\lfloor \frac{d+1}{2} \rfloor$, such that $V \cap V_j \neq \emptyset$, $j = 1, 2$. Denote by μ (resp., ν) the cardinality of $V \cap V_1$ (resp., $V \cap V_2$). Considering ζ as a small positive parameter, let $\gamma_1(t_{i_1}; \zeta), \gamma_1(t_{i_2}; \zeta), \dots, \gamma_1(t_{i_\mu}; \zeta)$ be the vertices in $V \cap V_1$, where $i_1 < i_2 < \dots < i_\mu$, and analogously, let $\gamma_2(s_{j_1}; \zeta), \gamma_2(s_{j_2}; \zeta), \dots, \gamma_2(s_{j_\nu}; \zeta)$ be the vertices in $V \cap V_2$, where $j_1 < j_2 < \dots < j_\nu$. Let $\mathbf{x} = (x_1, x_2, \dots, x_{d+1})$ and define the $(d + 2) \times (d + 2)$ determinant $F_V(\mathbf{x}; \zeta)$ as

$$F_V(\mathbf{x}; \zeta) = - \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \mathbf{x} & \gamma_1(t_{i_1}; \zeta) & \gamma_1(t_{i_1}^\varepsilon; \zeta) & \dots & \gamma_1(t_{i_\mu}^\varepsilon; \zeta) & \gamma_2(s_{j_1}; \zeta) & \gamma_2(s_{j_1}^\varepsilon; \zeta) & \dots & \gamma_2(s_{j_\nu}^\varepsilon; \zeta) \end{vmatrix}. \tag{57}$$

The equation $F_V(\mathbf{x}; \zeta) = 0$ is the equation of a hyperplane in \mathbb{E}^{d+1} that passes through the points in V . We claim that for all vertices $\mathbf{v} \in (V_1 \cup V_2) \setminus V$, we have $F_V(\mathbf{v}; \zeta) > 0$ for sufficiently small ζ .

To prove our claim, observe that

$$\lim_{\zeta \rightarrow 0^+} F_V(\mathbf{v}; \zeta) = F_U(\mathbf{u}; 0) = H_U(\mathbf{u}), \tag{58}$$

where $\mathbf{u} = \lim_{\zeta \rightarrow 0^+} \mathbf{v}$ is the projection of $\mathbf{v} \in V_i \setminus V$ on the curve $\gamma_i(t; 0)$, $i = 1, 2$, and $H_U(\mathbf{u})$ is the determinant in relation (55) in the proof of Lemma 20, which is positive due to the way we have chosen τ^* . Clearly, $F_V(\mathbf{v}; \zeta)$ is a polynomial function in ζ . Since $H_U(\mathbf{u}) > 0$, relation (58) implies that there exists some $\zeta_0 > 0$ depending on \mathbf{v} and V such that, for all $\zeta \in (0, \zeta_0)$, $F_V(\mathbf{v}; \zeta) > 0$.

We choose a value ζ^* for ζ that is small enough so that, for any $V \subseteq V_1 \cup V_2$ with $V \cap V_j \neq \emptyset$, $j = 1, 2$, and for all $\mathbf{v} \in (V_1 \cup V_2) \setminus V$, the determinant $F_V(\mathbf{v}; \zeta^*)$ is strictly positive. Since the number of such determinants is finite, we conclude that for ζ equal to ζ^* , every subset V of $V_1 \cup V_2$, where $|V| = \lfloor \frac{d+1}{2} \rfloor$ and $V \cap V_j \neq \emptyset$, $j = 1, 2$, defines a face of P ; this means that P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$ -bineighborly. \square

We are now ready to state the second main theorem of this section that concerns the tightness of our upper bounds on the number of k -faces of the Minkowski sum of two d -polytopes for all $0 \leq k \leq d - 1$ and for all odd dimensions $d \geq 3$.

Theorem 22 *Let $d \geq 3$ and d is odd. There exist two $\lfloor \frac{d}{2} \rfloor$ -neighborly d -polytopes P_1 and P_2 with n_1 and n_2 vertices, respectively, such that, for all $1 \leq k \leq d$,*

$$f_{k-1}(P_1 + P_2) = f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $C_d(n)$ stands for the cyclic d -polytope with n vertices.

6 Summary and Open Problems

In this paper we have computed the maximum number of k -faces, $f_k(P_1 + P_2)$, $0 \leq k \leq d - 1$, of the Minkowski sum of two d -polytopes P_1 and P_2 as a function of the number of vertices n_1 and n_2 of these two polytopes. In even dimensions $d \geq 2$, these maximal values are attained if P_1 and P_2 are cyclic d -polytopes with disjoint vertex sets. In odd dimensions $d \geq 3$, the construction that achieves the upper bounds is more intricate. Denoting by $\gamma_1(t; \zeta)$ and $\gamma_2(t; \zeta)$ the d -dimensional moment-like curves $(t, \zeta t^d, t^2, t^3, \dots, t^{d-1})$ and $(\zeta t^d, t, t^2, t^3, \dots, t^{d-1})$, where $t > 0$ and $\zeta > 0$, we have shown that these maximum values are attained if P_1 and P_2 are the d -polytopes with vertex sets $V_1 = \{\gamma_1(\alpha_i \tau^*; \zeta^*) \mid i = 1, \dots, n_1\}$ and $V_2 = \{\gamma_2(\beta_j; \zeta^*) \mid j = 1, \dots, n_2\}$, respectively, where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n_1}, 0 < \beta_1 < \beta_2 < \dots < \beta_{n_2}$, and τ^*, ζ^* are appropriately chosen, sufficiently small, positive parameters.

The obvious next step is to extend the results in this paper for the Minkowski sum of r d -polytopes in \mathbb{E}^d for $r \geq 3$ and $d \geq 4$. The case $r = 3$ and $d \geq 2$ has already been resolved by the authors of this paper in collaboration with Konaxis [18, 19], while recently Adiprasito and Sanyal [1] have resolved the general problem for any $r, d \geq 2$, as well as for summands of different dimensions. The Adiprasito and Sanyal approach uses tools from Combinatorial Commutative Algebra, through a newly developed powerful theory called the *relative Reisner–Stanley theory* for simplicial complexes. An alternative proof for the case $r < d$ that is based on geometric arguments has very recently been proposed by the authors of this paper in [16, 17].

A related problem is to express the number of k -faces of the Minkowski sum of r d -polytopes in terms of the number of facets of these polytopes. Results in this direction are known for $d = 2$ and $d = 3$ only (see the introductory section and [5] for the 3-dimensional case). We would like to derive such expressions for any $d \geq 4$ and any number r of summands.

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Appendix

Proof of Lemma 3

Proof McMullen [20] in his original proof of the Upper Bound Theorem for polytopes proved that for any d -polytope P the following relation holds:

$$(k + 1)h_{k+1}(\partial P) + (d - k)h_k(\partial P) = \sum_{v \in \text{vert}(P)} h_k(\partial P/v), \quad 0 \leq k \leq d - 1. \tag{59}$$

Multiplying both sides of (59) by t^{d-k-1} and summing over all $0 \leq k \leq d$, we get

$$\begin{aligned} & \sum_{k=0}^d (k + 1)h_{k+1}(\partial P)t^{d-k-1} + \sum_{k=0}^d (d - k)h_k(\partial P)t^{d-k-1} \\ &= \sum_{k=0}^d \sum_{v \in \text{vert}(P)} h_k(\partial P/v)t^{d-k-1}. \end{aligned} \tag{60}$$

For the right-hand side of (60), we have

$$\begin{aligned} \sum_{k=0}^d \sum_{v \in \text{vert}(P)} h_k(\partial P/v)t^{d-k-1} &= \sum_{v \in \text{vert}(P)} \sum_{k=0}^d h_k(\partial P/v)t^{d-1-k} \\ &= \sum_{v \in \text{vert}(P)} h(\partial P/v; t), \end{aligned} \tag{61}$$

whereas for the left-hand side of (60), we get

$$\begin{aligned} & \sum_{k=0}^d (k + 1)h_{k+1}(\partial P)t^{d-k-1} + \sum_{k=0}^d (d - k)h_k(\partial P)t^{d-k-1} \\ &= \sum_{k=0}^d kh_k(\partial P)t^{d-k} + \sum_{k=0}^d (d - k)h_k(\partial P)t^{d-k-1} \\ &= d \sum_{k=0}^d h_k(\partial P)t^{d-k} + (1 - t) \sum_{k=0}^d (d - k)h_k(\partial P)t^{d-k-1} \\ &= d h(\partial P; t) + (1 - t) h'(\partial P; t). \end{aligned} \tag{62}$$

Substituting in (60), from (61) and (62), we recover the relation in the statement of the lemma. □

Proof of an Identity

In this section we prove the following identity, used in Sect. 4, to prove the upper bound for $f_{k-1}(\mathcal{F})$ (see relations (50) and (51)).

Lemma 23 *For any $d \geq 2$ and any sequence of numbers α_i , where $0 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$, we have*

$$\begin{aligned} & \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \alpha_i + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \alpha_i \\ &= \sum_{i=0}^{\frac{d+1}{2} *} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i. \end{aligned}$$

Proof We start by recalling the definition of the symbol $\sum_{i=0}^{\frac{\delta}{2} *} T_i$. This symbol denotes the sum of the elements $T_0, T_1, \dots, T_{\lfloor \frac{\delta}{2} \rfloor}$, where the last term is halved if δ is even. More precisely:

$$\sum_{i=0}^{\frac{\delta}{2} *} T_i = \begin{cases} T_0 + T_1 + \dots + T_{\lfloor \frac{\delta}{2} \rfloor - 1} + \frac{1}{2} T_{\lfloor \frac{\delta}{2} \rfloor} & \text{if } \delta \text{ is even,} \\ T_0 + T_1 + \dots + T_{\lfloor \frac{\delta}{2} \rfloor - 1} + T_{\lfloor \frac{\delta}{2} \rfloor} & \text{if } \delta \text{ is odd.} \end{cases}$$

Let us now first consider the case d odd. In this case $d + 1$ is even, and we have

$$\begin{aligned} & \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \alpha_i + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \alpha_i \\ &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \alpha_i + \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor - 1} \binom{i}{k-d-1+i} \alpha_i \\ &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor - 1} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i + \binom{d+1 - \lfloor \frac{d+1}{2} \rfloor}{k - \lfloor \frac{d+1}{2} \rfloor} \alpha_{\lfloor \frac{d+1}{2} \rfloor} \\ &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor - 1} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i \\ & \quad + \frac{1}{2} \left(\binom{d+1 - \lfloor \frac{d+1}{2} \rfloor}{k - \lfloor \frac{d+1}{2} \rfloor} + \binom{\lfloor \frac{d+1}{2} \rfloor}{k-d-1 + \lfloor \frac{d+1}{2} \rfloor} \right) \alpha_{\lfloor \frac{d+1}{2} \rfloor} \\ &= \sum_{i=0}^{\frac{d+1}{2} *} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i. \end{aligned}$$

The case d even is even simpler to prove. In this case $d + 1$ is odd; hence

$$\begin{aligned}
 & \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \alpha_i + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \alpha_i \\
 &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \alpha_i + \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{i}{k-d-1+i} \alpha_i \\
 &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i \\
 &= \sum_{i=0}^{\frac{d+1}{2}} * \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \alpha_i
 \end{aligned}$$

This completes the proof. □

Proof of Lemma 19

Before proceeding with the proof of Lemma 19, we need to introduce Vandermonde and generalized Vandermonde determinants. Given a vector of $n \geq 2$ real numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the *Vandermonde determinant* $\text{VD}(\mathbf{x})$ of \mathbf{x} is the $n \times n$ determinant

$$\text{VD}(\mathbf{x}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

From the above expression, it is readily seen that if the elements of \mathbf{x} are in strictly increasing order, then $\text{VD}(\mathbf{x}) > 0$. A generalization of the Vandermonde determinant is the generalized Vandermonde determinant: if, in addition to \mathbf{x} , we specify a vector of exponents $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, where we require that $0 \leq \mu_1 < \mu_2 < \dots < \mu_n$, we can define the *generalized Vandermonde determinant* $\text{GVD}(\mathbf{x}; \boldsymbol{\mu})$ as the $n \times n$ determinant:

$$\text{GVD}(\mathbf{x}; \boldsymbol{\mu}) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \dots & x_n^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \dots & x_n^{\mu_2} \\ x_1^{\mu_3} & x_2^{\mu_3} & \dots & x_n^{\mu_3} \\ \vdots & \vdots & & \vdots \\ x_1^{\mu_n} & x_2^{\mu_n} & \dots & x_n^{\mu_n} \end{vmatrix}.$$

It is a well-known fact that, if the elements of \mathbf{x} are in strictly increasing order, then $\text{GVD}(\mathbf{x}; \boldsymbol{\mu}) > 0$ (for example, see [8] for a proof of this fact).

To prove Lemma 19 we exploit the *Cauchy–Binet formula* (cf. [3]). Let M be a $n \times n$ square matrix factorized into a product LR of an $n \times m$ and an $m \times n$ matrix L and R respectively, with $m \geq n$. If J is a subset of $\{1, 2, \dots, m\}$ of size n , we denote by $L_{[n],J}$ the $n \times n$ matrix whose columns are the columns of L at indices from J and by $R_{J,[n]}$ the $n \times n$ matrix whose rows are the rows of R at indices from J . The Cauchy–Binet theorem states that

$$\det(M) = \det(LR) = \sum_{J \in \binom{[m]}{n}} \det(L_{[n],J}) \det(R_{J,[n]}), \tag{63}$$

where $\binom{[m]}{n}$ denotes the set of subsets of $[m]$ of size n .

Proof The determinant $D_{k,l}(\tau)$ is clearly a polynomial function of τ . To prove our lemma, it suffices to show that the coefficient of the minimum exponent of τ in $D_{k,l}(\tau)$ is strictly positive.

In order to apply the Cauchy–Binet formula in our case, we factorize the matrix $\Delta_{k,l}(\tau)$, corresponding to the determinant $D_{k,l}(\tau)$, into the product of an $(m + 3) \times 2(m + 1)$ and a $2(m + 1) \times (k + l)$ matrix L and R , respectively, as follows (recall that $m + 3 = k + l$):

$$\begin{aligned} &\Delta_{k,l}(\tau) \\ &= LR = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & I_{m-1} & 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} \text{VD}(\tau x) & 0 \\ 0 & \text{VD}(y) \end{pmatrix} \\ &= \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & \dots & m+1 & \hat{1} & \hat{2} & \hat{3} & \hat{4} & \dots & m+1 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \\ m+3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ x_1 \tau & x_2 \tau & \dots & x_k \tau & 0 & 0 & \dots & 0 \\ x_1^2 \tau^2 & x_2^2 \tau^2 & \dots & x_k^2 \tau^2 & 0 & 0 & \dots & 0 \\ x_1^3 \tau^3 & x_2^3 \tau^3 & \dots & x_k^3 \tau^3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^m \tau^m & x_2^m \tau^m & \dots & x_k^m \tau^m & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & y_1 & y_2 & \dots & y_\ell \\ 0 & 0 & \dots & 0 & y_1^2 & y_2^2 & \dots & y_\ell^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & y_1^3 & y_2^3 & \dots & y_\ell^3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & y_1^m & y_2^m & \dots & y_\ell^m \end{pmatrix} \end{matrix} \tag{64} \end{aligned}$$

The numbers over and sideways of L indicate the column and row indices, respectively, with $\hat{k} := k + m + 1$. We partition the index set J into $J_1 \cup J_2$ where $J_1 \subseteq \{1, \dots, m + 1\}$ and $J_2 \subseteq \{\hat{1}, \hat{2}, \dots, \hat{m} + 1\}$. Notice that a term $\det(L_{[m+3],J}) \det(R_{J,[k+l]})$ in the Cauchy–Binet expansion of $D_{k,l}(\tau)$ vanishes in the following two cases:

- (i) $i \in J_1$ and $\hat{i} \in J_2$ for some $3 \leq i \leq m + 1$; in this case the i -th and \hat{i} -th columns of $L_{[m+3],J}$ are identical, and thus $\det(L_{[m+3],J}) = 0$.

(ii) $|J_1| \neq k$ or $|J_2| \neq l$; in this case $R_{J,[k+l]}$ is a block-diagonal square matrix with non-square non-zero blocks. The determinant of such a matrix is always zero.⁶

Furthermore, notice that for any non-vanishing term in the Cauchy–Binet expansion of $D_{k,l}(\tau)$, we have

$$\begin{aligned} \det(R_{J,[k+l]}) &= \text{GVD}(\tau \mathbf{x}; \boldsymbol{\mu}_1) \text{GVD}(\mathbf{y}; \boldsymbol{\mu}_2), \\ \tau \mathbf{x} &= (\tau x_1, \dots, \tau x_k), \\ \mathbf{y} &= (y_1, \dots, y_l), \end{aligned}$$

where $\boldsymbol{\mu}_1$ (resp., $\boldsymbol{\mu}_2$) is the vector consisting of the elements in $\{i - 1 \mid i \in J_1\}$ (resp., $\{i - (m + 1) - 1 \mid i \in J_2\}$) ordered increasingly. The parameter τ appears only in $\text{GVD}(\tau \mathbf{x}; \boldsymbol{\mu}_1)$ and can be factored out (see 64). We thus have

$$\det(R_{J,[k+l]}) = \tau^{M(J)} \text{GVD}(\mathbf{x}; \boldsymbol{\mu}_1) \text{GVD}(\mathbf{y}; \boldsymbol{\mu}_2) > 0, \quad M(J) = \sum_{i \in J_1} (i - 1),$$

since $\text{GVD}(\mathbf{x}; \boldsymbol{\mu}_1)$ and $\text{GVD}(\mathbf{y}; \boldsymbol{\mu}_2)$ are positive due to the way we have chosen \mathbf{x} and \mathbf{y} .

Among all possible index sets $J = J_1 \cup J_2$ for which the product $\det(L_{[m+3],J}) \det(R_{J,[k+l]})$ does not vanish, we have to find the index set that gives the minimum exponent for τ . Recall that, in view of condition (ii), the size of J_1 is k and the size of J_2 is l . The minimum exponent $M(J)$ is then attained when $J_1 = J_1^* := \{1, \dots, k\}$. In view of condition (i), we have $J_2 \subseteq \widehat{\{1, 2, \dots, m + 1\}} \setminus \widehat{\{3, \dots, k\}}$, which leaves no other choice but $J_2 = J_2^* := \{\widehat{1}, \widehat{2}, \widehat{k + 1}, \dots, \widehat{m + 1}\}$.

For $J^* = J_1^* \cup J_2^*$, the matrix $L_{[m+3],J^*}$ is

$$L_{[m+3],J^*} = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_l \end{pmatrix}.$$

Clearly, $L_{[m+3],J^*}$ becomes the identity matrix after $2k$ row swaps. Hence $\det(L_{[m+3],J^*}) = 1$, which further implies that

$$\text{sign}(D_{k,l}(\tau)) = \text{sign}(R_{J^*,[k+l]}) > 0.$$

□

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