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# Indecomposable Coverings with Homothetic Polygons

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**Abstract** We prove that for any convex polygon $S$ with at least four sides, or a concave one with no parallel sides, and any $m > 0$, there is an $m$-fold covering of the plane with homothetic copies of $S$ that cannot be decomposed into two coverings.

**Keywords** Homothetic copy · Multiple covering · Decomposable

## 1 Introduction

Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a collection of planar sets. It is an *m-fold covering* if every point in the plane is contained in at least $m$ members of $\mathcal{C}$. A 1-fold covering is simply called a *covering*.

A planar set $S$ is said to be cover-decomposable if there is a constant $m = m(S)$ such that every $m$-fold covering of the plane with translates of $S$ can be decomposed into two coverings. Pach [3] proposed the problem of determining all cover-decomposable sets in 1980. He conjectured that all planar convex sets are cover-decomposable. The conjecture has been verified, in several steps, for all convex polygons [9] (see also [4,11]). However, very recently, Pálvölgyi proved that the unit disk is not cover-decomposable [8]. His result holds also for convex sets with smooth boundary.

The problem of determining cover-decomposable sets has been generalized in many directions, see [5] for a survey.

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A *homothetic* transformation is the composition of a translation and a scaling. Keszegh and Pálvölgyi [1] proved that any 12-fold covering of the plane with homothetic copies of a fixed triangle $T$ can be decomposed into two coverings. In this note we prove that, with a few possible exceptions, this result cannot be extended to other polygons.

**Theorem 1** *Let $S$ be a convex polygon with at least four sides, or a concave polygon with no parallel sides, and let $m > 0$. There is an $m$-fold covering of the plane with homothetic copies of $S$ that cannot be decomposed into two coverings.*

For *convex* polygons we can keep the sizes of the homothetic copies "almost equal."

**Theorem 2** *Let $S$ be a convex polygon with at least four sides, and let $\varepsilon > 0$ and $m > 0$. There is a collection of homothetic copies of $S$, each of them with scaling factor between $1 - \varepsilon$ and $1 + \varepsilon$, which forms an $m$-fold covering of the plane that cannot be decomposed into two coverings.*

Our method is based on the ideas of Pálvölgyi [7,8].

## 2 Preparations

Most of the papers about cover-decomposability investigate the problem in its *dual form*.

Suppose that $\mathcal{H} = \{S_i \mid i \in I\}$ is collection of *translates* of $S$ that form an $m$-fold covering of the plane. For every $i \in I$, let $c_i$ be the center of gravity of $S_i$. Let $\mathcal{H}' = \{c_i \mid i \in I\}$ be the set of the centers. For any point $a$, let $-S(a)$ be a translate of $-S$ whose center of gravity is $a$. Then $a \in S_i$ if and only if $c_i \in -S(a)$. Therefore, the collection $\mathcal{H}$ can be decomposed into two coverings if and only if the points of the set $\mathcal{H}'$ can be colored with two colors such that every translate of $S$ contains points of both colors. This idea is originally due to Pach [4].

If we have homothetic copies, then the dual version of the problem is not equivalent to the original one. However, in this paper we give a tricky definition of the dual form.

Fix a coordinate system and let $o$ be the origin. If it does not lead to confusion, for any point $p$, we denote its position vector $\overrightarrow{op}$ also by $p$. For any $\alpha$ real, set $S$, and point $p$, let

$$\alpha \cdot S(p) = \{\alpha \cdot x + p \mid x \in S\}.$$

The Minkowski sum of any convex polygons $S$ and $T$ is defined as

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

Let $S$ be a fixed convex polygon of at least four sides, $o \in S$. It is well known [10] that for any $\alpha, \beta \geq 0$

$$\alpha \cdot S + \beta \cdot S = (\alpha + \beta) \cdot S.$$

As an easy consequence, we get the following statement.

**Statement 1** *Let* $\alpha, \beta \geq 0$, $p, q \in \mathbb{R}^2$. $(\alpha+\beta)\cdot S(p)$ *contains* $q$ *if and only if* $\alpha \cdot S(p)$ *and* $-\beta \cdot S(q)$ *intersect each other.*

First, for every pair $(k, l)$, we will construct a collection of homothetic copies of $S$, $\mathcal{X}_{k,l}$ and a collection of translates of $-S$, $\mathcal{Y}_{k,l}$ with the property that for every red-blue coloring of the elements of $\mathcal{X}_{k,l}$, there is an element of $\mathcal{Y}_{k,l}$ which intersects exactly $k$ elements, all of which are red (resp. exactly $l$ elements, all of which are blue).

Then we "dualize" this construction, for $m = k = l$, as follows. Replace each element of $\mathcal{X}_{m,m}$ by a larger homothetic copy and let $\mathcal{X}'_{m,m}$ be the new collection. Replace each element of $\mathcal{Y}_{m,m}$ by a point and let $\mathcal{Y}'_{m,m}$ be the set of these points.

By Statement 1, $\mathcal{X}'_{m,m}$ and $\mathcal{Y}'_{m,m}$ have the following property.

For every red-blue coloring of the elements of $\mathcal{X}'_{m,m}$, there is an element (point) of $\mathcal{Y}'_{m,m}$ which is contained in exactly $m$ elements of $\mathcal{X}'_{m,m}$, all of which are of the same color.

So, for every $m$, $\mathcal{X}'_{m,m}$ forms a non-decomposable $m$-fold covering of the points in $\mathcal{Y}'_{m,m}$. Finally, we extend it to a non-decomposable $m$-fold covering of the whole plane.

## 3 Proof of Theorems 1 and 2

Let $S$ be a fixed convex polygon of at least four sides, $o \in S$. We say that $o$ is the *center* of $S$. We can assume that $S$ is contained in the unit disk of center $o$. By definition, $-S$ denotes the reflection of $S$ about the origin. Let $v_1, v_2, \ldots, v_n$ be the vertices of $-S$, ordered clockwise. Indices are understood mod $n$, that is, $v_{n+1}$ means $v_1$.

**Definition 1** For every $i$, $1 \leq i \leq n$, let $E^i$ denote the convex wedge whose apex is at the origin and its bounding halflines are the translates of $\overrightarrow{v_i v_{i-1}}$ and $\overrightarrow{v_i v_{i+1}}$. $E^i$ is called the wedge that belongs to vertex $v_i$ of $-S$.

Choose a direction $d$ which is *not parallel* to the sides of $S$, and the two vertices $v_a$ and $v_b$, where $S$ can be touched by a line parallel to $d$, are not adjacent. Assume without loss of generality that $d$ is horizontal, and $v_a$ is the highest and $v_b$ is the lowest vertices of $S$. Let $Q$ be a quadrilateral created from $S$ by extending the sides at $v_a$ and $v_b$. Let $v_r$ and $v_l$ be the rightmost and the leftmost vertices of $Q$, respectively. See Fig. 1. We can assume without loss of generality that $v_l$ is not lower than $v_r$. Indeed, if $v_l$ is lower than $v_r$, then we can apply a reflection of $S$ about the $y$-axis. Let $\delta > 0$ be a very small constant.

For every pair $(k, l)$, $k, l \geq 1$, we will construct a triple $\mathcal{T}_{k,l} = (\mathcal{X}_{k,l}, \mathcal{E}^a_{k,l}, \mathcal{E}^b_{k,l})$, where $\mathcal{X}_{k,l} = \{\varepsilon_i \cdot S(p_i) \mid i \in I_{k,l}\}$, $\varepsilon_i > 0$, a collection of homothetic copies of $S$, and $\mathcal{E}^a_{k,l} = \{E^a(q_j) \mid j \in J^a_{k,l}\}$, $\mathcal{E}^b_{k,l} = \{E^b(r_j) \mid j \in J^b_{k,l}\}$ are collections of translates of the wedges $E^a$ and $E^b$, respectively, for some $I_{k,l}$, $J^a_{k,l}$, $J^b_{k,l}$ index sets. $\mathcal{T}_{k,l}$ will have the following properties.

**Property 1** *For every red-blue coloring of the elements of* $\mathcal{X}_{k,l}$, *either there is an element of* $\mathcal{E}^a_{k,l}$ *which intersects exactly* $k$ *elements of* $\mathcal{X}_{k,l}$, *all of which are red, or there is an element of* $\mathcal{E}^b_{k,l}$ *which intersects exactly* $l$ *elements of* $\mathcal{X}_{k,l}$, *all of which are blue.*

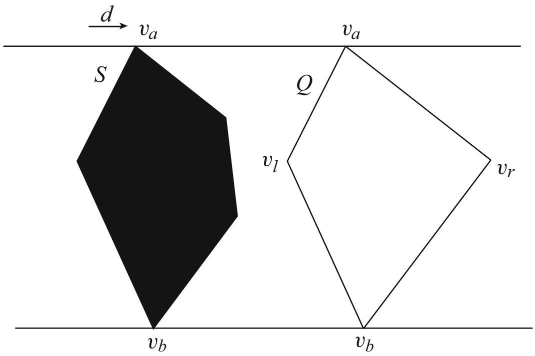



**Fig. 1** $S$, $Q$ and the corresponding vertices

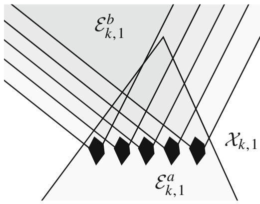



**Fig. 2** The construction of $\mathcal{T}_{k,1}$

**Property 2** *There is a disk $D_{k,l}$ of radius $\delta$ which contains all apices of the wedges in $\mathcal{E}^a_{k,l}$ and $\mathcal{E}^b_{k,l}$, and all elements of $\mathcal{X}_{k,l}$.*

First we define $\mathcal{T}_{k,1}$ and $\mathcal{T}_{1,l}$. For arbitrary $k$, let $\mathcal{X}_{k,1}$ be $k$ very small homothetic copies of $S$, very close to each other on a horizontal line. $\mathcal{E}^a_{k,1}$ contains one translate of the wedge $E^a$ that intersects all $k$ homothetic copies, and $\mathcal{E}^b_{k,1}$ contains $k$ translates of the wedge $E^b$, each intersecting exactly one of the $k$ homothetic copies, but each intersecting a different one. See Fig. 2. We define the triple $\mathcal{T}_{1,l}$ similarly for any $l$.

Suppose now that we have already defined $\mathcal{T}_{k,l-1}$ and $\mathcal{T}_{k-1,l}$. Take a translate of $\mathcal{T}_{k,l-1}$ so that the center of $D_{k,l-1}$ is $(0, 0)$ and a translate of $\mathcal{T}_{k-1,l}$ so that the center of $D_{k-1,l}$ is $(1, 3\delta)$. Place a suitable homothetic copy $S' = \varepsilon \cdot S$ of $S$ between points $(0, 0)$ and $(1, 3\delta)$ such that

(i) $S'$ intersects all wedges in $\mathcal{E}^b_{k,l-1}$, and all wedges in $\mathcal{E}^a_{k-1,l}$,
(ii) $S'$ does not intersect any of the wedges in $\mathcal{E}^a_{k,l-1}$, and any of the wedges in $\mathcal{E}^b_{k-1,l}$.

See Fig. 3. Let

$$\mathcal{X}_{k,l} = \mathcal{X}_{k-1,l} \cup \mathcal{X}_{k,l-1} \cup \{S'\},$$
$$\mathcal{E}^a_{k,l} = \mathcal{E}^a_{k-1,l} \cup \mathcal{E}^a_{k,l-1},$$
$$\mathcal{E}^b_{k,l} = \mathcal{E}^b_{k-1,l} \cup \mathcal{E}^b_{k,l-1}.$$

Apply a suitable scaling so that Property 2 is satisfied. We claim that Property 1 is also satisfied. Color the elements of $\mathcal{X}_{k,l}$ by red and blue. Suppose that $S'$ is red. In the subconfiguration that corresponds to $\mathcal{T}_{k-1,l}$, either there is a translate of $E^a$ that

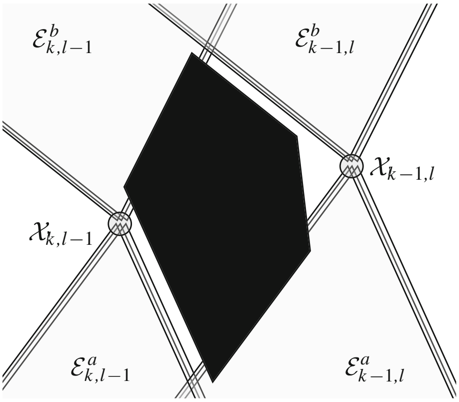



**Fig. 3** The induction step

intersects exactly $k-1$ elements of $\mathcal{X}_{k-1,l}$, all of which are red, or there is a translate of $E^b$ that intersects exactly $l$ elements of $\mathcal{X}_{k-1,l}$, all of which are blue. In the first case, the corresponding translate of $E^a$ intersects exactly one more element of $\mathcal{X}_{k,l}$, $S'$, and it is red, so we are done. In the second case, the corresponding translate of $E^b$ does not intersect any other element of $\mathcal{X}_{k,l}$, so we are done again. We can argue the same way if $S'$ is colored blue. Consequently, Property 1 is satisfied.

To obtain a non-decomposable $m$-fold covering, consider $\mathcal{T}_{m,m} = (\mathcal{X}_{m,m}, \mathcal{E}^a_{m,m}, \mathcal{E}^b_{m,m})$. $\mathcal{X}_{m,m} = \{\varepsilon_i \cdot S(p_i) \mid i \in I_{m,m}\}$, $\varepsilon_i > 0$, a collection of homothetic copies.

Replace each element of $\mathcal{E}^a_{m,m}$ (resp. $\mathcal{E}^b_{m,m}$) by a translate of $-S$ such that its vertex $v_a$ (resp. $v_b$) moves to its apex. We obtain a collection of translates of $-S$, $\mathcal{Y}_{m,m} = \{-S(q_j) \mid j \in J_{m,m}\}$, with the property that for every red-blue coloring of the elements of $\mathcal{X}_{m,m}$, there is an element of $\mathcal{Y}_{m,m}$ which intersects exactly $m$ elements of $\mathcal{X}_{m,m}$, all of the same color.

Let $\mathcal{X}'_{m,m} = \{(1+\varepsilon_i) \cdot S(p_i) \mid i \in I_{m,m}\}$, a collection of homothetic copies of $S$, and let $\mathcal{Y}'_{m,m} = \{q_j \mid j \in J_{m,m}\}$, a collection of points. By Statement 1, for every red-blue coloring of the elements of $\mathcal{X}'_{m,m}$, there is an element (point) of $\mathcal{Y}'_{m,m}$ which is contained in exactly $m$ elements of $\mathcal{X}'_{m,m}$, all of the same color.

That is, $\mathcal{X}'_{m,m}$ forms a non-decomposable $m$-fold covering of the points in $\mathcal{Y}'_{m,m}$. Moreover, for any $\varepsilon > 0$, if we choose $\delta$ small enough, then the scaling factor of each member of $\mathcal{X}'_{m,m}$ is between $1-\varepsilon$ and $1+\varepsilon$.

Now we extend $\mathcal{X}'_{m,m}$ to a non-decomposable $m$-fold covering of the whole plane as follows. We will add homothetic copies of $S$ to $\mathcal{X}'_{m,m}$ that do not contain any point in $\mathcal{Y}'_{m,m}$, but each point in the plane will be covered at least $m$ times. If we allow arbitrary small copies in the covering, then the extension is trivial since $\mathcal{Y}'_{m,m}$ is a finite point set. Just add *all* homothetic copies of $S$ that do not contain any point of $\mathcal{Y}'_{m,m}$.

If we want to keep the sizes almost equal, we have to be more careful. Points in $\mathcal{Y}'_{m,m}$ are of two types, type $a$ (resp. type $b$) is the set of those which come from a wedge in $\mathcal{E}^a_{m,m}$ (resp. $\mathcal{E}^b_{m,m}$). Observe that any two points of the same type determine a line which is almost horizontal. In fact, we can take two horizontal lines $\ell_a$ and $\ell_b$ at distance $\mathrm{Vert}(v_a v_b)$ such that all points of type $a$ (resp. type $b$) are at distance at most $\delta$ from $\ell_a$ (resp. $\ell_b$). See Figs. 4 and 5.

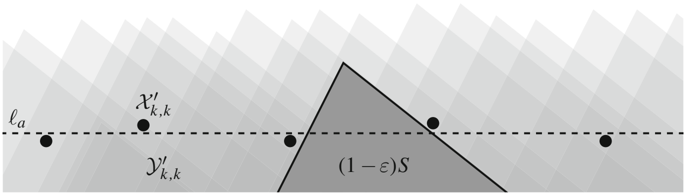
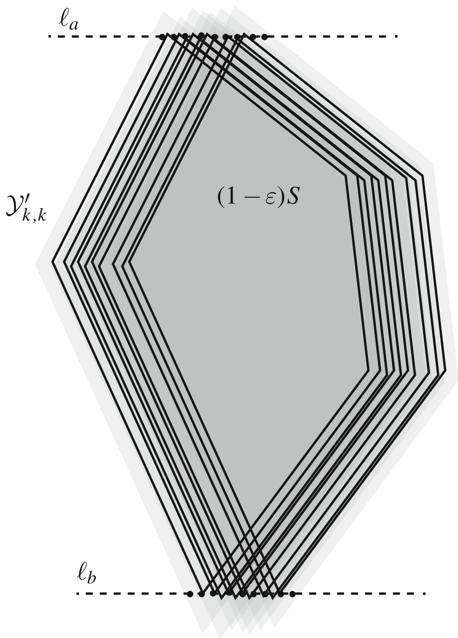



**Fig. 4** The points of type $a$ in $\mathcal{X}'_{m,m}$ are almost on the line $\ell_a$

**Fig. 5** Extending the cover by translates of $(1-\varepsilon)S$




Add all translates of $(1-\varepsilon)S$ which avoid the points in $\mathcal{Y}'_{m,m}$. Now it is not hard to see that the resulting collection is an $m$-fold covering of the whole plane, and by the construction of $\mathcal{X}'_{m,m}$ and $\mathcal{Y}'_{m,m}$, it is not decomposable. This concludes the proof of Theorem 2 and also the proof of Theorem 1 in the special case when $S$ is convex.

Now suppose that $S$ is concave with no parallel sides and let $m > 0$. Pálvölgyi [7] constructed a collection $\mathcal{X}'_{m,m}$ of *translates* of $S$ and a set $\mathcal{Y}'_{m,m}$ of points such that $\mathcal{X}'_{m,m}$ forms a non-decomposable $m$-fold covering of the points in $\mathcal{Y}'_{m,m}$. Add *all* homothetic copies of $S$ that do not contain any point of $\mathcal{Y}'_{m,m}$. The resulting collection is clearly an $m$-fold covering of the plane, and just like in the previous argument, it is not decomposable. This finishes the proof of Theorem 1.

*Remark 1* The dual version of this problem is still open. Let $S$ be a polygon of at least four sides. Is there an $m = m(S)$ with the following property? Any point set $\mathcal{P}$ can be colored with two colors such that if a homothetic copy of $S$ contains at least $m$ points of $\mathcal{P}$, then it contains points of both colors. If $S$ is concave and has no parallel sides, then the answer is NO to this question, even if we use only translates instead of homothetic copies, by the result of Pálvölgyi [7].

On the other hand, if $S$ is convex and we use only translates, then the answer is YES, by Pálvölgyi and Tóth [9]. If we do not allow arbitrarily large and arbitrarily small homothetic copies, then the answer is still YES, and the proof in [9] works also in this case. But if we allow all homothetic copies, then the problem is unsolved. See [2] for related results.

*Remark 2* We can define a hypergraph $\mathcal{H}_{k,l}$ to the pair $(\mathcal{X}_{k,l}, \mathcal{Y}_{k,l})$ in a natural way: elements of $\mathcal{X}_{k,l}$ correspond to the vertices and elements of $\mathcal{Y}_{k,l}$ correspond to the hyperedges—a hyperedge contains a vertex if and only if the corresponding elements intersect each other. The same hypergraph was used by Pálvölgyi in [7,8] to show that some concave polygons and the unit disk are not cover-decomposable.

*Remark 3* It was shown in [6] that for every $m$, there exists an $m$-fold covering of the plane with axis-parallel rectangles that cannot be decomposed into two coverings. We can slightly strengthen this result.

**Theorem 3** *For any $m > 0$, there is an $m$-fold covering of the plane with axis-parallel rectangles, each with unit horizontal side, that cannot be decomposed into two coverings.*

The proof is almost identical to the proof of Theorem 1. The main difference is that in the induction step, instead of a very small copy of $S$, we add a very short vertical segment. We omit the details.

*Remark 4* We believe that Theorem 2 can be extended to concave polygons with no parallel sides.

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