A Quantitative Variant of the Multi-colored Motzkin–Rabin Theorem

Zeev Dvir · Christian Tessier-Lavigne

Abstract We prove a quantitative version of the multi-colored Motzkin–Rabin theorem in the spirit of Barak et al. (Proceedings of the National Academy of Sciences, 2012): Let $V_1, \ldots, V_n \subset \mathbb{R}^d$ be n disjoint sets of points (of n 'colors'). Suppose that for every V_i and every point $v \in V_i$ there are at least $\delta|V_i|$ other points $u \in V_i$ so that the line connecting v and u contains a third point of another color. Then the union of the points in all n sets is contained in a subspace of dimension bounded by a function of n and δ alone.

1 Introduction

The Motzkin–Rabin (MR) theorem (see [3]) states that in a non-collinear set of points in the Euclidean plane, each colored blue or red, there always exists a monochromatic line (a line passing through at least two points and all points on the line are of the same color). Another way to state this theorem uses the following definition which we shall later generalize.

Definition 1.1 (*MR configuration*) Let $V_1, V_2 \subset \mathbb{R}^2$ be disjoint, finite sets of points in the plane. The pair V_1, V_2 is called an *MR-configuration* if every line L with $|L \cap (V_1 \cup V_2)| \geq 2$ must intersect both sets V_1 and V_2.

Z. Dvir
Department of Computer Science, Princeton University, Princeton, NJ, USA
e-mail: zeev.dvir@gmail.com

Z. Dvir · C. Tessier-Lavigne
Department of Mathematics, Princeton University, Princeton, NJ, USA

C. Tessier-Lavigne
e-mail: ctessierlavigne@gmail.com

The Motzkin–Rabin theorem can now be stated equivalently as:

Theorem 1.2 (Motzkin–Rabin Theorem) *Let V_1, $V_2 \subset \mathbb{R}^2$ be an MR-configuration. Then all points in $V_1 \cup V_2$ must belong to a single line.*

It is easy to see that one can replace $\mathbb{R}^2$ with $\mathbb{R}^d$ and that the theorem will still hold in this case (take a generic projection to the plane). This theorem answers a question first raised by Graham [7]. Theorem 1.2 was discovered independently by Motzkin [8] and Rabin (cf. Chakerian [5]. See [4, p. 306] for more references and history).

We will denote by $\mathsf{adim}(S)$ the dimension of the affine span (the smallest affine subspace containing the points) of a point set $S \subset \mathbb{R}^d$ and for a family of sets $S_1, \ldots, S_r$ we will write $\mathsf{adim}(S_1, \ldots, S_r) = \mathsf{adim}(S_1 \cup \cdots \cup S_r)$. Then, the conclusion of the MR theorem, namely all points in V_1, V_2 being on a line, can be stated as $\mathsf{adim}(V_1, V_2) \leq 1$. Hence, we can view the MR theorem as converting partial information about collinearity in the sets V_1, V_2 (the line through every pair of points of the same color contains a third point of a different color) into a global bound on the dimension of the entire configuration. A closely related theorem is the Sylvester–Gallai theorem which is a 'one color' version of the MR theorem: in every non-collinear set of points there is a line containing only two of the points.

Shannon [9] (see also [2]) proved an n-color variant of this theorem showing that if a family of n sets $V_1, \ldots, V_n$ spans $\mathbb{R}^n$ then they must define at least one monochromatic line. In this work we extend this result to the setting where the information about collinearities is only given for *many* of the lines passing through two points of the same color. To be precise we will give the following definition:

Definition 1.3 (δ, n)-*MR configuration*) Let $V_1, V_2, \ldots, V_n$ be disjoint sets of points in $\mathbb{R}^d$, and let $V = V_1 \cup V_2 \cup \cdots \cup V_n$. We say that $V_1, V_2, \ldots, V_n$ is a (δ, n)-*MR configuration* if for each V_i and for each $v \in V_i$, there are at least $\delta|V_i|$ points $u \in V_i \setminus \{v\}$ for which the line determined by v and u contains a third point in $V \setminus V_i$. For convenience we will always assume that $|V_1| \geq |V_2| \geq \cdots \geq |V_n|$.

Our main theorem gives a dimension bound for (δ, n)-MR configuration that depends only on n and δ. We do not believe our bound to be tight and conjecture that a bound of $\mathsf{poly}(n/\delta)$ holds in general.

Theorem 1.4 (Main theorem) *Let $V = V_1, V_2, \ldots, V_n \subset \mathbb{R}^d$ be a (δ, n)-MR configuration. Then, for any $0 < \varepsilon < \delta$ we have*

$$\mathsf{adim}(V) \leq \frac{C}{\varepsilon^2} \cdot \left(1 + \frac{1}{\delta - \varepsilon}\right)^n,$$

with $C > 0$ an absolute constant.[1]

Theorem 1.4 is a multi-colored version of recent results of [1,6], which give a similar 'δ-version' of the Sylvester–Gallai theorem ([1] also establishes the $n = 2$

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[1] One could set $\varepsilon = \delta/2$ to get a simpler (but worse, in some cases) bound.

case of Theorem 1.4). In fact, our proof uses one of the main results of [1,6] as its principal tool. This result, given below as Theorem 2.2, gives a lower bound on the rank of matrices whose pattern of zeros and non-zeros satisfies a certain 'design-like' condition. As the results of [1,6] work also over the complex numbers, our results (in particular, Theorem 1.4) hold also when one replaces $\mathbb{R}^d$ with $\mathbb{C}^d$ (with the same bounds).

In the next section we state some preliminaries from [1,6] that will be used in the proof of Theorem 1.4. The proof itself is given in Sect. 3.

2 Preliminaries

The main tool in the proof is a rank lower bound for *design-matrices* defined in [1]. For a vector $R \in \mathbb{F}^n$ we denote the *support* of R by $\mathsf{supp}(R) = \{i \in [n] | R_i \neq 0\}$.

Definition 2.1 (*Design matrix*) Let A be an $m \times n$ matrix over a field $\mathbb{F}$. Let $R_1, \ldots, R_m \in \mathbb{F}^n$ be the rows of A and let $C_1, \ldots, C_n \in \mathbb{F}^m$ be the columns of A. We say that A is a (q, k, t)-*design matrix* if the following three conditions are satisfied:

1. For all $i \in [m]$, $|\mathsf{supp}(R_i)| \leq q$.
2. For all $j \in [n]$, $|\mathsf{supp}(C_j)| \geq k$.
3. For all $j_1 \neq j_2 \in [n]$, $|\mathsf{supp}(C_{j_1}) \cap \mathsf{supp}(C_{j_2})| \leq t$.

The following is a quantitative improvement of a bound originally proved in [1].

Theorem 2.2 [6] *Let A by an $m \times n$ complex matrix. If A is a (q, k, t) design matrix then*

$$rank(A) \geq n - \frac{ntq(q-1)}{k}.$$

Another lemma we will use is the following lemma whose proof is a simple consequence of the existence of diagonal Latin squares.

Lemma 2.3 [1, Lemma 2.1] *Let $r \geq 3$. Then there exists a set $T \subset [r]^3$ of $r^2 - r$ triples that satisfies the following properties.*

1. *Each triple $(t_1, t_2, t_3) \in T$ consists of three distinct elements.*
2. *For each $i \in [r]$ there are exactly $3(r-1)$ triples in T that contain i as an element.*
3. *For every pair $i, j \in [r]$ of distinct elements there are at most 6 triples in T which contain both i and j as elements.*

3 Proof of the Main Theorem

Before giving the proof of Theorem 1.4 we prove some useful lemmas. The first is the technical heart of the proof and its proof utilizes the rank bound for design matrices (Theorem 2.2). In the following we will denote by $\dim(S)$ the dimension of the subspace spanned by a set S. Notice that, since $\mathsf{adim}(S) \leq \dim(S)$, we can bound $\dim(V)$ instead of $\mathsf{adim}(V)$.

Lemma 3.1 *Let* $V = \bigcup_{i=1}^{n} V_i$ *be a* (δ, n)*-MR configuration in* $\mathbb{R}^d$*. Let* x, y *be indices with* $0 \leq x < y \leq n$*. Let* $P_1 = \bigcup_{i=1}^{x} V_i$*, let* $P_2 = \bigcup_{i=x+1}^{y} V_i$*, and let* $P_3 = \bigcup_{i=y+1}^{n} V_i$ *(P_1 and P_3 might be empty if $x = 0$ or $y = n$). Suppose that for some constants $c_1, c_2 > 0$ the following two inequalities hold:*

$$|V_y| \geq c_1 |P_2|, \tag{1}$$

$$(\delta - c_2)|V_y| \geq |P_3|. \tag{2}$$

Then $\dim(P_2) \leq \dim(P_1) + 12/(c_1 c_2)$*.*

Proof We start by noting that, since $|V_1| \geq |V_2| \geq \cdots \geq |V_n|$, inequalities (1) and (2) in the lemma statement, $|V_y| \geq c_1|P_2|$ and $(\delta - c_2)|V_y| \geq |P_3|$, also hold when $|V_y|$ is replaced with $|V_i|$, for $i < y$.

We will call a line L *extraordinary* with respect to the configuration V if (1) L passes through at least one point of P_2 and (2) L passes through at least three points of $P_1 \cup P_2$. We will refer to the points of $P_1 \cup P_2$ that lie on some extraordinary line L as the *points associated with L* (such a line L might contain additional points from P_3 which are not associated with it).

Let $L_1, L_2, \ldots, L_k$ be an enumeration of the extraordinary lines of our configuration and let ℓ_i denote the number of points associated with L_i, for $1 \leq i \leq k$.

For each extraordinary line L_i we construct, using Lemma 2.3, a set T_i of $\ell_i^2 - \ell_i$ triples of points so that (1) each triple in T_i consists of three distinct points associated with L_i; (2) for any point v associated with L_i, there are exactly $3(\ell_i - 1)$ triples in T_i that contain v; and (3) for any two points $u \neq v$ associated with L_i, there are at most 6 triples in T_i that contain both u and v. Let

$$T = \bigcup_{i=1}^{k} T_i.$$

Next, let $m = |V|$ and let M be the $m \times d$ matrix whose rows are defined by the points of V (in some choice of coordinates for $\mathbb{R}^d$). We will now define a matrix A that will satisfy $A \cdot M = 0$. Each triple in T will correspond to one row of A. Every triple $t = (t_1, t_2, t_3) \in T$ consists of three distinct points in $P_1 \cup P_2$ that are collinear. Since they are collinear, there are coefficients h_1, h_2, h_3, not all zero, such that

$$h_1 t_1 + h_2 t_2 + h_3 t_3 = 0$$

(treating the points as vectors). We set the t'th row of A to have entries h_1, h_2, h_3 in the positions corresponding to the three points t_1, t_2, t_3 (we can do that since the columns of A are indexed by V) and zero elsewhere. Observe that A is a $|T| \times m$ matrix, since there is a bijection between the elements of T and the rows of A. Since the product of any row of A with M is 0, we must also have that

$$A \cdot M = 0.$$

There is a bijection between the rows of the matrix M and the points in the set V. Therefore, any subset of the set V corresponds to a submatrix of the matrix M, obtained by taking only those rows that correspond to the points in the subset. Let M_1 denote the submatrix of M corresponding to the point set P_1, and likewise let M_2 and M_3 be the submatrices corresponding to P_2 and P_3. Let A_1 be the submatrix of A obtained by taking those *columns* of A whose indices match the indices of the *rows* of M_1 (that is, with indices corresponding to elements of P_1). Define A_2 and A_3 analogously (with columns in P_2 and P_3 respectively). Observe that $A_1 M_1$, $A_2 M_2$, and $A_3 M_3$ are all valid matrix products, and that

$$A_1 M_1 + A_2 M_2 + A_3 M_3 = AM = 0.$$

From the definition of the matrix A we have that the column corresponding to any given point in P_3 contains only 0's; therefore $A_3 = 0$, and so $A_3 M_3 = 0$. Hence $A_1 M_1 + A_2 M_2 = 0$ which gives

$$\mathsf{rank}(A_2 M_2) = \mathsf{rank}(A_1 M_1) \leq \mathsf{rank}(M_1) = \dim(P_1). \tag{3}$$

(If $|P_1|$ is empty we get $A_2 M_2 = 0$ and the rest of the proof is the same.)

We now claim that:

Claim 3.2 *A_2 is a $(3, 3c_1 c_2 |P_2|, 6)$-design matrix.*

Proof By the construction of A each row contains at most three non-zero terms. Since A_2 is a submatrix of A, each row of A_2 can contain at most three non-zero terms. Similarly, by the construction of A, any two columns can share at most six non-zero locations; and again this holds for A_2 as well. Finally, we claim that each column of A_2 contains at least $3c_1 c_2 |P_2|$ non-zero entries.

Consider a column C of A_2. This column corresponds to a point p in P_2. The number of non-zero entries of C is exactly equal to the number of triples in T that contain the point p. Suppose that $p \in V_i \subset P_2$ for some i. We claim that there must be at least $\delta |V_i| - |P_3|$ points $q \neq p$ that lie on extraordinary lines through p. Observe that this quantity is at least $c_2 |V_i|$ by inequality (2). Indeed, there are at least $\delta |V_i|$ points $q \neq p$ in V_i, for which the line through q, p contains a point from some V_j, with $j \neq i$, because the configuration is (δ, n)-MR. Let us denote this set of at least $\delta |V_i|$ points by S. For each point q in S, either the line through q, p contains a third point from $P_1 \cup P_2$, and is therefore an extraordinary line, *or* (1) it contains no other points from $P_1 \cup P_2$, and (2) it contains some point r from P_3.

Thus, each point $q \in S$ that is not associated with any of the extraordinary lines passing through p corresponds to some point $r \in P_3$. Since no two $q_1 \neq q_2 \in S$ can correspond to the same r, at most $|P_3|$ of the points in S are *not* associated with any of the extraordinary lines passing through p. Thus, the remaining $\delta |V_i| - |P_3|$ points are associated with one of the extraordinary lines passing through p.

Now, if a given extraordinary line L passes through p, and if there are ℓ points associated with L besides p, then that line contributes 3ℓ triples to T that contain p. Therefore, since we showed that there are at least $c_1 c_2 |P_2|$ points other than p that lie

on the extraordinary lines passing through p, there must be at least $3c_1c_2|P_2|$ triples in T that contain p.

We conclude that that the point $p \in V_i$ is in at least $3c_2|V_i|$ triples; and since $|V_i| \geq c_1|P_2|$ (by inequality (1)), this quantity is at least $3c_1c_2|P_2|$ such points and so A_2 is indeed a $(3, 3c_1c_2|P_2|, 6)$ design matrix as claimed.   □

Applying Theorem 2.2 we have that

$$\mathsf{rank}(A_2) \geq |P_2| - 12/(c_1c_2).$$

Now, using basic linear algebra, we get that

$$\mathsf{rank}(A_2M_2) \geq \mathsf{rank}(M_2) - (|P_2| - \mathsf{rank}(A_2)) \geq \mathsf{rank}(M_2) - 12/(c_1c_2).$$

Using Eq. (3) we immediately get

$$\mathsf{rank}(M_2) \leq \dim(P_1) + 12/(c_1c_2),$$

which implies $\dim(P_2) \leq \dim(P_1) + 12/(c_1c_2)$ as was required. This completes the proof of Lemma 3.1.   □

To state the next lemma we will need the following definition.

Definition 3.3 (ε-*Large and ε-small indices*) Let $V_1, \ldots, V_n \subset \mathbb{R}^d$ be a (δ, n)-MR configuration and let $c_\varepsilon = 1/(\delta - \varepsilon)$ with $0 < \varepsilon < \delta$ some real number. We call an index $k \in [n]$ an ε-*large* index if

$$|V_k| \geq c_\varepsilon(|V_{k+1}| + |V_{k+2}| + \cdots + |V_n|),$$

otherwise we say that k is ε-*small*. By convention, we say that n is always ε-large.

Lemma 3.4 *Let $V_1, V_2, \ldots, V_n \subset \mathbb{R}^d$ be a (δ, n)-MR configuration, and suppose x and y are integers with $0 \leq x < y \leq n$ such that y is an ε-large index, and each of the indices $x+1, x+2, \ldots, y-2, y-1$ is ε-small. Then, for each i with $0 \leq i \leq y-x-1$ we have*

$$\sum_{j \geq y-i} |V_j| \leq 2(1 + c_\varepsilon)^i \cdot |V_y|.$$

Proof We will prove the lemma by induction on i. To prove the base case, $i = 0$, we need to show that

$$\sum_{j \geq y} |V_j| \leq 2|V_y|.$$

Since y is an ε-large index, we have

$$|V_y| \geq c_\varepsilon \sum_{j > y} |V_j|$$

and so $\sum_{j>y} |V_j| \leq 1/c_\varepsilon |V_y|$. By adding $|V_y|$ to both sides we immediately have that

$$\sum_{j\geq y} |V_j| \leq (1 + 1/c_\varepsilon)|V_y|,$$

which gives the desired bound since $c_\varepsilon > 1$ and so $1 + 1/c_\varepsilon < 2$.

Now suppose the claim holds for $i = k$. We wish to show that it also holds for $i = k+1$, assuming that $k+1 \leq y - x - 1$. From the induction we have that

$$\sum_{j\geq y-k} |V_j| \leq 2(1 + c_\varepsilon)^k |V_y|.$$

We also know that $y - (k+1)$ is an ε-small index, so

$$|V_{y-(k+1)}| \leq c_\varepsilon \sum_{j\geq y-k} |V_j|.$$

Substituting the first inequality into the second gives

$$|V_{y-(k+1)}| < 2c_\varepsilon(1 + c_\varepsilon)^k |V_y|.$$

Then adding this inequality to the first inequality yields the desired result. □

Corollary 3.5 *Under the same notations and conditions as Lemma 3.4, we have*

$$|V_y| \geq \frac{1}{2(1 + c_\varepsilon)^{y-x-1}} \sum_{j=x+1}^{y} |V_j|.$$

Proof Apply Lemma 3.4 with $i = y - x - 1$ to get that

$$\sum_{j=x+1}^{n} |V_j| \leq 2(1 + c_\varepsilon)^{y-x-1} \cdot |V_y|,$$

hence

$$\sum_{j=x+1}^{y} |V_j| \leq 2(1 + c_\varepsilon)^{y-x-1} \cdot |V_y|$$

and the corollary follows. □

3.1 Proof of Theorem 1.4

Let $d_1 < d_2 < \cdots < d_k = n$ be the ε-large indices determined by V (see Definition 3.3) and let us define $d_0 = 0$. We define

$$W_1 = V_1 \cup V_2 \cup \cdots \cup V_{d_1};$$
$$W_2 = V_{d_1+1} \cup V_{d_1+2} \cup \cdots \cup V_{d_2}$$

etc. for $1 \leq i \leq k$. Let

$$m_i = \dim(W_1 \cup W_2 \cup \cdots \cup W_i),$$

for $1 \leq i \leq k$ and set $m_0 = 0$.

Consider W_i for some $1 \leq i \leq k$. Since d_i is an ε-large index, we have that

$$|V_{d_i}| \geq c_\varepsilon(|V_{d_i+1}| + |V_{d_i+2}| + \cdots + |V_n|)$$

with $c_\varepsilon = 1/(\delta - \varepsilon)$. Hence

$$(\delta - \varepsilon)|V_{d_i}| \geq |V_{d_i+1}| + |V_{d_i+2}| + \cdots + |V_n|.$$

Furthermore, each of $d_{i-1}+1, d_{i-1}+2, \ldots, d_i - 2, d_i - 1$ are ε-small indices, so by Corollary 3.5 we have

$$|V_{d_i}| \geq \frac{1}{2(1+c_\varepsilon)^{d_i - d_{i-1}-1}}(|V_{d_{i-1}+1}| + |V_{d_{i-1}+2}| + \cdots + |V_{d_i-1}| + |V_{d_i}|).$$

Therefore our configuration satisfies the conditions of Lemma 3.1, with $x = d_{i-1}$, $y = d_i$, $c_1 = \frac{1}{2(1+c_\varepsilon)^{d_i-d_{i-1}-1}}$, and $c_2 = \varepsilon$. For these values of x and y, the set P_1 defined in the lemma equals $W_1 \cup W_2 \cup \cdots \cup W_{i-1}$, and the set P_2 equals W_i. Therefore we get that

$$\dim(W_i) \leq m_{i-1} + (24/\varepsilon) \cdot (1 + c_\varepsilon)^{d_i-d_{i-1}-1}.$$

Now, since $m_i \leq m_{i-1} + \dim(W_i)$, we have that

$$m_i \leq 2m_{i-1} + (24/\varepsilon) \cdot (1 + c_\varepsilon)^{d_i-d_{i-1}-1}.$$

Claim 3.6 *For all $0 \leq i \leq k$ we have*

$$m_i \leq \frac{24}{\varepsilon} \sum_{1 \leq j \leq i} 2^{i-j}(1+c_\varepsilon)^{d_j-d_{j-1}-1}.$$

Proof We prove the claim by induction on i. The base case, $i = 0$, holds since $m_0 = 0$. Suppose the claim holds for $i = h$ and consider the case $i = h + 1$. By induction we have that

$$m_h \leq \frac{24}{\varepsilon} \sum_{1 \leq j \leq h} 2^{h-j}(1+c_\varepsilon)^{d_j-d_{j-1}-1}.$$

We also showed that

$$m_{h+1} \le 2m_h + \frac{24}{\varepsilon}(1 + c_\varepsilon)^{d_{h+1}-d_h-1}.$$

Substituting the first inequality into the second we find that

$$m_{h+1} \le \frac{24}{\varepsilon} \sum_{1 \le j \le h} 2^{h+1-j}(1 + c_\varepsilon)^{d_j - d_{j-1} - 1} + \frac{24}{\varepsilon}(1 + c_\varepsilon)^{d_{h+1}-d_h-1}$$

which gives the desired result. $\square$

Using the claim for $i = k$ we get

$$m_k \le \frac{24}{\varepsilon} \sum_{1 \le j \le k} 2^{k-j}(1 + c_\varepsilon)^{d_j - d_{j-1} - 1}.$$

Observe that for all j, $d_j - d_{j-1} \le n - k + 1$. This follows from the fact that the d_j are strictly increasing, $d_0 = 0$, and $d_k = n$. Therefore, the summand $2^{k-j}(1+c_\varepsilon)^{d_j-d_{j-1}-1}$ is at most $2^{k-j}(1 + c_\varepsilon)^{n-k}$, which in turn is at most $(1 + c_\varepsilon)^n \cdot \left(\frac{2}{1+c_\varepsilon}\right)^k$. Adding these together we get that

$$m_k \le \frac{24 \cdot k}{\varepsilon}(1 + c_\varepsilon)^n \cdot \left(\frac{2}{1 + c_\varepsilon}\right)^k.$$

Observe that, since $c_\varepsilon = 1/(\delta - \varepsilon) > 1/(1 - \varepsilon) > 1 + \varepsilon$, we have $2/(1 + c_\varepsilon) < 2/(2 + \varepsilon)$ and so we get that

$$m_k \le \frac{24 \cdot k}{\varepsilon} \left(\frac{2}{2 + \varepsilon}\right)^k (1 + c_\varepsilon)^n.$$

The expression $\frac{24 \cdot k}{\varepsilon}\left(\frac{2}{2+\varepsilon}\right)^k$ is maximized when $k = -\frac{1}{\ln(2/(2+\varepsilon))} = O(1/\varepsilon)$ and so we get

$$m_k \le \frac{C}{\varepsilon^2} \cdot (1 + c_\varepsilon)^n$$

For some absolute constant C. Since $m_k = \dim(W_1 \cup W_2 \cup \cdots \cup W_n) = \dim(V_1, \ldots, V_n)$, the proof of the theorem is complete.

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