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A New Lower Bound Based on Gromov's Method of Selecting Heavily Covered Points

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Abstract Boros and Füredi (for $d = 2$) and Bárány (for arbitrary d) proved that there exists a positive real number c_d such that for every set P of n points in $\mathbf{R}^d$ in general position, there exists a point of $\mathbf{R}^d$ contained in at least $c_d\binom{n}{d+1}$ d-simplices with vertices at the points of P. Gromov improved the known lower bound on c_d by topological means. Using methods from extremal combinatorics, we improve one of the quantities appearing in Gromov's approach and thereby provide a new stronger lower bound on c_d for arbitrary d. In particular, we improve the lower bound on c_3 from 0.06332 to more than 0.07480; the best upper bound known on c_3 being 0.09375.

Keywords Flag algebras · Covering points by simplicies · Cofilling profiles · Boros–Füredi–Bárány–Pach–Gromov theorem

1 Introduction

We study an extremal graph theory problem linked to a classical geometric problem through a recent work of Gromov [8]. The geometric result that initiated this work is

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a theorem of Bárány [2], which extends an earlier generalization of Carathéodory's theorem due to Boros and Füredi [4].

Theorem 1 (Bárány [2]) *Let d be a positive integer. There exists a positive real number c such that for every set P of points in $\mathbf{R}^d$ that are in general position, there is a point of $\mathbf{R}^d$ that is contained in at least*

$$c \cdot \binom{|P|}{d+1} - O\big(|P|^d\big) \tag{1}$$

d-dimensional simplices spanned by the points in P.

Define c_d to be the supremum of all the real numbers that satisfy (1) in Theorem 1 for the dimension d.

Bukh, Matoušek and Nivasch [6] established that

$$c_d \le \frac{(d+1)!}{(d+1)^{d+1}}$$

by constructing suitable configurations of n points in $\mathbf{R}^d$. On the lower bound side, Boros and Füredi [4] proved that $c_2 \ge 2/9$, which matches the upper bound; so $c_2 = 2/9$ (another proof was given by Bukh [5]). Bárány's proof [2] yields $c_d \ge (d+1)^{-d}$. Wagner [18] improved this lower bound to

$$c_d \ge \frac{d^2+1}{(d+1)^{d+1}}.$$

Further improvements of the lower bound for c_3 were established by Basit et al. [3] and by Matoušek and Wagner [15].

Gromov [8] developed a topological method for establishing lower bounds on c_d (Matoušek and Wagner [15] provided an exposition of the combinatorial components of his method, while Karasev [13] managed to simplify Gromov's approach). His method yields a bound that matches the optimal bound for $d = 2$ and is better than that of Basit et al. [3] for $d = 3$. We need several definitions to state Gromov's lower bound. Fix a positive integer d and a finite set V. A *d-system E on V* is a family of d-element subsets of V. The *density* of the system E is $\|E\| := |E|/\binom{|V|}{d}$. The *coboundary* δE of a d-system E on V is the $(d+1)$-system composed of those $(d+1)$-element subsets of V that contain an odd number of sets of E. The coboundary operator δ commutes with the symmetric difference, i.e., $\delta(A \triangle B) = (\delta A)\triangle(\delta B)$. It is not hard to show that $\delta\delta E = \mathbf{0}$ for any d-system E where $\mathbf{0}$ is the empty $(d+2)$-system. In fact, the converse also holds: a d-system E is a coboundary of a $(d-1)$-system if and only if $\delta E = \mathbf{0}$.

A d-system E on V is *minimal* if $\|E\| \le \|E'\|$ for any d-system E' on V with $\delta E = \delta E'$. This is equivalent to saying that $\|E\| \le \|E \triangle \delta D\|$ for every $(d-1)$-system D on V. Let $\mathcal{M}_d(V)$ be the set of all minimal d-systems on V and define the following function:

$$\varphi_d(\alpha) := \liminf_{|V|\to\infty} \min\big\{\|\delta E\| \mid E \in \mathcal{M}_d(V) \text{ and } \|E\| \ge \alpha\big\}.$$

It is easy to observe that the functions φ_d are defined for $\alpha \in [0, 1/2]$ and $\varphi_1(\alpha) = 2\alpha(1-\alpha)$. It can also be shown that $\varphi_d(\alpha) \geq \alpha$.

Gromov's lower bound on the quantity c_d is given in the next theorem.

Theorem 2 (Gromov [8]) *For every positive integer d, we have*

$$c_d \geq \varphi_d\left(\frac{1}{2}\varphi_{d-1}\left(\frac{1}{3}\varphi_{d-2}\left(\cdots\frac{1}{d}\varphi_1\left(\frac{1}{d+1}\right)\cdots\right)\right)\right). \tag{2}$$

Plugging $\varphi_1(\alpha) = 2\alpha(1-\alpha)$ and the bound $\varphi_d(\alpha) \geq \alpha$ in (2), we obtain

$$c_d \geq \frac{2d}{(d+1)!(d+1)}. \tag{3}$$

Improvements of the bound in (3) can be obtained by proving stronger lower bounds on the functions φ_d. The first step in this direction has been done by Matoušek and Wagner.

Theorem 3 (Matoušek and Wagner [15])

- *For all $\alpha \in [0, 1/4]$, we have*

$$\varphi_2(\alpha) \geq \frac{3}{4}\bigl(1-\sqrt{1-4\alpha}\bigr)(1-4\alpha).$$

- *For all sufficiently small $\alpha > 0$, we have*

$$\varphi_3(\alpha) \geq \frac{4}{3}\alpha - O\bigl(\alpha^2\bigr).$$

Our main result asserts a stronger lower bound on $\varphi_2(\alpha)$ for $\alpha \in [0, 2/9]$, which are the values appearing in Theorem 2.

Theorem 4 *For all $\alpha \in [0, 2/9]$, we have*

$$\varphi_2(\alpha) \geq \frac{3}{4}\alpha\bigl(3-\sqrt{8\alpha+1}\bigr).$$

When plugged into Theorem 2, our bound yields $c_3 > 0.07433$. For comparison, the earlier bounds of Wagner [18], Basit et al. [3], Gromov [8] and Matoušek and Wagner [15] are $c_3 \geq 0.03906$, $c_3 \geq 0.05448$, $c_3 \geq 0.0625$ and $c_3 \geq 0.06332$, respectively. However, the bound on c_3 can be further improved as we now explain.

Matoušek and Wagner [15] improved the bound on c_3 through a combinatorial argument, which uses bounds on φ_2 and φ_3 as black-boxes. The proof employs a structure called pagoda (of dimension 3) consisting of a 4-system G (which is referred to as the *top* of the pagoda), 3-systems F_{ijk} (with $1 \leq i < j < k \leq 4$), 2-systems E_{ij} (with $1 \leq i < j \leq 4$) and 1-systems V_i (with $1 \leq i \leq 4$). For a precise definition of these sets and their interplay, we refer the reader to [15, Sect. 6]. Any lower bound on the density of G in a pagoda is also a lower bound on c_3. Gromov's approach is applicable to pagodas and it yields $\|G\| \geq \frac{1}{16} = 0.0625$ (using the trivial bounds on φ_2 and φ_3). Matoušek and Wagner investigated pagodas with $\|G\| = 0.0625 + \varepsilon$ and they obtain a contradiction for $\varepsilon \leq 0.00082$; this proves that $\|G\| \geq 0.06332$.

We can improve the bound using our Theorem 4 on φ_2 by investigating pagodas with density $3(3-\sqrt{2})/64 + \varepsilon$. This leads to the following system of inequalities:

$$\varphi_2(0.125 + 2\cdot\varepsilon_0) \leq 2\cdot\left(\frac{3(3-\sqrt{2})}{64} + \varepsilon\right)$$

$$\varphi_1(0.25 + \varepsilon_1) \leq 3\cdot(0.125 + 2\varepsilon_0)$$

$$4\cdot\varphi_1(0.25 - 3\varepsilon_1) \geq 4\left(\frac{3}{8} - \varepsilon_2\right)$$

$$2\cdot\left(\frac{3(3-\sqrt{2})}{64} + \varepsilon\right) \geq -6\varepsilon_0 + 24\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2 - \frac{27}{4}\varepsilon_1 - \frac{3}{2}\varepsilon_2 + \frac{3}{16}.$$

This system of inequalities together with the exact value of φ_1, Theorem 4 and the trivial (linear) bound on φ_3 yields a contradiction for every $\varepsilon \leq 0.00047$. This leads to the lower bound $c_3 \geq 0.07480$.

The definition of the function φ_2 can naturally be cast in the language of graphs. A *cut* of a graph G is a partition of the vertices of G into two (disjoint) parts; a (non-)edge that crosses the partition is said to be *contained* in the cut. A graph is *Seidel-minimal* if no cut contains more edges than non-edges. It is straightforward to see that a graph G with vertex set V is Seidel-minimal if and only if its edge-set viewed as a 2-system is minimal. Let $\mathcal{S}_n(\alpha)$ be the set of all Seidel-minimal graphs on n vertices with density at least α, i.e., with at least $\alpha\binom{n}{2}$ edges. Further, let $\mathcal{S}(\alpha)$ be the union of all $\mathcal{S}_n(\alpha)$.

A triple T of vertices of a graph G is *odd* if the subgraph of G induced by T contains precisely either one or three edges. Finally, let $\varphi_g(G)$ for a graph G be the density of odd triples in G, i.e.,

$$\varphi_g(G) = \frac{|\{T \in \binom{V(G)}{3} \mid T \text{ is odd}\}|}{\binom{|V(G)|}{3}}.$$

It is not hard to show that for every $\alpha \in [0, 1/2]$,

$$\varphi_2(\alpha) = \liminf_{n\to\infty} \min\bigl\{\varphi_g(G) \mid G \in \mathcal{S}_n(\alpha)\bigr\}.$$

Using this reformulation to the language of graph theory, we show that $\varphi_2(\alpha) \geq \frac{3}{4}\alpha(3 - \sqrt{8\alpha+1})$ for $\alpha \in [0, 2/9]$. Our proof is based on the notion of flag algebras developed by Razborov [16], which builds on the work of Lovász and Szegedy [14] on graph limits and of Freedman et al. [7]. The notion was further applied, e.g., in [1, 9–12, 17]. We do not use the full strength of this notion here and we survey the relevant parts in Sect. 2 to make the paper as much self-contained as possible. In Sect. 3, we establish a weaker bound $\varphi_2(\alpha) \geq \frac{9}{7}\alpha(1-\alpha)$ using just some of the methods presented in Sect. 2. The purpose of Sect. 3 is to get the reader acquainted with the notation. Our main result is proved in Sect. 4.

2 Flag Algebras

In this section, we review some of the theory related to flag algebras, which were introduced by Razborov [16]. We focus on the concepts that are relevant to our proof. The reader is referred to the seminal paper of Razborov [16] for a complete and detailed exposition of the topic.

Fix $\alpha > 0$ and consider a sequence of graphs $(G_i)_{i\in\mathbf{N}}$ from $\mathcal{S}(\alpha)$ such that

$$\lim_{i\to\infty} \left|V(G_i)\right| = \infty \quad \text{and} \quad \lim_{i\to\infty} \varphi_g(G_i) = \varphi_2(\alpha).$$

Let $p(H, H_0)$ be the probability that a randomly chosen subgraph of H_0 with $|V(H)|$ vertices is isomorphic to H. The sequence G_i must contain a subsequence $(G_{i_j})_{j\in\mathbf{N}}$ such that $\lim_{j\to\infty} p(H, G_{i_j})$ exists for every graph H. Define $q_\alpha(H) := \lim_{j\to\infty} p(H, G_{i_j})$. Observe that the definition of q_α implies that $q_\alpha(K_2) \geq \alpha$ and $q_\alpha(\overline{P_3}) + q_\alpha(K_3) = \varphi_2(\alpha)$ where $\overline{P_3}$ is the complement of the 3-vertex path.

The values of $q_\alpha(H)$ for various graphs H are highly correlated. Let $\mathcal{F}$ be the set of all graphs and $\mathcal{F}_\ell$ the set of graphs with ℓ vertices. Extend the mapping $q_\alpha(H)$ from $\mathcal{F}$ to $\mathbf{R}\mathcal{F}$ by linearity, where $\mathbf{R}\mathcal{F}$ is the linear space of formal linear combinations of the elements of $\mathcal{F}$ with real coefficients. Next, let $\mathcal{K}$ be the subspace of $\mathbf{R}\mathcal{F}$ generated by the elements of the form

$$H_0 - \sum_{H\in\mathcal{F}_\ell} p(H_0, H)H$$

for all graphs H_0 and all $\ell > |V(H_0)|$. Since the quantity $p(H_0, G)$ and the sum $\sum_{H\in\mathcal{F}_\ell} p(H_0, H)p(H, G)$ are equal for any graph G with at least ℓ vertices, $\mathcal{K}$ is a subset of the kernel of q_α, i.e., $q_\alpha(F) = q_\alpha(F + F')$ for every $F \in \mathbf{R}\mathcal{F}$ and $F' \in \mathcal{K}$.

Let $p(H_1, H_2; H_0)$ be the probability that two randomly chosen disjoint subsets V_1 and V_2 with cardinalities $|V(H_1)|$ and $|V(H_2)|$ induce in H_0 subgraphs isomorphic to H_1 and H_2, respectively. For two graphs H_1 and H_2, define their product to be

$$H_1 \times H_2 := \sum_{H_0\in\mathcal{F}_\ell} p(H_1, H_2; H_0)H_0$$

where $\ell = |V(H_1)| + |V(H_2)|$. The product operator can be extended to $\mathbf{R}\mathcal{F} \times \mathbf{R}\mathcal{F}$ by linearity. Since the product operator defined in this way is consistent with the equivalence relation on the elements of $\mathbf{R}\mathcal{F}$ induced by $\mathcal{K}$, we can consider the quotient $\mathcal{A} := \mathbf{R}\mathcal{F}/\mathcal{K}$ as an algebra with addition and multiplication. Since q_α is consistent with $\mathcal{K}$, the function q_α naturally gives rise to a mapping from $\mathcal{A}$ to $\mathbf{R}$, which is in fact a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. In what follows, we use q_α for this homomorphism exclusively. To simplify our notation, we will use $q_\alpha(F)$ for $F \in \mathbf{R}\mathcal{F}$ but we also keep in mind that F stands for a representative of the equivalence class of $\mathbf{R}\mathcal{F}/\mathcal{K}$.

A homomorphism $q : \mathcal{A} \to \mathbf{R}$ is *positive* if $q(F) \geq 0$ for every $F \in \mathcal{F}$. Positive homomorphisms are precisely those corresponding to the limits of convergent graph sequences. We write $F \geq 0$ for $F \in \mathcal{A}$ if $q(F) \geq 0$ for any positive homomorphism q.

Such $F \in \mathcal{A}$ form the semantic cone $\mathcal{C}_{\mathrm{sem}}(\mathcal{A})$. Razborov [16] developed various general and deep methods for proving that $F \geq 0$ for $F \in \mathcal{A}$. Here, we will use only one of them, which we now present. The reader may also check the paper [17] for the exposition of the method in a more specific context.

Consider a graph σ and let $\mathcal{F}^\sigma$ be the set of graphs G equipped with a mapping $\nu : \sigma \to V(G)$ such that ν is an embedding of σ in G, i.e., the subgraph induced by the image of ν is isomorphic to σ. We can extend the definitions of the quantities $p(H, H_0)$ and $p(H_1, H_2; H_0)$ to this "labeled" case by requiring that the randomly chosen sets always include the image of ν and preserve the mapping ν. In particular, $p(H_1, H_2; H_0)$ is the probability that two randomly chosen supersets of the image of σ in H_0 with sizes $V(H_1)$ and $V(H_2)$ that intersect exactly on σ induce subgraphs of H_0 isomorphic to H_1 and H_2; Similarly as before, one can define $\mathcal{K}^\sigma$, $\mathcal{A}^\sigma = \mathbf{R}\mathcal{F}^\sigma / \mathcal{K}^\sigma$ as an algebra with addition and multiplication, positive homomorphisms, etc.

The intuitive interpretation of homomorphisms from $\mathcal{A}^\sigma$ to $\mathbf{R}$ is as follows: for a fixed embedding ν of σ, the value $q_\nu(F)$ for $F \in \mathcal{F}^\sigma$ is the probability that a randomly chosen superset of the image of ν induces a subgraph isomorphic to F. A positive homomorphism q from $\mathcal{A}$ to $\mathbf{R}$ gives rise to a unique probability distribution on positive homomorphisms q^σ from $\mathcal{A}^\sigma$ to $\mathbf{R}$ such that this probability distribution is the limit of the probability distributions of homomorphisms q_ν from $\mathcal{A}^\sigma$ to $\mathbf{R}$ given by random choices of ν in the graphs in any convergent sequence corresponding to q, see [16, Sect. 3.2] for details.

Consider a graph H with an embedding ν of σ in G. Define $[\![H]\!]_\sigma$ to be the element $p \cdot H$ of $\mathcal{A}$ where p is the probability that a randomly chosen mapping ν from $V(\sigma)$ to $V(H)$ is an embedding of σ in H. Hence, the operator $[\![\cdot]\!]_\sigma$ maps elements of $\mathcal{F}^\sigma$ to $\mathcal{A}$ and it can be extended from $\mathcal{F}^\sigma$ to $\mathcal{A}^\sigma$ by linearity. For a positive homomorphism q from $\mathcal{A}$ to $\mathbf{R}$, the value of $q([\![H]\!]_\sigma)$ for $H \in \mathcal{A}^\sigma$ is the expected value of $q^\sigma(H)$ with respect to the probability distribution on q^σ corresponding to q. In particular, if $q^\sigma(H) \geq 0$ with probability one, then $q([\![H]\!]_\sigma) \geq 0$.

2.1 Example

As an example of the introduced formalism, we prove that $\varphi_2(\alpha) \geq \alpha$. The following notation is used: K_n is the complete graph with n vertices, P_n is the n-vertex path and $\overline{K}_n$ and $\overline{P}_n$ are their complements, respectively. We also use 1 for K_1 to simplify the notation. The following elements of $\mathcal{A}^1$ will be of particular interest to us: $P_3^{1,b}$ is P_3 with 1 embedded to the end vertex of the path and $P_3^{1,c}$ is P_3 with 1 embedded to the central vertex; $\overline{P_3}^{1,b}$ and $\overline{P_3}^{1,c}$ are their complements, respectively. See Fig. 1 for an illustration of this notation.

Consider the homomorphism q_α from $\mathcal{A}$ to $\mathbf{R}$. Recall that $\alpha \leq q_\alpha(K_2)$. Since we have

$$K_2 - \frac{1}{3}\overline{P_3} - \frac{2}{3}P_3 - K_3 \in \mathcal{K},$$

we obtain

$$\alpha \leq q_\alpha\left(\frac{1}{3}\overline{P_3} + \frac{2}{3}P_3 + K_3\right). \tag{4}$$

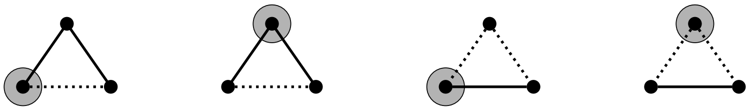


Fig. 1 Four elements of $\mathcal{A}^1$

We now use that the graphs in the sequence defining q_α are Seidel-minimal. Let G_i be a graph in this sequence, n the number of its vertices and v an arbitrary vertex of G_i. Let A be the neighbors of v and B its non-neighbors. Since G is Seidel-minimal, the number of edges between A and B does not exceed the number of non-edges between A and B (increased by $O(n)$ for the inclusion of v in one or the other side of the cut; however, this term will vanish in the limit). So, if $\sigma = 1$ is an embedding of K_1, we have $q_\alpha^1(\overline{P_3}^{1,b} - P_3^{1,b}) \geq 0$ with probability one (the term $q_\alpha^1(\overline{P_3}^{1,b})$ represents the number of non-edges between neighbors and non-neighbors of the target vertex of σ and $q_\alpha^1(P_3^{1,b})$ the number of edges). Therefore, we obtain

$$0 \leq q_\alpha\big(\llbracket \overline{P_3}^{1,b} - P_3^{1,b} \rrbracket_1\big). \tag{5}$$

Applying the operator $\llbracket \cdot \rrbracket_1$ in (5) yields

$$0 \leq q_\alpha\left(\frac{2}{3}\overline{P_3} - \frac{2}{3}P_3\right). \tag{6}$$

Summing (4) and (6) (recall that q_α is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$), we obtain

$$\alpha \leq q_\alpha(\overline{P_3} + K_3) = \varphi_2(\alpha).$$

This completes the proof.

A similar argument applied to the algebra based on d-uniform hypergraphs yields $\varphi_d(\alpha) \geq \alpha$. However, since we do not want to introduce additional notation not necessary for the exposition in the rest of the paper, we omit further details.

3 First Bound

To become more acquainted with the method, we now present a bound that is both weaker and simpler than our main result. Fix the enumeration of 4-vertex graphs as in Fig. 2. To simplify our formulas, $q_\alpha(\sum_{i=1}^{11} \xi_i F_i)$ shall simply be written $q_\alpha(\xi_1, \dots, \xi_{11})$.

Theorem 5 *For every* $\alpha \in [0, 2/9]$, *it holds that* $q_\alpha(\overline{P_3} + K_3) \geq \frac{9}{7}\alpha(1-\alpha)$.

Proof We first establish three inequalities on the values taken by q_α for various elements of $\mathcal{A}$. The choice of the graphs in the sequence defining q_α implies that

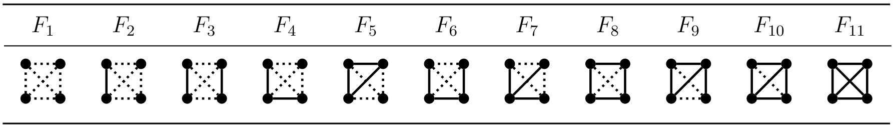
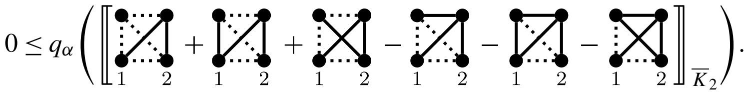


Fig. 2 The eleven non-isomorphic graphs with 4 vertices

$\alpha \le q_\alpha(K_2)$. As $q_\alpha(\overline{K}_2) = 1 - q_\alpha(K_2)$ and $q_\alpha(K_2) \in [0, 1/2]$, we infer that

$$\alpha(1-\alpha) \le q_\alpha(K_2)q_\alpha(\overline{K}_2) = q_\alpha(K_2 \times \overline{K}_2) = q_\alpha\left(0, \frac{1}{6}, 0, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2}, 0, \frac{1}{3}, \frac{1}{6}, 0\right). \tag{7}$$

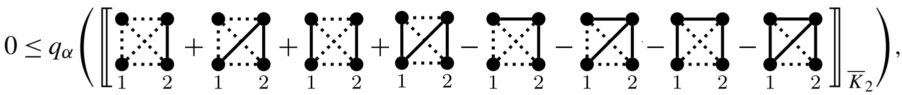
The other two inequalities follow from the Seidel-minimality of graphs in the sequence defining q_α. Consider a graph G_i and two non-adjacent vertices v_1 and v_2 (the target vertices of an embedding of $\overline{K_2}$ in elements of $\mathcal{F}^{\overline{K_2}}$ are marked by the numbers 1 and 2). Let A be the set of their common neighbors and B the set of the remaining vertices. Applying the Seidel-minimality to the cut given by A and B, we obtain the following inequality in the limit (the elements of $\mathcal{F}_4^{\overline{K_2}}$ with a non-edge between a common neighbor of 1 and 2 and a vertex that is not their common neighbor appear with the coefficient $+1$, those with an edge between two such vertices with the coefficient -1).



Evaluating the operator $[\![\cdot]\!]_{\overline{K}_2}$ yields

$$0 \le q_\alpha\left(0, 0, 0, \frac{1}{6}, 0, \frac{1}{3}, -\frac{1}{2}, 0, -\frac{1}{3}, 0, 0\right). \tag{8}$$

Now, let A' be the neighbors of v_2 and B' its non-neighbors. The Seidel-minimality of cuts of this type (the elements of $\mathcal{F}_4^{\overline{K_2}}$ with a non-edge between a neighbor of 2 and a non-neighbor of 2 appear with the coefficient $+1$ and those with an edge with the coefficient -1) yields



which subsequently implies that

$$0 \le q_\alpha\left(0, \frac{1}{3}, \frac{2}{3}, 0, 0, 0, -\frac{1}{2}, 0, -\frac{1}{6}, 0, 0\right). \tag{9}$$

The sum of (7), (8) and (9) with coefficients 9/7, 3/7 and 6/7 is the following inequality:

$$\frac{9}{7}\alpha(1-\alpha) \le q_\alpha\left(0, \frac{1}{2}, \frac{4}{7}, \frac{1}{2}, \frac{9}{14}, \frac{5}{14}, 0, 0, \frac{1}{7}, \frac{3}{14}, 0\right). \tag{10}$$

Since q_α is positive, we infer from (10) that

$$\frac{9}{7}\alpha(1-\alpha) \le q_\alpha\left(0, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1\right) = q_\alpha(\overline{P_3} + K_3). \qquad \square$$

4 Improved Bound

This section is devoted to the proof of Theorem 4. We equivalently prove the following.

Theorem 6 *For every* $\alpha \in [0, 2/9]$, *it holds that* $q_\alpha(\overline{P_3} + K_3) \ge \frac{3}{4}\alpha(3 - \sqrt{8\alpha + 1})$.

Proof Let $\beta := q_\alpha(K_2)$. Note that $\beta \in [\alpha, 1/2]$. We first derive two equalities using the fact that q_α is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. The first equation is a trivial corollary of this fact.

$$1 = q_\alpha(K_1) = q_\alpha(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1). \tag{11}$$

The choice of β implies that

$$\beta = q_\alpha(K_2) = q_\alpha\left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6}, 1\right). \tag{12}$$

The next equality is little bit more tricky. We use that $q_\alpha(K_2) - \beta = 0$.

$$0 = \big(q_\alpha(K_2) - \beta\big) q_\alpha(\overline{K}_2) = q_\alpha(K_2 \times \overline{K}_2 - \beta \overline{K}_2). \tag{13}$$

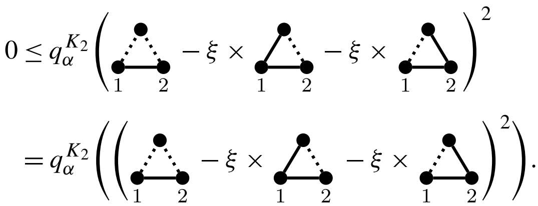
Again, we express (13) in terms of the four-vertex graphs:

$$0 = q_\alpha\left(-\beta, \frac{1-5\beta}{6}, \frac{-2\beta}{3}, \frac{1-2\beta}{3}, \frac{1-\beta}{2}, \frac{1-3\beta}{6}, \frac{1-\beta}{2}, \frac{-\beta}{3}, \frac{1-\beta}{3}, \frac{1-\beta}{6}, 0\right). \tag{14}$$

The next inequality is the inequality (9) established in the proof of Theorem 5. We copy the inequality to ease the reading.

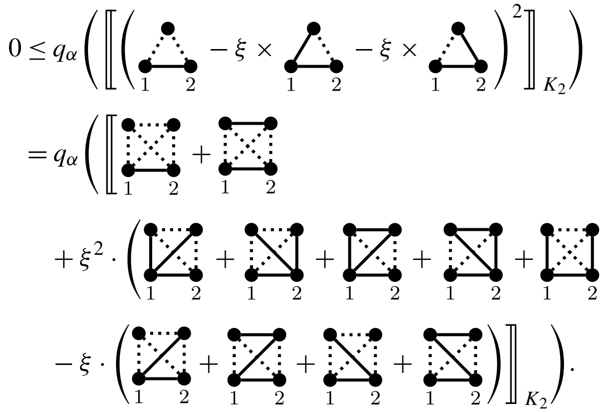
$$0 \le q_\alpha\left(0, 0, 0, \frac{1}{6}, 0, \frac{1}{3}, -\frac{1}{2}, 0, -\frac{1}{3}, 0, 0\right). \tag{15}$$

The final inequality is obtained by considering random homomorphisms $q_\alpha^{K_2}$. Since $q_\alpha^{K_2}$ is a homomorphism, it holds for every choice of $q_\alpha^{K_2}$ and every $\xi \in \mathbf{R}$ that



(16)

Hence,



(17)

Evaluating the operator $[\![\cdot]\!]_{K_2}$ yields the following inequality:

$$0 \le q_\alpha\left(0, \frac{1}{6}, \frac{1}{3}, \frac{-\xi}{3}, 0, \frac{\xi^2-2\xi}{6}, \frac{\xi^2}{2}, \frac{2\xi^2}{3}, \frac{\xi^2}{6}, 0, 0\right). \tag{18}$$

As an example of the evaluation, consider the third coordinate: the only four-vertex graph with two non-incident edges appears with the coefficient one in the sum. The probability that a randomly chosen pair of vertices in the four-vertex graph formed by two non-incident edges shows that this term of the sum is $1/3$ which is the third coordinate of the final vector.

Now, let us sum the equations and inequalities (11), (12), (14), (15) and (18) with coefficients $\frac{3\beta}{\sqrt{1+8\beta}}$, $\frac{3}{4}\cdot(3-\frac{5+8\beta}{\sqrt{1+8\beta}})$, $\frac{3}{\sqrt{1+8\beta}}$, $\frac{3}{\sqrt{1+8\beta}}$ and $\frac{3}{4}\cdot(1+\frac{1+4\beta}{\sqrt{1+8\beta}})$, respectively, and substitute $\xi = \frac{\sqrt{1+8\beta}-1}{2\beta} - 1$. Note that the coefficients for the inequalities (15) and (18) are non-negative. So, we eventually deduce that

$$\begin{aligned}&\frac{3}{4}\beta\bigl(3-\sqrt{8\beta+1}\bigr)\\&\quad\le q_\alpha\Biggl(0, \frac{1}{2}, 1-\frac{1}{\sqrt{1+8\beta}}, \frac{1}{2}, \frac{9}{8}-\frac{3+12\beta}{8\sqrt{1+8\beta}}, \frac{1}{2},\\&\qquad 0, 0, \frac{9}{8}-\frac{15+12\beta}{8\sqrt{1+8\beta}}, \frac{15}{8}-\frac{21+20\beta}{8\sqrt{1+8\beta}}, \frac{9}{4}-\frac{15+12\beta}{4\sqrt{1+8\beta}}\Biggr).\end{aligned} \tag{19}$$

Finally, since q_α is positive, we derive from (19) (the fifth, ninth, tenth and eleventh coordinates are decreasing for $\beta \in [0, 1]$ since their derivates are positive in this range) that

$$\frac{3}{4}\beta\bigl(3-\sqrt{8\beta+1}\bigr) \le q_\alpha\left(0, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1\right) = q_\alpha(\overline{P_3}+K_3). \tag{20}$$

Observe that the function $x \mapsto \frac{3}{4}x(3-\sqrt{8x+1})$ is increasing on the interval $[0,2/9]$ and that

$$\frac{3}{4}x\bigl(3-\sqrt{8x+1}\bigr) \geq 2/9 = \frac{3}{4}\cdot\frac{2}{9}\bigl(3-\sqrt{8\cdot 2/9+1}\bigr)$$

for $x \in [2/9, 1/2]$. Hence, the left hand side of (20) is at least $\frac{3}{4}\alpha(3-\sqrt{8\alpha+1})$ for $\alpha \in [0, 2/9]$ as asserted in the statement of the theorem. □

5 Conclusion

Using more sophisticated methods, we have been able to further improve the bounds on $\varphi_2(\alpha)$. However, the proof becomes extremely complicated and since we have not been able to prove that

$$\varphi_2(\alpha) = \frac{3\alpha(1+\sqrt{1-4\alpha})}{4},$$

which is the bound given by the best known example, we have decided not to further pursue our work in this direction. To show the limits of our current approach, let us mention that Theorem 4 asserts that $\varphi_2(1/12) \geq 0.10681$ and we can push the bound to $\varphi_2(1/12) \geq 0.11099$; the simple bound is 0.08333 and the expected bound is 0.11353 for this value.

We have also attempted together with Andrzej Grzesik to apply this method for improving bounds on φ_3. Though we have been able to obtain some improvements, e.g., we can show that $\varphi_3(1/20) \geq 0.05183$, the level of technicality of the argument seems to be too large for us to be able to report on our findings in an accessible way at this point.

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