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# Notes on the Roots of Ehrhart Polynomials

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Dedicated to Peter Gruber on the occasion of his 65th birthday

**Abstract.** We determine lattice polytopes of smallest volume with a given number of interior lattice points. We show that the Ehrhart polynomials of those with one interior lattice point have largest roots with norm of order $n^2$, where $n$ is the dimension. This improves on the previously best known bound $n$ and complements a recent result of Braun where it is shown that the norm of a root of a Ehrhart polynomial is at most of order $n^2$.

For the class of 0-symmetric lattice polytopes we present a conjecture on the smallest volume for a given number of interior lattice points and prove the conjecture for crosspolytopes.

We further give a characterisation of the roots of Ehrhart polyomials in the three-dimensional case and we classify for $n \leq 4$ all lattice polytopes whose roots of their Ehrhart polynomials have all real part $-\frac{1}{2}$. These polytopes belong to the class of reflexive polytopes.

## 1. Introduction

Let $\mathcal{P}^n$ be the set of all convex lattice $n$-polytopes in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with respect to the standard lattice $\mathbb{Z}^n$, i.e., all vertices of $P \in \mathcal{P}^n$ have integral coordinates and $\dim(P) = n$. The lattice point enumerator of a set $S \subset \mathbb{R}^n$ is denoted by $\mathrm{G}(S)$, i.e., $\mathrm{G}(S) = \#(S \cap \mathbb{Z}^n)$.

In 1962 Ehrhart [13] showed that for $k \in \mathbb{N}$ the lattice point enumerator $\mathrm{G}(kP)$, $P \in \mathcal{P}^n$, is a polynomial of degree $n$ in $k$ where the coefficients $\mathrm{G}_i(P)$, $0 \leq i \leq n$, depend only on $P$:

$$G(kP) = \sum_{i=0}^{n} G_i(P)k^i. \tag{1.1}$$

Moreover in [15] he proved his famous "reciprocity law"

$$G(\mathrm{int}(kP)) = (-1)^n \sum_{i=0}^{n} G_i(P)(-k)^i, \tag{1.2}$$

where int() denotes the interior. Two of the $n+1$ coefficients $G_i(P)$ are obvious, namely, $G_0(P) = 1$ and $G_n(P) = \mathrm{vol}(P)$, where vol() denotes the volume, i.e., the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. Also the second leading coefficient admits a simple geometric interpretation as the normalized surface area of $P$ which we present in detail in (4.1). All other coefficients $G_i(P)$, $1 \le i \le n-2$, have no such known geometric meaning, except for special classes of polytopes (see, e.g., [3], [6], [12], [19], [25], [26], [28], [29] and [33]).

A sometimes more convenient representation of $G(kP)$ is given by a change from the monomial basis $\{x^i\colon\ i = 0, \dots, n\}$ to the basis $\{\binom{x+n-i}{n}\colon\ i = 0, \dots, n\}$:

$$G(kP) = \sum_{i=0}^{n} a_i(P)\binom{k+n-i}{n}. \tag{1.3}$$

In view of (1.1) and (1.2) we get

$$\begin{gathered} a_0(P) = 1, \qquad a_1(P) = G(P) - (n+1), \qquad a_n(P) = G(\mathrm{int}(P)), \\ a_0(P) + a_1(P) + \cdots + a_n(P) = n!\,\mathrm{vol}(P), \end{gathered} \tag{1.4}$$

and all $a_i(P)$ are integers. Due to Stanley's famous non-negativity theorem [36] they are also non-negative, in contrast to the $G_i(P)$'s which might be negative.

In recent years the Ehrhart polynomial was not only regarded as a polynomial for integers $k$, but as a formal polynomial of a complex variable $s \in \mathbb{C}$ (see [2], [35] and [40]). Therefore, for $P \in \mathcal{P}^n$ and $s \in \mathbb{C}$ we set

$$G(s, P) = \sum_{i=0}^{n} G_i(P)s^i = \prod_{i=1}^{n}\left(1 + \frac{s}{\gamma_i(P)}\right),$$

where $-\gamma_i(P) \in \mathbb{C}$, $1 \le i \le n$, are the roots of the Ehrhart polynomial $G(s, P)$. In particular, for their geometric and arithmetic means we have

$$\left(\prod_{i=1}^{n}\gamma_i(P)\right)^{1/n} = \left(\frac{1}{\mathrm{vol}(P)}\right)^{1/n}, \qquad \frac{1}{n}\sum_{i=0}^{n}\gamma_i(P) = \frac{1}{n}\frac{G_{n-1}(P)}{\mathrm{vol}(P)}. \tag{1.5}$$

Here we are interested in geometric interpretations of the roots and in their size. Since the volume of lattice polytopes without interior lattice points might be arbitrary large for $n \ge 3$ the norm of the roots $|\gamma_i(P)|$ might be arbitrary small (see (1.5)). On the other hand, the volume of a lattice polytope with $l \ge 1$ interior lattice points is bounded (see [18],

[24], [32] and [41]) by a constant depending only on $l$ and $n$. Thus, up to unimodular transformations, there are only finitely many different of those lattice polytopes and in this case $|\gamma_i(P)|$ cannot be too small. Therefore, it seems to be reasonable to distinguish lattice polytopes with or without interior lattice points, and we define

**Definition 1.1.** For $l \in \mathbb{N}$ let $\mathcal{P}^n(l)$ be the set of lattice polytopes $P \in \mathcal{P}^n$ having exactly $l$ interior lattice points, i.e., $\mathrm{G}(\mathrm{int}(P)) = l$. Moreover, the set of all 0-symmetric lattice polytopes $P \in \mathcal{P}^n$ is denoted by $\mathcal{P}^n_o$.

The best upper bound on the volume of $P \in \mathcal{P}^n(l)$, $l \geq 1$, is due to Pikhurko [32]; he showed that

$$\mathrm{vol}(P) \leq \mathrm{c}_n l,$$

where $\mathrm{c}_n$ is a constant depending only on $n$. Hence we get

$$\left(\prod_{i=1}^{n} \gamma_i(P)\right)^{1/n} \geq (\mathrm{c}_n)^{-1/n} l^{-1/n}.$$

Candidates of lattice polytopes $P \in \mathcal{P}^n(l)$, $l \geq 1$, of maximal volume are certain simplices $T(n,l) \in \mathcal{P}^n(l)$, introduced by Zaks et al. [41] with $\mathrm{vol}(T(n,l)) \geq (l+1)/n!2^{2^{n-1}}$.

In order to present a lower bound on the volume in terms of the number of interior lattice points we define for $l \in \mathbb{N}$ the simplices

$$S_n(l) = \mathrm{conv}\left\{e_1, \ldots, e_n, -l\sum_{i=1}^{n} e_i\right\},$$

where $e_i$ denotes the $i$th unit vector. Observe that $\mathrm{G}(\mathrm{int}(S_n(l))) = l$ and $\mathrm{vol}(S_n(l)) = (nl+1)/n!$.

**Theorem 1.2.** *Let $P \in \mathcal{P}^n$. Then*

$$\mathrm{vol}(P) \geq \frac{n\mathrm{G}(\mathrm{int}\, P) + 1}{n!}, \tag{1.6}$$

*and the bound is best possible for any number of interior lattice points. For* $\mathrm{G}(\mathrm{int}\, P) = 1$ *equality holds if and only if $P$ is unimodular isomorphic to the simplex $S_n(1)$.*

The theorem above implies that for $P \in \mathcal{P}^n(l)$ the geometric mean of the roots is bounded from above by

$$\left(\prod_{i=1}^{n} \gamma_i(P)\right)^{1/n} \leq (n!)^{1/n}(nl+1)^{-1/n}.$$

In the two-dimensional case Theorem 1.2 is a direct consequence of Pick's identity $G(P) = \mathrm{vol}(P) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\, P) + 1$, where $\mathrm{bd}\, P$ denotes the boundary of $P$ [31]. In particular, equality is attained in (1.6) iff $P$ is a lattice triangle whose vertices are the only lattice points contained in the boundary. This also shows that for $\mathrm{G}(\mathrm{int}\, P) > 1$ the extremal cases in (1.6) are not necessarily unimodular equivalent. We remark, however, that all extremal cases have the same Ehrhart polynomial.

**Proposition 1.3.** *Let* $P \in \mathcal{P}^n(l)$, $l \geq 1$, *with* $\mathrm{vol}(P) = (nl+1)/n!$. *Then* $\mathrm{a}_i(P) = \mathrm{a}_i(S_n(l)) = l$, $1 \leq i \leq n$.

For 0-symmetric lattice polytopes $P \in \mathcal{P}^n_o$ there is a classical upper bound on the volume due to Blichfeldt and van der Corput (see p. 51 of [16]):

$$\mathrm{vol}(P) \leq 2^{n-1}(\mathrm{G}(\mathrm{int}\, P) + 1).$$

Lattice boxes

$$Q_n(2l-1) = \{x \in \mathbb{R}^n\colon\ |x_1| \leq l, |x_i| \leq 1, 2 \leq i \leq n\} \in \mathcal{P}^n_o, \qquad l \in \mathbb{N},$$

show that the bound is tight. As an analogue to Theorem 1.2 in the 0-symmetric case we conjecture

**Conjecture 1.1.** *Let* $P \in \mathcal{P}^n_o$. *Then*

$$\mathrm{vol}(P) \geq \frac{2^{n-1}}{n!}\,(\mathrm{G}(\mathrm{int}\, P) + 1)\,.$$

Again for $n = 2$ the inequality follows immediately from Pick's identity and the inequality is tight for any parallelogram whose vertices are the only lattice points on the boundary. It seems to be quite likely that certain crosspolytopes, i.e., $P \in \mathcal{P}^n_o$ with $2n$ vertices, are the extremal cases for the inequality above; for the family of 0-symmetric crosspolytopes we can prove the conjecture.

**Proposition 1.4.** *For* $P \in \mathcal{P}^n_o$ *with* $2n$ *vertices Conjecture* 1.1 *is true.*

One way to prove that proposition is based on the following lemma which might be of some interest in its own.

**Lemma 1.5.** *Let* $P \in \mathcal{P}^n_o$ *with* $2n$ *vertices. Then*

$$a_i(P) + a_{n-i}(P) \geq \binom{n}{i}\,(a_n(P) + a_0(P))\,, \qquad i = 0, \ldots, n.$$

Observe that on account of (1.4) Lemma 1.5 implies Proposition 1.4. As a side effect of the proof of that lemma we get

**Remark 1.6.** Let $P \in \mathcal{P}^n_o$. Then $\mathrm{a}_i(P) \geq \binom{n}{i}$ for $0 \leq i \leq n$.

The lower bounds on $\mathrm{a}_i(P)$ in the remark above also follow from a much deeper and much more general result of Stanley [37] on the $h$-vector of "symmetric" Cohen–Macaulay simplical complexes in conjunction with a result of Betke and McMullen [7] relating the coefficients $\mathrm{a}_i(P)$ with the $h$-vector of a triangulation of the polytope. Here we give a quite elementary proof which follows the method presented by Beck and Sottile in [5].

The regular unit crosspolytope $C_n^\star = \mathrm{conv}\{\pm e_i\colon 1 \leq i \leq n\}$ plays a special role in the context of the roots of Ehrhart polynomials. To our knowledge it was firstly shown by Kirschenhofer et al. [21] that the real part of $-\gamma_i(C_n^\star)$ is equal to $-\frac{1}{2}$, $1 \leq i \leq n$. This was independently proven by Bump et al. in [9, Theorem 4] and follows also from a more general result of Rodriguez-Villegas [35]. In [4, Open problem 2.41] the authors ask for other classes of lattice polytopes such that all roots of their Ehrhart polynomials have real part $-\frac{1}{2}$. Since $C_n^\star$ has minimal volume among all 0-symmetric lattice polytopes an obvious candidate is the simplex $S_n(1)$ in the non-symmetric case (see Theorem 1.2).

**Theorem 1.7.** *All roots of the polynomial* $\mathrm{G}(s, S_n(1))$ *have real part* $-\frac{1}{2}$. *If* $\alpha_n$ *is a root of* $\mathrm{G}(s, S_n(1))$ *with maximal norm, then*

$$|\alpha_n + \tfrac{1}{2}| = \frac{n(n+2)}{2\pi} + O(1),$$

*as n tends to infinity.*

In a recent paper Braun [8] proved that the roots of an Ehrhart polynomial lie inside the disc with center $-\frac{1}{2}$ and radius $n(n - \frac{1}{2})$. The above theorem shows that this bound is essentially tight and improves on the former best known lower bound of order $n$ [2, Theorem 1.3].

It seems to be quite likely that $\mathrm{G}(s, S_n(1))$ possesses the roots of maximal norm among all Ehrhart polynomials of polytopes with interior points. In the case $n = 2$ this follows from Theorem 2.2 of [2] and for a verification of this statement in the three-dimensional case see Theorem 1.10.

Looking at geometric properties of lattice polytopes $P$ whose roots have all real part $-\frac{1}{2}$ leads immediately to the class of reflexive lattice polytopes. Here $P \in \mathcal{P}^n$ with $0 \in \mathrm{int}\, P$ is called reflexive if

$$P^\star = \{y \in \mathbb{R}^n\colon\ xy \leq 1,\ \text{ for all } x \in P\} \in \mathcal{P}^n,$$

i.e., the polar polytope is again a lattice polytope. They play an important role in toric geometry since they are in one-to-one correspondence with Gorenstein toric Fano varieties. Reflexive polytopes have been extensively studied and exhibit many surprising properties (see [1], [19], [30] and the references within). In particular, Hibi [19] showed that the coefficients $\mathrm{a}_i(P)$ of a lattice polytope are symmetric, i.e., $\mathrm{a}_i(P) = \mathrm{a}_{n-i}(P)$, if and only if $P$ is reflexive. Kreuzer and Skarke [22], [23] classified all reflexive polytopes in dimensions $\leq 4$. For $n = 2, 3, 4$ there are respectively 16, 4319 and 473,800,776 reflexive polytopes (up to unimodular equivalence).

**Proposition 1.8.** *Let* $P \in \mathcal{P}^n$. *If all roots of* $\mathrm{G}(s, P)$ *have real part* $-\frac{1}{2}$ *then, up to an unimodular translation,* $P$ *is a reflexive polytope of volume* $\leq 2^n$.

It is easy to check that for $n \leq 3$ the converse is also true but not for $n \geq 4$. All in all, for $n \leq 4$ we have the following characterization:

**Proposition 1.9.** *Let $P \in \mathcal{P}^n$ be a reflexive polytope. Then all roots of* $\mathrm{G}(s, P)$ *have real part* $-\frac{1}{2}$

(i) *iff* $\mathrm{vol}(P) \leq 2^n$ *and* $n \leq 3$,
(ii) *iff* $(\mathrm{G}(P) - 1 - 4\,\mathrm{vol}(P))^2 \geq 16\,\mathrm{vol}(P)$, $2\mathrm{G}(P) \leq 9\,\mathrm{vol}(P) + 18$ *and* $n = 4$.

A classification of the roots of two-dimensional lattice polygons is given in the papers [2, Theorem 2.2] and [17, Theorem 1.9]. For $n = 3$ we know less and the basic properties are subsumed in the next theorem. For more detailed properties of Ehrhart polynomials of three-dimensional lattice polytopes we refer to Section 4.

**Theorem 1.10.** *The roots of the Ehrhart polynomials of three-dimensional lattice polytopes are contained in*

$$[-3, -1] \cup \left\{a + \mathrm{i}b\colon -1 \leq a < 1,\, a^2 + b^2 \leq 3\right\}$$

*and the bounds on a and $a^2 + b^2$ are tight. For $P \in \mathcal{P}_3(l)$, $l \geq 1$, the upper bound $\sqrt{3}$ on the norm of the complex roots is only attained by the roots of the Ehrhart polynomial of the simplex $S_3(1)$.*

The paper is organized as follows. In Section 2 we prove Theorem 1.2, Theorem 1.7 and what we know in the 0-symmetric case regarding Conjecture 1.1. Section 3 deals with reflexive polytopes and Ehrhart polynomials whose roots have all real part $-\frac{1}{2}$. Section 4 studies the Ehrhart polynomials and their roots for three-dimensional lattice polytopes.

## 2. Volume and Interior Lattice Points

The proof of Theorem 1.2 is based on a subdivision of $P$ with respect to the interior lattice points contained in $P$.

*Proof of Theorem* 1.2. Let $l = \mathrm{G}(\mathrm{int}(P))$. If $l = 0$ there is nothing to show since any lattice polytope has at least volume $1/n!$. So let $l > 0$ and let $y_1, \ldots, y_l$ be the interior lattice points of $P$. Obviously, it suffices to show that $P$ can be subdivided with the points $y_1, \ldots, y_k$ into at least $nk + 1$ lattice polytopes for $k = 1, \ldots, l$.

First we build the convex hulls of $y_1$ with all facets of $P$ yielding at least $n+1$ lattice polytopes. So let us assume that we have already dissected $P$ into $P_1, \ldots, P_{nk+1}$ lattice polytopes and let $y_{k+1}$ be contained in the relative interior of a $j$-dimensional face of $P_1$, say. Since $y_{k+1}$ is an interior point it is also contained in the relative interior of a $j$-face of at least $n - j$ further polytopes $P_2, \ldots, P_r$, say, $r \geq n - j + 1$. Subdividing each $P_s$ by building the convex hull of $y_{k+1}$ with all facets of $P_s$ not containing $y_{k+1}$ gives at least $j + 1$ new polytopes for each $P_s$, $s = 1, \ldots, r$. Thus this new subdivision of $P$ consists of at least

$$r \cdot (j+1) + nk + 1 - r \geq (n-j+1)j + nk + 1$$

lattice polytopes. Since $j \geq 1$ this number is at least $n(k+1)+1$.

The simplices $S_n(l)$ show that the bound is attained for any number of interior lattice points and the proof above shows that equality can only be achieved by simplices. So let us assume that we have a lattice simplex $S$ with only one interior lattice point $y_1$ and equality in (1.6). Without loss of generality let $y_1 = 0$ and let $v_1, \ldots, v_{n+1}$ be the vertices of $S$. Let $F_i$ be the facet of $S$ not containing $v_i$, $1 \leq i \leq n+1$. Subdividing $S$ into the $n+1$ simplices $\text{conv}\{0, F_i\}$, $1 \leq i \leq n+1$, gives

$$\frac{n+1}{n!} = \text{vol}(S) = \sum_{i=1}^{n+1} \text{vol}(\text{conv}\{0, F_i\}).$$

Since $\text{vol}(\text{conv}\{0, F_i\}) \geq 1/n!$ we must have $\text{vol}(\text{conv}\{0, F_i\}) = 1/n!$, or equivalently, any choice of $n$ vectors out of the vertices form a basis of the lattice $\mathbb{Z}^n$. Thus, up to a unimodular transformation, we may assume $v_i = e_i$, $1 \leq i \leq n$, and the absolute value of each coordinate of $v_{n+1}$ is 1. Finally, since 0 is contained in the interior of $S$ we must have $v_{n+1} = (-1, \ldots, -1)^\intercal$. ☐

We remark that inequality (1.6) can also be deduced from a result of Hibi [20] where it is shown that

$$\mathrm{a}_i(P) \geq \mathrm{a}_1(P), \qquad 1 \leq i \leq n-1, \tag{2.1}$$

for $P \in \mathcal{P}^n(l)$ with $l \geq 1$. Together with (1.4) this implies (1.6).

From (2.1) we also get Proposition 1.3, i.e., the uniqueness of the Ehrhart polynomials of $P \in \mathcal{P}^n(l)$, $l \geq 1$, with minimal volume. By (2.1) we know $\mathrm{a}_i(P) \geq l$ for $1 \leq i \leq n$ (see (1.4)) and since $P$ has minimal volume we also have

$$n!\ \text{vol}(S_n(l)) = 1 + nl = n!\ \text{vol}(P) = \mathrm{a}_0(P) + \mathrm{a}_1(P) + \cdots + \mathrm{a}_n(P).$$

Hence $\mathrm{a}_i(P) = \mathrm{a}_i(S_n(l)) = l$, $1 \leq i \leq n$, and the Ehrhart polynomial of $P \in \mathcal{P}_n(l)$ with minimal volume is uniquely determined.

We believe that the crosspolytopes

$$C_n^\star(2l-1) = \text{conv}\{\pm l e_1, \pm e_2, \ldots, \pm e_n\}, \qquad l \geq 1,$$

with $2l-1$ interior lattice points form the 0-symmetric counterpart to the simplices $S_n(l)$, i.e., they have minimal volume among all 0-symmetric polytopes with $2l-1$ interior lattice points. In Theorem 2.6 of [4] it is shown that the coefficients $\mathrm{a}_i(P)$ of a bipyramid $P = \text{conv}\{Q, \pm e_n\}$, where $Q$ is an $(n-1)$-dimensional lattice polytopes embedded in the hyperplane $\{x \in \mathbb{R}^{n-1} : x_n = 0\}$ and containing the origin, satisfy the recursion $\mathrm{a}_i(P) = \mathrm{a}_i(Q) + \mathrm{a}_{i-1}(Q)$. Hence we conclude

$$\mathrm{a}_i(C_n^\star(2l-1)) = \binom{n}{i} + \binom{n-1}{i-1}(2l-2),$$

and so

$$\mathrm{a}_i(C_n^\star(2l-1)) + \mathrm{a}_{n-i}(C_n^\star(2l-1)) = \binom{n}{i}\left(\mathrm{a}_n(C_n^\star(2l-1)) + 1\right), \qquad 0 \le i \le n.$$

Lemma 1.5 shows that $\binom{n}{i}(\mathrm{a}_n(P)+1)$ is a lower bound on $\mathrm{a}_i(P) + \mathrm{a}_{n-i}(P)$ for any lattice crosspolytope $P$.

*Proof of Lemma* 1.5. Let $P = \mathrm{conv}\{\pm v_j\colon 1 \le j \le n\}$ be a lattice crosspolytope in $\mathbb{R}^n$. For any of the $2^n$ subsets $W_k = \{w_1, \ldots, w_n\}$ with $w_j \in \{v_j, -v_j\}$ we consider the simplicial cone $C_k = \mathrm{cone}\{(w_1, 1)^\intercal, \ldots, (w_n, 1)^\intercal, e_{n+1}\} \subset \mathbb{R}^{n+1}$ and the open parallelepiped

$$Q_k = \left\{ \sum_{j=1}^{n} \lambda_j \binom{w_j}{1} + \lambda_{n+1} e_{n+1}\colon\ 0 < \lambda_j < 1, 1 \le j \le n+1 \right\}.$$

The cones $C_k$ form a triangulation of the cone $C = \mathrm{cone}\{(\pm v_j, 1)^\intercal\colon 1 \le j \le n\}$. In a recent paper Beck and Sottile [5] introduced a new method for "calculating" the numbers $\mathrm{a}_i(\cdot)$ of an arbitrary lattice polytope. In order to apply their approach we choose a vector $s = (s_1, \ldots, s_{n+1})^\intercal \in \mathbb{R}^{n+1}$ such that $C \cap \mathbb{Z}^{n+1} = (s + C) \cap \mathbb{Z}^{n+1}$ and none of the shifted cones $s + C_k$ contains a lattice point on its boundary. Obviously, we must have $s_{n+1} < 0$ and the vector $s$ can be chosen arbitrarily short. For $i = 0, \ldots, n$ we denote by

$$\alpha_i(s + Q_k) = \#\left\{(s + Q_k) \cap \{z \in \mathbb{Z}^{n+1}\colon\ z_{n+1} = i\}\right\}$$

the number of lattice points in $s + Q_k$ having last coordinate $i$. Then we have (see Proof of Theorem 2 of [5] and Proof of Theorem 3.12 of [4])

$$\mathrm{a}_i(P) = \sum_k \alpha_i(s + Q_k). \tag{2.2}$$

In particular we have $\mathrm{a}_n(P)$ many lattice points with last coordinate $n$ contained in the parallelepipeds $s + Q_k$. Let $(w, n)^\intercal$ be one of them and let it be given by

$$\binom{w}{n} = s + \sum_{j=1}^{n} \lambda_j \binom{w_j}{1} + \lambda_{n+1} e_{n+1}. \tag{2.3}$$

Now we fix an $i \in \{1, \ldots, n-1\}$. For a subset $I \subset \{1, \ldots, n\}$ of cardinality $i$ we denote by $I^{\mathrm{c}}$ its complement and let

$$\lambda_I := \lambda_{n+1} + 2\sum_{j\in I}(\lambda_j - 1).$$

With this notation we may write

$$\begin{aligned} f(w, I) &:= \binom{w}{n} - \sum_{j\in I}\binom{w_j}{1} \\ &= s + \sum_{j\in I}(1-\lambda_j)\binom{-w_j}{1} + \sum_{j\in I^{\mathrm{c}}}\lambda_j\binom{w_j}{1} + \lambda_I e_{n+1} \end{aligned}$$

Hence, if the scalar $\lambda_I$ is positive then the lattice point $f(w, I)$ is contained in some $s + Q_{k'}$, say, and therefore, it contributes to $\mathrm{a}_{n-i}(P)$ (see (2.2)).

Since $s_{n+1} < 0$ we have $\sum_{j=1}^{n+1} \lambda_j > n$ (see (2.3)) and so we get either $\lambda_I > 0$ or $\lambda_{I^c} > 0$. In other words, either $f(w, I)$ contributes to $\mathrm{a}_{n-i}(P)$ or $f(w, I^c)$ contributes to $\mathrm{a}_i(P)$. Since this argument works for any subset $I$ of cardinality $i$ the lattice point $(w, n)^\intercal$ "produces" in this way a contribution of $\binom{n}{i}$ to the sum $\mathrm{a}_i(P) + \mathrm{a}_{n-i}(P)$.

Next we have to check that for two different points

$$\binom{w}{n} = s + \sum_{j=1}^{n} \lambda_j \binom{w_j}{1} + \lambda_{n+1} e_{n+1} \quad \text{and} \quad \binom{\widetilde{w}}{n} = s + \sum_{j=1}^{n} \mu_j \binom{\widetilde{w}_j}{1} + \mu_{n+1} e_{n+1},$$

the lattice points $f(w, I)$ and $f(\widetilde{w}, \widetilde{I})$, $\#I = \#\widetilde{I} = i$, are also different, provided both of them contribute to $\mathrm{a}_{n-i}(P)$. Suppose the opposite, i.e., $f(w, I) = f(\widetilde{w}, \widetilde{I})$. Since both of them contribute to $\mathrm{a}_{n-i}(P)$ the two points $f(w, I)$, $f(\widetilde{w}, \widetilde{I})$ lie in the same cone $s + C_{k'}$, say, and since any lattice point in $s + C$ is contained in exactly one of the simplicial cones $s + C_k$ we conclude

$$\{-w_j \colon j \in I\} \cup \{w_j \colon j \in I^c\} = \{-\widetilde{w}_j \colon j \in \widetilde{I}\} \cup \{\widetilde{w}_j \colon j \in \widetilde{I}^c\}.$$

If $\{-w_j \colon j \in I\} = \{-\widetilde{w}_j \colon j \in \widetilde{I}\}$ then we must also have $\{w_j \colon j \in I^c\} = \{\widetilde{w}_j \colon j \in \widetilde{I}^c\}$. Each point in a simplicial cone, however, has an unique representation with respect to the generators and so we get the contradiction $(w, n)^\intercal = (\widetilde{w}, n)^\intercal$. Therefore, we may assume that there exists a $j_1 \in I \cap \widetilde{I}^c$ and a $j_2 \in \widetilde{I} \cap I^c$. Thus $1 - \lambda_{j_1} = \mu_{j_1}$ and $1 - \mu_{j_2} = \lambda_{j_2}$ and so

$$\mu_{j_1} + \mu_{j_2} + \lambda_{j_1} + \lambda_{j_2} = 2. \tag{2.4}$$

On the other hand, since $\sum_{i=1}^{n+1} \lambda_i, \sum_{i=1}^{n+1} \mu_i > n$ and $\lambda_i, \mu_i < 1$ we have $\lambda_{j_1} + \lambda_{j_2}, \mu_{j_1} + \mu_{j_2} > 1$ contradicting (2.4).

So far we have shown that

$$\mathrm{a}_i(P) + \mathrm{a}_{n-i}(P) \geq \binom{n}{i} \mathrm{a}_n(P). \tag{2.5}$$

Now there is one special point $\binom{\widehat{w}}{n}$ which contributes to $\mathrm{a}_i(P)$ as well as to $\mathrm{a}_{n-i}(P)$. Since the origin $0 \in \mathbb{R}^{n+1}$ is contained in one of the cones $s + C_k$, say, we can find a representation of the form

$$0 = s + \sum_{j=1}^{n} \mu_j \binom{w_j}{1} + \mu_{n+1} e_{n+1},$$

$\mu_i > 0$. Choosing the vector $s$ sufficiently small we may assume that

$$\mu_{n+1} + 2 \sum_{j=1}^{n} \mu_j < 1. \tag{2.6}$$

Hence the vector

$$\binom{\widehat{w}}{n} = \sum_{j=1}^{n} \binom{-w_j}{1} = s + \sum_{j=1}^{n} (1 - \mu_j) \binom{-w_j}{1} + \left( \mu_{n+1} + 2 \sum_{j=1}^{n} \mu_j \right) e_{n+1}$$

is contained in some $s+Q_{k'}$, say. On account of (2.6) the vectors $f(\widehat{w}, I)$ and $f(\widehat{w}, I^{\mathrm{c}})$ are contained in some of these parallelepipeds for all subsets $I \subset \{1, \ldots, n\}$ of cardinality $i$. Thus the vector $(\widehat{w}, n)^{\intercal}$ gives a contribution of $\binom{n}{i}$ to $\mathrm{a}_i(P)$ and to $\mathrm{a}_{n-i}(P)$. Together with (2.5) this proves the lemma. □

For the proof of the inequalities in Remark 1.6 we just observe that the last part of the proof above where the vector $(\widehat{w}, n)^{\intercal}$ is considered, in particular implies that $\mathrm{a}_i(P) \geq \binom{n}{i}$ for any lattice crosspolytope. Now any $n$-dimensional 0-symmetric lattice polytope $\tilde{P}$ contains a 0-symmetric lattice crosspolytopes $P$ and by Stanley's Monotonicity Theorem [38] (see also [5]) we have $\mathrm{a}_i(\tilde{P}) \geq \mathrm{a}_i(P)$.

Next we come to the roots of the polynomial $\mathrm{G}(s, S_n(1))$ which on account of Proposition 1.3 is given by

$$\mathrm{G}(s, S_n(1)) = \sum_{i=0}^{n} \binom{s+n-i}{n} = \binom{s+n+1}{n+1} - \binom{s}{n+1}. \tag{2.7}$$

*Proof of Theorem* 1.7. One way to see that all roots of $\mathrm{G}(s, S_n(1))$ have real part $-\frac{1}{2}$ is to apply a theorem of Rodriguez-Villegas [35]. In our setting it says that if all roots of the polynomial $f(s, P) = \sum_{i=0}^{n} \mathrm{a}_i(P)s^i$ lie on the unit circle then all roots of $\mathrm{G}(s, P)$ have real part $-\frac{1}{2}$. In our case we have $f(s, S_n(1)) = \sum_{i=0}^{n} s^i$ and so the norm of each root of that polynomial is 1.

Now let $s_0 = -\frac{1}{2} + \mathrm{i}b = r_0 \mathrm{e}^{\mathrm{i}\alpha_0}$ be a point on the line with real part $-\frac{1}{2}$ where we assume $b \geq 0$. Furthermore, for $m = 1, \ldots, n$ let $s_0 - m = r_m \mathrm{e}^{\mathrm{i}\alpha_m}$. Since $|s_0 - m| = |s_0 + m + 1|$, $m = 0, \ldots, n$, we also know that $s_0 + m + 1 = r_m \mathrm{e}^{\mathrm{i}(\pi - \alpha_m)}$. From the right hand side of (2.7) we conclude that $s_0$ is a root of $\mathrm{G}(s, S_n(1))$ if and only if

$$(s_0 + n + 1)(s_0 + n) \times \cdots \times (s_0 + 1) = s_0(s_0 - 1) \times \cdots \times (s_0 - n).$$

Substituting the polar representations leads to

$$(-1)^{n+1} = \mathrm{e}^{\mathrm{i}(2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_n)}.$$

Replacing the angle $\alpha_m$ by $\pi/2 + \overline{\alpha_m}$ with $\overline{\alpha_m} \in (0, \pi/2]$, $m = 0, \ldots, n$, yields $1 = \mathrm{e}^{\mathrm{i}(2\overline{\alpha_0} + 2\overline{\alpha_1} + \cdots + 2\overline{\alpha_n})}$ and thus we must have

$$\overline{\alpha_0} + \overline{\alpha_1} + \cdots + \overline{\alpha_n} = k\pi,$$

for an integer $k \in \{1, \ldots, \lfloor (n+1)/2 \rfloor\}$. Observe, that we have assumed $b \geq 0$. By construction we have $\cot \overline{\alpha_m} = 2b/(2m+1)$, $m = 0, \ldots, n$, and so we get that $s_0 = -\frac{1}{2} + \mathrm{i}b$ is a root of $\mathrm{G}(s, S_n(1))$ if and only if

$$h(b) := \sum_{m=0}^{n} \cot^{-1}\left(\frac{2b}{2m+1}\right) \in \{\pi, 2\pi, \ldots, \lfloor (n+1)/2 \rfloor \pi\},$$

where we require $\cot^{-1}() \in (0, \pi/2]$. Since $h(b)$ is a monotonously decreasing function in $b$ the imaginary part $b_n$ of the root of maximal norm is determined by the equation $h(b_n) = \pi$. Since $\cot^{-1}(t) = \tan^{-1}(1/t)$ "the inverse" of the cotangent has the power series representation $\cot^{-1}(t) = \sum_{k=0}^{\infty}(-1)^k/(2k+1)(1/t)^{2k+1}$ for $t > 1$. So we have $1/t > \cot^{-1}(t) > 1/t - 1/(3t^3)$. Hence for $b > n + 1/2$ we may write

$$\frac{1}{b}\frac{(n+1)^2}{2} > h(b) > \frac{1}{b}\frac{(n+1)^2}{2} - \mathrm{c}\frac{n^4}{b^3} \tag{2.8}$$

for a suitable constant c. The left-hand side gives

$$h\left((n+1)^2/(2\pi)\right) < \pi,$$

and since $h(\cdot)$ is a decreasing function we obtain $b_n \leq (n+1)^2/(2\pi)$. For $b = (n+1)^2/(2\pi) - 4c\pi$ the right-hand side of (2.8) is not less than $\pi$ for all sufficiently large $n$. By the monotonicity of $h(\cdot)$ we conclude that $b_n \geq (n+1)^2/(2\pi) - 4c\pi$ if $n$ is large. Thus we have shown $b_n = n(n+2)/(2\pi) + O(1)$. □

## 3. Reflexive Polytopes

As mentioned in the Introduction reflexive polytopes are of particular interests in many different branches of mathematics and have a lot of nice geometric properties. Some of them are collected in the following lemma for which we refer to [1] and [19].

**Lemma 3.1.** *Let $P \in \mathcal{P}^n$ with $0 \in \mathrm{int}(P)$. Then $P$ is relexive if and only if*:

(i) $P^\star \in \mathcal{P}^n$.
(ii) $\mathrm{a}_i(P) = \mathrm{a}_{n-i}(P)$, $0 \leq i \leq n$.
(iii) $\mathrm{G}(kP) = \mathrm{G}((k+1)\,\mathrm{int}(P))$ *for* $k \in \mathbb{N}$.
(iv) $\mathrm{vol}(P) = (n/2)\mathrm{G}_{n-1}(P)$, *i.e.*, *the origin lies in an adjacent lattice hyperplane to any facet*.

In particular, by (iii) (and $k = 0$) we see that the origin is the only interior lattice point of a reflexive polytope. Moreover, (iii) together with (1.2) implies that reflexive polytopes are precisely those lattice polytopes satisfying the functional equation:

$$\mathrm{G}(s, P) = (-1)^n \mathrm{G}(-(1+s), P), \qquad s \in \mathbb{C}.$$

Hence in any odd dimension the Ehrhart polynomials of reflexive polytopes have the real root $-\frac{1}{2}$.

Now let $P \in \mathcal{P}^n$ be a lattice polytope such that the real part of all roots $-\gamma_i(P)$ of its Ehrhart polynomial is $-\frac{1}{2}$. Then from (1.5) we immediately get

$$\frac{n}{2}\mathrm{G}_{n-1}(P) = \mathrm{vol}(P) \leq 2^n$$

which by Lemma 3.1(iv) verifies Proposition 1.8.

In dimension 2 any lattice polygon $P$ whose only interior lattice point is the origin is reflexive and its Ehrhart polynomial is given by (see (1.3))

$$\mathrm{G}(s, P) = \mathrm{vol}(P)\left(s^2 + s + \frac{1}{\mathrm{vol}(P)}\right).$$

Thus all roots have real part $-\frac{1}{2}$ if and only if $\mathrm{vol}(P) \leq 4$. Among the well-known 16 reflexive polytopes in $\mathbb{R}^2$ (see, e.g., [34]) there is only one with volume bigger than 4, namely the simplex $S = -(1,1)^\intercal + \mathrm{conv}\{0, 3e_1, 3e_2\}$ of volume $\frac{9}{2}$. By Theorem 1.2 we know that the reflexive polygon of minimal volume is $S_2(1)$ of volume $\frac{3}{2}$. Hence the Cartesian product $S \times S_2(1)$ is an example of a four-dimensional reflexive polytope of volume less than $2^n$ ($n = 4$), but not all roots of its Ehrhart polynomial have real part $-\frac{1}{2}$.

*Proof of Proposition* 1.9. First we check that all roots of the Ehrhart polynomial of a three-dimensional reflexive polytope $P$ have real part $-\frac{1}{2}$ if and only if its volume is not bigger than 8. By Lemma 3.1(ii) we have $\mathrm{a}_1(P) = \mathrm{a}_2(P)$ and so (see (1.3) and (1.4))

$$\begin{aligned} \mathrm{G}(s, P) &= \tfrac{1}{6}[(2\mathrm{a}_1(P)+2)s^3 + (3\mathrm{a}_1(P)+3)s^2 + (13\mathrm{a}_1(P)+1)s + 6] \\ &= \mathrm{vol}(P)\left[s^3 + \frac{3}{2}s^2 + \left(\frac{1}{2} + \frac{2}{\mathrm{vol}(P)}\right)s + \frac{1}{\mathrm{vol}(P)}\right] \\ &= \mathrm{vol}(P)\left[(s+\tfrac{1}{2})\left(s^2 + s + \frac{2}{\mathrm{vol}(P)}\right)\right]. \end{aligned}$$

Hence all roots have real part $-\frac{1}{2}$ iff $\mathrm{vol}(P) \leq 8$.

Now let $P$ be a four-dimensional reflexive polytope. Again by Lemma 3.1 we have $\mathrm{a}_1(P) = \mathrm{a}_3(P)$ and so we find

$$\begin{aligned} \mathrm{G}(s, P) &= \tfrac{1}{24}[(2\mathrm{a}_1(P) + \mathrm{a}_2(P) + 2)s^4 + (4\mathrm{a}_1(P) + 2\mathrm{a}_2(P) + 4)s^3 \\ &\quad + (10\mathrm{a}_1(P) - \mathrm{a}_2(P) + 46)s^2 + (8\mathrm{a}_1(P) - 2\mathrm{a}_2(P) + 44)s + 24] \\ &= \mathrm{vol}(P)\left[s^3 + 2s^3 + (2\mu+1)s^2 + (2\mu)s + \frac{1}{\mathrm{vol}(P)}\right], \end{aligned}$$

where $\mu = (1 + \frac{1}{4}\mathrm{a}_1(P))/\,\mathrm{vol}(P) - 1$. Further we set $\beta = 1/\,\mathrm{vol}(P)$ and obtain

$$\mathrm{G}(s, P) = \mathrm{vol}(P)[(s^2 + s + \mu + \sqrt{\mu^2 - \beta}) \cdot (s^2 + s + \mu - \sqrt{\mu^2 - \beta})].$$

Thus all roots have real part $-\frac{1}{2}$ if and only if $\mu^2 \geq \beta$ and $\mu - \sqrt{\mu^2 - \beta} \geq \frac{1}{4}$. The first condition translates into $(2 + (\frac{1}{2})\mathrm{a}_1(P) - 2\,\mathrm{vol}(P))^2 \geq 4\,\mathrm{vol}(P)$ and the seond becomes $2\mathrm{a}_1(P) \leq 9\,\mathrm{vol}(P) + 8$. Since $\mathrm{a}_1(P) = \mathrm{G}(P) - 5$ we get the inequalities stated in Proposition 1.9. □

Thanks to the classification of Kreuzer and Skarke (see `http://hep.itp.tuwien.ac.at/~kreuzer/CY/`) one can check that among the 4319 reflexive polytopes in dimension 3 only 64 have volume bigger than 8 and that there are only 33 different Ehrhart polynomials corresponding to $\mathrm{a}_1(P) \in \{1, \ldots, 35\}\backslash\{33, 34\}$.

In dimension 4 we have just made some calculations with the 1561 reflexive simplices (see [10]). Here the Ehrhart polynomials of 574 of them have roots with real part $-\frac{1}{2}$. Finally we present two four-dimensional reflexive simplices which show that both conditions in Proposition 1.9 are necessary. The first simplex is given by the inequalities $E_1 = \{x \in \mathbb{R}^4 \colon x_i \geq -1, 1 \leq i \leq 3, -x_3 - 2x_4 \leq 2, x_1 + x_2 + 2x_3 + 2x_4 \leq 1\}$. With the help of the computer program `latte` [11], which we have used for all our calculations, one (the computer) can easily determine the Ehrhart polynomial of such a polytope and here we find

$$\mathrm{G}(s, E_1) = \tfrac{27}{2}s^4 + 27s^3 + 21s^2 + \tfrac{15}{2}s + 1.$$

Thus we have $\mathrm{G}(E_1) = 70$ and hence $(\mathrm{G}(E_1) - 1 - 4\,\mathrm{vol}(E_1))^2 \geq 16\,\mathrm{vol}(E_1)$ but $2\mathrm{G}(E_1) > 9\,\mathrm{vol}(E_1) + 18$. Next let $E_2 = \{x \in \mathbb{R}^4\colon -x_1 \leq 1, -x_2 \leq 1, -2x_1 - 3x_2 - 4x_3 \leq 1, -4x_1 - 5x_2 - 8x_4 \leq 1, 10x_1 + 9x_2 + 4x_3 + 8x_4 \leq 1\}$. Then

$$\mathrm{G}(s, E_2) = \tfrac{4}{3}s^4 + \tfrac{8}{3}s^3 + \tfrac{8}{3}s^2 + \tfrac{4}{3}s + 1.$$

In this case we have $\mathrm{G}(E_2) = 9, 2\mathrm{G}(E_2) \leq 9\,\mathrm{vol}(E_2) + 18$ but $(\mathrm{G}(E_2) - 1 - 4\,\mathrm{vol}(E_2))^2 < 16\,\mathrm{vol}(E_2)$.

## 4. Three-Dimensional Lattice Polytopes

In this section we study the roots of Ehrhart polynomials of three-dimensional lattice polytopes. To this end we distinguish polytopes with and without interior lattice points.

**Theorem 4.1.** *Let $\Gamma(3, 0)$ be the set of all roots of Ehrhart polynomials of three-dimensional lattice polytopes $P \in \mathcal{P}^3(0)$, i.e., without interior lattice points.*

- (i) $\Gamma(3, 0) \cap \mathbb{R} = \{-3, -2\} \cup (-2, 1)$. *Moreover,* 1 *is a cluster point and there are infinitely many roots in the interval* $(-2, -1)$.
- (ii) $\{a + \mathrm{i}b \in \Gamma(3, 0)\colon b \neq 0\} \subset W := \{a + \mathrm{i}b\colon (a + 1)^2 + b^2 \leq 2$ *and* $a \geq -1\}$.
- (iii) *On the boundary of the semicircle $W$ lie exactly* 33 *pairs of zeros.* $-1 \pm \mathrm{i}\sqrt{2}$, $-1 \pm \mathrm{i}/\sqrt{2}$, $-1 \pm \mathrm{i}$ *and* $-1 \pm \mathrm{i}/\sqrt{5}$ *are the only complex roots in* $\Gamma(3, 0)$ *with real part* $-1$.

For the proof we need the following proposition

**Proposition 4.2.** *Let $P \in \mathcal{P}_n$ and let $k \in \mathbb{N}$ be the smallest positive integer with* $\mathrm{G}(k\,\mathrm{int}\,P) \neq 0$. *Then*

$$\mathrm{G}_{n-1}(P) \leq \frac{nk}{2}\mathrm{G}_n(P) = \frac{nk}{2}\,\mathrm{vol}(P).$$

*Proof.* Let $P = \{x \in \mathbb{R}^n\colon u_j^\intercal x \leq b_j, 1 \leq j \leq m\}$ be a lattice polytope with facets $F_j$ corresponding to the outer normal vector $u_j$. It was already shown by Ehrhart [14] that

$$\mathrm{G}_{n-1}(P) = \frac{1}{2}\sum_{i=1}^{m} \frac{\mathrm{vol}_{n-1}(F_i)}{\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^n)}. \tag{4.1}$$

Here $\mathrm{aff}\,F_i$ denotes the affine hull of the facet $F_i$, $\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^n)$ denotes the determinant of the $(n-1)$-dimensional sublattice $\mathrm{aff}\,F_i \cap \mathbb{Z}^n$ of $\mathbb{Z}^n$, and $\mathrm{vol}_{n-1}(F_i)$ is the $(n-1)$-dimensional volume of the facet $F_i$, i.e., the volume of $F_i$ with respect to the space $\mathrm{aff}\,F_i$.

Since $P$ is a lattice polytope we can assume $u_j \in \mathbb{Z}^n$, $0 \in P$, $b_j \in \mathbb{N}$, and that the vectors $u_j$ are primitive, i.e., $\mathrm{conv}\{0, u_j\} \cap \mathbb{Z}^n = \{0, u_j\}$. Hence $\det(\mathrm{aff}\, F_j \cap \mathbb{Z}^n) = \|u_j\|$ (see, e.g., Proposition 1.2.9 of [27]), where $\|\cdot\|$ denotes the Euclidean norm. By the choice of $k$ we can find a $z \in \mathbb{Z}^n$ such that $(1/k)z \in \mathrm{int}\, P$ and so we have $|u_j^{\intercal}(1/k)z - b_j| \geq 1$, for $1 \leq j \leq m$. In view of (4.1) we find

$$\begin{aligned}\mathrm{vol}(P) &= \frac{1}{n}\sum_{i=1}^{m} \mathrm{vol}_{n-1}(F_i)\frac{|u_j^{\intercal}(1/k)z - b_j|}{\|u_j\|} \geq \frac{2}{nk}\frac{1}{2}\sum_{i=1}^{m}\frac{\mathrm{vol}_{n-1}(F_i)}{\|u_j\|} \\ &= \frac{2}{nk}\mathrm{G}_{n-1}(P). \qquad \square\end{aligned}$$

We remark that we always have $k \leq n+1$ and thus by Proposition 4.2 $\mathrm{G}_{n-1}(P) \leq \binom{n+1}{2}\mathrm{vol}(P)$ which is a special case of another series of inequalities proved in [7]. The case $k = 1$ and thus $\mathrm{G}_{n-1}(P) \leq (n/2)\,\mathrm{vol}(P)$ was already shown in [39]. So with the notation of Proposition 4.2 we have, for three-dimensional polytopes $P$,

$$1 \leq \mathrm{G}_2(P) \leq k\tfrac{3}{2}\,\mathrm{vol}(P), \tag{4.2}$$

where the lower bound follows from (4.1) and the fact that for any facet $\mathrm{vol}_{n-1}(F_i)/\det(\mathrm{aff}\, F_i \cap \mathbb{Z}^n) \geq 1/(n-1)!$.

*Proof of Theorem* 4.1. From Theorem 1.2 and Proposition 4.7 of [2] it follows that all real roots of Ehrhart polynomials of three-dimensional polytopes are within $[-3, 1)$ and in Theorem 1.7 of [17] it was shown that 1 is cluster point of $\Gamma(3, 0)$. Next we observe that $-1$ is a root of $\mathrm{G}(s, P)$ for any polytope without interior lattice points (see (1.2)). Hence, denoting for short the coefficients $\mathrm{G}_i(P)$ by $\mathrm{g}_i$ we have $\mathrm{g}_3 - \mathrm{g}_2 + \mathrm{g}_1 - 1 = 0$ and so may write

$$\mathrm{G}(s, P) = \mathrm{g}_3 s^3 + \mathrm{g}_2 s^2 + \mathrm{g}_1 s + 1 = \mathrm{g}_3(s+1)\left(s^2 + \frac{\mathrm{g}_2 - \mathrm{g}_3}{\mathrm{g}_3}s + \frac{1}{\mathrm{g}_3}\right).$$

For the two remaining roots $-\gamma_{1,2}$ we find

$$-\gamma_{1,2} = -\frac{\mathrm{g}_2 - \mathrm{g}_3}{2\mathrm{g}_3} \pm \sqrt{\left(\frac{\mathrm{g}_2 - \mathrm{g}_3}{2\mathrm{g}_3}\right)^2 - \frac{1}{\mathrm{g}_3}}. \tag{4.3}$$

Now we want to show that there are no real roots in $(-3, -2)$. Suppose $-2$ is another root of $\mathrm{G}(s, P)$, then for the third root $\gamma$, say, we get $\gamma = -1/(2\mathrm{g}_3)$. Since $6\mathrm{g}_3$ is an integer we conclude that $\gamma = -3$ or $\gamma \geq -\frac{3}{2}$. Hence if there is an Ehrhart polynomial having a real root in $(-3, -2)$ then we know $\mathrm{G}(2\,\mathrm{int}\, P) \neq 0$ and so by (4.2) $\mathrm{g}_2 \leq 3\mathrm{g}_3$. For given $\mathrm{g}_3$ the right-hand side in (4.3) becomes minimal if $\mathrm{g}_2$ is as large as possible. Thus $-\gamma_{1,2} \geq -1 \pm \sqrt{1 - 1/\mathrm{g}_3} > -2$. Observe that $\mathrm{g}_3 \geq 1$ since we have assumed that all roots are real and $\mathrm{g}_2 \leq 3\mathrm{g}_3$.

For (i) it remains to show that there are infinitely many real roots in $(-2, -1)$. To this end we consider for an integer $q$ the pyramids $P(q) = \mathrm{conv}\{0, 2e_1, qe_2, 2e_1 + qe_2, e_3\}$. Then one gets $\mathrm{G}_3(P(q)) = \frac{2}{3}q$ and $\mathrm{G}_2(P(q)) = \frac{3}{2}q$ which shows by (4.3) that for $q$ large $\mathrm{G}(s, P(q))$ has a real root in $(-2, -1)$ depending on $q$.

For (ii) we assume that the roots $-\gamma_{1,2}$ in (4.3) are complex. Writing $-\gamma_{1,2} = a \pm \mathrm{i}b$ leads to $b^2 = 1/\mathrm{g}_3 - a^2$. Since $1/\mathrm{g}_3 = (1-2a)/\mathrm{g}_2$ we may rewrite this as

$$\left(a + \frac{1}{\mathrm{g}_2}\right)^2 + b^2 = \left(\frac{1}{\mathrm{g}_2}\right)^2 + \frac{1}{\mathrm{g}_2}.$$

By (4.2) we know $\mathrm{g}_2 \geq 1$ and it is not hard to see that all the circles above are contained in the disk given by the largest one, i.e., we have $(a+1)^2 + b^2 \leq 2$. Since we assume that the roots $-\gamma_{1,2}$ are complex we have $\mathrm{G}(2 \operatorname{int} P) \neq 0$, because otherwise $-2$ would be a root. Thus from (4.2) we conclude $\mathrm{g}_2 \leq 3\mathrm{g}_3$ which is equivalent to $a = -(\mathrm{g}_2 - \mathrm{g}_3)/(2\mathrm{g}_3) \geq -1$.

Now we come to part (iii). Let $\mathrm{g}_3 = \mathrm{vol}(P) = k/6$, $k \in \mathbb{N}$. All complex roots on the semicircle satisfy $\mathrm{g}_2 = 1$ and

$$0 > \left(\frac{\mathrm{g}_2 - \mathrm{g}_3}{2\mathrm{g}_3}\right)^2 - \frac{1}{\mathrm{g}_3} = \left(\frac{3}{k} - \frac{1}{2}\right)^2 - \frac{6}{k} = \frac{1}{4k^2}(k^2 - 36k + 36).$$

Hence $k$ is restricted to the 33 integers $k = 2, \ldots, 34$ and the Reeve simplices $T(k) = \mathrm{conv}\{0, e_1, e_2, (1,1,k)^\intercal\}$ form a family of simplices whose Ehrhart polynomials

$$\mathrm{G}(S, T(k)) = \frac{k}{6}s^3 + s^2 + \frac{12-k}{6}s + 1$$

have these roots on the semicircle.

Finally we consider the case that the complex roots have real part $-1$. Then $\mathrm{g}_2 = 3\mathrm{g}_3$ and the Ehrhart polynomial of such a polytope $P$ is of the type

$$\mathrm{G}(s, P) = \mathrm{g}_3 s^3 + 3\mathrm{g}_3 s^2 + (2\mathrm{g}_3 + 1)s + 1.$$

The roots of that polynomial are given by $-1, -1 \pm \sqrt{1 - 1/\mathrm{g}_3}$. Again let $\mathrm{g}_3 = k/6$, $k \in \mathbb{N}$. Since $1 - 1/\mathrm{g}_3$ has to be negative and since $\mathrm{g}_3 = \mathrm{g}_2/3 \geq \frac{1}{3}$ we just have to consider the cases $k = 2, \ldots, 5$. Moreover, we note that for such a polytope $P$ all roots of $2P$ have real part $-\frac{1}{2}$ and so $2P$ has to be a reflexive polytope (see Proposition 1.8). Hence all possible candidates are contained in the database of Kreuzer and Skarke of three-dimensional reflexive polytopes.

An example for $k = 2$ is given by the Reeve simplex $T(2)$ with Ehrhart polynomial $\frac{1}{3}s^3 + s^2 + \frac{5}{3}s + 1$ and with complex roots $-1 \pm \mathrm{i}\sqrt{2}$. For $k = 3, 4$ we found respectively the simplices $\mathrm{conv}\{0, e_1, e_2, (2,2,3)^\intercal\}$ with complex roots $-1 \pm \mathrm{i}$ and for $k = 4$ the simplex $\mathrm{conv}\{0, e_1, e_2, (2,3,4)^\intercal\}$ and complex roots $-1 \pm 1/\sqrt{2}\mathrm{i}$. For $k = 5$, i.e., $\mathrm{g}_3 = \frac{5}{6}$, there does not exist a simplex with the required Ehrhart polynomial. However, the pyramid over a quadrangle given by $\mathrm{conv}\{0, e_1, 2e_2, 2e_1 + e_2, e_3\}$ has the Ehrhart polynomial $\frac{5}{6}s^3 + \frac{5}{2}s^2 + \frac{16}{6}s + 1$ with complex roots $-1 \pm \mathrm{i}/\sqrt{5}$. □

Next we come to three-dimensional polytopes with interior lattice points. For those lattice polytopes we know by Proposition 4.2 that

$$\mathrm{G}_2(P) \leq \tfrac{3}{2}\mathrm{G}_3(P). \tag{4.4}$$

First we state some simple properties on the real parts of the roots.

**Proposition 4.3.** *Let $P \in \mathcal{P}_3(l)$, $l \geq 1$.*

(i) *If all roots of* $\mathrm{G}(s, P)$ *are real then either all roots are contained in* $(-1, 0)$ *or one belongs to* $(-1, 0)$ *and the two others are in* $(0, 1)$.
(ii) *If* $\mathrm{G}(s, P)$ *has only one real root* $\gamma$ *then* $\gamma \in (-1, 0)$ *and the real parts of the complex roots are contained in* $(-\frac{3}{4}, \frac{1}{2})$.

*Proof.* Let us assume that all roots are real. The point of inflexion of the real polynomial $\mathrm{G}(t, P)$, $t \in \mathbb{R}$, is given by $-\mathrm{G}_2(P)/(3\mathrm{G}_3(P))$ which by (4.4) is contained in $[-\frac{1}{2}, 0)$. Furthermore the derivative of that polynomial at 0 is given by $\mathrm{G}_1(P)$ and we also know that $\mathrm{G}(-1, P) = -l < 0$, $\mathrm{G}(1, P) > 0$. Thus, the polynomial always has a real root in $(-1, 0)$. If all roots are real then two cases occur. If $G_1(P) \geq 0$ then all of them are in $(-1, 0)$ and otherwise one root is contained in $(-1, 0)$ and the positive roots are strictly less than 1.

Now suppose we have one real root $\gamma$ and the two complex roots $a \pm \mathrm{i}b$. Since $\frac{1}{3}(2a + \gamma) = -\mathrm{G}_2(P)/(3\mathrm{G}_3(P)) \in [-\frac{1}{2}, 0)$ and $\gamma \in (-1, 0)$ we must have $-\frac{3}{4} < a < \frac{1}{2}$. □

For the proof of Theorem 1.10 we also need the following lemma.

**Lemma 4.4.** *Let $P \in \mathcal{P}_3(l)$, $l \geq 1$. Then*

(i) $\mathrm{G}_1(P) \leq \mathrm{G}_2(P) + \mathrm{G}_3(P) + \frac{2}{3} \leq \frac{5}{2}\mathrm{G}_3(P) + \frac{2}{3}$,
(ii) $\mathrm{G}(-1/(3\,\mathrm{vol}(P)), P) \geq 0$,

*and both bounds are tight. In particular, equality in* (ii) *is only attained if $P$ is unimodular equivalent to $S_3(1)$.*

*Proof.* By (1.2) we have $\mathrm{G}_1(P) = l - \mathrm{G}_3(P) + \mathrm{G}_2(P) + 1$. By Theorem 1.2 we also know $l \leq 2\mathrm{G}_3(P)/2 - \frac{1}{3}$ and thus $\mathrm{G}_1(P) \leq \mathrm{G}_2(P) + \mathrm{G}_3(P) + \frac{2}{3}$. The second inequality in (i) is a consequence of (4.4).

For the proof of (ii) we write for short $\mathrm{g}_i$ instead of $\mathrm{G}_i(P)$. On account of (i) we get

$$\begin{aligned} \mathrm{G}\left(-\frac{1}{3\,\mathrm{vol}(P)}, P\right) &= -\frac{1}{27(\mathrm{g}_3)^2} + \frac{\mathrm{g}_2}{9(\mathrm{g}_3)^2} - \frac{\mathrm{g}_1}{3\mathrm{g}_3} + 1 \\ &\geq -\frac{1}{27(\mathrm{g}_3)^2} + \frac{\mathrm{g}_2}{9(\mathrm{g}_3)^2} - \frac{\mathrm{g}_2 + \mathrm{g}_3 + \frac{2}{3}}{3\mathrm{g}_3} + 1 \\ &= \frac{1}{27(\mathrm{g}_3)^2}\,(3\mathrm{g}_2 - 1 - 6\mathrm{g}_3) - \frac{\mathrm{g}_2}{3\mathrm{g}_3} + \frac{2}{3}. \end{aligned}$$

With $\mathrm{g}_2 \geq 1$ (see (4.2)) and $\mathrm{g}_2/(3\mathrm{g}_3) \leq \frac{1}{2}$ (see (4.4)) we obtain

$$\mathrm{G}\left(-\frac{1}{3\,\mathrm{vol}(P)}, P\right) \geq \frac{1}{27(\mathrm{g}_3)^2}\,(2 - 6\mathrm{g}_3) + \frac{1}{6} = \frac{2}{3}\left(\frac{1}{3\mathrm{g}_3} - \frac{1}{2}\right)^2 \geq 0.$$

Now as the inequalities show we have equality in (i) if and only if $\mathrm{vol}(P) = (3l + 1)6$, i.e., if we have equality in Theorem 1.2. In (ii) we have equality if and only if in addition $\mathrm{vol}(P) = \frac{2}{3}$, i.e., $l = 1$ and so $P = S_3(1)$. □

*Proof of Theorem* 1.10. In view of Theorem 4.1 and Proposition 4.3 it remains to show that the norm of each complex root of the Ehrhart polynomial of a polytope with interior lattice points is bounded by $\sqrt{3}$. Let $-\gamma_1$ be the real root and let $a \pm \mathrm{i}b$ be the complex roots with $b \neq 0$. Since $\mathrm{G}(-1, P) = -l < 0$ we get from Lemma 4.4(i) that $\gamma_1 \geq 1/(3\,\mathrm{vol}(P))$. On the other hand we know that $\gamma_1 \cdot (a^2 + b^2) = 1/\mathrm{vol}(P)$ (see (1.5)) and thus $(a^2 + b^2) \leq 3$.

Among the polytopes $P \in \mathcal{P}_3(l)$, $l \geq 1$, the bound on the norm is attained if and only if the polynomial has two complex roots $a \pm \mathrm{i}b$ and one real root $-\gamma_1$ (see Proposition 4.3). Since $\gamma_1 \cdot (a^2 + b^2) = 1/\mathrm{vol}(P)$ we get $-\gamma_1 = -1/(3\,\mathrm{vol}(P))$. Thus by Lemma 4.4(ii) we conclude that this is only the case for a polytope unimodular equivalent to $S_3(1)$. By Proposition 1.3 we have $\mathrm{G}(s, S_3(1)) = \frac{2}{3}s^3 + s^2 + \frac{7}{3}s + 1$ with roots $-\frac{1}{2}, -\frac{1}{2} \pm \mathrm{i}\sqrt{11}/2$. □

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