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# Note on an Inequality of Wegner*

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**Abstract.** Wegner [10] gave a geometric characterization of all so-called Groemer packing of $n \geq 2$ unit discs in $\mathbb{E}^2$ that are densest packings of $n$ unit discs with respect to the convex hull of the discs. In this paper we provide a number theoretic characterization of all $n$ satisfying that such a "Wegner packing" of $n$ unit discs exists, and show that the proportion of these $n$ is $\frac{23}{24}$ among all natural numbers.

## 1. Introduction

Given $n \geq 2$, we dicuss packings of $n$ unit Euclidean discs in $\mathbb{R}^2$, and investigate the minimum of the area of the convex hull of the discs. Thue proved a certain lower bound for a special type of packings around 1900 (see [7]–[9]), which estimate actually yields that the packing density of the unit discs is $\pi/2\sqrt{3}$. Much later Groemer [5] and Oler [6] proved that if the convex compact set $C$ contains the centres of $n$ non-overlapping unit discs then

$$\frac{1}{2\sqrt{3}}A(C) + \frac{1}{4}P(C) + 1 \geq n. \tag{1}$$

Here $A(C)$ denotes the area and $P(C)$ the perimeter of $C$. We note that Oler [6] generalized (1) to Minkowski planes, and his inequality is known as the *Oler inequality*. Therefore we call (1) the Thue–Groemer inequality. For other proofs of the Thue–Groemer inequality, see [4] or [1].

Actually Groemer [5] described the equality case in (1); namely, either $C$ is a segment of length $2(n-1)$, or $C$ is a polygon that can be triangulated using $n$ vertices into regular triangles of edge length two. In the latter case $C$ is a hexagon. The corresponding packing of $n$ unit discs is known as the *Groemer packing*.

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Now the packing minimizing the area of the convex hull of $n$ unit discs is probably a Groemer packing. On the other hand the Groemer packings of highest densities are the ones where $P(C_n)$ is minimal. Since $C_n$ has at most six sides, we deduce $A(C_n) \leq (\sqrt{3}/24)P(C_n)^2$ according to the isoperimetric inequality for hexagons. Therefore the Thue–Groemer inequality (1) yields that $P(C_n) \geq 2\left\lceil\sqrt{12n-3}-3\right\rceil$. If in addition $P(C_n) = 2\left\lceil\sqrt{12n-3}-3\right\rceil$ then the Groemer packing is called a *Wegner packing*. A typical example is when $C_n$ is a regular hexagon. On the one hand, there exist two non-congruent Wegner packings say of 18 unit discs (see [10]), on the other hand, there may not exist any Wegner packing for a given $n$ (see Theorem 1.3) where the smallest such $n$ is 121 (see [10]).

Fejes Tóth conjectured in [3] that if $n = 6\binom{k}{2} + 1$ for some $k \geq 2$ then the optimal packing of $n$ unit discs is given by the regular hexagon of side length $2(k-1)$. Wegner [10] proved this conjecture in the following general form:

**Theorem 1.1** (Wegner Inequality). *If $D_n$ is the convex hull of n non-overlapping unit discs then*

$$A(D_n) \geq 2\sqrt{3}\cdot(n-1) + (2-\sqrt{3})\cdot\left\lceil\sqrt{12n-3}-3\right\rceil + \pi.$$

*Equality holds if and only if the packing is a Wegner packing.*

The lower bound of Theorem 1.1 is a very good estimate even if strict inequality holds, as we prove.

**Theorem 1.2.** *For any $n \geq 2$, there exists a Groemer packing of n unit discs whose convex hull $D_n$ satisfies*

$$A(D_n) \leq 2\sqrt{3}\cdot(n-1) + (2-\sqrt{3})\cdot\left\lceil\sqrt{12n-3}-2\right\rceil + \pi.$$

Finally we characterize all $n$ such that a Wegner packing of $n$ unit discs exists, and show that these numbers constitute 95.83... % of $\mathbb{N}$:

**Theorem 1.3.**

(i) *Given $n \geq 2$, a Wegner packing of n unit discs exists if and only if $\left\lceil\sqrt{12n-3}\right\rceil^2 + 3 - 12n \neq (3k-1)\cdot 9^m$ for positive $k, m \in \mathbb{Z}$.*
(ii) *Given $N \geq 2$, let $f(N)$ be the number of $2 \leq n \leq N$ such that there exists a Wegner packing of n unit discs. Then*

$$\lim_{N\to\infty}\frac{f(N)}{N} = \frac{23}{24}.$$

The results above support the following conjecture:

**Conjecture 1.4.** *The packing of n unit discs minimizing the area of the convex hull of the discs is the Groemer packing of minimal perimeter. In other words, if $D_n$ is the optimal convex hull of n non-overlapping unit discs then equality holds either in Theorem* 1.1 *or* 1.2.

## 2. Proofs

We start with Theorem 1.3. Let $C_n$ be the convex hull of the centres in a Groemer packing of $n$ unit discs. If the lengths of three non-neighbouring sides are $2a, 2b, 2c$ then the other three non-neighbouring sides are of lengths $2(a+d), 2(b+d), 2(c+d)$, and the latter three sides determine a regular triangle of side length $2(a+b+c+d)$ for some integer $d$. It follows that the perimeter of $C_n$ is $2p$ for $p = 2a+2b+2c+3d$,

$$n = \frac{1}{2\sqrt{3}} A(C_n) + \frac{1}{2} p + 1,$$
$$A(C_n) = \sqrt{3} \cdot ((a+b+c+d)^2 - a^2 - b^2 - c^2).$$

Thus the isoperimetric deficit $p^2 - 2\sqrt{3}\, A(C_n)$ can be expressed as

$$(p+3)^2 + 3 - 12n = p^2 - 2\sqrt{3}\, A(C_n) = (2a-b-c)^2 + 3(b-c)^2 + 3d^2.$$

Here the left side is never of the form $3k-1$, while $x^2+3y^2+3z^2$, $x, y, z \in \mathbb{Z}$, is never of the form $(3k-1)\cdot 9^m$. Therefore we conclude the necessity condition in (i).

If $n \le 30$ then the condition in (i) is also sufficient, as can be checked by hand. So let $n \ge 31$, and assume that $p = \lceil \sqrt{12n-3} - 3 \rceil$ is such that $A = (p+3)^2 + 3 - 12n$ is not of the form $(3k-1)\cdot 9^m$. Then $3A$ is not of the form $(9k-3)\cdot 9^m$, thus $3A$ can be written in the form

$$3A = 3x^2 + \tilde{y}^2 + \tilde{z}^2 \tag{2}$$

for some $x, \tilde{y}, \tilde{z} \in \mathbb{Z}$ (see p. 97 of [2]). Checking remainders modulo 3, we deduce that $\tilde{y} = 3y$ and $\tilde{z} = 3z$ for some $y, z \in \mathbb{Z}$, and hence

$$(p+3)^2 + 3 - 12n = x^2 + 3y^2 + 3z^2.$$

Changing $x$ to $-x$ if necessary, we may assume that $p \equiv x \pmod 3$. Since a square is 0 or 1 (mod 4), and the roles of $y$ and $z$ are symmetric, we may assume that $p \equiv z \pmod 2$, and hence $x \equiv y \pmod 2$. We define

$$a = \frac{p+2x-3z}{6}, \qquad b = \frac{p+3y-x-3z}{6}, \qquad c = \frac{p-3y-x-3z}{6}, \qquad d = z.$$

The divisibility properties imply that $a, b, c, d$ are integers. We want to make a hexagon (namely $\frac{1}{2}C_n$) with sides $a, b, c, a+d, b+d$ and $c+d$; to this end we need that they are all positive. This is equivalent to

$$2|x| + 3|z| < p \quad \text{and} \quad |x| + 3|y| + 3|z| < p. \tag{3}$$

We have $p < \sqrt{12n-3} - 2$, which in turn yields that

$$x^2 + 3y^2 + 3z^2 < (p+3)^2 - (p+2)^2 = 2p+5. \tag{4}$$

Here $p \ge 17$ follows by $n \ge 31$, and hence the conditions in (3) follow by (4) and the Cauchy–Schwarz inequality. Therefore there exists a Groemer packing of $n$ unit discs with $p$ boundary discs, completing the proof of (i).

Turning to (ii), let $g(N) = N - f(N)$. It is sufficient to show that for any $\varepsilon > 0$ and for $N$ large,

$$g(N) = \frac{1 + O(\varepsilon)}{24} \cdot N, \tag{5}$$

where the implied constant in $O(\cdot)$ is some absolute constant. Given $n \geq 2$, we set $s = \lceil \sqrt{12n-3} \rceil$ and define $t$ by the formula

$$12n - 3 = s^2 - t.$$

The condition $s = \lceil \sqrt{12n-3} \rceil$ is equivalent to $s^2 - t > (s-1)^2$, and hence to $s > (t+1)/2$. On the other hand, $n \leq N$ is equivalent to $s \leq \sqrt{12N - 3 + t}$. Therefore $g(N)$ is the number of "good" pairs $s, t \in \mathbb{N}$ such that $t = l \cdot 9^m$ for some positive $l, m \in \mathbb{Z}$ with $l \equiv -1 \pmod 3$, $n = (s^2 - t + 3)/12$ is an integer, and

$$\frac{t+1}{2} < s \leq \sqrt{12N - 3 + t}. \tag{6}$$

If $N$ is large then $t < (1+\varepsilon)2\sqrt{12N}$ and $m < (1+\varepsilon)\log_9 2\sqrt{12N}$ follows by (6). Now $n$ is an integer if and only if either $l \equiv -4 \pmod{12}$ and $s \equiv \pm 3 \pmod{12}$, or $l \equiv -1 \pmod{12}$ and $s \equiv 0, 6 \pmod{12}$. We observe that if $t$ is fixed and $t < (1-\varepsilon)2\sqrt{12N}$ then a "good" pair $s$ of $t$ occurs uniformly and with density $\frac{1}{6}$. Therefore given large $N$ and $1 \leq m \leq (1-\varepsilon)\log_9 2\sqrt{12N}$, the number of "good" pairs $s, t \in \mathbb{N}$ is

$$(1 + O(\varepsilon)) \sum_{\substack{2 \leq l \leq (1-\varepsilon)2\sqrt{12N}/9^m \\ l \equiv -1, -4 \pmod{12}}} \frac{1}{6} \cdot \left( \sqrt{12N} - \frac{l \cdot 9^m}{2} \right) = (1 + O(\varepsilon)) \cdot \frac{N}{3 \cdot 9^m}.$$

We conclude that

$$g(N) = (1 + O(\varepsilon)) \cdot \sum_{m \geq 1} \frac{N}{3 \cdot 9^m} = (1 + O(\varepsilon)) \cdot \frac{N}{24},$$

completing the proof of Theorem 1.3.

Finally we prove Theorem 1.2. If there exists no Wegner packing for some $n$ then $p = \lceil \sqrt{12n-3} - 3 \rceil$ is divisible by 3 according to Theorem 1.3(i). Thus $(p+1+3)^2 + 3 - 12n$ is not equal to $(3k-1) \cdot 9^m$ for any positive $k, m \in \mathbb{Z}$, and the proof of the sufficiency of the condition in Theorem 1.3(i) yields the existence of a Groemer packing of $n$ unit discs with $p+1$ boundary discs. In turn we conclude Theorem 1.2.

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