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Stability of the Minkowski Measure of Asymmetry for Convex Bodies

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Abstract. Given a convex body $C \subset R^n$ (i.e., a compact convex set with nonempty interior), for $x \in int(C)$, the interior, and a hyperplane H with $x \in H$, let H_1, H_2 be the two support hyperplanes of C parallel to H. Let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1, H_2. Then the quantity

$$As(C) := \inf_{x \in int(C)} \sup_{H \ni x} r(H, x)$$

is called the Minkowski measure of asymmetry of C.

$As(\cdot)$ can be viewed as a real-valued function defined on the family of all convex bodies in R^n. It has been known for a long time that $As(\cdot)$ attains its minimum value 1 at all centrally symmetric convex bodies and maximum value n at all simplexes.

In this paper we discuss the stability of the Minkowski measure of asymmetry for convex bodies. We give an estimate for the deviation of a convex body from a simplex if the corresponding Minkowski measure of asymmetry is close to its maximum value. More precisely, the following result is obtained:

Let $C \subset R^n$ be a convex body. If $As(C) \geq n - \varepsilon$ for some $0 \leq \varepsilon < 1/8(n+1)$, then there exists a simplex S_0 formed by $n+1$ support hyperplanes of C, such that

$$(1 + 8(n+1)\varepsilon)^{-1} S_0 \subset C \subset S_0,$$

where the homethety center is the (unique) Minkowski critical point of C. So

$$d_{\mathrm{BM}}(C, S) \leq 1 + 8(n+1)\varepsilon$$

holds for all simplexes S, where $d_{\mathrm{BM}}(\cdot, \cdot)$ denotes the Banach–Mazur distance.

1. Introduction

A measure of asymmetry (or symmetry) for convex bodies, according to Grünbaum's definition in his well-known paper [1], is a nonnegative real-valued function defined on the family of all convex bodies of a finite-dimensional (affine) space, which satisfies some additional conditions and therefore can be used to describe how close (or far) a convex body is to (or from) a centrally symmetric one. In [1] some known measures of symmetry were summarized and discussed, among which the Minkowski measure of asymmetry (see Section 2 in this paper or [5] for the definition) is probably the simplest and best known one. After [1] some new ones have also been studied.

Many important properties of such measures (especially the Minkowski measure) have been studied by different authors from different points of view (see [2]–[11]). Among these studies, Groemer [3] focuses on the stabilities of two measures of asymmetry that characterize cones as the most asymmetric convex bodies. More precisely, he established stability estimates for these (two) measures that provide information on the deviation of a convex body from a cone if the corresponding measure of symmetry is close to its maximum value. Naturally the same stability problems for other measures of asymmetry (or symmetry) should be studied too.

In this paper we study the stability of the Minkowski measure of asymmetry. We give an estimate for the deviation of a convex body from a simplex if the corresponding Minkowski measure of asymmetry is close to its maximum value.

The main result in this paper is the following stability theorem (the proof will be found in Section 4):

Theorem A. *Let $C \subset R^n$ be a convex body. If its Minkowski measure of asymmetry $As(C) \geq n - \varepsilon$ for some $0 \leq \varepsilon \leq 1/8(n+1)$, then*

$$d_{\mathrm{BM}}(C, S) \leq 1 + 8(n+1)\varepsilon$$

holds for all simplexes S, where $d_{\mathrm{BM}}(\cdot, \cdot)$ denotes the Banach–Mazur distance.

(See Section 2 for the definition of the Banach–Mazur distance). More precisely,

Theorem A*. *Let $C \subset R^n$ be a convex body. If its Minkowski measure of asymmetry $As(C) \geq n - \varepsilon$ for some $0 \leq \varepsilon \leq 1/8(n+1)$, then there exists a simplex S_0 formed by $n+1$ support hyperplanes of C, such that*

$$(1 + 8(n+1)\varepsilon)^{-1} S_0 \subset C \subset S_0,$$

where the homothety center is the (unique) Minkowski critical point of C.

Remark. The constant $8(n+1)$ here may not be the best, but there are examples showing that in general the term ε cannot be replaced by ε^k for any $k > 1$.

2. Preliminaries

In this paper we work mainly on the n-dimensional Euclidean spaces ($n \geq 1$) and work with affine maps (functions) instead of linear ones since it turns out that the affine setting works more suitably than the linear one.

Denote by R^n the standard Euclidean space, by C, D, etc., the convex sets in R^n and by $\mathfrak{K}^n$ the family of all convex bodies (i.e., the compact convex sets with nonempty interiors) in R^n. The convex body of the form $conv(x_1, \ldots, x_{n+1})$ is called a simplex (in this case, $x_1, \ldots, x_{n+1}$ must be affinely independent), where $conv$ denotes the convex hull. Notice that a simplex can be expressed as an intersection of suitable $n+1$ closed half-spaces. Denote by T, S, etc., the maps from R^n to R^m.

Given $C \subset R^n$ and $T : R^n \to R^m$, denote by $T_*(C)$ the image set

$$\{Tx \in R^m \mid x \in C\}.$$

A map $T : R^n \to R^m$ is called affine if $T(\sum_{i=1}^k \lambda_i x_i) = \sum_{i=1}^m \lambda_i T x_i$ for any $x_i \in R^n$ and $\lambda_i \in R$ $(1 \le i \le k)$ with $\sum_{i=1}^m \lambda_i = 1$. Especially, if $m = 1$, T is called an affine function. We usually denote functions by f, g, etc.

We denote

$$\mathcal{A}\!f\!f(R^n, R^m) := \{T : R^n \to R^m \mid T \text{ is affine}\},$$

$$\mathit{aff}(R^n) := \{f \mid f \text{ is an affine function on } R^n\}.$$

$\mathit{aff}(R^n)$ is called the affine dual space of R^n, which is a linear space under the ordinary addition and scalar multiplication of functions and can be identified with R^{n+1} in a natural way. We write $\mathcal{A}\!f\!f(R^n)$ in brief for $\mathcal{A}\!f\!f(R^n, R^n)$.

If we denote the linear dual space of $\mathit{aff}(R^n)$ by $\left(\mathit{aff}(R^n)\right)'$, then R^n can be embedded as an affine subspace into $\left(\mathit{aff}(R^n)\right)'$. More precisely, under the correspondence $R^n \ni x \leftrightarrow x'' \in \left(\mathit{aff}(R^n)\right)'$ defined by $x''(f) = f(x)$ for $f \in \mathit{aff}(R^n)$, we have

$$R^n = \{z'' \in \left(\mathit{aff}(R^n)\right)' \mid z''(\mathbf{1}) = 1\}, \tag{2.1}$$

where $\mathbf{1}$ denotes the constant affine function 1 on R^n.

For any $\lambda \in R$, $x \in R^n$, and $C \subset R^n$, we denote

$$\lambda_x C := \{x + \lambda(y - x) \mid y \in C\}.$$

We recall the well-known Banach–Mazur distance defined on $\mathfrak{K}^n$.

Definition 1. Given $C, D \in \mathfrak{K}^n$, the Banach–Mazur distance $d_{\mathrm{BM}}(C, D)$ between C, D is defined as

$$d_{\mathrm{BM}}(C, D) := \inf\left\{\lambda > 0 \mid T_*(C) \subset D \subset \lambda_x\big(T_*(C)\big)\right\},$$

where the infimum is taken over all $x \in R^n$ and all nonsingular $T \in \mathcal{A}\!f\!f(R^n)$.

Remark 1. It is easy to see that $d_{\mathrm{BM}}(C, D) = 1$ iff C and D are affinely equivalent, i.e., there exists a nonsingular $T \in \mathcal{A}\!f\!f(R^n)$ such that $D = T_*(C)$. Thus d_{BM} (or more precisely $\log d_{\mathrm{BM}}$) is actually a metric on $\mathfrak{K}^n/\sim$, where "$\sim$" denotes the affine equivalence relation.

In this paper the main objects we study are the so-called (affinely invariant) measures of asymmetry for convex bodies. So we recall here some measures of asymmetry (for a general definition of measures of asymmetry, see [1] or [5]).

Definition 2 (see [1]). Given $C \in \mathfrak{K}^n$, for $x \in int(C)$, the interior, and a hyperplane H with $x \in H$, let H_1, H_2 be the two support hyperplanes of C parallel to H. Let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1, H_2. Now if we denote

$$r(x) = r(C, x) := \sup_{H \ni x} r(H, x),$$

then the Minkowski measure of asymmetry of C is defined as

$$As(C) := \inf_{x \in int(C)} r(x).$$

Any point $c \in int(C)$ satisfying $r(c) = As(C)$ is called a (Minkowski) critical point of C.

Remark 2. (1) It is easy to see that $As(\cdot)$ is affinely invariant, i.e., $As(T_*(C)) = As(C)$ for any $C \in \mathfrak{K}^n$ and any nonsingular $T \in \mathcal{Aff}(R^n)$.

(2) It is easy to check that the function $r(x)$ defined on $int(C)$ is convex (see Remark 4) and $\lim_{x \to \partial C} r(x) = \infty$, where ∂C denotes the boundary of C, so $\inf_{x \in int(C)} r(x)$ is attained, i.e., for each C there exists at least one critical point.

(3) There are several other equivalent definitions. For instance, for any $x \in int(C)$ and chord l of C passing through x, let $r'(l, x)$ be the ratio, not less than 1, in which x divides the length of l. Then it is known that $r(x) = \sup_{l \ni x} r'(l, x)$, and so

$$As(C) = \inf_{x \in int(C)} \sup_{l \ni x} r'(l, x).$$

For other equivalent definitions see [6] or [9].

(4) It has been known for a long time that $1 \le As(C) \le n$ ($\forall C \in \mathfrak{K}^n$) and $As(C) = 1$ iff C is centrally symmetric; $As(C) = n$ iff C is a simplex (see [1] or [2]).

The following measure of symmetry, introduced for the first time in [9], is equivalent in some sense to the Minkowski one and is more practical and useful. To introduce this measure of symmetry, we first give the concept of an affine dual set of a given $C \in \mathfrak{K}^n$ (see [9]).

Definition 3 (see [9]). Given $C \in \mathfrak{K}^n$, we define the affine dual set $C^A_{[0,1]}$ of C by

$$C^A_{[0,1]} := \{f \in \mathit{aff}(R^n) \mid f_*(C) \subset [0, 1]\}.$$

We also denote

$$C^a_{[0,1]} := \{f \in \mathit{aff}(R^n) \mid f_*(C) = [0, 1]\},$$

which is a subset of $\partial C^A_{[0,1]}$.

Remark 3. $C^a_{[0,1]}$ consists of the main part of $\partial\big(C^A_{[0,1]}\big)$ in the sense that $C^A_{[0,1]} = conv\big(0, 1, C^a_{[0,1]}\big)$, where $0, 1$ denote the constant affine functions $\mathbf{0}, \mathbf{1}$ on R^n, respectively.

Now we have the following

Definition 4 (see [9] or [10]). Given $C \in \mathfrak{K}^n$. For each $x \in int(C)$, denote

$$\nu(x) = \nu(C, x) = \inf_{f \in C^a_{[0,1]}} f(x),$$

and we define the measure $as(C)$ of symmetry of C by

$$as(C) := \sup_{x \in int(C)} \nu(C, x).$$

Any point x satisfying $\nu(C, x) = as(C)$ is called an as-critical point of C.

Remark 4. (1) For each $C \in \mathfrak{K}^n$, the function $\nu(C, x)$ defined on C is concave since it is the infimum of a family of affine functions, and $\nu(C, x)_{|\partial C} = 0$. So $as(C)$ is attained, i.e., there exists at least one as-critical point for each C.

(2) It is not hard to see that for any $C \in \mathfrak{K}^n$ and $x \in int(C)$, we have

$$r(x) = \frac{1}{\nu(x)} - 1 \quad \text{or} \quad \nu(x) = \frac{1}{r(x) + 1},$$

and consequently

$$As(C) = \frac{1}{as(C)} - 1 \quad \text{or} \quad as(C) = \frac{1}{As(C) + 1}$$

from which and (4) in Remark 2 it follows that C is symmetric iff $as(C) = \frac{1}{2}$, C is a simplex iff $as(C) = 1/(n + 1)$, and $c \in int(C)$ is a Minkowski critical point iff c is an as-critical point.

We also point out that, with the help of the relation between $r(x)$ and $\nu(x)$ above, the convexity of the function $r(x)$ in Definition 2 can be easily derived from the concavity of the function $\nu(x)$.

We use $\mathcal{C}_C$ to denote the set of all as critical (or Minkowski) points of C. By the above, $\mathcal{C}_C$ is a nonempty convex set.

3. Some Properties of the Critical Affine Functionals

Given a convex body $C \in \mathfrak{K}^n$ and $c \in \mathcal{C}_C$, an affine function $f \in C^a_{[0,1]}$ is called a critical function (with respect to c) of C if $f(c) = as(C)$.

We point out that if f is a critical function, then the ratio, not less than 1, in which $H := \{f = as(C)\}$ divides the distance between $H_0 := \{f = 0\}$ and $H_1 := \{f = 1\}$, is exactly the Minkowski measure $As(C)$. Therefore the hyperplane $\{f = 0\}$ is called a critical support hyperplane (with respect to c) of C if f is a critical function (with respect to c) of C.

Denote, for a fixed $c \in \mathcal{C}_C$, by $\mathbb{E}_C(c)$ the family of all critical functions with respect to c, i.e.,

$$\mathbb{E}_C(c) = \{f \in C^a_{[0,1]} \mid f(c) = as(C)\}.$$

In this section we mainly discuss the properties of $\mathbb{E}_C(c)$. We show that $\mathbb{E}_C(c)$ contains "enough" elements to represent some constant functions, which in turn implies that, under some conditions, there exist "enough" critical support hyperplanes to form a simplex which is "close to" C.

Lemma 1. *$C^a_{[0,1]}$ is bounded and closed, and so is its convex hull $conv(C^a_{[0,1]})$.*

Proof. The boundedness is obvious. So we need only show the closedness.

Suppose $f_0 \in \mathit{aff}(R^n)$, $f_k \in C^a_{[0,1]}$ $(k = 1, 2, \ldots)$ satisfy

$$\lim_{k\to\infty} f_k(x) = f_0(x) \qquad \text{for all} \quad x \in R^n.$$

First, it is clear that $f_{0*}(C) \subset [0, 1]$. To show the converse inclusion, we choose, for each k, $x_k, y_k \in C$ such that $f_k(x_k) = 0$, $f_k(y_k) = 1$. Then by the compactness of C, we may find $x_0, y_0 \in C$ such that (by passing to subsequences)

$$x_k \to x_0, \qquad y_k \to y_0, \qquad k \to \infty.$$

Then by the equicontinuity of f_k on C, it follows that we must have $f_0(x_0) = \lim_{k\to\infty} f_k(x_0) = \lim_{k\to\infty} f_k(x_k) = 0$ and similarly $f_0(y_0) = 1$. Therefore $f_{0*}(C) \supset [0, 1]$. Thus $f_{0*}(C) = [0, 1]$, i.e., $f_0 \in C^a_{[0,1]}$. □

If we regard each number in R as a constant affine function on R^n, then we have the following:

Lemma 2. *For any $C \in \mathfrak{K}^n$, we have*

$$conv(C^a_{[0,1]}) \cap R = [as(C), 1 - as(C)].$$

Proof. We first observe that $\mu \in conv\,(C^a_{[0,1]})$ iff $1 - \mu \in conv\,(C^a_{[0,1]})$. In fact, if $\mu \in conv\,(C^a_{[0,1]})$, i.e., $\mu = \sum_{i=1}^N \alpha_i f_i$ for some $f_i \in C^a_{[0,1]}$ and $\alpha_i > 0$ with $\sum_{i=1}^N \alpha_i = 1$, then $1 - \mu = \sum_{i=1}^N \alpha_i(1 - f_i) \in conv\,(C^a_{[0,1]})$ since if $f \in C^a_{[0,1]}$, then $1 - f \in C^a_{[0,1]}$. Then we just need to change the roles of μ and $1 - \mu$.

Now suppose $\mu \in conv\,(C^a_{[0,1]})$, i.e., $\mu = \sum_{i=1}^N \alpha_i f_i$ for some $f_i \in C^a_{[0,1]}$ and $\alpha_i > 0$ with $\sum_{i=1}^N \alpha_i = 1$, then by choosing $c \in \mathcal{C}_C$, we get that $\mu = \sum_{i=1}^N \alpha_i f_i(c) \geq \sum_{i=1}^N \alpha_i \ as(C) = as(C)$. Thus by the above observation, we proved in fact that

$$conv\,(C^a_{[0,1]}) \cap R \subset [as(C), 1 - as(C)].$$

To prove the converse inclusion, by the same observation above, we now need only show that $as(C) \in conv(C^a_{[0,1]})$. To see this, we first make another observation that the constant function $\frac{1}{2} \in conv(C^a_{[0,1]})$ since $\frac{1}{2} = \frac{1}{2}f + \frac{1}{2}(1 - f)$ for any $f \in C^a_{[0,1]}$.

Now suppose $as(C) \notin conv(C^a_{[0,1]})$, then by the Hahn–Banach theorem, there is $x'' \in \left(\mathit{aff}(R^n)\right)'$ such that

$$x''(as(C)) < \inf_{f \in conv(C^a_{[0,1]})} x''(f). \tag{3.1}$$

Notice that $x''(\mathbf{1}) \neq 0$, otherwise we will have $x''(\lambda) = \lambda x''(\mathbf{1}) = 0$ for all $\lambda \in R$ and especially $x''(as(C)) = x''(\frac{1}{2}) = 0$ which contradicts (3.1) since $\frac{1}{2} \in conv(C^a_{[0,1]})$.

We denote $x^* = x''/x''(\mathbf{1})$. Thus we have that $x^*(\mathbf{1}) = 1$, which ensures by (2.1) that $x^* \in R^n$, and that either

$$as(C) = x^*(as(C)) < \inf_{f \in C^a_{[0,1]}} x^*(f) = \inf_{f \in C^a_{[0,1]}} f(x^*) \tag{3.2}$$

or

$$as(C) = x^*(as(C)) > \sup_{f \in C^a_{[0,1]}} x^*(f) = \sup_{f \in C^a_{[0,1]}} f(x^*). \tag{3.3}$$

Since $x^* \in R^n$, then either $x^* \in C$ or $x^* \notin C$. If $x^* \notin C$, then by the Hahn–Banach theorem, in case (3.2), we may choose $f_0 \in C^a_{[0,1]}$ such that $f_0(x^*) < 0$ which leads to the contradiction that $as(C) < 0$, and, in case (3.3), we may choose $f_1 \in C^a_{[0,1]}$ such that $f_1(x^*) > 1$ which leads to the contradiction that $as(C) > 1$. So we must have $x^* \in C$.

However, if $x^* \in C$, then (3.2) implies that

$$as(C) < \nu(C, x^*),$$

which is a contradiction to the definition of $as(C)$. Equation (3.3) implies the contradiction that $as(C) > \frac{1}{2}$ since we can always choose $f_2 \in C^a_{[0,1]}$ such that $f_2(x^*) \geq \frac{1}{2}$. Therefore we must have that $as(C) \in conv(C^a_{[0,1]})$. This completes the proof. □

Remark 5. Since $1/(n+1) \leq as(C) \leq \frac{1}{2}$ for $C \in \mathfrak{K}^n$, by Lemma 2, we get that

$$\{\tfrac{1}{2}\} \subset conv(C^a_{[0,1]}) \cap R \subset \left| \frac{1}{n+1}, \frac{n}{n+1} \right|.$$

Lemma 3. *If $as(C) = \sum_{i=1}^N \alpha_i f_i$ for $f_i \in C^a_{[0,1]}$ and $\alpha_i > 0$ with $\sum_{i=1}^N \alpha_i = 1$, then for any $c \in \mathcal{C}_C$, $f_i \in \mathbb{E}_C(c)$ for all i.*

Proof. For each $c \in \mathcal{C}_C$, we have

$$as(C) = \sum_{i=1}^N \alpha_i f_i(c) \geq \sum_{i=1}^N \alpha_i \ as(C) = as(C)$$

since $f_i(c) \geq as(C)$ for all i, which in turn implies that $f_i(c) = as(C)$ for all i. □

By Lemmas 2 and 3, we immediately get

Corollary 1. *For any $C \in \mathfrak{K}^n$ and $c \in \mathcal{C}_C$, we have*

$$as(C) \in conv(\mathbb{E}_C(c)).$$

Furthermore, we have the following:

Lemma 4. *For any $c \in \mathcal{C}_C$, there exist $\{f_i\}_{i=1}^{n+1} \subset \mathbb{E}_C(c)$ and $\alpha_1 \geq 0, \ldots, \alpha_{n+1} \geq 0$ with $\sum_{i=1}^{n+1} \alpha_i = 1$ such that*

$$as(C) = \sum_{i=1}^{n+1} \alpha_i f_i.$$

Proof. By Corollary 1, $as(C) \in conv(\mathbb{E}_C(c))$. However, $\mathbb{E}_C(c)$ is a subset of the n-dimensional affine space $\{f \in aff(R^n) \mid f(c) = as(C)\}$, thus by the Carathéodory theorem there exist $\{f_i\}_{i=1}^{n+1} \subset \mathbb{E}_C(c)$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n+1} \alpha_i = 1$ such that

$$as(C) = \sum_{i=1}^{n+1} \alpha_i f_i. \qquad \square$$

Lemma 5. *If $as(C) = \sum_{i=1}^{N} \alpha_i f_i$ for $f_i \in C^a_{[0,1]}$ and $\alpha_i > 0$ with $\sum_{i=1}^{N} \alpha_i = 1$, then $\alpha_i \leq as(C)$ for all i. Therefore $\alpha_i \geq 1 - (N-1)\ as(C)$ for all i.*

Proof. For each i $(1 \leq i \leq N)$, choose $x_i \in \partial C$ such that $f_i(x_i) = 1$, then (notice that $f_j(x_i) \geq 0$ for all i, j) we have for each i,

$$as(C) = \sum_{j=1}^{N} \alpha_j f_j(x_i) = \alpha_i + \sum_{j \neq i} \alpha_j f_j(x_i) \geq \alpha_i,$$

and for each i,

$$\alpha_i = 1 - \sum_{j \neq i}^{N} \alpha_j \geq 1 - (N-1)\ as(C). \qquad \square$$

Lemma 6. *If $as(C) < 1/n$ and if $as(C) = \sum_{i=1}^{n+1} \alpha_i f_i$ for some $f_i \in \mathbb{E}_C(c)$ (where c is the unique critical point of C) and $\alpha_i \geq 0$ with $\sum_{i=1}^{n+1} \alpha_i = 1$, then the set $\Delta := \bigcap_{i=1}^{n+1} \{f_i \geq 0\}$ is bounded, i.e., Δ is a simplex.*

Proof. Before we prove the lemma, we point out that, under the assumption that $as(C) < 1/n$, the uniqueness of the critical point is ensured by a theorem in [2] which states that $\dim \mathcal{C}_C + as(C)^{-1} \leq n+1$, where dim denotes the dimension.

To complete the proof, we first notice that, by Lemma 5, $\alpha_i \geq 1 - n\ as(C) > 1 - n(1/n) = 0$ for all i.

Thus, suppose Δ is not bounded, then there exists a ray $l \subset \Delta$. Now if there exists some $1 \leq i_0 \leq n+1$, say $i_0 = 1$, such that f_1 is not a constant on l, then we will have $\sup_{x \in l} f_1(x) = +\infty$ since f_1 is an affine function and $f_1(x) \geq 0$ on $l \subset \Delta$, which, together with the fact that for any $x \in l$, $as(C) = \alpha_1 f_1(x) + \sum_{i=2}^{n+1} \alpha_i f_i(x) \geq \alpha_1 f_1(x)$ since $f_i(x) \geq 0$ for all i and $x \in l$, leads to the contradiction that $as(C) \geq \sup_{x \in l} \alpha_1 f_1(x) = +\infty$. So all f_i are constants on l.

Now we consider the $(n-1)$-dimensional space $R^n/[l] \cong R^{n-1}$ where $[l]$ denotes the one-dimensional affine space containing l. Let $P : R^n \to R^{n-1}$ be the corresponding projection, and denote $\widetilde{C} = P(C) = \{\tilde{x} \mid \tilde{x} = P(x), x \in C\}$. It is clear that $\widetilde{C}$ is a convex body in R^{n-1}. Now if we define $\tilde{f}_i \in aff(R^{n-1})$ $(1 \le i \le n+1)$ by $\tilde{f}_i(\tilde{x}) := f_i(x)$ for $\tilde{x} \in R^{n-1}$, then it is easy to see that $\tilde{f}_i \in \widetilde{C}^a_{[0,1]}$ $(1 \le i \le n+1)$ and that $as(C) = \sum_{i=1}^{n+1} \alpha_i \tilde{f}_i$, i.e., $as(C) \in conv(\widetilde{C}^a_{[0,1]})$, which, together with the fact that $as(C) < 1/n \le 1/2$, implies that $\frac{1}{n} \in conv(\widetilde{C}^a_{[0,1]})$, a contradiction to Lemma 2 (see Remark 5). This finishes the proof. □

Now, by Lemmas 4 and 6, we have the following:

Corollary 2. *Let $C \in \mathfrak{K}^n$. If $as(C) < 1/n$, then there exist $f_i \in \mathbb{E}_C(c)$ $(1 \le i \le n+1)$, where (the unique) $c \in \mathcal{C}_C$, such that $\bigcap_{i=1}^{n+1}\{f_i \ge 0\}$ is bounded, i.e., $\bigcap_{i=1}^{n+1}\{f_i \ge 0\}$ is a simplex.*

4. Proofs of the Main Theorems

We first show a simple property of the critical functions of a simplex.

Let S be a simplex with facets F_i $(1 \le i \le n+1)$. We denote by $f_{F_i} \in S^a_{(0,1]}$ the affine function vanishing on F_i. Then we have

Lemma 7. *If S is a simplex with facets F_i $(1 \le i \le n+1)$, then*

$$\sum_{i=1}^{n+1} f_{F_i} = 1.$$

Proof. If we denote by x_i the unique point in $\{f_{F_i} = 1\} \cap S$ $(1 \le i \le n+1)$, i.e., x_i $(1 \le i \le n+1)$ are all the vertices of S, then we have that $f_{F_i}(x_i) = 1$ and $f_{F_i}(x_j) = 0$ for $j \ne i$. Furthermore, it is known that for each $x \in R^n$, there exist $\beta_i \in R$ $(1 \le i \le n+1)$ such that $\sum_{i=1}^{n+1} \beta_i = 1$ and $x = \sum_{i=1}^{n+1} \beta_i x_i$. Therefore

$$\begin{aligned}\sum_{i=1}^{n+1} f_{F_i}(x) &= \sum_{j=1}^{n+1} f_{F_j}\left(\sum_{i=1}^{n+1} \beta_i x_i\right) = \sum_{i,j=1}^{n+1} \beta_i f_{F_j}(x_i) \\ &= \sum_{i=1}^{n+1} \beta_i f_{F_i}(x_i) = \sum_{i=1}^{n+1} \beta_i = 1.\end{aligned}$$ □

Notice that the functions f_{F_i} in Lemma 7 are actually critical functions of the simplex. So we give the following stability version of Lemma 7 for general convex bodies.

Lemma 8. *Let $C \in \mathfrak{K}^n$, $c \in \mathcal{C}_C$, if $as(C) \le (1/(n+1))(1+\varepsilon)$ for some $0 \le \varepsilon < 1/2(n+1)$, then for any $n+1$ affine functions $f_1, \ldots, f_{n+1} \in \mathbb{E}_C(c)$ satisfying that $\bigcap_{i=1}^{n+1}\{f_i \ge 0\}$ is bounded (i.e., a simplex), we have*

$$1 \le \sum_{i=1}^{n+1} f_i(x) \le 1 + 2(n+1)\varepsilon \qquad \text{for all} \quad x \in \bigcap_{i=1}^{n+1}\{f_i \ge 0\}.$$

Proof. Denote by S the simplex $\bigcap_{i=1}^{n+1}\{f_i \geq 0\}\,(\supset C)$, and denote by g_i $(1 \leq i \leq n+1)$, where g_i satisfies $\{g_i = 0\} = \{f_i = 0\}$, the critical functional of S. Then by Lemma 7 and the fact that $g_i(x) \leq f_i(x)$ for all $x \in S$, it follows that

$$1 = \sum_{i=1}^{n+1} g_i(c) \leq \sum_{i=1}^{n+1} f_i(c) = (n+1)\ as(C) \leq 1+\varepsilon.$$

So

$$0 \leq \sum_{i=1}^{n+1} (f_i(c) - g_i(c)) \leq \varepsilon.$$

Thus, for any $1 \leq i \leq n+1$,

$$g_i(c) \geq f_i(c) - \varepsilon.$$

Therefore, for all $x \in S$ (notice that $\{g_i = 0\} = \{f_i = 0\}$), we have

$$\begin{aligned} g_i(x) &= \frac{g_i(c)}{f_i(c)} f_i(x) \geq \frac{f_i(c)-\varepsilon}{f_i(c)} f_i(x) \\ &= \left(1 - \frac{1}{f_i(c)}\varepsilon\right) f_i(x) \geq (1-(n+1)\varepsilon) f_i(x) \end{aligned}$$

(here we use the fact that $f_i(c) = as(C) \geq 1/(n+1)$) followed by

$$f_i(x) \leq \big(1-(n+1)\varepsilon\big)^{-1} g_i(x) \leq \big(1+2(n+1)\varepsilon\big) g_i(x),$$

where we use the fact that $(1-t)^{-1} \leq 1+2t$ for $0 \leq t \leq \frac{1}{2}$. Now it follows that, for all $x \in S$,

$$\begin{aligned} 1 &= \sum_{i=1}^{n+1} g_i(x) \leq \sum_{i=1}^{n+1} f_i(x) \\ &\leq \big(1+2(n+1)\varepsilon\big) \sum_{i=1}^{n+1} g_i(x) = 1+2(n+1)\varepsilon. \end{aligned}$$

□

The next theorem, stated as a stability theorem for the *as*-measure of asymmetry, is equivalent to Theorem A*.

Theorem 1. *Given $C \in \mathfrak{K}^n$. If $as(C) \leq (1/(n+1))(1+\varepsilon)$ for some $0 \leq \varepsilon < 1/4(n+1)^2$, then there exists a simplex S_0 formed by the critical support hyperplanes such that*

$$\left(1 + 4(n+1)^2 \frac{\varepsilon}{n}\right)_c^{-1} S_0 \subset C \subset S_0.$$

Proof. We will show actually that for all simplexes S of the form $\bigcap_{i=1}^{n+1}\{f_i \geq 0\}$, where $f_i \in \mathbb{E}_C(c)$ $(1 \leq i \leq n+1)$, we have

$$\left(1 + 4(n+1)^2 \frac{\varepsilon}{n}\right)_c^{-1} S \subset C \subset S,$$

which together with Corollary 2 (notice that the assumption on ε ensures that $(1/(n+1))(1+\varepsilon) < 1/n$) will finish the proof.

Suppose a simplex $S_0 = \bigcap_{i=1}^{n+1}\{f_i \geq 0\}$ for some $f_i \in \mathbb{E}_C(c)$ $(1 \leq i \leq n+1)$, then by Lemma 8, for all $x \in C$,

$$1 \leq \sum_{i=1}^{n+1} f_i(x) \leq 1 + 2(n+1)\varepsilon.$$

Therefore, for any fixed $1 \leq j \leq n+1$, if we choose $y_j \in C \cap \{f_j = 1\}$, then we have

$$\sum_{i\neq j} f_i(y_j) + 1 \leq 1 + 2(n+1)\varepsilon \quad \text{or} \quad \sum_{i\neq j} f_i(y_j) \leq 2(n+1)\varepsilon$$

followed by

$$f_i(y_j) \leq 2(n+1)\varepsilon, \qquad i \neq j.$$

Now set $S_1 = \bigcap_{i=1}^{n+1}\{f_i \geq 2(n+1)\varepsilon\}$. Then clearly S_1 is bounded since $S_1 \subset S_0$, and $int(S_1)/\varphi$ since $c \in int(S_1)$ derived from the fact that $f_i(c) \geq 1/(n+1) > 2(n+1)\varepsilon$ for all i. So S_1 is a simplex. It is also clear that the facets $\{f_i = 2(n+1)\varepsilon\}$ of S_1 are parallel to the facets $\{f_i = 0\}$ of S_0.

Furthermore, since for any $1 \leq i \leq n+1$,

$$\frac{f_i(c)}{f_i(c) - 2(n+1)\varepsilon} = 1 + \frac{2(n+1)}{f_i(c) - 2(n+1)\varepsilon}\varepsilon < 1 + 4(n+1)^2\varepsilon$$

(where we used the fact that $f_i(c) - 2(n+1)\varepsilon > 1/(n+1) - 2(n+1)/4(n+1)^2 = 1/2(n+1)$), we have

$$\lambda_c^{-1} S_0 \subset S_1,$$

where $\lambda = 1 + 4(n+1)^2\varepsilon$.

Notice further that (since $f_i(y_j) \leq 2(n+1)\varepsilon$ for all $1 \leq j \leq n+1, i \neq j$)

$$S_1 \subset conv\{y_j \mid 1 \leq j \leq n+1\} \subset C,$$

we finally get

$$\lambda_c^{-1} S_0 \subset S_1 \subset C \subset S_0.$$

This completes the proof. □

Now we can finish the proof for Theorem A^*:

Proof of Theorem A^*. By Remark 4, it is easy to see that if $As(C) \geq n - \varepsilon$, then $as(C) \leq (1/(n+1))(1+\varepsilon/n)$, and $\varepsilon/n < 1/4(n+1)^2$ when $\varepsilon < 1/8(n+1)$. So by Theorem 1 and the fact that $1 + 4(n+1)^2(\varepsilon/n) \leq 1 + 8(1+n)\varepsilon$ when $\varepsilon < 1/8(n+1)$, there is a simplex S_0 formed by the critical support hyperplanes of C such that

$$\left(1 + 8(1+n)\varepsilon\right)_c^{-1} S_0 \subset \left(1 + 4(n+1)^2\frac{\varepsilon}{n}\right)_c^{-1} S_0 \subset C \subset S_0. \qquad \square$$

The following estimate immediately comes from Theorem A.

Corollary 3. *Let* $C, D \in \mathfrak{K}^n$. *If* $As(C) \geq n - 1/8(n+1)$, $As(D) \geq n - 1/8(n+1)$, *then*

$$d_M(C, D) \leq 1 + c_n \max\{n - As(C), n - As(D)\},$$

where $c_n = 24(n+1)$.

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