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Anchored Hyperplane Location Problems

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Abstract. The *anchored hyperplane location problem* is to locate a hyperplane passing through some given points $\mathcal{P} \subseteq \mathbb{R}^n$ and minimizing either the sum of weighted distances (*median problem*), or the maximum weighted distance (*center problem*) to some other points $\mathcal{Q} \subseteq \mathbb{R}^n$.

This problem of computational geometry is analyzed by using nonlinear programming techniques. If the distances are measured by a norm, it will be shown that in the median case there exists an optimal hyperplane that passes through at least $n - k$ affinely independent points of $\mathcal{Q}$, if k is the maximum number of affinely independent points of $\mathcal{P}$. In the center case, there exists an optimal hyperplane which is at maximum distance to at least $n - k + 1$ affinely independent points of $\mathcal{Q}$. Furthermore, if the norm is a smooth norm, *all* optimal hyperplanes satisfy these criteria. These results generalize known results about unrestricted hyperplane location problems.

1. Introduction

Approximating a set of given points $\mathcal{Q}$ in $\mathbb{R}^n$ by a linear function is known as the *linear fit problem* or the *hyperplane location problem*. The goal is to find a hyperplane (represented by a linear function) minimizing the sum of weighted distances to the points in $\mathcal{Q}$, or minimizing the maximum weighted distance to the points in $\mathcal{Q}$, respectively. As distance measure any norm is possible, but also gauges and metrics have recently been discussed [S3], [PC]. In this paper a restricted version of the hyperplane location problem—the so-called *anchored hyperplane location problem*—is analyzed, namely, the hyperplane is additionally forced to pass through some given points $p \in \mathcal{P}$.

Hyperplane location problems appear in several mathematical disciplines, where they have mainly been studied with the Euclidean and the rectangular distance. In *robust statistics* variants of the hyperplane location problem are known as absolute errors regression, median problems, L_1 regression, L_∞ regression, and orthogonal/vertical

L_1-fit or L_∞-fit problems, respectively, depending on the type of distance measure and on the objective function used. Related investigations go back to the 18th century (see, e.g., [B]). A motivation of these problems is given in [RL]. In *numerical mathematics*, certain approximation problems, for example the approximation of a given function by a linear function, lead in a natural way to the same type of problems, see, e.g., [R]. In *computational geometry*, hyperplane location problems are known as linear L_1 or L_∞ approximation problems. Here, in particular, the time complexity of the Euclidean variant of the problem was investigated by several authors, see e.g., [KM] and [HII$^+$]. The two-dimensional version of the problem has also been studied in *operations research*, known as the *line facility location problem* which is a special case of location problems. Line location problems in the plane were first discussed in [W], and later by many other authors, e.g., [MN1], [MT], and [LW]. Recently, line and hyperplane location problems have also been discussed for distance measures apart from the Euclidean and the rectangular distance, namely, for block norms in [S1], arbitrary norms in [S2] and [MS1], smooth norms (see [MS2]), and also for gauges (see [PC]).

If distances are measured by a norm, the main results for unrestricted hyperplane location problems are the following incidence criteria. There always exists an optimal hyperplane for the median problem that passes through n affinely independent points of $\mathcal{Q}$, and, in the center case, there always exists an optimal hyperplane which is at maximum distance from $n + 1$ affinely independent points of $\mathcal{Q}$. This has first been noted for the Euclidean distance, independently by many authors (see, for example, [W], [MN1], [LW], [MN2], [KM], and [HII$^+$]) and has later been generalized to block norms and even to distances derived from arbitrary norms [S2], [MS1], [S3]. A slightly weaker condition for gauges in the case of the median objective function has recently been developed in [PC]. Furthermore, if and only if the norm is a smooth norm, *all* optimal hyperplanes in the median problem pass through n affinely independent points of $\mathcal{Q}$, and in the center problem *all* optimal hyperplanes are at maximum distance from $n + 1$ affinely independent points of $\mathcal{Q}$ [MS2].

In this paper a nonlinear programming technique is developed to transfer both incidence properties to restricted hyperplane location problems in the following sense. If the hyperplane approximating $\mathcal{Q}$ is forced to contain k affinely independent points of a set $\mathcal{P}$, then there exists a median hyperplane passing through at least $n - k$ affinely independent points of $\mathcal{Q}$, and a center hyperplane which is at maximum distance from $n-k+1$ affinely independent points of $\mathcal{Q}$. Sharper results for smooth norms will also be developed. For $k = 0$ this directly yields analytical proofs for the incidence properties of unrestricted hyperplane location problems.

2. The Anchored Hyperplane Location Problems

Let d be a distance derived from a norm, i.e.,

$$d(x, y) = \gamma(y - x) \qquad \text{for all} \quad x, y \in \mathbb{R}^n$$

for some norm γ. We consider the following problems (AMH) and (ACH):

Given two finite sets $\mathcal{P}$ and $\mathcal{Q}$ of points in $\mathbb{R}^n$, find a hyperplane H passing through all points in $\mathcal{P}$ and minimizing

$$\text{(AMH)}\quad f(H) = \sum_{q\in\mathcal{Q}} w_q\, d(H, q), \quad \text{or}$$

$$\text{(ACH)}\quad g(H) = \max_{q\in\mathcal{Q}} w_q\, d(H, q), \quad \text{respectively,}$$

where $w_q \geq 0$ for all $q \in \mathcal{Q}$ are nonnegative weights, d is a distance measure derived from a norm, and

$$d(H, q) = \min_{x\in H} d(q, x).$$

Note that, although the results of this paper remain true for zero weights, points q with weight $w_q = 0$ can simply be deleted from $\mathcal{Q}$.

An optimal hyperplane H for problem (AMH) is called an *anchored median hyperplane*, and an *anchored center hyperplane* is an optimal solution of problem (ACH).

First, note that the anchored hyperplane location problem is only feasible if there exists a hyperplane containing the whole set $\mathcal{P}$. In particular, (AMH) and (ACH) are feasible if and only if the maximum number of affinely independent points in $\mathcal{P}$ is smaller than or equal to n. Let $k = dim_{\text{aff}}(\mathcal{P})$ denote the maximum number of affinely independent points in $\mathcal{P}$. Without loss of generality, we therefore assume in the following that

$$k \leq n.$$

Furthermore, note that if $k + dim_{\text{aff}}(\mathcal{Q}) \leq n$, then there exists a (clearly optimal) hyperplane containing *all* points in $\mathcal{P}$ and in $\mathcal{Q}$. To avoid this trivial case, we also assume that

$$dim_{\text{aff}}(\mathcal{Q}) > n - k.$$

As mentioned before, both problems (AMH) and (ACH) have been studied extensively in their unrestricted versions, i.e., with $\mathcal{P} = \emptyset$ (see, e.g., [MS1], [S3], and the references therein). The restricted version of (AMH) has been discussed in [MN2] in a planar setting, i.e., $n = 2$ and $\mathcal{P}$ consisting of one single point. For the Euclidean distance it has been shown that all optimal lines for problem (AMH) contain at least one point from the set $\mathcal{Q}$. The same has been noted in [KM].

In the following we generalize the incidence properties of unrestricted hyperplane location problems to the restricted versions of (AMH) and (ACH). The proofs we present combine the techniques developed in [S3] and in [PC]. They are based on the minimization of quasiconcave functions, which contain the sum, or the maximum of piecewise affine linear functions. Note that an affine linear function $f: \mathbb{R}^n \to \mathbb{R}$ is given by some vector $a \in \mathbb{R}^n$ and a real number b such that $f(x) = \langle a, x\rangle + b$. If $b = 0$ the function f is linear. We first turn to the case of median hyperplanes.

3. Anchored Median Hyperplanes

Lemma 1. *Let* $\mathcal{M} = \{1, 2, \ldots, M\}$, $M \geq n$, *and let* $h: \mathbb{R}^n \to \mathbb{R}^+$ *be a strictly positive and convex function*, *and* $f_m: \mathbb{R}^n \to \mathbb{R}$, $m \in \mathcal{M}$, *be affine linear and nonconstant* *functions. Consider the problem to minimize*

$$f(x) = \frac{1}{h(x)} \sum_{m \in \mathcal{M}} |f_m(x)|.$$

Furthermore, suppose that a minimum of f *exists. Then there exists an optimal solution* x^* *and a subset* $\mathcal{M}^* \subseteq \mathcal{M}$ *with* $|\mathcal{M}^*| \geq n$ *such that*

$$f_m(x^*) = 0 \qquad \textit{for all} \quad m \in \mathcal{M}^*.$$

Moreover, if h *is strictly convex, all optimal solutions* x^* *satisfy* $f_m(x^*) = 0$ *for at least* n *of the functions* f_m.

Proof. The following set of hyperplanes

$$H_m := \{x \in \mathbb{R}^n \colon f_m(x) = 0\}, \qquad m \in \mathcal{M},$$

partitions $\mathbb{R}^n$ into polyhedral (full-dimensional) cells. On each cell C all the functions $|f_m|$, $m \in \mathcal{M}$, are affine linear, and therefore $\sum_{m \in \mathcal{M}} |f_m(x)|$ is also an affine linear function. Since h is convex, f is a quasiconcave function on each cell. Minimizing over each of the cells separately yields a minimum at a cell vertex (see, e.g., page 109, Theorem 3.5.3, of [BSS]) whenever it exists. Using that a finite optimal solution exists, a global optimum x^* is then obtained as the best of all these minima. Hence, x^* is also a cell vertex, yielding that it lies on the intersection of at least n hyperplanes H_m. Defining

$$\mathcal{M}^* := \{m \in \mathcal{M} \colon x^* \in H_m\}$$

shows the first result.

For the second part of the lemma, note that an affine linear function divided by a strictly convex function is a strictly quasiconcave function, meaning that on each cell C the objective f is strictly quasiconcave and therefore attains its minima only at cell vertices. □

In the following we describe a hyperplane by its normal vector $s \in \mathbb{R}^n \backslash \{0\}$ and its intercept $b \in \mathbb{R}$, i.e.,

$$H_{s,b} = \{x \in \mathbb{R}^n \colon \langle s, x \rangle + b = 0\}.$$

We can now state the main result for anchored median hyperplanes.

Theorem 1. *Let* d *be a distance derived from a norm* γ, *and let* $k \leq n$ *be the number of affinely independent points in* $\mathcal{P}$. *Then there exists an anchored median hyperplane passing through at least* $n - k$ *affinely independent points of* $\mathcal{Q}$.

Proof. First note that hyperplanes lying too far away from $\mathcal{Q}$ need not be considered, and since the length of the normal vector s can be assumed to be bounded, we can restrict the problem to a compact feasible set yielding the existence of a minimum s^*, b^*. Furthermore, since $s \neq 0$ is required for a hyperplane, we conclude that the normal vector s^* of an optimal solution to problem (AMH) has at least one nonzero component. Without loss of generality assume that this is the first one, i.e., $s_1^* \neq 0$. Dividing all coefficients by s_1^* yields an optimal solution with the first component of the normal vector equal to 1. Restricting the optimization problem (AMH) to hyperplanes with normal vectors s satisfying $s_1 = 1$ hence yields the same optimal solution s^*, b^*. We will therefore additionally require $s_1 = 1$.

According to Mangasarian [M] or, independently, Plastria and Carrizosa [PC] the distance between a point $q \in \mathbb{R}^n$ and a hyperplane $H_{s,b}$ can be calculated by

$$d(q, H_{s,b}) = \frac{|\langle s, q\rangle + b|}{\gamma^0(s)},$$

where γ^0 denotes the dual (or polar) norm of γ, defined by

$$\gamma^0(x) = \max\{\langle x, y\rangle \colon\ \gamma(y) \leq 1\}.$$

For $\mathcal{P} = \{p_1, p_2, \ldots, p_l\}$ problem (AMH) can now be reformulated as the following nonlinear programming problem in $n+1$ variables:

$$\min_{s,b} \frac{1}{\gamma^0(s)} \sum_{q\in\mathcal{Q}} w_q |\langle s, q\rangle + b|$$

$$\text{such that} \quad \langle s, p_i\rangle + b = 0 \quad \text{for} \quad i = 1, 2, \ldots, l,$$
$$s_1 = 1.$$

Since k is the maximum number of affinely independent points of $\mathcal{P}$ we conclude that the linear dimension

$$dim \left\{ \begin{pmatrix} p_1 \\ 1 \end{pmatrix}, \begin{pmatrix} p_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} p_l \\ 1 \end{pmatrix} \right\} = k,$$

where $\binom{p_i}{1} \in \mathbb{R}^{n+1}$, $i = 1, 2, \ldots, l = |\mathcal{P}|$. Consequently, the coefficient matrix of

$$\langle s, p_i\rangle + b = 0 \qquad \text{for} \quad i = 1, 2, \ldots, l$$

has rank k, meaning that there exist k variables which can be substituted and eliminated in the objective function. Together with $s_1 = 1$ (which is linearly independent of the equations $\langle s, p_i\rangle + b = 0$) there remain $(n+1) - k - 1 = n - k$ variables, denoted by $t \in \mathbb{R}^{n-k}$. This leads to the following equivalent problem in $\mathbb{R}^{n-k}$:

$$\min f(t) = \frac{1}{\gamma^0(s(t))} \sum_{q\in\mathcal{Q}} w_q |\langle s(t), q\rangle + b(t)|$$

with s and b affine linear functions,

$$s\colon\ \mathbb{R}^{n-k} \to \mathbb{R}^n \quad \text{and}$$
$$b\colon\ \mathbb{R}^{n-k} \to \mathbb{R}.$$

Note that the n-vector $s(t)$ contains all components of the smaller vector t; hence the mapping s is injective. Since the composition of a convex and an affine linear function still remains convex, and the functions $f_q(t) = w_q(\langle s(t), q\rangle + b(t))$ are affine linear we can apply Lemma 1 in dimension $n - k$ and conclude that there exists an optimal solution t^* and a set $\mathcal{Q}^* \subseteq \mathcal{Q}$ with $|\mathcal{Q}^*| \geq n - k$ such that

$$\langle s(t^*), q\rangle + b(t^*) = 0$$

for all $q \in \mathcal{Q}^*$. Defining $s^* = s(t^*)$ and $b^* = b(t^*)$ it follows that for all $q \in \mathcal{Q}^*$,

$$q \in H_{s^*,b^*},$$

meaning that the optimal hyperplane $H^* := H_{s^*,b^*}$ passes through at least $n - k$ points in $\mathcal{Q}$.

It remains to show that $\mathcal{Q}^*$ can be chosen in such a way that $k = dim_{\text{aff}}(\mathcal{Q}^*) = n - k$. To this end, let $\mathcal{Q}^*$ be the set of at least $n - k$ points of $\mathcal{Q}$ contained in an optimal hyperplane H^*. Define

$$\bar{\mathcal{P}} := \mathcal{P} \cup \mathcal{Q}^*,$$
$$\bar{\mathcal{Q}} := \mathcal{Q} \setminus \mathcal{Q}^*,$$

and note that each optimal solution (AMH) with $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ is not worse than H^* and hence also solves the original problem (AMH). If $\bar{k} = dim_{\text{aff}}(\mathcal{Q}^*) < n - k$ we get that $dim_{\text{aff}}(\bar{\mathcal{P}}) < n$ and, due to the above result, an optimal solution H' exists passing through at least one more point of Q than H^*. This procedure can be continued until the optimal hyperplane contains $n - k$ affinely independent points of $\mathcal{Q}$. □

Two remarks should be added:

- Note that the optimal hyperplane may pass through more than $n - k$ affinely independent points since points in Q that lie within the affine hull of $\mathcal{P}$ will automatically be covered.
- For $\mathcal{P} = \emptyset$ a *halving property* has been shown for locating a median hyperplane in normed spaces (see [S3]), i.e., if $Q^+(H)$, $Q^-(H)$, and $Q^0(H)$ are the points of Q lying below, underneath, and on the hyperplane H, respectively, then *all* median hyperplanes satisfy

$$\left| \sum_{q \in Q^+} w_q - \sum_{q \in Q^-} w_q \right| \leq \sum_{q \in Q^0} w_q.$$

  In general, this property does not hold if $|\mathcal{P}| \geq 1$, not even for $n = 2$ and the Euclidean distance, as the following example demonstrates:

  Let $\mathcal{P} = \{0\}$, $\mathcal{Q} = \{(10, 1), (11, 0), (-10, 1)\}$, and assume equal weights for the points in $\mathcal{Q}$. Then the optimal anchored line l^* passes through 0 and $(10, 1)$, with the two remaining points of $\mathcal{Q}$ lying on the same side of l^*.

4. Anchored Center Hyperplanes

In this section, too, we first derive a result for minimizing a quasiconcave function. This time, we deal with a function g, which is given as the maximum of piecewise affine linear functions, all divided by the same convex denominator.

Lemma 2. *Let $\mathcal{M} = \{1, 2, \ldots, M\}$, $M > n$, and let $h\colon \mathbb{R}^n \to \mathbb{R}^+$ be a strictly positive and convex function, and $f_m\colon \mathbb{R}^n \to \mathbb{R}$, $m \in \mathcal{M}$, be affine linear functions. Consider the problem to minimize*

$$g(x) = \frac{1}{h(x)} \max_{m\in\mathcal{M}} |f_m(x)|.$$

Furthermore, suppose that a minimum of g exists. Then there exists an optimal solution x^* and a subset $\mathcal{M}^* \subseteq \mathcal{M}$ with $|\mathcal{M}^*| \geq n+1$ such that

$$g(x^*) = \frac{|f_m(x^*)|}{h(x^*)} \qquad \textit{for all} \quad m \in \mathcal{M}^*.$$

Moreover, if h is strictly convex, all optimal solutions x^* satisfy $g(x^*) = |f_m(x^*)|/h(x^*)$ for at least $n+1$ of the indices $m \in \mathcal{M}$.

Proof. First suppose that there exists some $x \in \mathbb{R}^n$ such that $f_m(x) = 0$ for all $m \in \mathcal{M}$. Then x is the optimal solution and the lemma is trivially true.

Otherwise, we can assume that $\max_{m\in\mathcal{M}}|f_m(x)| > 0$ for all $x \in \mathbb{R}^n$. Define for all $m \in \mathcal{M}$ two cells given by

$$\begin{aligned} C_m^+ &= \left\{x \in \mathbb{R}^n\colon\ f_m(x) = \max_{k\in\mathcal{M}}|f_k(x)|\right\}, \quad \text{and} \\ C_m^- &= \left\{x \in \mathbb{R}^n\colon\ -f_m(x) = \max_{k\in\mathcal{M}}|f_k(x)|\right\}. \end{aligned}$$

Then each of the C_m^+, C_m^- is either empty or a polyhedral set given by linear inequalities of type $f_m(x) \geq f_k(x)$ or $f_m(x) \leq -f_k(x)$ for all $k \in \mathcal{M}$, $k \neq m$. That is, the boundaries of the cells are given by the hyperplanes

$$\begin{aligned} H_{km}^+ &= \{x \in \mathbb{R}^n\colon\ f_m(x) = f_k(x)\} \quad \text{and} \\ H_{km}^- &= \{x \in \mathbb{R}^n\colon\ f_m(x) = -f_k(x)\} \end{aligned}$$

for $k \neq m$. On each cell, $\max_{k\in\mathcal{M}}|f_k(x)|$ is affine linear and, hence, g is quasiconcave on each cell. Since a minimum exists, we conclude that there exists an optimal cell vertex x^*, e.g., of the cell $C_{\bar{m}}^+$ (again, see [BSS]). Due to the assumption that $\max_{m\in\mathcal{M}}|f_m(x)| > 0$ for all $x \in \mathbb{R}^n$, we know that $H_{k\bar{m}}^+ \cap H_{k\bar{m}}^- \cap C_{\bar{m}}^+ = \emptyset$ for all $k \in \mathcal{M}$ and hence each cell vertex of $C_{\bar{m}}$ is the intersection of n of the hyperplanes $H_{k\bar{m}}^+, H_{k\bar{m}}^-$, $k \neq \bar{m}$, with pairwise different indices k. Consequently, the cardinality of

$$\mathcal{M}^* = \{k \in \mathcal{M}\colon\ x^* \in H_{k\bar{m}}^+ \text{ or } x^* \in H_{k\bar{m}}^-\} \cup \{\bar{m}\}$$

is greater than or equal to $n+1$, proving the first part of the lemma.

In case of a strictly convex function h, we get that g is strictly quasiconcave on each cell C_m, and hence *all* optimal solutions are attained at cell vertices. □

Note that in [D] the following reduction result for minimizing *convex* functions f_m, $m \in \mathcal{M}$, has been provided: there exists an optimal solution x^* and a subset $\mathcal{M}^* \subseteq \mathcal{M}$ with $|\mathcal{M}^*| \leq n+1$ such that x^* is the optimal solution for the reduced problem

$$\min \max_{m \in \mathcal{M}^*} f_m(x),$$

i.e., also in this (opposite) case, an optimal solution can be found by looking only at subsets with cardinality of at most $n+1$.

Now the main result for finding anchored center hyperplanes can be shown.

Theorem 2. *Let d be a distance derived from a norm γ, and let $k \leq n$ be the number of affinely independent points in $\mathcal{P}$. Then there exists an anchored center hyperplane which is at maximum distance from at least $n-k+1$ affinely independent points of Q.*

Proof. The proof works along the lines of the proof to Theorem 1. Assuming $s_1^* = 1$ and using

$$d(q, H_{s,b}) = \frac{|\langle s, q\rangle + b|}{\gamma^0(s)},$$

for the distance between a point q and a hyperplane $H_{s,b}$, (ACP) can be rewritten as

$$\min g(t) = \frac{1}{\gamma^0(s(t))} \max_{q \in \mathcal{Q}} w_q |\langle s(t), q\rangle + b(t)|$$

with s and b affine linear functions, $\gamma^0(s(t))$ convex, and $t \in \mathbb{R}^{n-k}$.

Applying Lemma 2 in dimension $n-k$ yields the existence of an optimal solution t^* and a set $\mathcal{Q}^* \subseteq \mathcal{Q}$ with $|\mathcal{Q}^*| \geq n-k+1$ such that for $s^* := s(t^*)$ and $b^* := b(t^*)$ we get that

$$g(H_{s^*,b^*}) = \frac{1}{\gamma^0(s^*)} \max_{q' \in \mathcal{Q}} w_{q'} |\langle s^*, q'\rangle + b^*| = \frac{w_q}{\gamma^0(s^*)} |\langle s^*, q\rangle + b^*|$$

for all $q \in \mathcal{Q}^*$. Hence, the optimal hyperplane $H^* := H_{s^*,b^*}$ is at maximum distance from at least $n-k+1$ points in $\mathcal{Q}$, and using the same iterative argument as in the proof of Theorem 1 shows that these points can assumed to be affinely independent. □

5. Anchored Hyperplanes and Smooth Norms

A smooth norm γ is defined as follows. Consider the unit ball of γ, given by

$$B_\gamma = \{x \in \mathbb{R}^n : \gamma(x) \leq 1\}.$$

The norm γ is called a *smooth norm* if B_γ is supported by exactly one hyperplane for any point $x \in \partial B_\gamma$ on its boundary. In this case we get the following stronger result.

Theorem 3. *Let d be a distance derived from a smooth norm γ, and let k be the number of affinely independent points in $\mathcal{P}$. Then* **all** *anchored median hyperplanes pass through at least $n-k$ affinely independent points of Q, and* **all** *anchored center hyperplanes are at maximum distance from at least $n-k+1$ affinely independent points of Q.*

Proof. Since γ is a smooth norm, the dual norm γ^0 is strictly convex, see, for example, [K]. Consequently, we derive the following objective functions (analogously to the proof of Theorem 1), but with a strictly convex denominator $\gamma^0(s(t))$:

$$\min f(t) = \frac{1}{\gamma^0(s(t))} \sum_{q \in \mathcal{Q}} w_q |\langle s(t), q \rangle + b(t)|$$

for the median problem, and

$$\min g(t) = \frac{1}{\gamma^0(s(t))} \max_{q \in \mathcal{Q}} w_q |\langle s(t), q \rangle + b(t)|$$

for the center problem, respectively. As in the proofs of Theorems 1 and 2, s and b are affine linear functions and $t \in \mathbb{R}^{n-k}$. Since the composition of a strictly convex function and an affine linear injective function still remains *strictly* convex we can apply the second parts of Lemmas 1 and 2, respectively, and conclude the result. □

6. Conclusion

Theorems 1 and 2 provide the basics for polynomial-time algorithms to solve anchored hyperplane location problems in fixed dimensions with median or center objective function, since an enumeration approach is possible in both cases. If k is the number of affinely independent points in $\mathcal{P}$ we have to check

- all $\binom{n-k}{|\mathcal{Q}|}$ hyperplanes passing through all points in $\mathcal{P}$ and through any (affinely independent) subset of $n-k$ points of $\mathcal{Q}$ in the median case, and
- all $\binom{n-k+1}{|\mathcal{Q}|}$ hyperplanes which are at maximum distance from any (affinely independent) subset of $n-k+1$ points of $\mathcal{Q}$ and contain all points of $\mathcal{P}$ in the center case.

Since evaluating a hyperplane can be done in $O(|\mathcal{Q}|)$, this yields the following complexity results.

Corollary 1. *If the distance has been derived from a norm, an anchored median hyperplane can be found in $O(|\mathcal{Q}|^{n-k+1})$ time, and an anchored center hyperplane can be found in $O(|\mathcal{Q}|^{n-k+2})$ time, assuming that a norm evaluation can be done in constant time. In the case that d has been derived from a smooth norm, the same time complexity is sufficient to determine* **all** *optimal anchored hyperplanes.*

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