Discrete Comput Geom 26:591–601 (2001)
DOI: 10.1007/s00454-001-0034-y

Discrete & Computational
Geometry

Disk-Like Self-Affine Tiles in $\mathbb{R}^2$*

C. Bandt[1] and Y. Wang[2]

[1]Institut für Mathematik und Informatik, Arndt-Universität,
17487 Greifswald, Germany
bandt@uni-greifswald.de

[2]School of Mathematics, Georgia Institute of Technology,
Atlanta, GA 30332-0160, USA
wang@math.gatech.edu

Abstract. We give simple necessary and sufficient conditions for self-affine tiles in $\mathbb{R}^2$ to be homeomorphic to a disk.

1. Introduction

Throughout this note we consider integral self-affine tiles with standard digit sets. Such are tiles $T := T(A, \mathcal{D})$ satisfying

$$A(T) = T + \mathcal{D} \tag{1}$$

or

$$T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d), \tag{2}$$

where A is an expanding 2×2 matrix of integers, and $\mathcal{D} \subset \mathbb{Z}^2$ with $|\mathcal{D}| = |\det A|$ is a complete set of coset representatives for $\mathbb{Z}^2/A\mathbb{Z}^2$. See [1]–[3], [5]–[8], [10], [13]–[15], and [19]–[21].

Moreover, we assume that $T(A, \mathcal{D})$ tiles by the lattice $\mathbb{Z}^2$, that is, $T + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$. Such tiles are called *self-affine* $\mathbb{Z}^2$*-tiles*. There are standard methods for checking this property [13]–[15], [20]. When the digit set $\mathcal{D}$ is primitive, only in special cases may the corresponding tile $T(A, \mathcal{D})$ not be a $\mathbb{Z}^2$-tile, see [14].

---

* The second author was supported in part by the National Science Foundation, Grant DMS-0070586 and a grant from the Center for Wavelets, Approximation and Information Processing in the National University of Singapore.

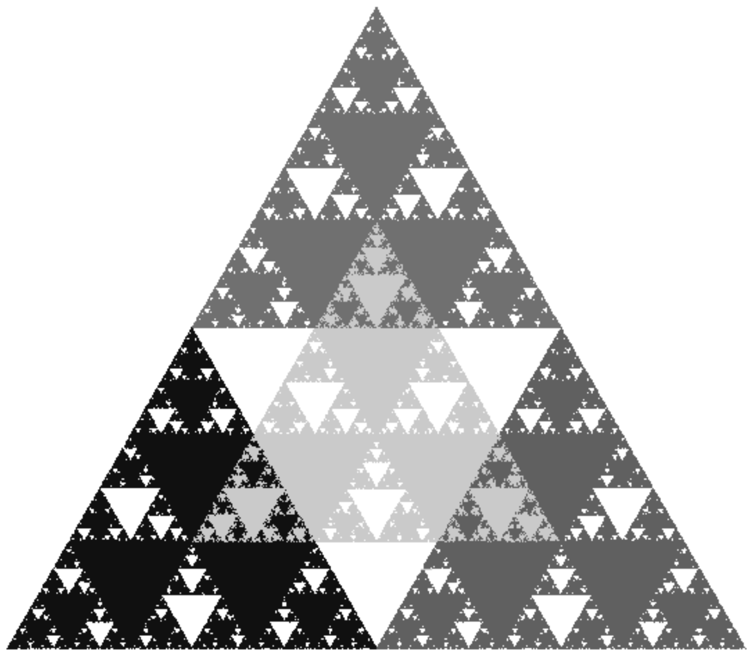

Fig. 1. Edge neighbors intersect in a Sierpinski gasket.

The simplest example of a self-affine $\mathbb{Z}^2$-tile is the unit square, divided into $n \times n$ small squares:

$$A = nI = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} \middle| \; i, j = 1, \ldots, n \right\}.$$

Figure 1 was obtained from this example, with $n = 2$, just replacing the residue $[1, 1]^T$ by $[-1, -1]^T$. (In order to get the symmetric picture, we have chosen coordinates which are not rectangular. The origin is in the center of Fig. 1 while the vertices of the triangle correspond to $[1, 0]^T$, $[0, 1]^T$ and $[-1, -1]^T$.) Figures 2 and 3 were obtained from the 4×4 square by replacing two residues in an obvious manner. There are infinitely many other ways in which residues can be exchanged but nearly all of them lead to tiles with holes or with disconnected interior.

Question. Given a self-affine $\mathbb{Z}^2$-tile $T(A, \mathcal{D})$, under what conditions is $T(A, \mathcal{D})$ homeomorphic to a disk?

Lattice tilings by topological disks must satisfy certain combinatorial properties. We state them here, and they are the keys to answering our question. We say that two tiles T' and T'' in a tiling are *neighbors* if $T' \cap T'' \neq \emptyset$. We call the tiles *vertex neighbors* if their intersection is a single point. They are *edge neighbors* if their intersection contains a point inside $\mathrm{int}(T' \cup T'')$, and hence uncountably many points (see Section 3). Note that there might be other types of neighbors.

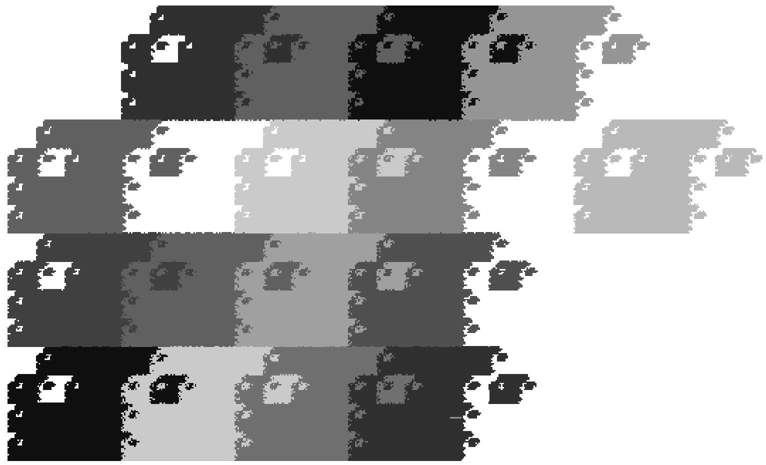

Fig. 2. A disconnected tile with six neighbors.

If the tiles are topological disks, an edge will be an arc, as usual. The tile in Fig. 1 will intersect an edge neighbor in a more complicated set (actually in a Sierpinski gasket).

It should be pointed out that for a given integral self-affine $\mathbb{Z}^2$-tile $T(A, \mathcal{D})$ there is a simple algorithm to determine its neighbors [19].

Proposition 1.1 [3, Lemma 5.1]. *Let Ω be a topological disk which tiles $\mathbb{R}^2$ by lattice translates of the lattice $\mathcal{L}$. Then in the tiling $\Omega + \mathcal{L}$ one of the following must be true*:

(i) *Ω has no vertex neighbors and six edge neighbors $\Omega \pm \alpha$, $\Omega \pm \beta$, $\Omega \pm (\alpha + \beta)$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

(ii) *Ω has four edge neighbors $\Omega \pm \alpha$, $\Omega \pm \beta$ and four vertex neighbors $\Omega \pm \alpha \pm \beta$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

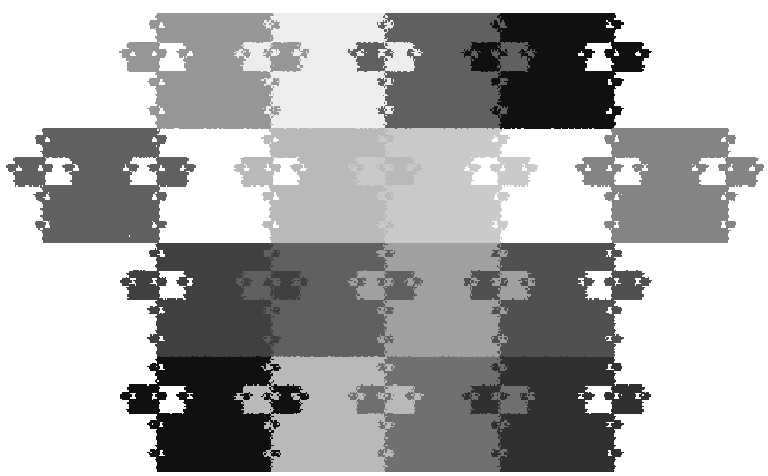

Fig. 3. A tile with eight neighbors which is not a disk.

Now let $\mathcal{F}$ be a finite subset of $\mathbb{Z}^2$. We say a subset $\mathcal{E} \subset \mathbb{Z}^2$ is $\mathcal{F}$-*connected* if for any $u, v \in \mathcal{E}$ there exist $u_0 = u, u_1, \ldots, u_n = v \in \mathcal{E}$ with $u_{i+1} - u_i \in \mathcal{F}$.

Proposition 1.2 (see [18]). *Let the self-affine $\mathbb{Z}^2$-tile $T = T(A, \mathcal{D})$ be a topological disk whose edge neighbors are $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. Then $\mathcal{D}$ is $\mathcal{F}$-connected.*

Proof. Note that $A(T) = T + \mathcal{D}$ is a topological disk. Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be the $\mathcal{F}$-connected components of $\mathcal{D}$ and assume that $k > 1$. Let $T_i = T + \mathcal{D}_i$. The set $T_1 \cap T_2$ is countable since $T + d_1$ and $T + d_2$ are not edge neighbors for $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$. The same is true for $T_i \cap T_j$ with $i \neq j$. Thus $A(T)$ becomes disconnected when a countable set is removed. This is not possible for a disk. □

2. Main Results

The main contribution of this paper is to show that the necessary conditions given in Propositions 1.1 and 1.2 are also sufficient. These seem to be the first sufficient conditions for tiles to be disk-like, and they solve a problem in [18]. It turns out that the type of neighbors is not essential, only their number and relative lattice position.

Theorem 2.1. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile. Suppose that T has not more than six neighbors $T + \mathcal{F}$. Then T is a topological disk if and only if $\mathcal{D}$ is $\mathcal{F}$-connected.*

Theorem 2.2. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile. Suppose that T has eight neighbors $T + \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha - \beta)\}$. Then T is a topological disk if and only if $\mathcal{D}$ is $\{\pm\alpha, \pm\beta\}$-connected.*

We give some examples to examine these conditions. The tile in Fig. 1 has six edge neighbors, and $\mathcal{D}$ is $\mathcal{F}$-connected. However, it is not a disk since there are six other vertex neighbors. Figure 2 shows a tile with six neighbors which is disconnected because $\mathcal{D}$ is not $\mathcal{F}$-connected. In Fig. 3 we have eight neighbors as assumed in Theorem 2.2, and the tile is connected. It is not a disk, however, since $\mathcal{D}$ is not connected with respect to the edge neighbors only.

Figure 4 shows that the mere assumption of eight neighbors in Theorem 2.2 would not suffice. Here $A = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$ and $\mathcal{D} = \{[0, 0]^T, [1, 0]^T, [-1, 0]^T\}$. The tile in the middle is T, and the three tiles of the middle row form $A(T)$. It is obvious that T has six edge neighbors $\pm\alpha = \pm[1, 0]^T$, $\pm\beta = \pm[-2, 1]^T$ and $\pm(\alpha + \beta)$. Moreover, the upper left and lower right neighbors $\pm\beta$ meet with their long narrow peaks in the center of T. This is only indicated by the picture, for a proof see 6.1 of [3]. Thus T is not a topological disk, and T has two more vertex neighbors $\pm 2\beta$.

For small numbers $m = |\mathcal{D}|$ of pieces, all possible disk-like $\mathbb{Z}^2$-tiles have been classified up to affine conjugacy. For $m = 2$ there are three and for $m = 3$ seven nonisomorphic cases [3], for $m = 4$ their number is twenty-nine [7], [18]. The proof

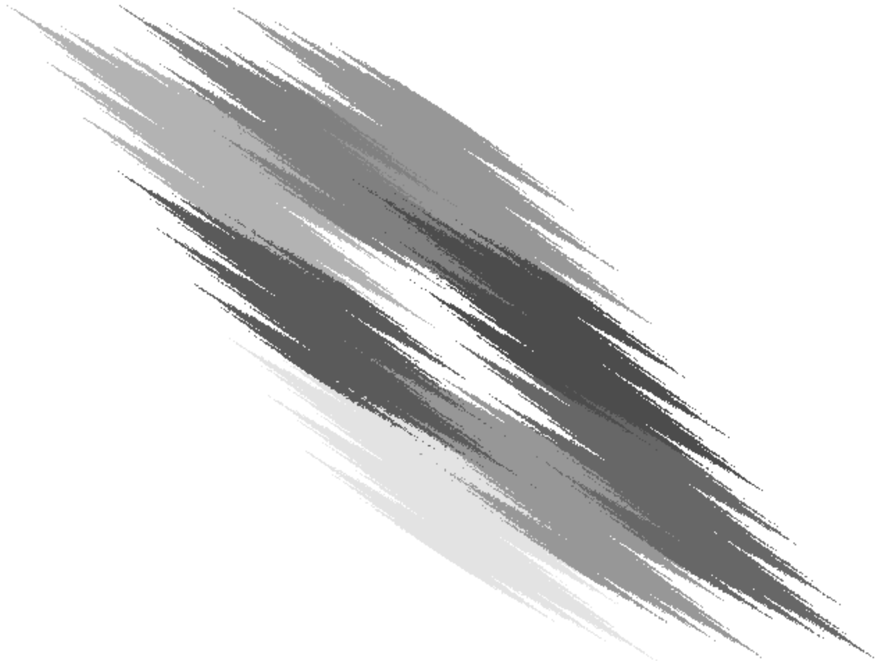

Fig. 4. A tile with eight neighbors and disconnected interior.

that the tiles are disk-like was given "by inspection." Even for tiles like the twindragon which are well known to be topological disks, no proof of this property seems to be published. Theorems 2.1 and 2.2, together with the algorithm in [19], now provide rigorous arguments.

Example. We just indicate the proof for the twindragon where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathcal{D} = \{[0,0]^T, [1,0]^T\}$. The neighbors are $\mathcal{F} = \{\pm[1,0]^T, \pm[0,1]^T, \pm[1,-1]^T\}$. Thus we have the six neighbors case of Theorem 2.1, and it is enough to see the first neighbor in order to conclude that $\mathcal{D}$ is $\mathcal{F}$-connected. Similarly, all cases for $m \leq 4$ can be checked.

Our technique also allows us to characterize the connectedness of a self-affine tile in n-dimensional space (see [8] and [11]).

Theorem 2.3. *Let $T(A, \mathcal{D})$ be a self-affine set in $\mathbb{R}^n$ for an integer matrix $A \in M_n(\mathbb{Z})$ and $\mathcal{D} \subset \mathbb{Z}^n$. Let $T + \mathcal{F}$ be the neighbors of T where $\mathcal{F} \subset \mathbb{Z}^n$. Then T is connected if and only if $\mathcal{D}$ is $\mathcal{F}$-connected.*

We note that in Theorem 2.3 we do not require $T(A, \mathcal{D})$ to be a tile. For general data $(A, \mathcal{D})$ the self-affine set $T(A, \mathcal{D})$ given by (1) may not be a tile. A neighbor of T is nevertheless well defined for general self-affine sets.

Theorems 2.1 and 2.3 combine to give

Corollary 2.4. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile with not more than six neighbors. Then T is a topological disk if and only if T is connected.*

3. A Topological Result

In this section we prove the key topological lemma for our theorems, as suggested by the referees. Our result is connected with two fundamental theorems of plane topology and geometry: the Jordan curve theorem [12], [22] and the Riemann mapping theorem. A more general statement with a more complicated proof was found recently by Luo and Tan [16].

Theorem 3.1. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile such that the interior* $\operatorname{int} T$ *is connected. Then T is a topological disk.*

Proof. We know already that (1) *T is the closure of its interior*, and (2) $\operatorname{int} T$ *is connected*. We now observe that (3) *T is simply connected*. That is, each simple closed curve $C \subset T$ contracts to a point within the set T, briefly, T contains no holes.

If there are holes, they must contain points of another tile $T + \alpha$ in the lattice tiling. Since the interior of $T + \alpha$ is connected, it must be completely surrounded by T, which is not possible. Note that $\operatorname{int} T$ is also simply connected by (1)–(3) as well as by the above argument.

Finally, we show that (4) *T is locally connected*, see [4]. The connectedness of T implies that each piece in the self-affine hierarchy of T is connected. Thus for each point $x \in T$ and each level k, the union of all level k pieces containing x forms a connected neighborhood $U_k(x)$ of x. The family of these (closed) neighborhoods is a neighborhood base of x.

It follows from classical results in plane topology that a compact set T with properties (1)–(4) must be a topological disk.

One way to deduce this from the literature is as follows. Since $\operatorname{int} T$ is simply connected, the Riemann mapping theorem provides a conformal homeomorphism $h_0\colon D \to \operatorname{int} T$ from the open unit disk D to $\operatorname{int} T$. Moreover, h_0 can be extended to a continuous mapping $h\colon \overline{D} \to T$ from the closed unit disk to T if and only if the boundary ∂T of T is locally connected [17, Theorem 2.1]. This condition follows from (3) and (4) by the Torhorst theorem, see p. 124 of [22]. Finally, a theorem of Carathéodory (see Theorem 2.6 of [17]) says that h is a homeomorphism if ∂T has no cutpoints, which follows from (3). □

4. Proof of the Theorems

Proof of Theorem 2.3. It is clear that $T + d$ and $T + d'$ are neighbors if and only if $d - d' \in \mathcal{F}$. It is known that the connectedness of a self-affine set can be expressed as the connectedness of the graph which has the pieces as vertices and edges between neighbors [4, Proposition 2] (see [11]). For $A(T)$ this means that $\mathcal{D}$ is $\mathcal{F}$-connected. □

The following lemma, as well as Theorem 4.2 and Lemma 4.3, does not use any self-similarity, and the structure of the edges may be as complicated as in our Fig. 3.

Lemma 4.1. *Let T be a $\mathbb{Z}^2$-tile with neighbors $T + \mathcal{F}$ for some $\mathcal{F} \subset \mathbb{Z}^2$. Let $\mathbb{Z}[\mathcal{F}]$ denote the subgroup of $\mathbb{Z}^2$ generated by $\mathcal{F}$. Then $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$.*

Proof. Call $\alpha_1, \alpha_2 \in \mathbb{Z}^2$ neighbors if $\alpha_1 - \alpha_2 \in \mathcal{F}$. Let $\mathcal{F}_0 = \{0\}$ and $\mathcal{F}_{n+1}$ be the neighbors of $\mathcal{F}_n$, $n \geq 0$. Define

$$\mathcal{G} = \bigcup_{n\geq 0} \mathcal{F}_n.$$

Clearly, $\mathcal{G} \subseteq \mathbb{Z}[\mathcal{F}]$ (in fact they are equal). Assume that $\mathcal{G} \neq \mathbb{Z}^2$. Then $\mathcal{H} = \mathbb{Z}^2 \backslash \mathcal{G}$ is nonempty. Set $\Omega = T + \mathcal{G}$ and $\Omega' = T + \mathcal{H}$. It follows that $\Omega \cap \Omega' = \emptyset$. However, both Ω and Ω' are closed sets and $\Omega \cup \Omega' = \mathbb{R}^2$. This contradicts the connectedness of $\mathbb{R}^2$. Therefore $\mathcal{G} = \mathbb{Z}^2$ and hence $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$. $\square$

To prove the sufficiency of $\mathcal{F}$-connectedness in Theorem 2.1 and of the stronger $\{\pm\alpha, \pm\beta\}$-connectedness in Theorem 2.2, we can now assume that T is connected (and hence arcwise connected [4]). It remains to show that the interior int T of T is connected. First we strengthen our assumptions in the case of not more than six neighbors.

Theorem 4.2. *Let T be a connected $\mathbb{Z}^2$-tile with at most six neighbors. Then there are α, β in $\mathbb{Z}^2$ such that the set of neighbors is $T + \mathcal{F}$ with $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$.*

Proof. Let $\Omega := T + \mathcal{F}$ and $\tilde{\Omega} := \mathbb{R}^2 \backslash (T \cup \Omega)$. The Hausdorff distance (see p. 65 of [6]), $d(T, \tilde{\Omega}) = \delta$ is positive since T is separated from $\tilde{\Omega}$. For $\varepsilon > 0$ let $B_\varepsilon(z)$ denote the open disk of radius ε centered at z. The collection of open disks $\{B_\varepsilon(z)\colon z \in T\}$ covers T. So by compactness we may find $z_1, \ldots, z_k \in T$ such that $T_\varepsilon = \bigcup_{j=1}^k B_\varepsilon(z_j)$ covers T. T_ε is connected because T is. Now T_ε is a finite union of disks, so ∂T_ε consists of a finite number of simple piecewise smooth closed Jordan curves. Assume that C is the Jordan curve of the outer boundary. For each $y \in \mathbb{Z}^2 \backslash \{0\}$ let $z_y \in C$ such that $\langle z_y, y\rangle = \max\{\langle z, y\rangle\colon z \in C\}$. It is easy to see that $z_y + y \in C + y$ is outside C and $d(z_y + y, C) \geq 1$. There exists a point $z' \in T + y$ with $d(z', z_y + y) < \varepsilon$, and this point z' must be outside C if $\varepsilon < \frac{1}{2}$.

Choose $\varepsilon < \min\{\delta/2, \frac{1}{2}\}$. Then for each $y \in \mathcal{F}$ the tile $T + y$ has points outside C. It also has points inside C because $T + y$ intersects T. Furthermore, $d(T_\varepsilon, \tilde{\Omega}) > \delta/2$. So $C \subset \operatorname{int} \Omega$. Because each neighbor of T has both points inside and outside C, and because T is connected, C must intersect all neighbors of T. Parametrize C by $z(t)$, $t \in [0, 1]$ with $z(0) = z(1)$. We now partitition $[0, 1]$ by $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that each segment $C_i = z([t_{i-1}, t_i])$ of the curve C has diam $(C_i) < \delta/2$. This partitition yields a sequence

$$y_{11}, \ldots, y_{1j_1}, y_{11}, \ldots, y_{2j_1}, \ldots, y_{k1}, \ldots, y_{kj_k}$$

in $\mathcal{F}$ such that $\{y_{ij}\colon 1 \leq j \leq j_i\}$ consists of all $y \in \mathcal{F}$ such that $(T + y) \cap C_i \neq \emptyset$. Pruning the sequence so that any two adjacent elements in the sequence are distinct we obtain a new sequence $y_1, \ldots, y_m, y_{m+1} = y_1$. Since each $y \in \mathcal{F}$ appears at least once in (y_{ij}) it must appear also at least once in the new sequence (y_i). Furthermore, points in two adjacent C_i's are less than δ apart so $d(T + y_i, T + y_{i+1}) < \delta$. Hence $y_{i+1} - y_i \in \mathcal{F}$.

Note that $\mathcal{F}$ must be centrally symmetric so $\mathcal{F}$ can only have two, four, or six elements. By Lemma 4.1 $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$. So $\mathcal{F}$ contains at least two linearly independent elements. This immediately rules out two elements for $\mathcal{F}$. If $\mathcal{F}$ has four elements, then $\mathcal{F} = \{\pm\alpha, \pm\beta\}$ with α and β independent. Thus one of $\pm\alpha$ must be followed by one of $\pm\beta$ somewhere in the sequence, yielding one of $\pm\alpha \pm \beta$ in $\mathcal{F}$, a contradiction. Hence $\mathcal{F}$ must have six elements. Again, in the sequence (y_j) there must be two adjacent elements α_1 and α_2 that are independent, yielding $\alpha_1 - \alpha_2 \in \mathcal{F}$. Therefore

$$\mathcal{F} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 - \alpha_2)\}.$$

The theorem is proved by setting $\alpha = \alpha_1$ and $\beta = -\alpha_2$. □

Lemma 4.3. *Let T be a connected $\mathbb{Z}^2$-tile with neighbors $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha - \beta)\}$, then $T + \{\pm\alpha, \pm\beta\}$ are edge neighbors. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$, then $T + \mathcal{F}$ are edge neighbors.*

Proof. Since $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathbb{Z}^2$ in both cases by Lemma 4.1 we may, without loss of generality, assume that $\alpha = [1, 0]^T$ and $\beta = [0, 1]^T$.

Let $\delta > 0$ denote the minimal distance between two disjoint tiles in the lattice tiling. Denote $S_1 = \mathrm{int}(T + \alpha\mathbb{Z})$. This is an open set near the x_1-axis which by the assumption of our lemma separates the set B_+ consisting of all tiles $T + m\beta + n\alpha$ with positive m from the set B_- consisting of all tiles with negative m. The distance between B_+ and B_- is $\geq \delta$. Take an integer k with $1/k < \delta/2$. Write $x = [x_1, x_2]^T$ and let

$$f(x_1) = \sup\{x_2\colon\ d(x, B_-) < \delta/2\}$$

for all $x_1 = n/k$ with $n \in \mathbb{Z}$. Thus the points $z = [x_1,\ f(x_1)]^T$ fulfill $d(z, B_-) = \delta/2 \leq d(z, B_+)$. We extend f as a linear function between these points and let

$$C_1 = \text{graph of } f = \{z(s) = [s,\ f(s)]^T\colon\ s \in \mathbb{R}\}.$$

Since $f(s + 1) = f(s)$, the polygonal line C_1 is periodic: $C_1 = C_1 + \alpha$. Now we prove that $C_1 \subset S_1$. Take $z(s)$ on a line segment of C_1 and let $z(x_1)$ be that vertex of the line segment for which $f(x_1) \leq f(s)$. For $x' = [x_1,\ f(s)]^T$ we have

$$\delta/2 \leq d(x', B_-) \leq |x_1 - s| + d(z(s), B_-) < \delta/2 + d(z(s), B_-),$$

which implies $d(z(s), B_-) > 0$. Similarly we see that $d(z(s), B_+) > 0$. The connectedness of T, and hence of B_- and B_+, now implies that all points of B_- lie below C_1 and all points of B_+ above. Hence $C_1 \subseteq S_1$.

Note that C_1 must cross from one tile into another, say from T to $T + m\alpha$. Clearly, $m = \pm 1$, or the two tiles are disjoint. Say $m = 1$. So part of C_1 must lie in $\mathrm{int}(T \cup (T + \alpha))$. Taking a point of C_1 in $T \cap (T + \alpha)$ we see that $T + \alpha$ is an edge neighbor of T.

The proofs for the other cases are identical. □

Let $T = T(A, \mathcal{D})$ be a self-affine tile satisfying (1). Iterating (1) yields

$$A^k(T) = T + \mathcal{D}_k, \qquad \text{where} \quad \mathcal{D}_k := \mathcal{D} + A\mathcal{D} + \cdots + A^{k-1}\mathcal{D}. \tag{3}$$

Note that $\mathcal{D}_k = \mathcal{D}_{k-1} + A^k\mathcal{D}$, with $D_0 := \{0\}$.

Lemma 4.4. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile with neighbors $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ and $\mathcal{D}$ is $\mathcal{F}$-connected, then so is $\mathcal{D}_k$ for all $k \geq 0$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha-\beta)\}$ and $\mathcal{D}$ is $\{\pm\alpha, \pm\beta\}$-connected, then so is $\mathcal{D}_k$ for all $k \geq 0$.*

Proof. In the six neighbors case note that $A^k(T) = T + \mathcal{D}_k$ and T is connected. By Theorem 2.3 $\mathcal{D}_k$ must be $\mathcal{F}$-connected.

In the eight neighbors case let $\mathcal{F}_0 = \{\pm\alpha, \pm\beta\}$. We prove $\mathcal{F}_0$-connectedness of D_k by induction on k. Observe that $\mathcal{D}_0 = \{0\}$ is clearly $\mathcal{F}_0$-connected, and $\mathcal{D}_k = \mathcal{D} + A\mathcal{D}_{k-1}$. We assume that $\mathcal{D}_{k-1}$ is $\mathcal{F}$-connected and show that $\mathcal{D}_k$ is $\mathcal{F}$-connected.

It is sufficient to show that for $u, u' \in \mathcal{D}_{k-1}$ with $u - u' \in \mathcal{F}$ there exist $d, d' \in \mathcal{D}$ such that $(d + Au) - (d' + Au')$ is also in $\mathcal{F}$. However, $u - u' \in \mathcal{F}$ means that $T + u$ and $T + u'$ are edge neighbors. Hence the larger tiles $A(T) + Au$ and $A(T) + Au'$ are also edge neighbors: they have uncountably many common points. Since $A(T) = \bigcup_{d \in \mathcal{D}} T + d$, there must exist $d, d' \in \mathcal{D}$ such that $T + d + Au$ and $T + d' + Au'$ also have uncountably many points. Thus they are edge neighbors and the difference of the vectors is in $\mathcal{F}$ by our assumptions. Lemma 4.4 is proved. $\square$

Lemma 4.5. *Under the assumptions of Theorem* 2.1 *or* 2.2, $\operatorname{int} T$ *is connected*.

Proof. We prove that $\operatorname{int} T$ is connected under the assumptions of Theorem 2.2. The other case is virtually identical (in fact a little simpler). Denote

$$\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha-\beta)\} \quad \text{and} \quad \mathcal{F}_0 = \{\pm\alpha, \pm\beta\}.$$

Let z_1 and z_2 be two points in $\operatorname{int} T$. We construct an arc from z_1 to z_2 within $\operatorname{int} T$. Let $K_0 \in \mathbb{Z}$ such that $K_0 > \max\{|x|: x \in T\}$ and let $R > 5K_0$. Choose k sufficiently large so that $B_R(A^k z_i) \subseteq A^k(\operatorname{int} T)$. It follows from $A^k(T) = T + \mathcal{D}_k$ and the $\mathcal{F}_0$-connectedness of $\mathcal{D}_k$ that we may find $y_0, y_1, \ldots, y_N \in \mathcal{D}_k$ such that $y_{i+1} - y_i \in \mathcal{F}_0$ and $A^k z_1 \in T + y_0$, $A^k z_2 \in T + y_N$. Hence $|A^k z_1 - y_0| < K_0$ and $|A^k z_2 - y_N| < K_0$. We prove there exists an arc connecting $A^k z_1$ and $A^k z_2$ that lies within $\operatorname{int}(A^k T)$.

Let $\delta > 0$ be the minimal distance between two disjoint tiles in the $\mathbb{Z}^2$-tiling and let T_ε be as in the proof of Theorem 4.2 with $\varepsilon < \min(1, \delta/4)$. Then the set

$$\Omega = \bigcup_{y \in \mathbb{Z}^2 \setminus \{y_i\}} (T_\varepsilon + y) \setminus \left(\overline{B_R(A^k z_1) \cup B_R(A^k z_2)} \right)$$

is an open set whose boundary consists of finitely many circular arcs. Furthermore, $\mathbb{R}^2 \setminus \overline{\Omega} \subseteq \operatorname{int}(A^k T)$. Assume that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ belong to the same connected component of $\mathbb{R}^2 \setminus \overline{\Omega}$. Then we can find an arc in $\mathbb{R}^2 \setminus \overline{\Omega}$ that connects $A^k z_1$ and $A^k z_2$. This arc is in $\operatorname{int}(A^k T)$. So we may connect z_1 and z_2 by an arc in $\operatorname{int} T$.

Now assume that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ belong to two different connected components of $\mathbb{R}^2 \setminus \overline{\Omega}$, say Ω_1 and Ω_2, respectively. We derive a contradiction. Choose a simple closed curve $C \subseteq \partial\Omega_1$ such that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ are on separate sides of C, and without loss of generality assume that $B_R(A^k z_1)$ is on the inside of C. We parametrize C by $x(t)$ where $t \in [0, 1]$ with $x(0) = x(1)$. As t varies from 0 to 1 the curve wraps around $B_R(A^k z_1)$. Take points $x_i = x(t_i)$ for $0 \le i \le m$ where $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $|x_{i+1} - x_i| < \delta/4$. Each x_i is in the closure of $T_\varepsilon + w_i$ for some $w_i \notin \{y_j\}$ with $w_0 = w_m$. It is easy to see that $d(T + w_{i+1}, T + w_i) < \delta$ for $0 \le i < m$. By removing redundant vertices, we may assume without loss of generality that $w_{i+1} \neq w_i$ for all $0 \le i \le m$. It follows that $w_{i+1} - w_i \in \mathcal{F}$.

Let C_1 be the closed piecewise linear curve with vertices $w_0, w_1, \ldots, w_m$. Since each $|x_i - w_i| \le K_0 + \varepsilon < 2K_0$, we must have $|w_i - A^k z_1| \ge 3K_0$ and $|w_i - A^k z_2| \ge 3K_0$. Therefore $d(A^k z_i, C_1) > 2K_0$. It follows that C_1 must wrap around $B_{K_0}(A^k z_1)$ as it traverses w_0 through w_m while leaving $B_{K_0}(A^k z_2)$ outside. Hence any path from y_0 to y_N must cross C_1. In particular, the piecewise arc C_2 with vertices $y_0, y_1, \ldots, y_N$ must intersect C_1. This means some line segment $\overline{w_i w_{i+1}}$ must intersect some line segment $\overline{y_j y_{j+1}}$. However, $y_{j+1} - y_j \in \mathcal{F}_0$ and $w_{i+1} - w_i \in \mathcal{F}_0$. It is easy to check that the only way the two line segments can intersect is that they share at least one common vertex. This contradicts the assumption that the y_j and the w_i are disjoint.

Therefore int T must be connected, and Theorem 3.1 applies. □

Acknowledgments

Most of this work was done when both authors were visiting the Institute for Mathematical Sciences and the Department of Mathematics of the Chinese University of Hong Kong. We would like to express our gratitude to the institute and the department for their support and hospitality. In particular we would like to thank Ka-Sing Lau for the invitations to visit, as well as for helpful discussions. We are also very grateful for the careful work and constructive criticism of both referees.

References

1. C. Bandt, Self-similar sets, 5: integer matrices and tilings of $\mathbb{R}^d$, *Proc. Amer. Math. Soc.* **112** (1991), 549–562.
2. C. Bandt, Self-similar tilings and patterns described by mappings, in *Mathematics of Aperiodic Order* (ed. R. Moody), Proc. NATO Advanced Study Institute C489, Kluwer, Dordrecht, 1997, pp. 45–83.
3. C. Bandt and G. Gelbrich, Classification of self-affine lattice tilings, *J. London Math. Soc.* (2) **50** (1994), 581–593.
4. C. Bandt and K. Keller, Self-similar sets, 2. A simple approach to the topological structure of fractals, *Math. Nachr.* **154** (1991), 27–39.
5. P. Duvall, J. Keesling and A. Vince, The Hausdorff dimension of the boundary of a self-similar tile, *J. London Math. Soc.*, to appear.
6. G. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.
7. G. Gelbrich, Crystallographic reptiles, *Geom. Dedicata* **51** (1994), 235–256.
8. K. Gröchenig and A. Haas, Self-similar lattice tilings, *J. Fourier Anal. Appl.* **1** (1994), 131–170.
9. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, NJ, 1948.

10. R. Kenyon, J. Li, R. S. Strichartz and Y. Wang, Geometry of self-affine tiles, II, *Indiana Univ. Math. J.* **48** (1999), 24–42.
11. I. Kirat and K.-S. Lau, On the connectedness of self-affine tiles, preprint.
12. K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, and PWN, Warszawa, 1968.
13. J. C. Lagarias and Y. Wang, Self-affine tiles in $\mathbb{R}^n$, *Adv. in Math.* **121** (1996), 21–49.
14. J. C. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^n$, I. Standard and nonstandard digit sets, *J. London Math. Soc.* **54** (1996), 161–179.
15. J. C. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^n$, II. Lattice tilings, *J. Fourier Anal. Appl.* **3** (1997), 83–101.
16. J. Luo and B. Tan, Topological structure of self-similar tilings, preprint, 2000.
17. C. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, New York, 1992.
18. H. J. Song and B. K. Kang, Disc-like lattice reptiles induced by exact polyominoes, *Fractals* **7**(1) (1999), 9–22.
19. R. S. Strichartz and Y. Wang, Geometry of self-affine tiles, I, *Indiana Univ. Math. J.* **48** (1999), 1–23.
20. A. Vince, Replicating tesselations, *SIAM J. Discrete Math.* **6** (1993), 501–521.
21. A. Vince, Self-replicating tiles and their boundary, *Discrete Comput. Geom.* **21** (1999), 463–476.
22. G. Whyburn and E. Duda, *Dynamic Topology*, Springer-Verlag, New York, 1979.

Received October 10, 2000, *and in revised form February* 16, 2001, *and April* 25, 2001.
Online publication July 25, 2001.