# Diameter and Treewidth in Minor-Closed Graph Families[1]

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**Abstract.** It is known that any planar graph with diameter $D$ has treewidth $O(D)$, and this fact has been used as the basis for several planar graph algorithms. We investigate the extent to which similar relations hold in other graph families. We show that treewidth is bounded by a function of the diameter in a minor-closed family, if and only if some apex graph does not belong to the family. In particular, the $O(D)$ bound above can be extended to bounded-genus graphs. As a consequence, we extend several approximation algorithms and exact subgraph isomorphism algorithms from planar graphs to other graph families.

**Key Words.** Local treewidth, Excluded minors, Apex graphs, $k$-Outerplanar graphs, Approximation algorithms, Subgraph isomorphism.

**1. Introduction.** Baker [2] implicitly based several planar graph approximation algorithms on the following result, which can be found more explicitly in [4]:

DEFINITION 1. A *tree decomposition* of a graph $G$ is a representation of $G$ as a subgraph of a chordal graph $G'$. The *width* of the tree decomposition is one less than the size of the largest clique in $G'$. The *treewidth* of $G$ is the minimum width of any tree decomposition of $G$.

LEMMA 1. *Let D denote the diameter of a planar graph G. Then a tree decomposition of G with width $O(D)$ can be found in time $O(Dn)$.*

The lemma can be proven by defining a chordal graph having cliques for certain three-leaf subtrees in a breadth first search tree of $G$. Such a subtree has at most $3D - 2 = O(D)$ vertices.

Baker used this method to find approximation schemes for the maximum independent set and many other covering and packing problems in planar graphs, improving previous results on planar graph approximation algorithms based on separator decomposition [9], [15]. Baker's basic idea was to remove the vertices in every $k$th level of the breadth first search tree of an arbitrary planar graph $G$; there are $k$ ways of choosing which set of levels to remove, at least one of which only decreases the size of the maximum independent set by a factor of $(k - 1)/k$. Then each remaining set of contiguous levels

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forms a graph with treewidth $O(k)$ (it is a subgraph of the graph with diameter $k$ formed by removing vertices in outer levels and contracting edges in inner levels), and the maximum independent set in each such component can be found by standard dynamic programming techniques [3], [19].

Other workers have developed parallel variants of these approximation schemes [6], [8], [10], applied Baker's method to exact subgraph isomorphism, connectivity, and shortest path algorithms [12], extended similar ideas to approximation algorithms in other classes of graphs [7], [20] or graphs equipped with a geometric embedding [13], and defined structural complexity classes based on these methods [14].

These results naturally raise the question, how much further can these algorithms be extended? To what other graph families do these techniques apply? Since the argument above about contiguous levels of the breadth first search tree being contained in a low-diameter graph is implicitly based on the concept of *graph minors*, we restrict our attention to *minor-closed* families; that is, graph families closed under the operations of edge deletion and edge contraction. Minor-closed families have been studied extensively by Robertson, Seymour, and others, and include such familiar graph families as the planar graphs, outerplanar graphs, graphs of bounded genus, graphs of bounded treewidth, and graphs embeddable in $\mathbb{R}^3$ without any linked or knotted cycles.

DEFINITION 2.    Define a family $\mathcal{F}$ of graphs to have the *diameter-treewidth property* if there is some function $f(D)$ such that every graph in $\mathcal{F}$ with diameter at most $D$ has treewidth $f(D)$.

Lemma 1 can be rephrased as showing that the planar graphs have the diameter-treewidth property with $f(D) = O(D)$. In this paper we exactly characterize the minor-closed families of graphs having the diameter-treewidth property, in a manner similar to Robertson and Seymour's characterization of the minor-closed families with bounded treewidth as being those families that do not include all planar graphs [16].

DEFINITION 3.    An *apex graph* is a graph $G$ such that for some vertex $v$ (the *apex*), $G - v$ is planar (Figure 1).
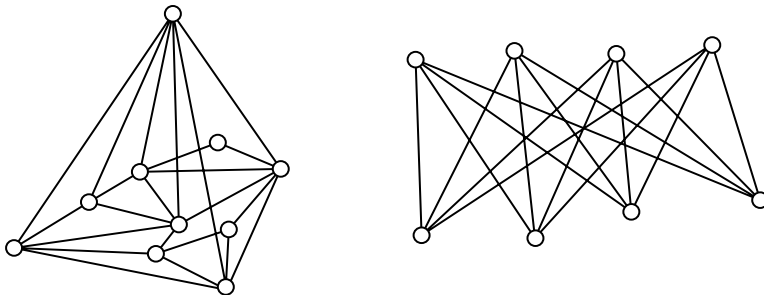


**Fig. 1.** The graph on the left is an apex graph; the topmost vertex is one of the possible choices for its apex. The graph on the right is not an apex graph.

Apex graphs have also been known as nearly planar graphs, and have been introduced to study linkless and knotless three-dimensional embeddings of graphs [17], [21].

The significance of apex graphs for us is that they provide examples of graphs without the diameter-treewidth property: let $G$ be an $n \times n$ planar grid, and let $G'$ be the apex graph formed by connecting some vertex $v$ to all vertices of $G$; then $G'$ has treewidth $n+1$ and diameter two. Therefore, the family of apex graphs does not have the diameter-treewidth property, nor does any other family containing all apex graphs. Our main result is a converse to this: any minor-closed family $\mathcal{F}$ has the diameter-treewidth property, if and only if $\mathcal{F}$ does not contain all apex graphs.

**2. Walls.**   Recall that the Euclidean plane can be exactly covered by translates of a regular hexagon, with three hexagons meeting at a vertex.

DEFINITION 4.   We say that a set of hexagons is *connected* if its union is a connected subset of the Euclidean plane. If $h_1$ and $h_2$ are two hexagons from a tiling of the plane by infinitely many regular hexagons, define the *distance* between the two hexagons to be the smallest integer $d$ for which there exists a connected subset of the infinite tiling, containing both $h_1$ and $h_2$, with cardinality $d + 1$.

Thus, any hexagon is at distance zero from itself; two hexagons meeting edge-to-edge are at distance one, and in general if $h_1 \neq h_2$ and $h_1$ is at distance $d$ from $h_2$, then $h_1$ meets edge-to-edge with some other hexagon at distance $d - 1$ from $h_2$.

DEFINITION 5.   Let $S$ be a finite connected subset of the hexagons in a tiling of the Euclidean plane by regular hexagons. Then we define the *graph of $S$* to be formed by creating a vertex at each point of the plane covered by the corner of at least one tile of $S$, and creating an edge along each line segment of the plane forming one of the six edges of at least one hexagon in $S$.

Observe that the graph of $S$ is planar and each of its vertices has degree at most three. Figure 2 shows an example of a set of hexagonal tiles and its graph.
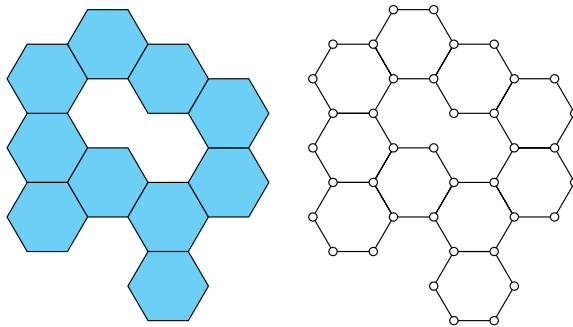


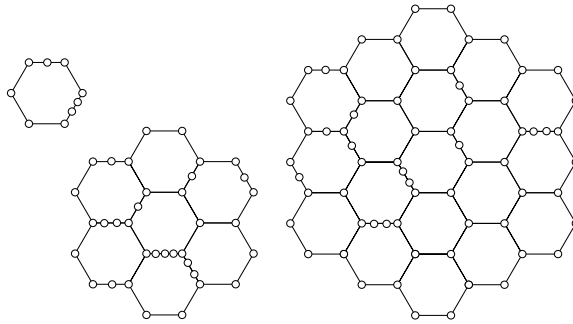**Fig. 2.** A set of hexagons and its graph.

**Fig. 3.** Walls of size one, two, and three.

DEFINITION 6. A *subdivision* of a graph $G$ is a graph $G'$ formed by replacing some or all edges of $G$ by paths of two or more edges. A *wall of size s* is a subdivision of the graph of $S$, where $S$ is the set of all hexagons within distance $s - 1$ from some given *central tile* in a tiling of the Euclidean plane by regular hexagons.

Note that since the definition of a wall depends only on the combinatorial structure of $S$, it is independent of the particular tiling or central tile chosen in the definition. Examples of walls are shown in Figure 3. Walls are very similar to (subdivisions of) grid graphs but have a slight advantage of having degree three. Thus we can hope to find them as subgraphs rather than as minors in other graphs.

LEMMA 2 [16]. *For any s there is a number $w = W(s)$ such that any graph of treewidth $w$ or larger contains as a subgraph a wall of size $s$.*

In a recent improvement to this lemma, Robertson et al. [18] showed that if $H$ is a planar graph, the family of graphs with no $H$-minor has treewidth at most $20^{2(2|V(H)|+4|E(H)|)^5}$. Since a wall of size $s$ is a planar graph with $O(s^2)$ edges and vertices, this implies that $W(s) \leq \exp(O(s^{10}))$.

LEMMA 3 [16]. *For any planar graph $G$ there is some $s = s(G)$ such that any wall of size s has $G$ as a minor.*

We will subsequently need to identify certain components of walls. To do this we need to use not just the graph-theoretic structure of a wall but its geometric structure as a subdivision of the graph of a set of hexagons. (This geometric structure is essentially unique for large walls, but not for walls of size two, and in any case we do not prove uniqueness here.)

DEFINITION 7. An *embedding* of a wall $G$ is the identification of $G$ as a subdivision of a graph of a set of hexagons meeting the requirements for the definition of a wall. A *t-inner* vertex of an embedded wall is a vertex incident to a hexagon within distance $t - 1$ of the wall's central hexagon (so all vertices in a wall of size $s$ are $s$-inner). An

*outer* vertex or edge of an embedded wall is a vertex or edge incident to the boundary of the union of the set of hexagons forming the embedding.

## 3. Routing Across Walls

DEFINITION 8.   An $(s, t)$ *routing problem* consists of an embedded wall $G$ of size $s + t$, an $(s - 1)$-inner vertex $v$, and a set $S$ of pairs of *terminals* (certain vertices of the wall), satisfying the following conditions:

1. Each terminal is either $v$ or a degree-three outer vertex of the wall.
2. Each outer vertex occurs at most once as a terminal in $S$; $v$ occurs at most three times as a terminal.
3. The graph formed by the pairs of terminals in $S$ has a planar embedding as a set of noncrossing curves within the interior of $U$, where $U$ denotes the union of the hexagons forming the embedding of the wall.
4. At most $t$ pairs of terminals do not involve $v$.

DEFINITION 9.   A *solution* to an $(s, t)$ routing problem consists of a vertex $v'$ and a set of $|S| + 1$ edge-disjoint paths in $G$, satisfying the following conditions:

1. Each pair in $S$ must correspond to one of the paths of the solution.
2. Each outer terminal of a pair in $S$ must be an endpoint of the corresponding path.
3. Each pair in $S$ involving vertex $v$ must correspond to a path having $v'$ as one of its endpoints.
4. The remaining path in the set, not corresponding to a pair in $S$, must have as its two endpoints $v$ and one of the vertices on a path involving $v'$.
5. All paths are disjoint from the outer edges of the wall.

A $(2, 2)$ routing problem (with five paths, three involving the inner vertex) and its six-path solution is depicted in Figure 4.

DEFINITION 10.   A pair $(x, y)$ of terminals in an $(s, t)$ routing problem is *splittable* if the curve corresponding to $(x, y)$ in the planar embedding of $S$ partitions $U$ into two regions $A$ and $B$ such that all terminals are incident to $A$ and only terminals $x$ and $y$ are incident to $B$.

In Figure 4 both pairs of outer terminals are splittable.

LEMMA 4.   *If an $(s, t)$ routing problem includes a pair of outer terminals, it includes a splittable pair.*

PROOF.   The planar embedding of the pairs of outer terminals in $S$, together with the boundary of the wall, forms an outerplanar graph (a planar graph in which all vertices are incident to the outer face). Because the weak dual of an outerplanar graph (the graph formed from the planar dual by removing the vertex corresponding to the outer face)
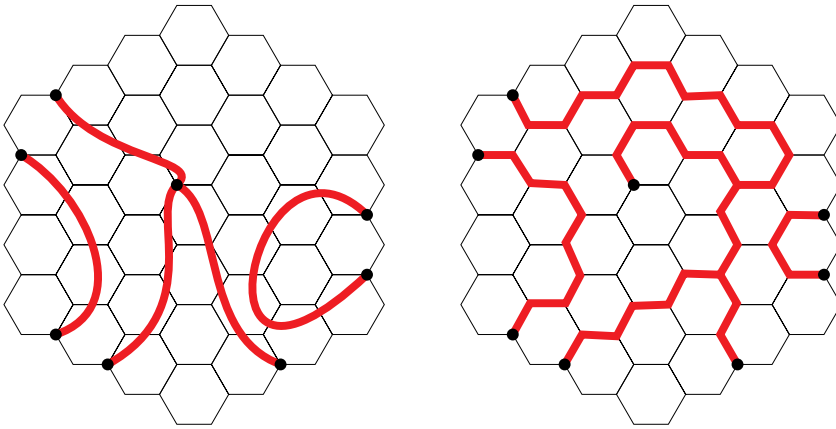
**Fig. 4.** A (2, 2) routing problem and one of its solutions.

is a tree, it has at least two leaves. Each leaf of this tree corresponds to a region of $U$ bounded by a curve in the planar embedding of $S$ and not containing any other outer terminals of $S$. At most one leaf contains the inner terminal $v$, so there is at least one leaf not containing any terminals.                                                                    □

LEMMA 5.    *Every $(s, t)$ routing problem has a solution.*

PROOF.    We use induction on $t$. If there are fewer than $t$ pairs of terminals involving $v$, the given problem is also an $(s + 1, t − 1)$ routing problem and the result follows from induction.

If $t > 0$, let $(x, y)$ be a splittable pair. Then we can extend an edge from each terminal of $S$ to an $(s + t − 1)$-inner vertex, on the boundary of a wall of size $(s + t − 1)$ within the original wall. We connect $x$ and $y$ by a path around the boundary of this smaller wall.

Next we connect each other outer terminal to a degree-three outer vertex of the smaller wall, one terminal at a time, starting from the terminal immediately counterclockwise of the pair $(x, y)$, and continuing counterclockwise from there. For each terminal $t$, we first attempt to extend a path clockwise around the inner wall's boundary to the next degree-three vertex. There are four possible situations that can arise in this extension:

1. We reach an unused degree-three vertex. This vertex will become the terminal of a smaller problem in the inner wall.
2. We reach a vertex that is part of a path extended from the other endpoint $u$ of a pair $(t, u)$ in $S$. In this case we have found a path connecting $(t, u)$ and will not continue using this pair in the smaller problem we form.
3. We reach a vertex that is part of a path extended from another terminal $u$, and both $(t, v)$ and $(u, v)$ are pairs in $S$. In this case we will form a smaller problem in which these two pairs have been replaced by a single pair $(w, v)$ where $w$ is the degree-three vertex reached from both $t$ and $u$.

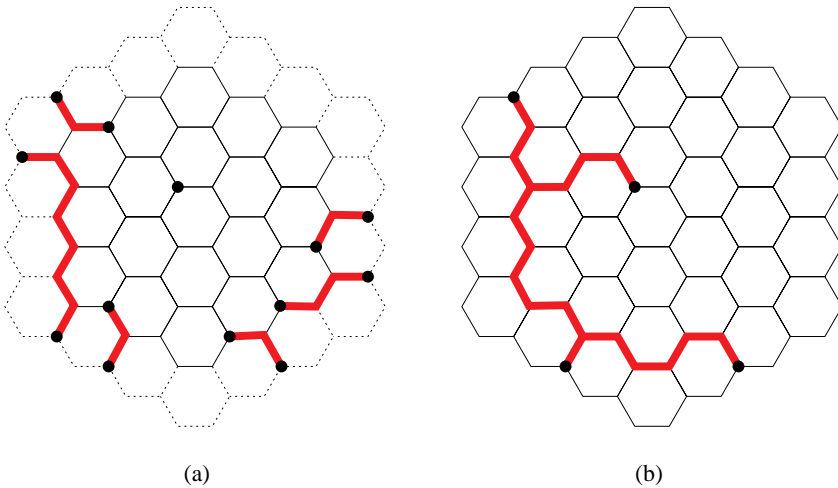(a)                                                    (b)

**Fig. 5.** Cases for solution of routing problems: (a) reduction from an $(s, t)$ problem to an $(s, t - 1)$ problem; (b) solution of an $(s, 0)$ problem by routing around the boundary of the inner wall.

4. The degree-three vertex we reach is already part of a path but cannot be connected to $t$. In this final case we instead extend a path counterclockwise from $t$ to the next degree-three vertex.

Note that the first time the counterclockwise extension of case 4 happens it can only be at one of the six points where two degree-two outer vertices of the wall are adjacent. Case 4 may then continue to happen as long as each successive degree-three vertex on the boundary of the wall is a terminal that cannot be connected to the previous terminal. However, by planarity, this can only happen if no two terminals in this sequence form pairs with each other or with $v$, for if they did we would have one of cases 2 or 3 instead. Therefore there are at most $t$ terminals in such a sequence, and we will escape from this counterclockwise case before we reach the next pair of two adjacent outer degree-two vertices of the wall. As a consequence, this case always succeeds in extending the path to an unused degree-three vertex.

The result of this path extension process is an $(s, t - 1)$ routing problem on the smaller wall (Figure 5(a)). By induction, this smaller problem has a solution which can be combined with the path extensions to solve the original $(s, t)$ routing problem.

Finally, if $t = 0$, we have at most three outer terminals on the boundary of a wall of size $s$ and one nonboundary vertex $v$. Again, we extend an edge from each terminal to a vertex on the boundary of a smaller wall of size $(s + t - 1)$. We connect these three vertices by paths. If $v$ is not already on one of these paths we add a path connecting it to the solution (Figure 5(b)).                                                                                  □

**4. Macrocells.**  The strategy for our proof that graphs without the diameter-treewidth property contain all apex graphs as minors will be first to place a given apex graph's vertices on a wall, and then solve many routing problems in order to show that the
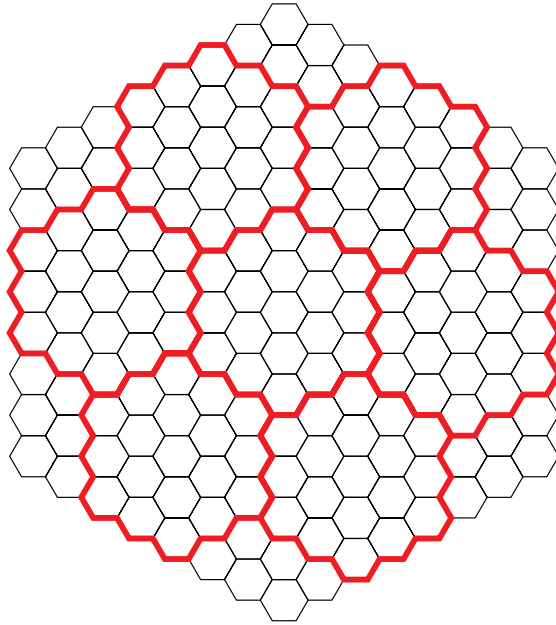
**Fig. 6.** Subdivision of a large wall into many smaller walls.

wall contains the appropriate connections between these vertices. To do this, we need to partition the one large wall into many smaller walls. As shown in Figure 6, the union of the hexagons of a wall forms a shape that can itself tile the plane, with a pattern of connectivity equivalent to that of the original hexagonal tiling. If the hexagons of a large wall are partitioned into smaller walls according to such a tiling, we call the smaller walls *macrocells*.

Note that while the macrocells are a partition of the hexagons of a wall, they are not a partition of the edges and vertices of the wall. We say that two macrocells are *adjacent* if they share some edges and vertices; if the macrocells are walls of size $s$ the shared vertices form a path of $O(s)$ corner vertices (and possibly many more path vertices). Define a *side* of a macrocell to be one of these shared paths.

LEMMA 6.     *Let $S$ be a set of $(s - t/2)$-inner corners of a wall of size $s$. Then one can partition the wall into macrocells of size $t$ so that $|S|/4$ members of $s$ are $t/2$-inner.*

PROOF.     The partition into macrocells is determined by the choice of one central hexagon for one macrocell. If one chooses this hexagon uniformly at random, the probability that any given corner in $S$ is $t/2$-inner is propertional to the area of the inner size-$t/2$ wall of a macrocell relative to the overall macrocell's area; this probability is therefore $1/4$. Thus choosing a random macrocell center gives an expected number of $t/2$-inner members of $S$ equal to $|S|/4$. The best macrocell center must give at least as many $t/2$-inner members of $S$ as this expectation.                                                                         $\square$
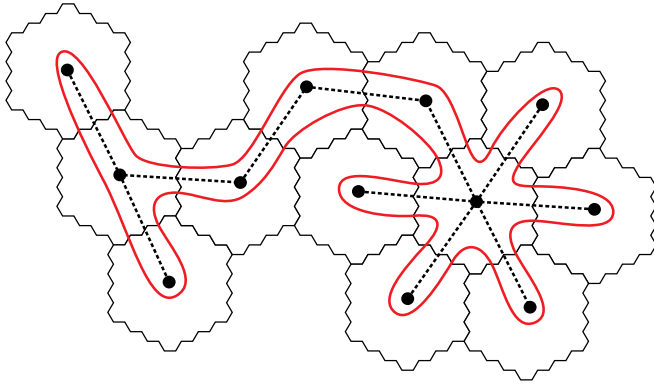
**Fig. 7.** Curve formed by doubling the spanning tree of a set of macrocells.

LEMMA 7.    *Let M be a set of macrocells of an embedded wall, such that one can connect any two macrocells in M by a chain of adjacent pairs of macrocells. Then there exists a non-self-intersecting curve in the plane that is contained in the union of M, that passes through all macrocells in M, such that the intersection of the curve with any macrocell has at most three connected components.*

PROOF.    Form a planar graph by placing a point at the center of each macrocell, and connecting pairs of points at the centers of adjacent macrocells. Then by assumption this graph is connected, so we can choose a spanning tree. A curve $C$ formed by thickening the edges of this tree and passing around the boundary of the thickened tree has two of the three properties we want: it is contained in the union of $M$ and passes through all macrocells in $M$ (Figure 7).

Now, suppose that some path $x$ in $C$ passes consecutively through three pairwise adjacent macrocells $m_i$, $m_j$, and $m_k$ (e.g., at points where the spanning tree edges form a $60°$ angle), and the intersection of $C$ with the middle macrocell $m_j$ has more than one component. Then we can simplify $C$ by replacing $x$ with a curve that passes directly from $m_i$ to $m_k$ (Figure 8). This simplification step maintains the two properties that $C$ is in the union of $M$ and passes through each macrocell. It is possible for such a simplification step to introduce a crossing, but only in the case that more than one path passes through the same triple of macrocells; to avoid this problem we always choose the innermost path when more than one path passes through the same triple of macrocells. Each simplification step reduces the total number of connected components formed by intersecting $C$ with macrocells, so the simplification process must terminate.

Once this simplification process has terminated, the components of an intersection of $C$ with a macrocell (if that intersection has multiple components) must connect nonadjacent pairs of macrocells, so there can be at most three components per macrocell.                                                                                           □
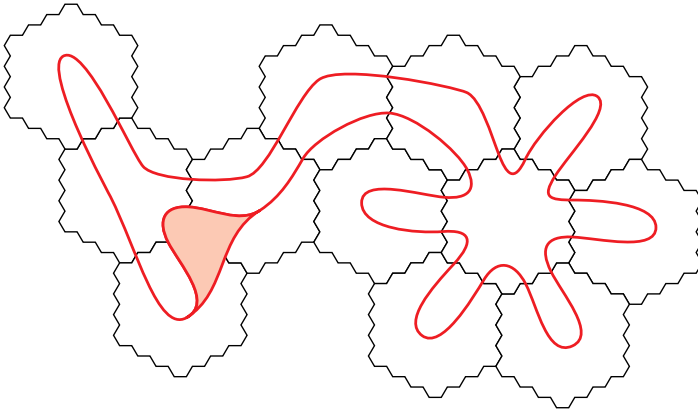
**Fig. 8.** Curve simplification by removal of a 60° angle.

**5. Monotone Embedding.**    We now show how to partition a graph into smaller pieces
that can be mapped onto a wall using $(s, t)$ routing problems. Specifically, we are con-
cerned with performing this sort of partition to walls, since any other planar graph can
be found as a minor of a wall (Lemma 3).

DEFINITION 11.    A planar graph $G$ is *monotone embedded* in the plane if no vertical
line crosses any edge more than once, and no vertical line contains more than one vertex.
The *monotone bandwidth* of $G$ is the maximum number of edges crossed by any vertical
line, minimized over all such embeddings.

LEMMA 8.    *A wall of size $s$ has monotone bandwidth $O(s)$.*

PROOF.    Draw the wall using regular hexagons, tilted slightly so that no edge is vertical;
this gives a monotone embedding. Any vertical line crosses $O(s)$ hexagons, and hence
$O(s)$ edges.                                                                                □

LEMMA 9.    *Let $W$ be a graph formed by connecting a sequence of macrocells of size $s$,
such that one can connect any two macrocells by a chain of adjacent pairs of macrocells,
and let $k$ of the macrocells in $W$ have a marked degree-three $s/2$-inner corner. Then $W$
contains as a minor any $k$-vertex trivalent graph $G$ with monotone bandwidth $s/6$, such
that each subset of vertices of $W$ that is collapsed to form each vertex of $G$ contains one
of the marked vertices.*

PROOF.    According to Lemma 7, we can find a curve $C$ contained in an embedding of
$W$, and passing through each macrocell between one and three times. Find a monotone
embedding of $G$, and form a correspondence with the marked corners of $W$ (ordered
by the positions along $C$ where $C$ first intersects each macrocell) and the vertices of $G$
(ordered according to the monotone embedding).

Then we form an $(2, (s-2)/3)$ routing problem for each component of an intersection of $C$ with a macrocell. If the macrocell does not contain a marked vertex, or if the component is not the first intersection of $C$ with the macrocell, the routing problem just consists of pairs of boundary vertices of the macrocell, with each pair placed on the two sides of the macrocell crossed by $C$; the number of pairs is chosen to match the number of edges cut by a vertical slice through the corresponding part of the monotone embedding. However, for the first intersection of $C$ with a marked macrocell, we instead form a routing problem in which the pattern of connections between the boundary vertices and the marked inner corner matches the pattern of connections in a vertical slice through the corresponding vertex of the monotone embedding.

This set of routing problems involves the placement of at most $3 + s/3$ terminals on any side of any macrocell. These vertices can be placed arbitrarily on that side, as long as they can be connected by disjoint paths along the side to the corresponding terminals of the adjacent macrocell.

The union of the at most three routing problems within each macrocell is an $(s/2, s/2)$ routing problem and therefore has a solution. Combining these solutions, and contracting the solution paths in each macrocell, forms the desired minor.                             □

Figure 9 depicts a set of routing problems on four macrocells, the solutions to which could be combined to form a complete graph on four vertices. For simplicity we have drawn the figure using macrocells of size eight, but (since $K_4$ has monotone bandwidth four) the lemma above only guarantees the existence of such a routing for macrocells of size 24.

We note that Lemma 3 follows as an easy consequence of Lemma 9: given any $n$-vertex planar graph $G$, expand its vertices into trees of degree-three vertices. The resulting $O(n)$-node graph has monotone bandwidth $O(n)$, so it can be found as a minor of a wall of size $O(n^{3/2})$, partitioned into a path of $O(n)$ smaller walls of size $O(n)$ as depicted in Figure 6.

## 6. The Main Result

THEOREM 1.    *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ has the diameter-treewidth property iff $\mathcal{F}$ does not contain all apex graphs.*

PROOF.    One direction is easy: we have seen that the apex graphs do not have the diameter-treewidth property, so no family containing all apex graphs can have the property.

In the other direction, we wish to show that if $\mathcal{F}$ does not have the diameter-treewidth property, then it contains all apex graphs. By Lemma 3 it will suffice to find a graph in $\mathcal{F}$ formed by connecting some vertex $v$ to all the vertices of a wall of size $n$, for any given $n$. If $\mathcal{F}$ does not have the diameter-treewidth property, there is some $D$ such that $\mathcal{F}$ contains graphs with diameter $D$ and with arbitrarily large treewidth.

Let $G$ be a graph in $\mathcal{F}$ with diameter $D$ and treewidth $W(N_1)$ for some large $N_1$ and for the function $W(N)$ shown to exist in Lemma 2. Then $G$ contains a wall of size $N_1$. We choose appropriate values $N_2$ and $N_3 = \Theta(N_1/N_2)$ and partition the wall into $N_3^2$
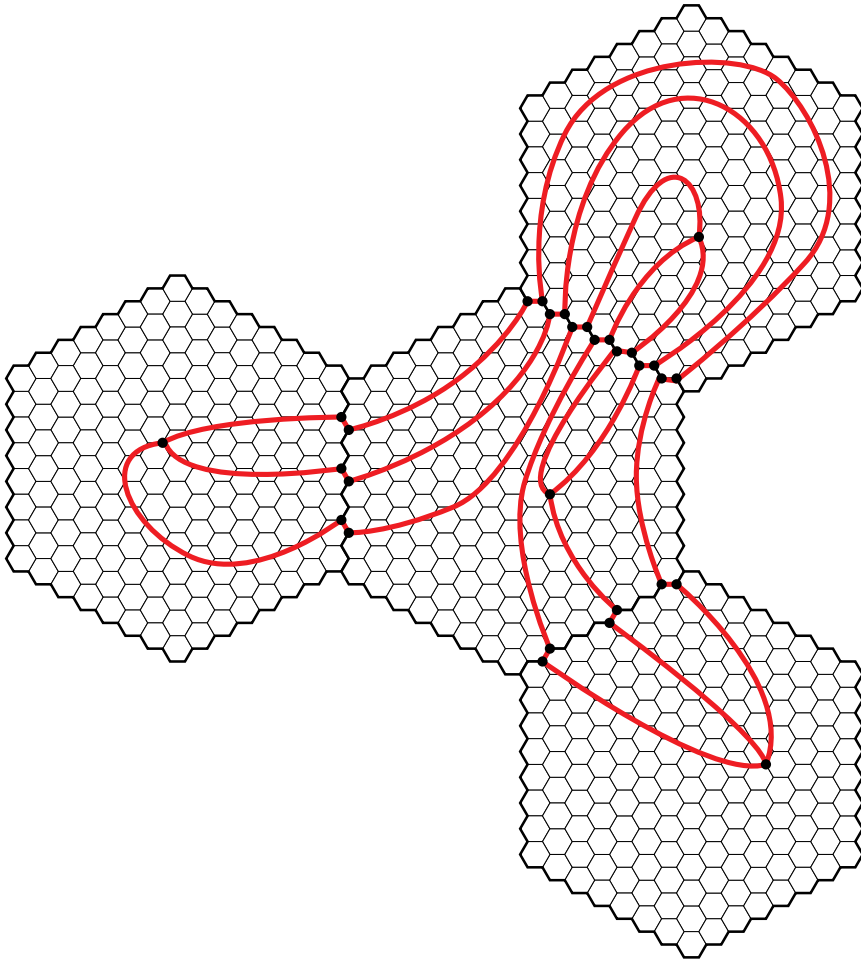
**Fig. 9.** The solutions to four routing problems on marked macrocells can be combined to form a $K_4$ minor.

macrocells of size $N_2$. Say a macrocell is *good* if it is not adjacent to the boundary of the wall.

Choose any vertex $v \in G$ and find a tree of shortest paths from $v$ to each vertex. We say that a macrocell is reached at level $i$ of the tree if some vertex of the macrocell is included in that level. Since $G$ has diameter $D$, the tree will have height $D$. Since all macrocells are reached at level $D$, and the number of macrocells reached at level zero is just one, there must be some intermediate level $\lambda$ of the tree for which the number $N_4$ of good macrocells reached is larger by a factor of $N_3^{2/D}$ than the number of good macrocells reached in all previous tree levels combined.

Let set $S$ be a set of corners of the wall formed by taking, in each good macrocell reached at level $\lambda$, a corner nearest to one of the vertices in that level of the tree. By Lemma 6, we can find a new partition into macrocells, and a set of $|S|/4$ corners that

are $N_2/2$-central for this partition. Each macrocell in this new partition contains $O(1)$ of these corners, so by removing corners that appear in the same macrocell we can mark a set $S'$ of $\Omega(N_4)$ inner corners of macrocells, at most one corner per macrocell. Note that the number of new macrocells reached at level $\lambda - 1$ is still $O(N_4/N_3^{2/D})$, since each old macrocell reached at that level can only contribute vertices to $O(1)$ new macrocells.

We then contract levels 1 through $\lambda - 1$ of the tree to a single vertex $v$. This gives a minor $G'$ of $G$ in which $v$ is connected to inner corners of $\Omega(N_4)$ distinct macrocells, and in which $O(N_4/N_3^{2/D})$ other macrocells are "damaged" by having a vertex included in the contracted portion of the tree. The adjacencies between damaged regions of the wall form a planar graph with $O(N_4/N_3^{2/D})$ vertices and so $O(N_4/N_3^{2/D})$ faces, and there must therefore be a face of this graph containing $\Omega(N_3^{2/D})$ members of $S'$. Let $S''$ denote this subset of $S'$.

Now $S''$ is part of a connected set of undamaged macrocells of size $N_2$, so by Lemma 9 we can find a wall of size $O(\min(N_3^{1/D}, N_2))$ as a minor of this set of undamaged macrocells. If $N_2 = \Omega(n)$ and $N_3^{2/D} = \Omega(n^2)$, we can find a wall of size $n$. These conditions can both be assured by letting $N_1 = \Omega(n)^{D+1}$. Combining this wall with the contracted vertex $v$ forms the apex graph minor we were seeking.

We can carry out this construction for any $n$, and since by Lemma 3 every apex graph can be found as a minor of graphs of the form of $M$, all apex graphs are minors of graphs in $\mathcal{F}$ and are therefore themselves graphs of $\mathcal{F}$.                                    □


Alternatively, instead of finding apex-grid graph minors, and using those to find all other apex graphs as minors, we can find any apex graph directly by following the proof of Lemma 3 sketched above after Lemma 9.


**7. Bounded Genus Graphs.**    The results above show that any minor-closed family excluding an apex graph has the diameter-treewidth property. For example, consider bounded genus graphs. It is not hard to show that, for any $g$, there is an apex graph with genus more than $g$: genus $g$ graphs have at most $3n + O(g)$ edges, while maximal apex graphs have $4n - 10$ edges, so choosing $n$ large gives an apex graph with too many edges to have genus $g$. Therefore, genus $g$ graphs have the diameter-treewidth property. However, this proof does not give us a very tight relation between diameter, genus, and treewidth. We can achieve a much better treewidth bound by proving the diameter-treewidth property more directly.

LEMMA 10.    *Let $G$ be embedded on a surface $S$ of genus $g$, with all faces of the embedding topologically equivalent to disks. Then there exists a subgraph $X$ of $G$, isomorphic to a subdivision of a graph $Y$ with $O(g)$ edges and vertices, such that the removal of the points of $X$ from $S$ leaves a set topologically equivalent to a disk.*

PROOF.    Let $X$ be a minimal connected subgraph of $G$ such that all components of $S - X$ are topological disks. Then there must be at most one such component, for multiple components could be merged by removing from $X$ an edge along which two adjacent components are connected; any such merger preserves the disk topology of the
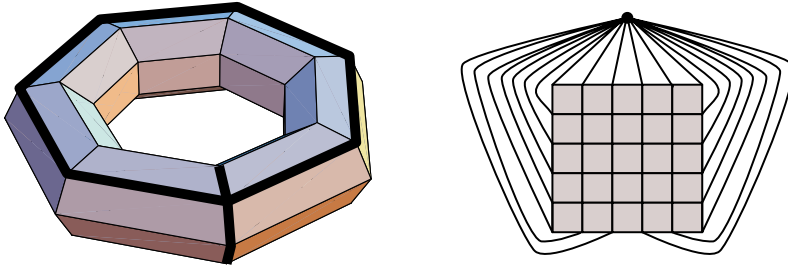
**Fig. 10.** Torus graph with subgraph $X$ highlighted, and planar graph formed by contracting $X$.

components and the connectivity of $X$ (since any path through the removed edge can be replaced by a path around the boundary of a component).

Thus $X$ is a graph bounding a single disk face. By Euler's formula, if $X$ has $n$ vertices, it has $n + O(g)$ edges. Let $T$ be a spanning tree of $X$; then $X - T$ has $O(g)$ edges. Note also that $X$ has no degree-one vertices, so each leaf of $T$ must be an endpoint of an edge in $X - T$ and there are $O(g)$ leaves. Any graph formed by adding $O(g)$ edges to a tree with $O(g)$ leaves must be a subdivision of a graph with $O(g)$ edges and vertices. $\qquad\square$

Figure 10 depicts a graph $X$ for an example in which $G$ is embedded on a torus.

THEOREM 2.   *Let $G$ have genus $g$ and diameter $D$. Then $G$ has treewidth $O(gD)$.*

PROOF.     Embed $G$ on a minimal-genus surface $S$, so that all its faces are topological disks. Choose a subgraph $X$ as in Lemma 10, having the minimum number of edges possible among all subgraphs satisfying the conditions of the lemma, and let $Y$ be a graph with $O(g)$ vertices and edges of which $X$ is a subdivision (as described in the lemma). Then each path in $X$ corresponding to an edge in $Y$ has $O(D)$ edges. For, if not, one could find a smaller $X$ by replacing part of a long path by the shortest path from its midpoint to the rest of $X$. Therefore, $X$ has $O(gD)$ edges and vertices.

Now contract $X$ forming a minor $G'$ of $G$. The result is a planar graph, since $G - X$ can remain embedded in its disk, with the vertex contracted from $X$ being connected to $G - X$ by edges that cross the boundary of this disk. The contraction can only reduce the diameter of $G$. Therefore, $G'$ has treewidth $O(D)$, and a tree decomposition of $G$ with treewidth $O(gD)$ can be formed by adjoining $X$ to each clique in a tree decomposition of $G'$. $\qquad\square$

## 8. Algorithmic Consequences

THEOREM 3.   *For any minor-closed family of graphs with the diameter-treewidth property, there exists a linear time approximation scheme for maximimum independent set, minimum vertex cover, maximum $H$-matching, minimum dominating set, and the other approximation problems solved by Baker [2].*

The method is the same as in [2]: we remove every $k$th level in a breadth first search tree, with one of $k$ different choices of the starting level, forming a collection of subgraphs each of which is induced by some $k-1$ contiguous levels of the tree. (For the minimum dominating set and vertex cover problems, we instead duplicate the vertices on every $k$th level, and form subgraphs induced by $k+1$ contiguous levels of the tree.) As Baker shows, one of these choices leads to a graph that approximates the optimum within a $1 + O(1/k)$ factor. We then use the diameter-treewidth property to show that each of these subgraphs has bounded treewidth. A tree decomposition of each subgraph can be found in linear time [5], after which the appropriate optimization problem can be solved in linear time in each subgraph by using dynamic programming techniques [3], [19].

We note that maximum independent set can also be approximated for all minor-closed families, using the results of Alon et al. [1] on separator theorems for such families, however, the separator algorithm of [1] takes superlinear time $O(k^{1/2}n^{3/2})$ (where $k$ is the number of vertices of the largest clique belonging to the family) and this approximation technique does not seem to apply to the other problems on the list above.

THEOREM 4. *Subgraph isomorphism or induced subgraph isomorphism for a fixed pattern $H$ in any minor-closed family of graphs with the diameter-treewidth property can be tested in time $O(n)$.*

The algorithm closely follows that of [12]. We again remove every $k$th level of the tree with one of $k$ different choices of the starting level, forming subgraphs of $k-1$ contiguous levels, where $k-1$ is the diameter of $H$. If $H$ occurs in $G$, it must occur in one of these subgraphs, which can be tested by finding a tree decompostion and performing dynamic programming.

**9. Conclusions and Open Problems.** We have characterized the minor-closed families with the diameter-treewidth property. However, some further work remains. Notably, the relation we showed between diameter and treewidth was not as strong as for planar graphs: for planar graphs (and bounded-genus graphs) $w = O(D)$ while for other minor-closed families our proof only shows that $w = W(c^{D+1}))$, where $c$ is a constant that depends on the family and $W(x)$ represents the rapidly growing function used by Robertson and Seymour to prove Lemma 2. Can we prove tighter bounds on treewidth for general minor-closed families?

Specifically, what relation between diameter and treewidth holds for the graphs having no $K_{3,a}$ minor for some fixed $a$? Note that $K_{3,a}$ is an apex graph, so these graphs have the diameter-treewidth property. $K_{3,a}$-free graphs are a generalization of planar graphs (which have no $K_{3,3}$ or $K_5$ minor) and have other interesting properties; notably, in connection with the subgraph isomorphism algorithms described above, a subgraph $H$ has an $O(n)$ bound on the number of times it can occur in $K_{3,a}$-free graphs, if and only if $H$ is 3-connected [11]. Any improved treewidth bounds would improve the running time and practicality of the subgraph isomorphism and approximation algorithms we described.

Also, are there natural families of graphs that are not minor-closed and that have the diameter-treewidth property (other than the bounded-degree graphs or other classes in

which a diameter bound imposes a limit on total graph size)? Although one could not then apply Baker's approximation technique [2], this would still lead to quadratic-time subgraph isomorphism algorithms based on testing bounded-radius neighborhoods of each vertex [12].

Finally, can we extend some of the same efficient subgraph isomorphism and approximation algorithms to graph families without the diameter-treewidth property? For instance, it is trivial to do so for apex graphs, by treating the apex specially and applying a modified algorithm in the remaining graph. What about other graph families containing the apex graphs, such as linkless and knotless embeddable graphs, or $K_{4,4}$-free graphs?

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