



On the DISTANCE IDENTIFYING SET Meta-problem and Applications to the Complexity of Identifying Problems on Graphs

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Received: 1 December 2018 / Accepted: 8 January 2020 / Published online: 27 January 2020
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Abstract

Numerous problems consisting in identifying vertices in graphs using distances are useful in domains such as network verification and graph isomorphism. Unifying them into a meta-problem may be of main interest. We introduce here a promising solution named DISTANCE IDENTIFYING SET. The model contains IDENTIFYING CODE (IC), LOCATING DOMINATING SET (LD) and their generalizations r -IC and r -LD where the closed neighborhood is considered up to distance r . It also contains METRIC DIMENSION (MD) and its refinement r -MD in which the distance between two vertices is considered as infinite if the real distance exceeds r . Note that while $IC = 1$ -IC and $LD = 1$ -LD, we have $MD = \infty$ -MD; we say that MD is not *local*. In this article, we prove computational lower bounds for several problems included in DISTANCE IDENTIFYING SET by providing generic reductions from (PLANAR) HITTING SET to the meta-problem. We focus on two families of problems from the meta-problem: the first one, called *local*, contains r -IC, r -LD and r -MD for each positive integer r while the second one, called *1-layered*, contains LD, MD and r -MD for each positive integer r . We have: (1) the 1-layered problems are NP-hard even in bipartite apex graphs, (2) the local problems are NP-hard even in bipartite planar graphs, (3) assuming ETH, all these problems cannot be solved in $2^{o(\sqrt{n})}$ when restricted to bipartite planar or apex graph, respectively, and they cannot be solved in $2^{o(n)}$ on bipartite graphs, and (4) except if $W[0] = W[2]$, they do not admit parameterized algorithms in $2^{O(k)} \cdot n^{O(1)}$ even when restricted to bipartite graphs. Here k is the solution size of a relevant identifying set. In particular, METRIC DIMENSION cannot be solved in $2^{o(n)}$ under ETH, answering a question of Hartung and Nichterlein (Proceedings of the 28th conference on computational complexity, CCC, 2013).

Keywords Identifying code · Resolving set · Metric dimension · Distance identifying set · Parameterized complexity · W-hierarchy · Meta-problem · Hitting set

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1 Introduction and Corresponding Works

Problems consisting in identifying each element of a combinatorial structure with a hopefully small number of elements have been widely investigated. Here, we study a meta identification problem which generalizes three of the most well-known identification problems in graphs, namely IDENTIFYING CODE (IC), LOCATING DOMINATING SET (LD) and METRIC DIMENSION (MD). These problems are used in network verification [3, 4], fault-detection in networks [24, 30], graph isomorphism [2] or logical definability of graphs [25]. The versions of these problems in hypergraphs have been studied under different names in [6–8].

Given a graph G with vertex set V , the classical identifying sets are defined as follows:

- IC: Introduced by Karposky et al. [24], a set C of vertices of G is said to be an *identifying code* if none of the sets $N[v] \cap C$ are empty, for $v \in V$ and they are all distinct.
- LD: Introduced by Slater [27, 28], a set C of vertices of G is said to be a *locating-dominating set* if none of the sets $N[v] \cap C$ are empty, for $v \in V \setminus C$ and they are all distinct. When not considering the dominating property ($N[v] \cap C$ may be empty), these sets have been studied in [2] as distinguishing sets and in [25] as sieves.
- MD: Introduced independently by Harary et al. [18] and Slater [26], a set C of vertices of G is said to be a *resolving set* if C contains one vertex from each connected component of G and, for every distinct vertices u and v of G , there exists a vertex w of C such that $d(w, u) \neq d(w, v)$. The *metric dimension* of G is the minimum size of its resolving sets.

The corresponding minimization problems of the previous identifying sets are defined as follows: given a graph G , compute a suitable set C of minimal size, if one exists. In this paper, we mainly focus on the computational complexity of these minimization problems.

Known results A wide collection of NP-hardness results has been proven for the problems.

For IC and LD, the minimization problems are indeed NP-hard [10, 11]. Charon et al. showed the NP-hardness when restricted to bipartite graphs [9], while Auger showed it for planar graphs with arbitrarily large girth [1]. For trees, there exists a linear algorithm [27].

METRIC DIMENSION is also NP-hard, even when restricted to Gabriel unit disk graphs [17, 21]. Epstein et al. [14] showed that MD is polynomial on several classes as trees, cycles, cographs, partial wheels, and graphs of bounded cyclo-matic number, but it remains NP-hard on split graphs, bipartite graphs, co-bipartite and line graphs of bipartite graphs. Diaz et al. [12] proved that MD is polynomial on outerplanar graphs whereas it remains NP-hard on bounded degree planar graphs. Additionally, Fernau et al. provided a linear algorithm for chain graphs [16] while Hoffmann et al. gave one for cactus block graphs [22]. Foucaud

et al. [15] also proved the NP-hardness of the three problems restricted to interval graphs and permutation graphs.

These notions may be considered under the parameterized point of view; see [13] for a comprehensive study of Fixed Parameter Tractability (FPT). In the following, the parameter k is chosen as the solution size of a suitable set.

For IC and LD, the parameterized problems are clearly FPT since the number of vertices of a positive instance is bounded by $2^k + k$ (k vertices may characterize 2^k neighbors).

Such complexity is not likely to be achievable in the case of MD, since it would imply $W[2] = FPT (= W[0])$. Indeed, Hartung et al. [19, 20] showed MD is $W[2]$ -hard for bipartite subcubic graphs. The problem is however FPT on families of graphs with degree Δ growing with the number of vertices because the size k of a resolving set must satisfy $\log_3(\Delta) < k$. Foucaud et al. also provided a FPT algorithm on interval graphs [15], which was generalized to graphs of bounded tree-length by Belmonte et al. [5] (Fig. 1).

Our contributions In order to unify the previous minimization problems, we introduce the concept of *distance identifying functions*. Given a distance identifying function f and a value r as a positive integer or infinity, the DISTANCE IDENTIFYING SET meta-problem consists in finding a minimal sized r -dominating set which distinguishes every couple of vertices of an input graph thanks to the function f . Here, we mainly focus on two natural subfamilies of problems of DISTANCE IDENTIFYING SET named *local*, in which a vertex cannot discern the vertices outside of its i -neighborhood, for i a fixed positive integer, and *1-layered*, where a vertex is able to separate its open neighborhood from the distant vertices.

With this approach, we obtain several computational lower bounds for problems included in DISTANCE IDENTIFYING SET by providing generic reductions from (PLANAR) HITTING SET to the meta-problem. The reductions rely on the classical set/element-gadget technique, the noteworthy adaptation of the clause/variable-gadget technique from SAT to HITTING SET.

We provide a *1-layered generic gadget* as well as a *local generic gadget*. However, the local planar reduction is slightly more efficient than its 1-layered counterpart: it indeed implies computational lower bounds for planar graphs whereas

	1-layered problems	r -local problems
Reduction using PLANAR HITTING SET	NP-hard on bipartite apex graphs	NP-hard on bipartite planar graphs
with ETH	no algorithm running in $2^{o(\sqrt{n})}$ time for relevant classes of graphs.	
Reduction using HITTING SET	NP-hard on bipartite graphs	
with ETH with $W[2] \neq W[0]$	no algorithm running in $2^{o(n)}$ for bipartite graphs. no parameterized algorithm in $2^{O(k)} \cdot n^{O(1)}$ for bipartite graphs.	

Fig. 1 The computational lower bounds implied by our generic reductions

the 1-layered reduction requires an auxiliary apex, limiting the consequences to *apex graphs* (see definition in Sect. 2.1).

The reductions in general graphs are designed to exploit the $W[2]$ -hardness of HITTING SET parameterized by the solution size k_{HS} of a hitting set, hereby using:

Theorem 1 (folklore) *Let n_{HS} and m_{HS} be the number of elements and sets of an HITTING SET instance, and k_{HS} be its solution size. A parameterized problem with parameter k admitting a reduction from HITTING SET verifying $k = \mathcal{O}(k_{HS} + \log(n_{HS} + m_{HS}))$ does not have a parameterized algorithm running in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time except if $W[2] = FPT$.*

Proof Given a reduction from HITTING SET to a parameterized problem Π such that the reduced parameter satisfies $k = \mathcal{O}(k_{HS} + \log(n_{HS} + m_{HS}))$ and the size of the reduced instance verifies $n = (n_{HS} + m_{HS})^{\mathcal{O}(1)}$, an algorithm for Π of running time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ is actually an algorithm for HITTING SET of running time $2^{\mathcal{O}(k_{HS})} \cdot (n_{HS} + m_{HS})^{\mathcal{O}(1)}$, meaning that HITTING SET is FPT, a contradiction to its $W[2]$ -hardness (otherwise $W[2] = FPT$). \square

Hence, as each gadget contributes to the resulting solution size of a distance identifying set, we set up a binary compression of the gadgets to limit their number to the logarithm order. To the best of our knowledge, this merging gadgets technique has never been used to lower bound the parameterized complexity of a problem within the framework of W -hierarchy.

The organization of the paper is as follows. After a short reminder of the computational properties of HITTING SET, Sect. 2 contains the definitions of distance identifying functions and sets, allowing us to precise the computational lower bounds we obtain. The Sect. 3 designs the supports of the reductions as *distance identifying graphs* and *compressed graph*. Finally, the gadgets needed for the reductions to apply are given in Sect. 4 as well as the proofs of the main theorems.

2 Definition of the Meta-problem and Related Concepts

2.1 Preliminaries

Notations Throughout the paper, we consider simple non oriented graphs.

Given a positive integer n , the set of positive integers smaller than n is denoted by $\llbracket n \rrbracket$. By extension, we define $\llbracket \infty \rrbracket = \mathbb{N}_{>0} \cup \{\infty\}$. Given two vertices u, v of a graph G , the distance between u and v corresponds to the number of vertices in the shortest path between u and v and is denoted $d(u, v)$. The *open neighborhood* of u is denoted by $N(u)$, its *closed neighborhood* is $N[u] = N(u) \cup \{u\}$, and for a value $r \in \llbracket \infty \rrbracket$, the *r -neighborhood* of u is $N_r[u]$, that is the set of vertices at distance less than $r + 1$ of u . For $r = \infty$, the ∞ -neighborhood of u is the set of

vertices in the same connected component than u . We recall that a subset D of V is called an r -dominating set of G if for all vertices u of V , the set $N_r[u] \cap D$ is non-empty. Thus an ∞ -dominating set of G contains at least a vertex for each connected component of G .

Given two subsets X and Y of V , the distance $d(X, Y)$ corresponds to the value $d(X, Y) = \min\{d(x, y) \mid x \in X, y \in Y\}$. For a vertex u , we will also use $d(u, X)$ and $d(X, u)$, defined similarly. The symmetric difference between X and Y is denoted by $X \Delta Y$, and the 2-combination of a set X is denoted $\mathcal{P}_2(X)$

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, H is an induced subgraph of G if $V_H \subseteq V_G$ and for all vertices u and v of V_H , $(u, v) \in E_G$ if and only if $(u, v) \in E_H$. We denote $H = G[V_H]$ and $V_G \setminus V_H$ by $V_{G \setminus H}$. Symmetrically, G is an induced supergraph of H . A graph $G = (V \cup \{a\}, E)$ is an apex graph if $G[V]$ is planar, the vertex a being its apex.

The (PLANAR) HITTING SET problem. Consider a universe of n elements denoted $\Omega = \{u_i \mid i \in \llbracket n \rrbracket\}$ and a set of m non-empty subsets of Ω denoted $\mathcal{S} = \{S_i \mid i \in \llbracket m \rrbracket\}$ such that every element belongs to at least a subset. Then, a subset of Ω intersecting every set of \mathcal{S} is called an hitting set of \mathcal{S} :

HITTING SET

Input: A universe Ω and a set \mathcal{S} of non-empty subsets of Ω whose union covers Ω .

Output: A minimal-sized hitting set C of \mathcal{S} , i.e. a subset of Ω satisfying $\forall S_i \in \mathcal{S}, S_i \cap C \neq \emptyset$.

The parameterized version HITTING SET(k) decides if there exists a hitting set of size k .

Theorem 2 (Downey and Fellows [13]) HITTING SET cannot be solved in $2^{o(n)}$ time under ETH even if $m = \mathcal{O}(n)$. Moreover, HITTING SET(k) is W[2]-hard.

HITTING SET may be translated into a dominating problem on bipartite graphs. Given an instance (Ω, \mathcal{S}) of HITTING SET, let us define $\phi(\Omega, \mathcal{S}) = (V_\Omega \cup V_\mathcal{S}, E)$ as the bipartite graph of size $n + m$ such that for each $i \in \llbracket n \rrbracket$, there exists a vertex v_i^Ω in V_Ω , for each $j \in \llbracket m \rrbracket$, there exists a vertex $v_j^\mathcal{S}$ in $V_\mathcal{S}$, and the edge $(v_i^\Omega, v_j^\mathcal{S})$ is present in E if and only if the element u_i belongs to the subset S_j . Therefore, a hitting set of \mathcal{S} is equivalent to a subset C of V_Ω that dominates $V_\mathcal{S}$. We call $\phi(\Omega, \mathcal{S})$ the associated graph of (Ω, \mathcal{S}) .

PLANAR HITTING SET

Input: An instance (Ω, \mathcal{S}) of HITTING SET such that $\phi(\Omega, \mathcal{S})$ is planar.

Output: A hitting set C of \mathcal{S} of minimal size.

We also consider the parameterized version PLANAR HITTING SET(k) of the latter problem.

Theorem 3 (folklore) There exists a reduction from the problem SAT to PLANAR HITTING SET(n) producing associated graphs of quadratic size in the number n of variables of the instances of SAT. Thus PLANAR HITTING SET cannot be solved in $2^{o(\sqrt{n})}$ under ETH even if $m = \mathcal{O}(n)$.

Proof Let Φ be the set of the n variables present in the set \mathcal{C} of clauses of an instance of SAT. For each variable φ of Φ , we add two new elements u_φ and \bar{u}_φ to the universe Ω_Φ representing the two possible affectations of variable φ , and we create a set $S_\varphi = \{u_\varphi, \bar{u}_\varphi\}$ that we append to the set $\mathcal{S}_\mathcal{C}$ of subsets of Ω_Φ . The independence of the sets S_φ implies that the existence of a hitting set of size strictly smaller than n is impossible. Reciprocally, a potential hitting set of size exactly n must define an affectation of the n variables of Φ . Finally, to determine if an affectation satisfies the set of clauses \mathcal{C} , for each clause $c \in \mathcal{C}$ we append to $\mathcal{S}_\mathcal{C}$ the set of elements representing each literal present in the clause c . The equivalence between the satisfiability of \mathcal{C} and the existence of a hitting set of $\mathcal{S}_\mathcal{C}$ of size n is immediate by construction. It remains to guarantee the planarity of the associated graph $\phi(\Omega_\Phi, \mathcal{S}_\mathcal{C})$. To do so, we actually apply the reduction on a restriction of SAT named SEPARATE SIMPLE PLANAR SAT (See [29] for a precise definition). Adding the *sparsification lemma* from [23], the reduction produces a graph of size linear in n , preserving the computational lower bound of SEPARATE SIMPLE PLANAR SAT. In particular, the latter problem is not solvable in $2^{o(\sqrt{n})}$ under ETH, □

2.2 The DISTANCE IDENTIFYING SET Meta-problem

Given a graph $G = (V, E)$ and $r \in \llbracket \infty \rrbracket$, the classical identifying sets may be rewritten:

- r -IC: a subset C of V is an r -identifying code of G if it is an r -dominating set and for every distinct vertices u, v of V , a vertex w in C verifies $w \in N_r[u] \Delta N_r[v]$.
- r -LD: a subset C of V is an r -locating dominating set of G if it is an r -dominating set and for every distinct vertices u, v of V , a vertex w in C verifies $w \in (N_r[u] \Delta N_r[v]) \cup \{u, v\}$.
- r -MD: a subset C of V is an r -resolving set of G if it is an r -dominating set and for every distinct vertices u, v of V , a vertex w in C verifies $w \in N_r[u] \cup N_r[v]$ and $d(u, w) \neq d(v, w)$.

It is not hard to check that IC = 1-IC, LD = 1-LD (= 1-MD) and MD = ∞ -MD.

A pattern clearly appears: the previous identifying sets only deviate on the criterion that the vertex w must verify. The pivotal idea is to consider an abstract version of the criterion which does not depend on the input graph. Hence:

Definition 1 (*identifying function*) A function f of type: $G \rightarrow (V \times \mathcal{P}_2(V) \rightarrow \{\text{true}, \text{false}\})$, is called an *identifying function*. Given three vertices u, v and w of a graph G such that $u \neq v$, we write $f_G[w](u, v)$ to get the resulting boolean. The notation $\mathcal{P}_2(V)$ implies that f_G is symmetric, that is $f_G[w](u, v) = f_G[w](v, u)$.

In the following, given an identifying function f and three vertices u, v, w of a graph G , we say that w f -distinguishes u and v if and only if $f_G[w](u, v)$ is true. By extension, given three vertex sets C, X and Y of G , we say that C f -distinguishes

X and Y if for every u in X and v in Y , either $u = v$ or there exists w in C verifying $f_G[w](u, v)$. Finally, a graph G of vertex set V is f -distinguished by C when C f -distinguishes V and V .

An identifying function may be defined using various criteria such as adjacency, coloring or vertex degree. Here, we focus on distance in order to mimic the proposed identifying sets.

Definition 2 (*distance identifying function*) A distance identifying function f is an identifying function such that for every graph G and all vertices u, v and w of G with $u \neq v$:

(α): $f_G[w](u, v)$ is false when $d(w, u) = d(w, v)$.

We can now define the distance identifying sets and their associated meta-problem.

Definition 3 (*(f, r)-distance identifying set*) Given a distance identifying function f and $r \in \llbracket \infty \rrbracket$, a (f, r) -distance identifying set of a graph G is an r -dominating set of G that f -distinguishes G .

DISTANCE IDENTIFYING SET

Input: A distance identifying function f and $r \in \llbracket \infty \rrbracket$. A graph G .

Output: An (f, r) -distance identifying set of G of minimal size, if one exists.

Given a distance identifying function f and $r \in \llbracket \infty \rrbracket$ as inputs of the meta-problem, the resulting problem is called (f, r) -DISTANCE IDENTIFYING SET and denoted (f, r) -DIS. We also consider the parameterized version DISTANCE IDENTIFYING SET(k).

An (f, r) -DIS problem is not necessarily NP-hard. For instance, let \perp be the function such that $\perp_G[w](u, v)$ is false for every inputs. Then, (\perp, r) -DIS can trivially be solved in constant time. We need to consider restrictions in which positive outputs appear.

Let $i \in \{0\} \cup \llbracket \infty \rrbracket$, we suggest two criteria. First, we may restrain the range of a vertex to its i -neighborhood: a vertex should not distinguish two vertices if they do not lie in its i -neighborhood but it should always distinguish them whenever exactly one of them lies in that i -neighborhood. Reciprocally, we may ensure that a vertex could distinguish the vertices of its i -neighborhood: a vertex should distinguish a vertex belonging to its i -neighborhood from all the other vertices, assuming the distances are different. Formally, we have:

Definition 4 (*i -local function*) For $i \in \{0\} \cup \llbracket \infty \rrbracket$, an i -local identifying function f is an identifying function such that for every graph G and all vertices u, v and w of G with $u \neq v$:

- (β_1): $f_G[w](u, v)$ is true when $d(w, u) \leq i < d(w, v)$ or, symmetrically, $d(w, v) \leq i < d(w, u)$.
- (β_2): $f_G[w](u, v)$ is false when $i < \min\{d(w, u), d(w, v)\}$.

Definition 5 (*i-layered function*) For $i \in \{0\} \cup \llbracket \infty \rrbracket$, an *i-layered identifying function* f is an identifying function such that for every graph G and all vertices u, v, w of G with $u \neq v$:

- (γ): $f_G[w](u, v)$ is true when $\min\{d(w, u), d(w, v)\} \leq i$ and $d(w, u) \neq d(w, v)$.

A problem (f, r) -DIS is said to be *i-layered* when the function f is *i-layered*, and it is said to be *i-local* when f is *i-local* and $r = i$. The problem is said local if it is *i-local* for an integer i . We observe that r -IC, r -LD and r -MD are r -local for each positive integer r , while MD is not local. However, for each $r \in \llbracket \infty \rrbracket$, r -MD is r -layered, thus 1-layered. We also notice that r -LD is 0-layered while r -IC is not, which is their unique difference.

2.3 Detailed Computational Lower Bounds

Using the DISTANCE IDENTIFYING SET meta-problem, we get the following lower bounds:

Theorem 4 For each 1-layered distance identifying function f and every $r \in \llbracket \infty \rrbracket$, the (f, r) -DISTANCE IDENTIFYING SET problem restricted to bipartite apex graphs is NP-hard, and does not admit an algorithm running in $2^{o(\sqrt{n})}$ time under ETH.

Theorem 5 The local problems restricted to bipartite planar graphs are NP-hard, and do not admit an algorithm running in $2^{o(\sqrt{n})}$ time under ETH.

Theorem 6 Let f be a 1-layered distance identifying function, $r \in \llbracket \infty \rrbracket$, and g be a q -local distance identifying function. Both the (f, r) - and (g, q) -DIS problems are NP-hard, and do not admit:

- algorithms running in $2^{o(n)}$ time, except if ETH fails,
- parameterized algorithms running in $2^{O(k)} \cdot n^{O(1)}$ time, except if $W[2] = \text{FPT}$.

even when restricted to bipartite graphs.

The parameter k denotes here the solution size of a relevant distance identifying set.

As a side result, Theorem 6 answers a question of Hartung and Nichterlein in [20]:

Corollary 1 Under ETH, METRIC DIMENSION cannot be solved in $2^{o(n)}$.

Finally, notice that the parameterized lower bound from Theorem 6 may be complemented by an elementary upper bound inspired from the kernel of IC and LD of size $2^k + k$:

Proposition 1 *For every r -local distance identifying function f , the (f, r) -DISTANCE IDENTIFYING SET problem has a kernel of size $(r + 1)^k + k$ where k is the solution size. Therefore, if f is computable in polynomial time, enumerating the $\binom{n}{k}$ sets of k vertices of an input yields a naive algorithm running in time $\mathcal{O}(n^{k+\mathcal{O}(1)}) \in \mathcal{O}^*(r^{k^2})$ for (f, r) -DIS.*

The proofs of the Theorems 4 to 6 will be given in Sect. 4.

3 The Supports of the Reductions for DISTANCE IDENTIFYING SET

3.1 The Distance Identifying Graphs

Consider the associated graph $\phi(\Omega, \mathcal{S})$ as defined in Sect. 2.1. The differences between the DISTANCE IDENTIFYING SET meta-problem and the dominating problem related to associated graphs actually raise two issues for a reduction based on these latter notions to be effective on DISTANCE IDENTIFYING SET. First, contrarily to the dominating problem where a vertex may only discern its close neighborhood, the meta-problem may allow a vertex to discern further than its direct neighborhood. In that case, we cannot certify that a vertex v_i^Ω does not distinguish a vertex v_j^S when u_i is not in S_j , the adjacency not remaining a sufficient argument. Secondly, one may object that a vertex v_i^Ω formally has to distinguish a vertex v_j^S from another vertex, but that distinguishing a single vertex is not defined.

To circumvent these problems, we suggest the following fix: rather than producing a single vertex for each $S_j \in \mathcal{S}$, the set V_S may contain two vertices v_j^S and \bar{v}_j^S . Then, the role of v_i^Ω would be to distinguish them if and only if $u_i \in S_j$. To ensure that the vertex v_i^Ω distinguishes v_j^S and \bar{v}_j^S when $u_i \in S_j$, we may use the properties (β_1) and (γ) of Definitions 4 and 5 for the r -local and 1-layered problems, respectively. Precisely, when $u_i \in S_j$, v_i^Ω should be at distance r to v_j^S (with $r = 1$ in the 1-layered cases) while \bar{v}_j^S should not be in the r -neighborhood of v_i^Ω . Similarly, to ensure that v_i^Ω cannot distinguish v_j^S and \bar{v}_j^S when $u_i \notin S_j$, we may use properties (α) or (β_2) of Definitions 2 and 4. Hence, when $u_i \notin S_j$, v_j^S should not be in the r -neighborhood of v_i^Ω , or $d(v_i^\Omega, v_j^S)$ and $d(v_i^\Omega, \bar{v}_j^S)$ should be equal.

That fix fairly indicates how to initiate the transformation of the associated graphs in order to deliver an equivalence between a hitting set formed by elements of Ω and the vertices of a distance identifying set included in V_Ω . However, it is clearly not sufficient since we also have to distinguish the couples of vertices of V_Ω for which nothing is required. To solve that problem, we suggest to append to each vertex of the associated graph a copy of some gadget with the intuitive requirement that the

gadget is able to distinguish the close neighborhood of its vertices from the whole graph. We introduce the notion of *B-extension*:

Definition 6 (*B-extension*) Let $H = (V_H, E_H)$ be a connected graph, and $B \subseteq V_H$. An induced supergraph $G = (V_G, E_G)$ is said to be a *B-extension* of H if it is connected and for every vertex v of $V_{G \setminus H}$, the set $N(v) \cap V_H$ is either equal to \emptyset or B .

A vertex v of $V_{G \setminus H}$ such that $N(v) \cap V_H = B$ is said to be *B-adjacent*. The *B-extensions* of H such that $V_{G \setminus H}$ contains exactly a *B-adjacent* vertex or two *B-adjacent* vertices but not connected to each other are called the *B-single-extension* and the *B-twin-extension* of H , respectively.

Here, the “border” B makes explicit the connections between a copy of a gadget H and a vertex outside the copy. In particular, a *B-single-extension* is formed by a gadget with its related vertex v_i^Ω , while a *B-twin-extension* contains a gadget with its two related vertices v_j^S and \bar{v}_j^S . Piecing all together, we may adapt the associated graphs to the meta-problem:

Definition 7 (*(H, B, r)-distance identifying graph*) Let $(\Omega = \{u_i \mid i \in \llbracket n \rrbracket\}, \mathcal{S} = \{S_j \mid j \in \llbracket m \rrbracket\})$ be an instance of HITTING SET. Let H be a connected graph, B a subset of its vertices, and r a positive integer. The *(H, B, r)-distance identifying graph* $\Phi[H, B, r](\Omega, \mathcal{S})$ is as follows.

- for each $i \in \llbracket n \rrbracket$, the graph $\Phi[H, B, r](\Omega, \mathcal{S})$ contains as induced subgraph a copy H_i^Ω of H together with a B_i^Ω -adjacent vertex v_i^Ω , where B_i^Ω denotes the copy of B .
- similarly, for each $j \in \llbracket m \rrbracket$, the graph $\Phi[H, B, r](\Omega, \mathcal{S})$ contains a copy H_j^S of H together with two B_j^S -adjacent vertices v_j^S and \bar{v}_j^S (the latter vertices are not adjacent) and where B_j^S denotes the copy of B .
- finally, for each $S_j \in \mathcal{S}$ and each $u_i \in S_j$, v_i^Ω is connected to v_j^S by a path of $r - 1$ vertices denoted l_{ij}^k with $d(v_i^\Omega, l_{ij}^k) = k$ for each $k \in \llbracket r - 1 \rrbracket$.

When the problem is not local, we prefer the following identifying graph:

Definition 8 (*(H, B)-apex distance identifying graph*) An *(H, B)-apex distance identifying graph* $\Phi^*[H, B](\Omega, \mathcal{S})$ is the union of an *(H, B, 1)-distance identifying graph* with an additional vertex a called *apex* such that:

- for each $u_i \in \Omega$, the apex a is B_i^Ω -adjacent to H_i^Ω , where B_i^Ω (resp. H_i^Ω) denotes the copy of B (resp. H).
- for each $S_j \in \mathcal{S}$, the apex a is adjacent to v_j^S and \bar{v}_j^S .

See Fig. 2 for an example of an *(H, B, r)-distance identifying graph* (on the left) and an example of *(H, B)-apex distance identifying graph* (on the right).

Proposition 2 Given an instance (Ω, \mathcal{S}) of PLANAR HITTING SET where $|\Omega| = n$, $|\mathcal{S}| = m$, the graphs $G = \Phi[H, B, r](\Omega, \mathcal{S})$ and $G' = \Phi^*[H, B](\Omega, \mathcal{S})$

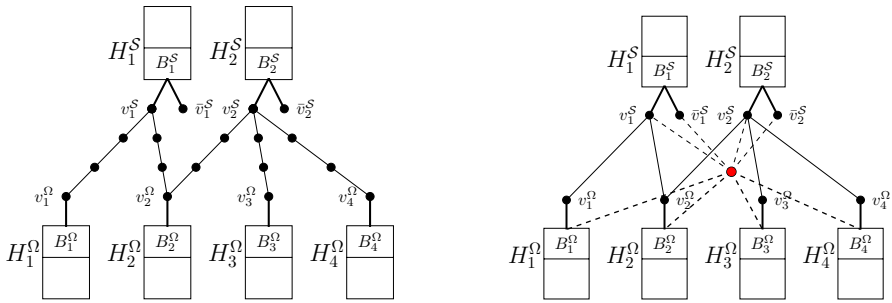


Fig. 2 An $(H, B, 3)$ -distance identifying graph and an (H, B) -apex distance identifying graph built on the planar instance formed by $\Omega = \{1, 2, 3, 4\}$ and $S = \{\{1, 2\}, \{2, 3, 4\}\}$

- are connected and have size bounded by $(|H| + 2r)(n + m)$, (with $r = 1$ for G'),
- may be built in polynomial time in their size,
- are bipartite if the B -single extension of H is bipartite,
- are respectively planar and an apex graph if the B -twin-extension of H is planar.

Proof The graph G is formed by the union of n B -single-extensions of H , m B -twin-extensions of H and all the possible paths of $r - 1$ vertices. As $\phi(\Omega, S)$ is a bipartite planar graph, the Euler formula implies that the number of paths is bounded by $2(n + m) - 4$. We conclude that the number of vertices of G is bounded by:

$$n(|H| + 1) + m(|H| + 2) + (r - 1)(2(n + m) - 4) = (|H| + 2r)(n + m) - n - 4(r - 1)$$

Furthermore, it is clear that G is connected if and only if the associated graph $\phi(\Omega, S)$ is connected. Additionally, we may consider that $\phi(\Omega, S)$ is connected since it is a property decidable in polynomial time, and that the instances corresponding to the distinct connected components of $\phi(\Omega, S)$ may be considered independently.

Finally, all the other items of the proposition are direct by construction. □

Having defined the (apex) distance identifying graphs, the main effort to obtain generic reduction from PLANAR HITTING SET is done. We now define relevant gadgets:

Definition 9 (*(f, r)-gadgets*) Let f be a distance identifying function and $r \in \llbracket \infty \rrbracket$. Let $H = (V_H, E_H)$ be a connected graph, and B, C be two subsets of V_H . We say that the triple (H, B, C) is a (f, r) -gadget if for every B -extension G of H :

- (p_h) C f -distinguishes V_H and V_G .
- (p_b) C f -distinguishes N_B and $V_{G \setminus H} \setminus N_B$, where N_B is the set of B -adjacent vertices of G .
- (p_d) C is an r -dominating set of $G[V_H \cup N_B]$.
- (p_s) For all (f, r) -distance identifying sets S of G , $|C| \leq |S \cap V_H|$.

Definition 10 (*local gadgets*) An (f, r) -gadget is a *local gadget*, if f is an r -local identifying function with $r \neq \infty$, and

(p_l) for every $k \in \llbracket r \rrbracket$, there exists $c \in C$ such that $d(c, B) = k - 1$.

Consistently, we say that an (f, r) -gadget (H, B, C) is *bipartite* if the B -single-extension of H is bipartite, and that it is *planar* if the B -twin-extension of H is planar.

Theorem 7 Let (Ω, \mathcal{S}) be an instance of HITTING SET such that $|\Omega| = n > 1$, $|\mathcal{S}| = m$. Let (H, B, C) be an (f, r) -gadget for a 1-layered identifying function f and let (H', B', C') be a local (g, q) -gadget. The following propositions are equivalent:

- There exists a hitting set of \mathcal{S} of size k .
- There exists an (f, r) -distance identifying set of $\Phi^*[H, B](\Omega, \mathcal{S})$ of size $k + |C|(n + m)$.
- There exists a (g, q) -distance identifying set of $\Phi[H', B', q](\Omega, \mathcal{S})$ of size $k + |C'|(n + m)$.

Proof We start by focusing on the equivalence between the first and second items.

Suppose first that P is a hitting set of (Ω, \mathcal{S}) of size k . By denoting C_i^Ω and C_j^S the copies of C associated to the copies H_i^Ω and H_j^S of H , we suggest the following set I of size $k + |C|(n + m)$ as an (f, r) -distance identifying set of $G = \Phi^*[H, B](\Omega, \mathcal{S})$:

$$I = \{v_i^\Omega : u_i \in P\} \cup \bigcup_{i \in \llbracket n \rrbracket} C_i^\Omega \cup \bigcup_{j \in \llbracket m \rrbracket} C_j^S.$$

Recall that by construction, G is a B_i^Ω -extension of H_i^Ω (respectively B_j^S -extension of H_j^S) for any $i \in \llbracket n \rrbracket$ (respectively $j \in \llbracket m \rrbracket$). This directly implies that I is an r -dominating set of G . Indeed, the condition (p_d) of Definition 9 implies that C_i^Ω (respectively C_j^S) r -dominates H_i^Ω plus v_i^Ω (respectively of H_j^S plus v_j^S, \bar{v}_j^S). The remaining apex is also r -dominated by any C_i^Ω , as it is B_i^Ω -adjacent for every $i \in \llbracket n \rrbracket$.

We now have to show that I f -distinguishes G . We begin with the vertices of the gadget copies because the condition (p_h) implies that $C_i^\Omega \subseteq I$ f -distinguishes the vertices of H_i^Ω and G for every $i \in \llbracket n \rrbracket$, and I f -distinguishes the vertices of H_j^S and G for every $j \in \llbracket m \rrbracket$. Thereby, we only have to study the vertices of the form $v_i^\Omega, v_j^S, \bar{v}_j^S$, and the apex a (there is no vertex of the form $l_{i,j}^k$ in an apex distance identifying graph). To distinguish them, we use the condition (p_b). Recall that $n > 1$. Then, for each distinct $i, i' \in \llbracket n \rrbracket$, we have:

- v_i^Ω is B_i^Ω -adjacent but not $B_{i'}^\Omega$ -adjacent,
- a is both B_i^Ω -adjacent and $B_{i'}^\Omega$ -adjacent,
- a vertex of the form v_j^S or \bar{v}_j^S is neither B_i^Ω -adjacent nor $B_{i'}^\Omega$ -adjacent.

Enumerating the relevant i and i' , we deduce that every couple of vertices is distinguished except when they are both of the form v_j^S or $\bar{v}_{j'}^S$ for $j, j' \in \llbracket m \rrbracket$. But we may distinguish v_j^S or $\bar{v}_{j'}^S$ for distinct j, j' by applying (p_b) on H_j^S .

It remains to distinguish v_j^S and \bar{v}_j^S for $j \in \llbracket m \rrbracket$. We now use the fact that P is a hitting set for (Ω, S) . By definition of a hitting set, for any set $S_j \in S$, there exists a vertex $u_i \in P$ such that $u_i \in S_j$. We observe that $d(v_i^\Omega, v_j^S) = 1 < d(v_i^\Omega, \bar{v}_j^S)$ by construction of G and that $v_i^\Omega \in I$ by definition of I . Since f is 1-layered, I f -distinguishes v_j^S and \bar{v}_j^S .

In the other direction, assume that I is a distance identifying set of G of size $k + |C|(n + m)$. As every set of S is not empty, we may define a function $\varphi : \llbracket m \rrbracket \rightarrow \llbracket n \rrbracket$ such that $u_{\varphi(j)} \in S_j$.

We suggest the following set P as a hitting set of S of size at most k :

$$P = \{u_i \in \Omega \mid v_i^\Omega \in I\} \cup \{u_{\varphi(j)} \in \Omega \mid v_j^S \in I \text{ or } \bar{v}_j^S \in I\}$$

We claim that the only vertices that may f -distinguish v_j^S and \bar{v}_j^S are themselves and the vertices v_i^Ω such that $u_i \in S_j$. To prove so, we apply propriety (α) of Definition 2:

- The apex a verifies $d(a, v_j^S) = 1 = d(a, \bar{v}_j^S)$.
- A vertex v_i^Ω such that $u_i \notin S_j$ verifies $d(v_i^\Omega, v_j^S) = 3 = d(v_i^\Omega, \bar{v}_j^S)$.
- A vertex v of H_i^Ω verifies $d(v, v_j^S) = 2 + d(v, B_i^\Omega) = d(v, \bar{v}_j^S)$.
- A vertex v of $H_{j'}^S$ with $j \neq j'$ verifies $d(v, v_j^S) = 3 + d(v, B_j^S) = d(v, \bar{v}_j^S)$.
- Both v_j^S and \bar{v}_j^S are B_j^S -adjacent, so they are at the same distance of any vertex of H_j^S .

We deduce that v_j^S and \bar{v}_j^S are f -distinguished only if either one on them belongs to I (in that case $u_{\varphi(j)} \in P \cap S_j$) or there exists $v_i^\Omega \in I$ such that $u_i \in S_j$ (and then $u_i \in P \cap S_j$).

It remains to show that $|P| \leq k$. By the condition (p_s) of Definition 9, we know that $|I \cap V_{H_i^\Omega}| \geq |C_i^\Omega|$ and $|I \cap V_{H_j^S}| \geq |C_j^S|$ for any $i \in \llbracket n \rrbracket$ and $j \in \llbracket m \rrbracket$, implying

$$\begin{aligned} k &= |I| - |C|(n + m) \\ &\geq \sum_{i \in \llbracket n \rrbracket} |I \cap \{v_i^\Omega\}| + \sum_{j \in \llbracket m \rrbracket} |I \cap \{v_j^S, \bar{v}_j^S\}| \\ &\geq \sum_{v_i^\Omega \in I} 1 + \sum_{I \cap \{v_j^S, \bar{v}_j^S\} \neq \emptyset} 1 = |P| \end{aligned}$$

Now, we prove the equivalence between the first and third items. Consider a q -local distance identifying function g , a local (g, q) -gadget (H, B, C) and an instance (Ω, S) of PLANAR HITTING SET such that $|\Omega| = n, |S| = m$. We denote the copies of H as H_i^Ω or H_j^S , the copies of C as C_i^Ω and C_j^S , and the copies of B as B_i^Ω and B_j^S for any $i \in \llbracket n \rrbracket$ and $j \in \llbracket m \rrbracket$.

In the first direction, suppose that P is a hitting set of (Ω, \mathcal{S}) of size k , the (g, q) -distance identifying set I of $G = \Phi[H, B, q](\Omega, \mathcal{S})$ is defined identically as in the equivalence of the first and second items of the current theorem:

$$I = \{v_i^\Omega : u_i \in P\} \cup \bigcup_{i \in [n]} C_i^\Omega \cup \bigcup_{j \in [m]} C_j^S$$

Using conditions (p_d) and (p_l) of Definitions 9 and 10, I is a q -dominating set of G . Indeed, by (p_d) every vertex belonging to a copy of the gadget is q -dominated. Additionally, every vertex outside of the copies of the gadgets is at distance at most q of a copy by construction, but there exists a vertex $b \in B \cap C$ (so a relevant copy in D) by (p_l) .

To prove that I g -distinguishes G , the strategy is differing from the previous equivalence only on the l_{ij}^k vertices and when distinguishing v_j^S and \bar{v}_j^S as we will see.

Recall that by construction, G is a B_i^Ω -extension of H_i^Ω (respectively B_j^S -extension of H_j^S) for any $i \in [n]$ (respectively $j \in [m]$). Distinguishing the vertices of the gadget copies is easy, as the condition (p_h) implies that $C_i^\Omega \subseteq I$ g -distinguishes the vertices of H_i^Ω and G for every $i \in [n]$, and similarly I g -distinguishes the vertices of H_j^S and G for every $j \in [m]$.

Thereby, we only have to study the vertices of the form $v_i^\Omega, v_j^S, \bar{v}_j^S$, and the vertices l_{ij}^k .

To distinguish them, we mainly use the condition (p_b) . We observe that for each distinct $i, i' \in [n]$ (they exist as $n > 1$) :

- The vertex v_i^Ω is B_i^Ω -adjacent but not $B_{i'}^\Omega$ -adjacent.
- For every $j \in [m]$, v_j^S or \bar{v}_j^S is neither B_i^Ω -adjacent nor $B_{i'}^\Omega$ -adjacent.
- For every $i \in [n]$, $j \in [m]$ and $k \in [r - 1]$, l_{ij}^k is neither B_i^Ω -adjacent nor $B_{i'}^\Omega$ -adjacent.

Thus I g -distinguishes v_i^Ω and G .

As the vertices of form l_{ij}^k are the only vertices to belong to both the q -neighbourhood of B_i^Ω and B_j^S , and as the vertices l_{ij}^k and $l_{ij}^{k'}$ with $k < k'$ are g -distinguished by the guaranteed vertex $c \in C_i^\Omega$ such that $d(c, B_i^\Omega) = q - k - 1$, I g -distinguishes l_{ij}^k and G for every relevant i, j and k .

It remains to distinguish v_j^S and $\bar{v}_{j'}^S$ for $j, j' \in [m]$. If j and j' are distinct we may use (p_b) on the copy H_j^S of the gadget H . We may assume $j = j'$. We now use the fact that P is a hitting set for (Ω, \mathcal{S}) . By definition of a hitting set, for any set $S_j \in \mathcal{S}$, there exists a vertex $u_i \in P$ such that $u_i \in S_j$. We observe that $d(v_i^\Omega, v_j^S) = q < d(v_i^\Omega, \bar{v}_j^S)$ (when $u_i \in S_j$) by construction of G and I indeed g -distinguishes v_j^S and \bar{v}_j^S because g is q -local.

In the other direction, assume that I is a distance identifying set of G of size $k + |C|(n + m)$. The hitting set P may now depend on l_{ij}^k . Let define $L_i = \{v_i^\Omega\} \cup \{l_{ij}^k \mid k \in [r - 1] \text{ and } u_i \in S_j\}$. As every set of \mathcal{S} is not empty, we may

define a function $\varphi : \llbracket m \rrbracket \rightarrow \llbracket n \rrbracket$ such that $u_{\varphi(j)} \in S_j$. We suggest the following set P as a hitting set of \mathcal{S} of size at most k :

$$P = \{u_i \in \Omega \mid I \cap L_i \neq \emptyset\} \cup \{u_{\varphi(j)} \in \Omega \mid v_j^S \in I \text{ or } \bar{v}_j^S \in I\}$$

Consider $j \in \llbracket m \rrbracket$, let us show that the only vertices that may g -distinguish the couple (v_j^S, \bar{v}_j^S) are themselves and the vertices from L_i (and not only v_i^{Ω}) such that $u_i \in S_j$. Every vertex from H_j^S is at the same distance to v_j^S and \bar{v}_j^S and thus cannot g -distinguish them because of the distance property (α). Every vertex not in H_j^S , not in L_i for every $i \in \llbracket n \rrbracket$ such that $u_i \in S_j$ and different from v_j^S and \bar{v}_j^S is at distance at least $q + 1$ of the two latter vertices. Thus, because of the propriety (β_2) of Definition 4 (a vertex cannot distinguish two vertices outside of its q -neighbourhood) any of these vertices does not q -distinguish v_j^S and \bar{v}_j^S . We deduce that v_j^S and \bar{v}_j^S are g -distinguished if and only if either one of them belongs to I (in that case $u_{\varphi(j)} \in P \cap S_j$) or there exists $i \in \llbracket n \rrbracket$ such that $u_i \in S_j$ and $I \cap L_i \neq \emptyset$.

The proof that $|P| \leq k$ is provided by (p_s) , we know that $|I \cap V_{H_i^\Omega}| \geq |C_i^\Omega|$ and $|I \cap V_{H_j^S}| \geq |C_j^S|$ for any $i \in \llbracket n \rrbracket$ and $j \in \llbracket m \rrbracket$. Considering the following partition of I

$$I = \left(\bigsqcup_{i \in \llbracket n \rrbracket} (I \cap H_i^\Omega) \right) \bigsqcup \left(\bigsqcup_{j \in \llbracket m \rrbracket} (I \cap H_j^S) \right) \bigsqcup \left(\bigsqcup_{i \in \llbracket n \rrbracket} (I \cap L_i) \right) \bigsqcup \left(\bigsqcup_{j \in \llbracket m \rrbracket} (I \cap \{v_j^S, \bar{v}_j^S\}) \right)$$

We get

$$\begin{aligned} |I| &\geq |C|(n + m) + \sum_{i \in \llbracket n \rrbracket} |I \cap L_i| + \sum_{j \in \llbracket m \rrbracket} |I \cap \{v_j^S, \bar{v}_j^S\}| \\ &\geq |C|(n + m) + \sum_{I \cap L_i \neq \emptyset} 1 + \sum_{I \cap \{v_j^S, \bar{v}_j^S\} \neq \emptyset} 1 \\ &= |C|(n + m) + |P| \end{aligned}$$

Because $|I| = k + |C|(n + m)$, we conclude that $|P| \leq k$.

Obviously, the second and third items are equivalent since they are both equivalent to the first item, which concludes the proof. \square

3.2 Binary Compression of Gadgets

The Theorem 7 is a powerful tool to get reductions, in particular in the planar cases. However, the number of involved gadgets does not allow to use Theorem 1. This limitation is due to the uses of a gadget per vertex to identify in the distance identifying graphs. Using power set, we may obtain a better order: given k gadgets, we may identify $2^k - 1$ vertices (we avoid to identify a vertex with the empty subset of gadgets). Thus, we will consider *binary representations* of integers as sequences of bits, with weakest bit at last position. For a positive integer n , we define the integer $b_n = 1 + \lceil \log_2(n) \rceil$ and introduce a new graph:

Definition 11 (*(H, B, r)-compressed graph*) Let (Ω, \mathcal{S}) be an instance of HITTING SET, with $\Omega = \{u_i \mid i \in \llbracket n \rrbracket\}$ and $\mathcal{S} = \{S_i \mid i \in \llbracket m \rrbracket\}$. Let H be a connected graph, B be a subset of its vertices, and r be a positive integer. The (H, B, r) -compressed graph $\Psi[H, B, r](\Omega, \mathcal{S})$ is defined as follows. $\Psi[H, B, r](\Omega, \mathcal{S})$ contains as induced subgraphs b_{n+1} copies of H denoted H_i^Ω for $i \in \llbracket b_{n+1} \rrbracket$ and b_m other copies of H denoted H_j^S for $j \in \llbracket b_m \rrbracket$. Then:

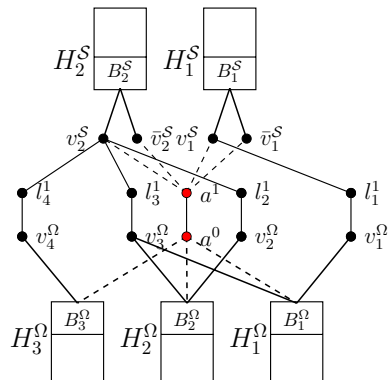
- For each $j \in \llbracket m \rrbracket$, we add two non-adjacent vertices v_j^S and \bar{v}_j^S . They are B_k^S -adjacent for each $k \in \llbracket b_m \rrbracket$ such that the k^{th} bit of the binary representation of j is 1.
- For each $i \in \llbracket n \rrbracket$, we add r vertices denoted l_i^{j-1} with $j \in \llbracket r \rrbracket$ to form a fresh path such that $d(v_i^{\Omega}, l_i^{j-1}) = j - 1$ where $v_i^{\Omega} = l_i^0$. We make v_i^{Ω} B_k^Ω -adjacent for each $k \in \llbracket b_{n+1} \rrbracket$ such that the k^{th} bit of the binary representation of i is 1.
- For each $S_j \in \mathcal{S}$ and each $u_i \in S_j$, we add the edge (l_i^{r-1}, v_j^S) .
- We add r vertices denoted a^{j-1} with $j \in \llbracket r \rrbracket$ to form the induced path such that $d(a^0, a^{j-1}) = j - 1$. The vertex a^0 is B_k^Ω -adjacent for every $k \in \llbracket b_{n+1} \rrbracket$, and we add the edges (a^{r-1}, v_j^S) and (a^{r-1}, \bar{v}_j^S) for each $j \in \llbracket m \rrbracket$.

By definition of b_{n+1} , for every $i \in \llbracket n \rrbracket$, one of the last b_{n+1} bits of the binary representation of i is 0. So, a^0 has a distinct characterization in the power set formed by the gadgets H_i^Ω . See Fig. 3 for an example of (H, B, r) -compressed graph.

Proposition 3 *The graph $G = \Psi[H, B, r](\Omega, \mathcal{S})$ built on an instance (Ω, \mathcal{S}) of HITTING SET:*

- is connected and has size at most $|H|(b_{n+1} + b_m) + r(n + 1) + 2m$, where $|\Omega| = n$ and $|\mathcal{S}| = m$,
- may be built in polynomial time in its size, and
- is bipartite if the B -single extension of H is bipartite.

Fig. 3 The $(H, B, 2)$ -compressed graph where $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1, 2\}, \{2, 3, 4\}\}$



Proof The graph G is formed by the union of $b_{n+1} + b_m$ copies of H , one vertex per variable, two vertices per clause, n paths of $r - 1$ vertices and one path of size r . Thus, in total, the number of vertices is

$$(b_{n+1} + b_m)|H| + n + 2m + n(r - 1) + r = |H|(b_{n+1} + b_m) + r(n + 1) + 2m$$

Finally, the two last items of the proposition are also direct by construction. □

Theorem 8 *Let (Ω, \mathcal{S}) be an instance of HITTING SET such that $|\Omega| = n, |\mathcal{S}| = m$. Let (H, B, C) be an (f, r) -gadget for a 1-layered identifying function f and let (H', B', C') be a local (g, q) -gadget. The following propositions are equivalent:*

- *There exists a hitting set of \mathcal{S} of size k .*
- *There exists an (f, r) -distance identifying set of the graph $\Psi[H, B, 1](\Omega, \mathcal{S})$ which size is $k + |C|(b_{n+1} + b_m)$.*
- *There exists a (g, q) -distance identifying set of the graph $\Psi[H', B', q](\Omega, \mathcal{S})$ which size is $k + |C'|(b_{n+1} + b_m)$.*

Proof Suppose again that P is a hitting set of (Ω, \mathcal{S}) of size k . By denoting C_i^Ω and C_j^S the copy of C (respectively C') associated to the copy H_i^Ω and H_j^S of H (respectively H'), we suggest the following set I of size $k + |C|(b_{n+1} + b_m)$ as an (f, r) -distance identifying set of $G = \Psi[H, B, 1](\Omega, \mathcal{S})$ (respectively (g, q) -distance identifying set of $G' = \Psi[H', B', q](\Omega, \mathcal{S})$):

$$I = \{v_i^\Omega : u_i \in P\} \cup \bigcup_{i \in \llbracket b_{n+1} \rrbracket} C_i^\Omega \cup \bigcup_{j \in \llbracket b_m \rrbracket} C_j^S.$$

By construction, G is a B_i^Ω -extension of H_i^Ω (respectively B_j^S -extension of H_j^S) for any $i \in \llbracket b_{n+1} \rrbracket$ (respectively $j \in \llbracket b_m \rrbracket$). This directly implies that I is an r -dominating set of G (respectively q -dominating set G').

We only have to show that I f -distinguishes G (respectively g -distinguishes G'). Distinguishing the vertices of the gadget copies is still easy using the first item of Definition 9. Thereby, we only have to study the vertices of the form $v_j^S, \bar{v}_j^S, l_i^k$, and a^k . To distinguish them, we mainly use the second item of Definition 9 together with the characteristic function of the power set of the gadgets. We deduce that every couple of vertices is distinguished except when the two vertices are of the form v_j^S or \bar{v}_j^S for $j \in \llbracket m \rrbracket$ (or if they are both of the form a^k or l_i^k for $k \in \llbracket q - 1 \rrbracket$ for G' , G not containing such vertices).

Let us now distinguish v_j^S and \bar{v}_j^S for $j \in \llbracket m \rrbracket$. To do so, we use the fact that P is a hitting set for (Ω, \mathcal{S}) . By definition of a hitting set, for any set $S_j \in \mathcal{S}$, there exists a vertex $u_i \in P$ such that $u_i \in S_j$. We observe that $d(v_i^\Omega, v_j^S) = 1 < d(v_i^\Omega, \bar{v}_j^S)$ by construction of G (respectively $d(v_i^\Omega, v_j^S) = q < d(v_i^\Omega, \bar{v}_j^S)$ by construction of G') and that $v_i^\Omega \in I$ by definition of I . Since f is 1-layered (respectively g is q -local), I f -distinguishes and g -distinguishes v_j^S and \bar{v}_j^S .

For G' , it remains to distinguish a^k and l_i^k for $k \in \llbracket q - 1 \rrbracket$ and $i \in \llbracket n \rrbracket$. We recall that in a (g, q) -local gadget (H', B', C') , there exists $c \in C'$ such that $d(c, B') = k - 1$

for each $k \in \llbracket q \rrbracket$. Then we may use the characteristic function of the power set together with property (β_1) of a q -local function to distinguish them.

In the other direction, assume that I is an (f, r) -distance identifying set of G of size $k + |C|(b_{n+1} + b_m)$ (respectively (g, q) -distance identifying set of G of size $k + |C'|(b_{n+1} + b_m)$). As every set of \mathcal{S} is not empty, we may define a function $\varphi : \llbracket m \rrbracket \rightarrow \llbracket n \rrbracket$ such that $u_{\varphi(j)} \in S_j$.

We suggest the following set P as a hitting set of \mathcal{S} of size at most k :

$$P = \{u_i \in \Omega \mid l_i^k \in I \text{ for any } k \in \llbracket r - 1 \rrbracket\} \cup \{u_{\varphi(j)} \in \Omega \mid v_j^S \in I \text{ or } \bar{v}_j^S \in I\}$$

The size of P is ensured by the fourth item of Definition 9 of a gadget.

We claim that the only vertices that may f -distinguish the couple (v_j^S, \bar{v}_j^S) are themselves and the vertices of form l_i^k such that $u_i \in S_j$. To prove so, we apply property (α) from Definition 2 (a vertex cannot distinguish two vertices at the same distance from it) on the following enumeration on G :

- The vertices a^k verify $d(a^k, v_j^S) = r - k = d(a^k, \bar{v}_j^S)$ for each $k \in \llbracket r - 1 \rrbracket$.
- A vertex l_i^k such that $u_i \notin S_j$ verifies $d(l_i^k, v_j^S) = 2 + r - k = d(l_i^k, \bar{v}_j^S)$ because of a^{r-1} .
- A vertex v of H_i^Ω verifies $d(v, v_j^S) = d(v, B_i^\Omega) + 1 + r = d(v, \bar{v}_j^S)$ because of the path formed by the vertices of form a^k .
- A vertex v of $H_{j'}^S$ with $j \neq j'$ verifies $d(v, v_j^S) = d(v, B_{j'}^S) + 3 = d(v, \bar{v}_j^S)$ because of a^{r-1} .
- Both v_j^S and \bar{v}_j^S are B_j^S -adjacent, so they are at the same distance of any vertex of H_j^S .

The enumeration on G' is identical when replacing r by q . We deduce that v_j^S and \bar{v}_j^S are f -distinguished if and only if either one on them belongs to I (in that case $u_{\varphi(j)} \in P \cap S_j$) or if there exists $l_i^k \in I$ such that $u_i \in S_j$ and $k + 1 \in \llbracket r \rrbracket$ (and then $u_i \in P \cap S_j$). □

4 On Providing Gadgets to Establish Generic Reductions

In this section, we finalize the reductions by furnishing some gadgets and combining them with the suitable theorems and propositions from Sect. 3. The existence of the gadgets rely on the following tool lemma:

Lemma 1 (Twins Lemma) *Let x and y be two vertices of a graph G such that $N(x) = N(y)$. Then any distance identifying set of G contains either x or y .*

Proof Because $N(x) = N(y)$, for every vertex u of G , if $u \notin \{x, y\}$, then $d(u, x) = d(u, y)$. Thus, by property (α) of a distance identifying set, u may distinguish x and y if and only if $u \in \{x, y\}$, implying that a distance identifying set must contain either x or y . □

The gadgets are defined as follows:

Definition 12 (*The 1-layered gadget*) Let H be the bipartite planar graph such that:

- Its ten vertices are denoted $b, \bar{b}, u_1, \bar{u}_1, u_2, \bar{u}_2, v_1, \bar{v}_1, v_2$ and \bar{v}_2 .
- The vertices u_1, u_2, \bar{u}_1 and \bar{u}_2 form a cycle as well as the vertices v_1, v_2, \bar{v}_1 and \bar{v}_2 .
- The vertices b and \bar{b} are adjacent to u_1, \bar{u}_1, v_1 and \bar{v}_1 .

We define the sets $B = \{b, \bar{b}\}$ and $C = \{b, u_1, u_2, v_1, v_2\}$.

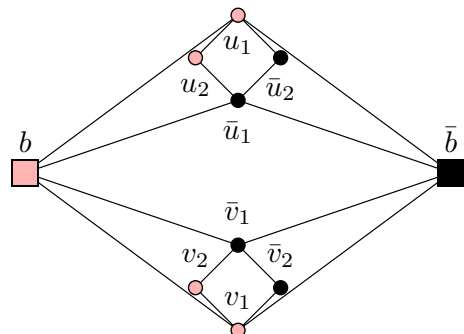
The triple (H, B, C) is called *the 1-layered gadget* (see Fig. 4).

Proposition 4 *The 1-layered gadget is a bipartite planar (f, r) -gadget for any 1-layered distance identifying function f and $r \in \llbracket \infty \rrbracket$.*

Proof We have to check the four conditions to be an (f, r) -gadget. Consider a B -extension G of H . Clearly, (p_d) is satisfied as C is even a 1-dominating set of $V_H \cup N_B$. The condition (p_s) is also easily verified using the Twins Lemma 1 on the distinct pairs $(b, \bar{b}), (u_1, \bar{u}_1), (u_2, \bar{u}_2), (v_1, \bar{v}_1)$ and (v_2, \bar{v}_2) . To prove (p_h) and (p_b) , we only have to study vertices not belonging to C . Remark that:

- The vertex \bar{u}_1 is the only one outside of C that is adjacent to u_2 .
- The vertex \bar{v}_1 is the only one outside of C that is adjacent to v_2
- The vertex \bar{u}_2 is the only one outside of C that is adjacent to u_1 and not adjacent to v_1 .
- The vertex \bar{v}_2 is the only one outside of C that is adjacent to v_1 and not adjacent to u_1 .
- The vertex \bar{b} is the only one outside C both adjacent to u_1 and v_1
- A B -adjacent vertex is not adjacent to u_1 nor to v_1 but is adjacent to b
- Finally, a vertex from $V_{G \setminus H}$ which is not B -adjacent is neither adjacent to u_1, v_1 nor b .

Fig. 4 The 1-layered gadget (H, B, C) . The set C contains the colored vertices and the set B contains the square vertices



Therefore properties (p_h) and (p_b) are satisfied by (H, B, C) which is a 1-layered gadget for f . □

Definition 13 (*r*-local gadget) Given a positive integer r , let H_r be the bipartite planar graph of size $(2r + 2)^2$ such that:

- Its vertices are denoted u_i^j and d_i^j for $i \in \llbracket 2r + 2 \rrbracket$ and $j \in \llbracket r + 1 \rrbracket$.
- For $i \in \llbracket 2r + 2 \rrbracket$ and $j \in \llbracket r + 1 \rrbracket$, we define $U_i = \{u_i^k \mid k \in \llbracket r + 1 \rrbracket\}$, $U^j = \{u_k^j \mid k \in \llbracket 2r + 2 \rrbracket\}$, $D_i = \{d_i^k \mid k \in \llbracket r + 1 \rrbracket\}$ and $D^j = \{d_k^j \mid k \in \llbracket 2r + 2 \rrbracket\}$. Furthermore, as the vertices u_i^j (resp. d_i^j) will form a cycle, we consider in the following that if $i \notin \llbracket 2r + 2 \rrbracket$, $u_i^1 = u_{1+(i-1) \pmod{2r+2}}^1$ (resp. $d_i^1 = d_{1+(i-1) \pmod{2r+2}}^1$).
- For each $i \in \llbracket 2r + 2 \rrbracket$, the vertex u_i^1 is adjacent to d_i^1 , the vertices of U_i form a path such that $d(u_i^1, u_i^j) = j - 1$ for each $j \in \llbracket r + 1 \rrbracket$ and the vertices of D_i form a path such that $d(d_i^1, d_i^j) = j - 1$ for each $j \in \llbracket r + 1 \rrbracket$.
- The vertices of U^1 form a cycle such that u_i^1 is adjacent to u_{i+1}^1 for each $i \in \llbracket 2r + 2 \rrbracket$, and the vertices of D^1 form a cycle such that d_i^1 is adjacent to d_{i+1}^1 for each $i \in \llbracket 2r + 2 \rrbracket$.

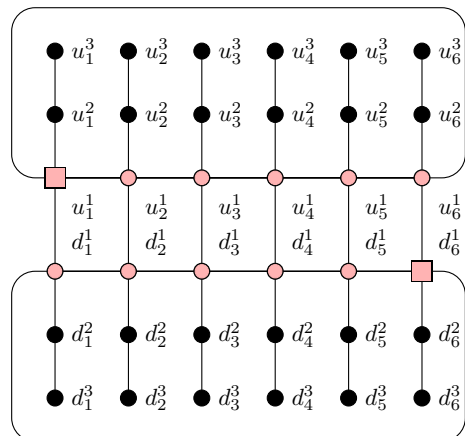
We define the sets $B_r = \{u_1^1, d_{2r+2}^1\}$ and $C_r = U^1 \cup D^1$.

The triple (H_r, B_r, C_r) is called *the r-local gadget* (see Fig. 5).

Proposition 5 *Let f be an r-local distance identifying function. Then the r-local gadget is an (f, r) -local gadget.*

Proof By construction, for each $i \in \llbracket 2r + 2 \rrbracket$, the r -neighborhood of u_i^{r+1} is U_i . An r -local distance identifying set must contain a vertex of U_i to r -dominate u_i^{r+1} . Symmetrically, it must contain a vertex of D_i to r -dominate d_i^{r+1} . Therefore, C_r has optimal size and (p_s) is verified. The condition (p_d) is trivially verified, and the condition (p_r) is also easily verified because of the vertices u_i^1 for $i \in \llbracket r + 1 \rrbracket$.

Fig. 5 The 2-local gadget. The set C_r contains the red and B_r contains the square vertices



We have to prove (p_h) and (p_b) . We first focus on vertices of H_r . We repeatedly use the property (β_1) of Definition 4 when two vertices present different intersections between C_r and their respective r -neighborhood. For $j \in \llbracket r + 1 \rrbracket$, let $uU_r(j) = 2r + 3 - 2j$ be the number of r -neighbors of u_i^j (with $i \in \llbracket 2r + 2 \rrbracket$) belonging to the line U^1 of the code. Similarly, let $dU_r(j) = \max(0, uU_r(j) - 2)$ be the number of r -neighbors of d_i^j belonging to U^1 , let $uD_r(j) = dU_r(j)$ be the number of r -neighbors of u_i^j belonging to D^1 and let $dD_r(j) = uU_r(j)$ be the number of r -neighbors of d_i^j belonging to D^1 .

As $uU_r > uD_r$ while $dU_r < dD_r$, C_r distinguishes u_i^j and $d_{i'}^{j'}$ for any relevant i, j, i' and j' . Now, as the graph is symmetrical, we only have to consider vertices of form u_i^j .

For integers j and j' such that $0 < j < j' < r + 2$, we have $uU_r(j) > uU_r(j')$, and therefore C_r distinguishes u_i^j and $u_{i'}^{j'}$, for any relevant i, i', j and j' such that $j \neq j'$.

It remains to distinguish u_i^j and $u_{i'}^j$ for $i \neq i'$. A vertex u_i^j is adjacent to every u_k^1 such that $i - r - 1 + j \leq k \leq i + r + 1 - j$, an interval of size $uU_r(j) < 2r + 2$. We deduce that the r -neighbors of u_i^j belonging to U^1 and the r -neighbors of $u_{i'}^j$ belonging to U^1 are different since they can be represented by two non-empty and non-covering intervals shifted by $i - i'$ (and $0 < i - i' < 2r + 2$).

We now have to consider the B_r -extensions of H_r . Let the graph G of vertex set V_G be a B_r -extension of the graph H_r of vertex set V_H , and let N_B be the set of B_r -adjacent vertices. Note that u_1^1 was already at distance 2 of d_{2r+2}^1 so the B_r -adjacent vertices do not introduce any short cut. To distinguish $u \in V_H$ and $v \in V_{G \setminus H}$, it is sufficient to observe that by symmetry of the graph H_r , the number of r -neighbors of v belonging to U^1 is equal to the number of r -neighbors of v belonging to D^1 , which is not the case for u (recall functions uU_r and uD_r , and functions dU_r and dD_r). Finally, to distinguish $u \in N_B$ and $v \in V_{G \setminus H} \setminus N_B$, we just have to use the guaranteed vertex $c \in C_r$ such that $d(c, B_r) = r - 1$ and property (β_1) . This concludes the proof of the conditions (p_h) and (p_b) .

We lastly observe that because H_r is a bipartite planar graph, the r -local gadget is planar since $|B_r| = 2$, and it is bipartite since $d(u_1^1, d_{2r+2}^1) = 2$. □

With Propositions 4 to 5, we can now prove the Theorems 4 to 6.

Proof (Theorems 4 and 6 for each 1-layered identifying function f and $r \in \llbracket \infty \rrbracket$).

We first suggest a reduction from PLANAR HITTING SET to (f, r) -DIS based on the bipartite planar 1-layered gadget (H, B, C) . Let (Ω, \mathcal{S}) be an instance of PLANAR HITTING SET with $|\Omega| = n$ and $|\mathcal{S}| = m$ such that $m = \mathcal{O}(n)$. According to Proposition 2, the bipartite apex graph $G = \Phi^*[H, B](\Omega, \mathcal{S})$ has size n' linear in $n + m = \mathcal{O}(n)$ and may be built in polynomial-time in its size. Recall that (H, B, C) is an (f, r) -gadget by Proposition 4. By Theorem 7, G admits an (f, r) -distance identifying set of size $k' = k + |C|(n + m)$ if and only if \mathcal{S} admits a hitting set of size k . Thus, an algorithm solving (f, r) -DIS in $2^{o(\sqrt{n'})}$ would solve PLANAR HITTING SET in time $2^{o(\sqrt{n})}$, a contradiction to Theorem 3 (assuming ETH).

We adapt the previous argumentation to get a reduction from HITTING SET to (f, r) -DIS, the instance (Ω, \mathcal{S}) belonging now to the HITTING SET problem. According

to Proposition 3, the bipartite graph $G = \Psi[H, B, 1](\Omega, \mathcal{S})$ has size n' linear in $n + m = \mathcal{O}(n)$ and may also be built in polynomial-time in its size. By Theorem 8, G admits an (f, r) -distance identifying set of size $k' = k + |C|(b_{n+1} + b_m)$ if and only if \mathcal{S} admits a hitting set of size k . Thus, an algorithm solving (f, r) -DIS in $2^{\mathcal{O}(n')}$ would solve HITTING SET in time $2^{\mathcal{O}(n)}$, contradicting Theorem 2 when assuming ETH. Moreover, a parameterized algorithm solving (f, r) -DIS in $2^{\mathcal{O}(k)} \cdot n'^{\mathcal{O}(1)}$ would be in contradiction with Theorem 1 when assuming $W[2] \neq \text{FPT}$. \square

Proof (Theorems 5 and 6 for each r -local identifying function f) First, we suggest a reduction from PLANAR HITTING SET to (f, r) -DIS based on the bipartite planar local (f, r) -gadget (H, B, C) . Let (Ω, \mathcal{S}) be an instance of PLANAR HITTING SET with $|\Omega| = n$ and $|\mathcal{S}| = m$ such that $m = \mathcal{O}(n)$. According to Proposition 2, the (bipartite) planar graph $G = \Phi[H, B, r](\Omega, \mathcal{S})$ has size n' linear in $n + m = \mathcal{O}(n)$ and may be built in polynomial-time in its size. By Theorem 7, G admits an (f, r) -distance identifying set of size $k' = k + |C|(n + m)$ if and only if \mathcal{S} admits a hitting set of size k . Thus, an algorithm solving (f, r) -DIS in $2^{\mathcal{O}(\sqrt{n'})}$ would solve PLANAR HITTING SET in time $2^{\mathcal{O}(\sqrt{n})}$, a contradiction to Theorem 3 (assuming ETH).

We adapt the previous argumentation to get a reduction from HITTING SET to (f, r) -DIS, the instance (Ω, \mathcal{S}) belonging now to the HITTING SET problem. In this case, we only have to assume the existence of a (bipartite) local (f, r) -gadget (H, B, C) . According to Proposition 3, the bipartite graph $G = \Psi[H, B, r](\Omega, \mathcal{S})$ has size n' linear in $n + m = \mathcal{O}(n)$ and may also be built in polynomial-time in its size. By Theorem 8, G admits an (f, r) -distance identifying set of size $k' = k + |C|(b_{n+1} + b_m)$ if and only if \mathcal{S} admits a hitting set of size k . Thus, an algorithm solving (f, r) -DIS in $2^{\mathcal{O}(n')}$ would solve HITTING SET in time $2^{\mathcal{O}(n)}$, contradicting Theorem 2 when assuming ETH. Moreover, a parameterized algorithm solving (f, r) -DIS in $2^{\mathcal{O}(k)} \cdot n'^{\mathcal{O}(1)}$ would be in contradiction with Theorem 1 when assuming $W[2] \neq \text{FPT}$. \square

5 Conclusion

In this paper, we showed generic tools to analyze identifying problems and their computational lower bounds. Questions about upper bounds naturally arise. While the general $2^{\mathcal{O}(n)}$ lower bound under ETH is clearly tight, because the naive algorithm consisting in enumerating all the subsets of vertices has a complexity of $2^{\mathcal{O}(n)}$, it is unknown whether the planar/apex $2^{\mathcal{O}(\sqrt{n})}$ lower bound is tight. Furthermore, there is still a gap between the computational lower bound provided by Theorem 6 and the elementary upper bound from Proposition 1 in the local cases. We wonder if local problems may be solved in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. Notice that a polynomial kernel would imply such a complexity (but the converse is not true). For non-local problems, an FPT upper bound is globally unknown. In particular, $W[2]$ -hard problems like MD cannot admit FPT algorithms unless $W[2] = \text{FPT}$. Then, which non-local problem is $W[2]$ -hard? We mention that we actually get an FPT reduction from HITTING SET to some scarce non-local problems (however including MD) proving their $W[2]$

-hardness, but the family of involved problems is not precise nor wide. Nevertheless, we remark that most of our reductions may be generalized to the oriented version of DISTANCE IDENTIFYING SET sometimes even for strongly connected graphs—this is due to the fact that the paths in our distance identifying graphs and gadgets may often be seen as oriented—.

Acknowledgements We want to thank Stéphane Bessy and Anaël Grandjean for their precious advice concerning the correctness of the claims, and the structure of the article.

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