

On the Value of Job Migration in Online Makespan Minimization

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Abstract Makespan minimization on identical parallel machines is a classical scheduling problem. We consider the online scenario where a sequence of n jobs has to be scheduled non-preemptively on m machines so as to minimize the maximum completion time of any job. The best competitive ratio that can be achieved by deterministic online algorithms is in the range $[1.88, 1.9201]$. Currently no randomized online algorithm with a smaller competitiveness is known, for general m . In this paper we explore the power of job migration, i.e. an online scheduler is allowed to perform a limited number of job reassignments. Migration is a common technique used in theory and practice to balance load in parallel processing environments. As our main result we settle the performance that can be achieved by deterministic online algorithms. We develop an algorithm that is α_m -competitive, for any $m \geq 2$, where α_m is the solution of a certain equation. For $m = 2$, $\alpha_2 = 4/3$ and $\lim_{m \rightarrow \infty} \alpha_m = W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2)) \approx 1.4659$. Here W_{-1} is the lower branch of the Lambert W function. For $m \geq 11$, the algorithm uses at most $7m$ migration operations. For smaller m , $8m$ to $10m$ operations may be performed. We complement this result by a matching lower bound: No online algorithm that uses $o(n)$ job migrations can achieve a competitive ratio smaller than α_m . We finally trade

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performance for migrations. We give a family of algorithms that is c -competitive, for any $5/3 \leq c \leq 2$. For $c = 5/3$, the strategy uses at most $4m$ job migrations. For $c = 1.75$, at most $2.5m$ migrations are used.

Keywords Scheduling · Makespan minimization · Online algorithm · Competitive analysis · Job migration

1 Introduction

Makespan minimization on identical machines is a fundamental scheduling problem that has received considerable research interest over the last forty years. Let $\sigma = J_1, \dots, J_n$ be a sequence of jobs that has to be scheduled non-preemptively on m identical parallel machines. Each job J_i is specified by a processing time p_i , $1 \leq i \leq n$. The goal is to minimize the makespan, i.e. the maximum completion time of any job in a schedule. In the offline setting all jobs are known in advance. In the online setting the jobs arrive one by one. Each job J_i has to be scheduled immediately on one of the machines without knowledge of any future jobs J_k , $k > i$. An online algorithm A is called c -competitive if, for any job sequence, the makespan of A 's schedule is at most c times the optimum makespan for that sequence [23].

Early work on makespan minimization studied the offline setting. Already in 1966, Graham [12] presented the *List* scheduling algorithm that schedules each job on a least loaded machine. *List* can be used both as an offline and as an online strategy and achieves a performance ratio of $2 - 1/m$. Hochbaum and Shmoys devised a famous polynomial time approximation scheme [15]. More recent research, published mostly in the 1990s, investigated the online setting. The best competitive factor that can be attained by deterministic online algorithms is in the range [1.88, 1.9201]. Due to this relatively high factor, compared to *List*'s ratio of $2 - 1/m$, it is interesting to consider scenarios where an online scheduler has more flexibility to serve the job sequence.

In this paper we investigate the impact of job migration. At any time an online algorithm may perform *reassignments*, i.e. a job already scheduled on a machine may be removed and transferred to another machine. Process migration is a well-known and widely used technique to balance load in parallel and distributed systems. It leads to improved processor utilization and reduced processing delays. Migration policies have been analyzed extensively in theory and practice.

It is natural to investigate makespan minimization with job migration. In this paper we present a comprehensive study and develop tight upper and lower bounds on the competitive ratio that can be achieved by deterministic online algorithms. It shows that even with a very limited number of migration operations, significantly improved performance guarantees are obtained.

Previous work We review the most important results relevant to our work. As mentioned above, *List* is $(2 - 1/m)$ -competitive. Deterministic online algorithms with a smaller competitive ratio were presented in [2, 4, 10, 11, 16]. The best algorithm currently known is 1.9201-competitive [10]. Lower bounds on the performance of deterministic strategies were given in [2, 3, 9, 14, 18, 19]. The best bound currently known is 1.88, for general m , see [18]. Randomized online algorithms cannot achieve

a competitive ratio smaller than $e/(e-1) \approx 1.58$ [6,21]. No randomized algorithm whose competitive ratio is provably below the deterministic lower bound is currently known, for general m . If job preemption is allowed, the best competitiveness of online strategies is equal to $e/(e-1) \approx 1.58$ [7].

Makespan minimization with job migration was first addressed by Aggarwal et al. [1]. They consider an offline setting. An algorithm is given a schedule, in which all jobs are already assigned, and a budget. The algorithm may perform job migrations up to the given budget. The authors design strategies that perform well with respect to the best possible solution that can be constructed with the budget. Online makespan minimization with migration on $m = 2$ machines was considered in [17,22]. The best competitiveness is $4/3$. Sanders et al. [20] study an online setting in which before the assignment of each job J_i , jobs up to a total processing volume of βp_i may be migrated, for some constant β . For $\beta = 4/3$, they present a 1.5-competitive algorithm. They also show a $(1 + \epsilon)$ -competitive algorithm, for any $\epsilon > 0$, where β depends exponentially on $1/\epsilon$. The algorithms are robust in that the stated competitive ratios hold after each job assignment. However in this framework, over time, $\Omega(n)$ migrations may be performed and jobs of total processing volume $\beta \sum_{i=1}^n p_i$ may be moved.

Englert et al. [8] study online makespan minimization if an algorithm is given a buffer that may be used to partially reorder the job sequence. In each step an algorithm assigns one job from the buffer to the machines. Then the next job in σ is admitted to the buffer. Englert et al. show that, using a buffer of size $\Theta(m)$, the best competitive ratio is $W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2))$, where W_{-1} is the Lambert W function.

Our contribution We investigate online makespan minimization with limited migration. The number of job reassignments does not depend on the length of the job sequence. We determine the exact competitiveness achieved by deterministic algorithms, for general m .

In Sect. 2 we develop an optimal algorithm. For any $m \geq 2$, the strategy is α_m -competitive, where α_m is the solution of an equation representing load in an ideal machine profile for a subset of the jobs. For $m = 2$, the competitive ratio is $4/3$. The ratios are non-decreasing and converge to $W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2)) \approx 1.4659$ as m tends to infinity. Again, W_{-1} is the lower branch of the Lambert W function. The algorithm uses at most $(\lceil (2 - \alpha_m)/(\alpha_m - 1)^2 \rceil + 4)m$ job migrations. For $m \geq 11$, this expression is at most $7m$. For smaller machine numbers it is $8m$ to $10m$. We note that the competitiveness of 1.4659 is considerably below the factor of roughly 1.9 obtained by deterministic algorithms in the standard online setting. It is also below the ratio of $e/(e-1)$ attainable if randomization or job preemption are allowed.

In Sect. 3 we give a matching lower bound. We show that no deterministic algorithm that uses $o(n)$ job migrations can achieve a competitive ratio smaller than α_m , for any $m \geq 2$. Hence in order to beat the factor of α_m , $\Theta(n)$ reassignments are required. Finally, in Sect. 4 we trade migrations for performance. We develop a family of algorithms that is c -competitive, for any constant c with $5/3 \leq c \leq 2$. Setting $c = 5/3$ we obtain a strategy that uses at most $4m$ job migrations. For $c = 1.75$, the strategy uses no more than $2.5m$ migrations.

Our algorithms rely on a number of new ideas. All strategies classify incoming jobs into small and large depending on a careful estimate on the optimum makespan. The algorithms consist of a job arrival phase followed by a migration phase. The optimal

algorithm, in the arrival phase, maintains a load profile on the machines with respect to jobs that are currently small. In the migration phase, the algorithm removes a certain number of jobs from each machine. These jobs are then rescheduled using strategies by Graham [12, 13]. Our family of algorithms partitions the m machines into two sets A and B . In the arrival phase the algorithms prefer to place jobs on machines in A so that machines in B are available for later migration. In general, the main challenge in the analyses of the various algorithms is to bound the number of jobs that have to be migrated from each machine.

We finally relate our contributions to some existing results. First we point out that the goal in online makespan minimization is to construct a good schedule when jobs arrive one by one. Once the schedule is constructed, the processing of the jobs may start. It is not stipulated that machines start executing jobs while other jobs of σ still need to be scheduled. This framework is assumed in all the literature on online makespan minimization mentioned above. Consequently it is no drawback to perform job migrations when the entire job sequence has arrived. Nonetheless, as for the algorithms presented in this paper, the machines can start processing jobs except for the up to 10 largest jobs on each machine. A second remark is that the algorithms by Aggarwal et al. [1] cannot be used to achieve good results in the online setting. The reason is that those strategies are designed to perform well relative to the best possible makespan attainable from an initial schedule using a given migration budget. The strategies need not perform well compared to a globally optimal schedule. The algorithms by Aggarwal et al. and ours are different, see [1].

On the other hand, our results exhibit similarities to those by Englert et al. [8] where a reordering buffer is given. The optimal competitive ratio of α_m is the solution of an equation that also arises in [8]. This is due to the fact that our optimal algorithm and that in [8] maintain a certain load profile on the machines. Our strategy does so w.r.t. jobs that are currently small while the strategy in [8] considers all jobs assigned to machines. In our framework the profile is harder to maintain because of *shrinking jobs*, i.e. jobs that are large at some time t but small at later times $t' > t$. In the job migration phase our algorithm reschedules jobs removed from some machines. This operation corresponds to the "final phase" of the algorithm in [8]. However, our algorithm directly applies policies by Graham [12, 13] while the algorithm in [8] computes a virtual schedule.

In general, an interesting question is if makespan minimization with limited migration is equivalent to makespan minimization with a bounded reordering buffer. We cannot prove this in the affirmative. As for the specific algorithms presented in [8] and in this paper, the following relation holds. All our algorithms can be transformed into strategies with a reordering buffer. The competitive ratios are preserved and the number of job migrations is equal to the buffer size. This transformation is possible because our algorithms are *monotone*: If a job does not have to be migrated at time t , assuming σ ended at time t , then there is no need to migrate it at times $t' > t$. Hence, at any time a buffer can store the candidate jobs to be migrated. On the other hand, to the best of our knowledge, the algorithms by Englert et al. [8] do not translate into strategies with job migration. All the algorithms in [8] use the given buffer of size cm , for some constant c , to store the cm largest jobs of the job sequence. However in our setting, a migration of the largest jobs does not generate good schedules. The problem

are again shrinking jobs, i.e. jobs that are among the largest jobs at some time t but not at later times. We cannot afford to migrate all shrinking jobs, unless we invest $\Theta(n)$ migrations. With limited job migration, scheduling decisions are final for almost all of the jobs. Hence the corresponding algorithms are more involved than in the setting with a reordering buffer.

2 An Optimal Algorithm

For the description of the algorithm and the attained competitive ratio we define a function $f_m(\alpha)$. Intuitively, $f_m(\alpha)$ represents accumulated normalized load in a “perfect” machine profile for a subset of the jobs. In such a profile the load fractions of the first $\lfloor m/\alpha \rfloor$ machines follow a Harmonic series of the form $(\alpha - 1)/(m - 1), \dots, (\alpha - 1)/(m - \lfloor m/\alpha \rfloor)$ while the remaining ratios are α/m . Summing up these ratios we obtain $f_m(\alpha)$. Formally, let

$$f_m(\alpha) = (\alpha - 1)(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil \alpha/m,$$

for any machine number $m \geq 2$ and real-valued $\alpha > 1$. Here $H_k = \sum_{i=1}^k 1/i$ denotes the k -th Harmonic number, for any integer $k \geq 1$. We set $H_0 = 0$. For any fixed $m \geq 2$, let α_m be the value satisfying $f_m(\alpha) = 1$. Lemma 1 below implies that α_m is well-defined. The algorithm we present is exactly α_m -competitive. By Lemma 2, the values α_m form a non-decreasing sequence. There holds $\alpha_2 = 4/3$ and $\lim_{m \rightarrow \infty} \alpha_m = W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2)) \approx 1.4659$, see also [8]. The following two technical lemmas are proven in the “Appendix”.

Lemma 1 *The function $f_m(\alpha)$ is continuous and strictly increasing in α , for any integer $m \geq 2$ and real number $\alpha > 1$. There holds $f_m(1 + 1/(3m)) < 1$ and $f_m(2) \geq 1$.*

Lemma 2 *The sequence $(\alpha_m)_{m \geq 2}$ is non-decreasing with $\alpha_2 = 4/3$ and $\lim_{m \rightarrow \infty} \alpha_m = W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2))$.*

We point out that α_m is a rational number, for any $m \geq 2$. This follows from the fact that, for any $\alpha > 1$, $H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}$ and $\lceil (1 - 1/\alpha)m \rceil/m$ are rational numbers so that the solution to $f_m(\alpha) = 1$ is a rational number as well.

2.1 Description of the Algorithm

Let $m \geq 2$ and M_1, \dots, M_m be the given machines. Furthermore, let α_m be as defined above. The algorithm, called $ALG(\alpha_m)$, operates in two phases, a *job arrival phase* and a *job migration phase*. In the job arrival phase all jobs of $\sigma = J_1, \dots, J_n$ are assigned one by one to the machines. In this phase no job migrations are performed. Once σ is scheduled, the job migration phase starts. First the algorithm removes some jobs from the machines. Then these jobs are reassigned to other machines.

Job arrival phase In this phase $ALG(\alpha_m)$ classifies jobs into small and large and, moreover, maintains a load profile with respect to the small jobs on the machines. At

any time the load of a machine is the sum of the processing times of the jobs currently assigned to it. Let *time* t be the time when J_t has to be scheduled, i.e. J_t has arrived but not yet been scheduled on any machine, $1 \leq t \leq n$.

In order to classify jobs $ALG(\alpha_m)$ maintains a lower bound L_t on the optimum makespan, assuming that J_t as already been placed in an optimal schedule. Let $p_t^+ = \sum_{i=1}^t p_i$ be the sum of the processing times of the first t jobs. Furthermore, for $i = 1, \dots, 2m + 1$, let p_t^i denote the processing time of the i -th largest job in J_1, \dots, J_t , provided that $i \leq t$. More formally, if $i \leq t$, let p_t^i be the processing time of the i -th largest job; otherwise we set $p_t^i = 0$. Obviously, when t jobs have been scheduled, the optimum makespan cannot be smaller than the average load $\frac{1}{m} p_t^+$ on the m machines. This lower bound was already used in [12]. A second lower bound, used e.g. in [2,4], is that the optimum makespan cannot be smaller than twice the processing time of the $(m + 1)$ -st largest job. We slightly generalize this bound. The optimum makespan is lower bounded by three times the processing time of $(2m + 1)$ -st largest job seen so far, i.e. it is at least $3p_t^{2m+1}$. Define

$$L_t = \max \left\{ \frac{1}{m} p_t^+, 3p_t^{2m+1} \right\}.$$

The estimates L_t are non-decreasing over time. Hence a job that is large relative to L_t might not be large relative to $L_{t'}$, where $t' > t$. Therefore we need a careful notion of small and large. A job J_i , with $i \leq t$, is *small at time* t if $p_i \leq (\alpha_m - 1)L_t$; otherwise it is *large at time* t . In particular J_t is small at time t if $p_t \leq (\alpha_m - 1)L_t$; otherwise J_t is large at time t . We introduce a final piece of notation. In the sequence p_t^1, \dots, p_t^{2m} of the $2m$ largest processing times up to time t we focus on those that are large at time t . More specifically, for $i = 1, \dots, 2m$, let $\hat{p}_t^i = p_t^i$ if $p_t^i > (\alpha_m - 1)L_t$; otherwise let $\hat{p}_t^i = 0$. Define

$$L_t^* = \frac{1}{m} (p_t^+ - \sum_{i=1}^{2m} \hat{p}_t^i).$$

Intuitively, L_t^* is the average machine load ignoring jobs that are large at time t . Note again that this includes job J_t . We verify that there exist at most $2m$ jobs that are large at time t : If there were at least $2m + 1$ such jobs, then $L_t \geq 3p_t^{2m+1} > 3(\alpha_m - 1)L_t \geq L_t$ because $\alpha_m \geq 4/3$, see Lemma 2.

We describe the scheduling steps in the job arrival phase. Initially, the machines are numbered in an arbitrary way and this numbering M_1, \dots, M_m remains fixed throughout the execution of $ALG(\alpha_m)$. As mentioned above the algorithm maintains a load profile on the machines as far as small jobs are concerned. Define

$$\beta(j) = \begin{cases} (\alpha_m - 1) \frac{m}{m-j} & \text{if } j \leq \lfloor m/\alpha_m \rfloor \\ \alpha_m & \text{otherwise.} \end{cases}$$

We observe that $f_m(\alpha_m) = \frac{1}{m} \sum_{j=1}^m \beta(j)$, taking into account that $m - \lfloor m/\alpha_m \rfloor = \lceil (1 - 1/\alpha_m)m \rceil$. For any machine M_j $1 \leq j \leq m$, let $\ell(j, t)$ denote its load at time t before J_t is assigned to a machine. Let $\ell_s(j, t)$ be the load caused by the jobs on M_j that are small at time t ; again this is the load before J_t is scheduled. The algorithm's

Algorithm $ALG(\alpha_m)$:

Job arrival phase. Each J_t , $1 \leq t \leq n$, is scheduled as follows.

- J_t is small at time t : Assign J_t to an M_j with $\ell_s(j, t) \leq \beta(j)L_t^*$.
- J_t is large at time t : Assign J_t to a least loaded machine.

Job migration phase.

- Job removal: Set $R := \emptyset$. While there exists an M_j with $\ell(j) > \max\{\beta(j)L^*, (\alpha - 1)L\}$, remove the largest job from M_j and add it to R .
- Job reassignment: $R' = \{J_i \in R \mid p_i > (\alpha_m - 1)L\}$. For $i = 1, \dots, m$, set P_i contains J_r^i , if $i \leq |R'|$, and J_r^{2m+1-i} , if $p_r^{2m+1-i} > p_r^i/2$ and $2m + 1 - i \leq |R'|$. Number the sets in order of non-increasing total processing time. For $i = 1, \dots, m$, assign P_i to a least loaded machine. Assign each $J_i \in R \setminus (P_1 \cup \dots \cup P_m)$ to a least loaded machine.

Fig. 1 The algorithm $ALG(\alpha_m)$

classification of small and large jobs ensures that at any time t there exists a machine M_j satisfying $\ell_s(j, t) \leq \beta(j)L_t^*$. Lemma 3 in Sect. 2.2 gives a formal proof.

For $t = 1, \dots, n$, each J_t is scheduled as follows. If J_t is small at its arrival time t , then it is scheduled on a machine with $\ell_s(j, t) \leq \beta(j)L_t^*$. In Lemma 3 we show that such a machine always exists. If J_t is large at its arrival time t , then it is assigned to a machine having the smallest load among all machines. At the end of the phase let $L = L_n$ and $L^* = L_n^*$.

Job migration phase This phase consists of a *job removal step* followed by a *job reassignment step*. At any time during the phase, let $\ell(j)$ denote the current load of M_j , $1 \leq j \leq m$. In the removal step $ALG(\alpha_m)$ maintains a set R of removed jobs. Initially $R = \emptyset$. During the removal step, while there exists a machine M_j whose load $\ell(j)$ exceeds $\max\{\beta(j)L^*, (\alpha_m - 1)L\}$, $ALG(\alpha_m)$ removes the job with the largest processing time currently residing on M_j and adds the job to R .

If $R = \emptyset$ at the end of the removal step, then $ALG(\alpha_m)$ terminates. If $R \neq \emptyset$, then the reassignment step is executed. Let $R' \subseteq R$ be the subset of the jobs that are large at the end of σ , i.e. whose processing time is greater than $(\alpha_m - 1)L$. As shown above, the can exist at most $2m$ such jobs. $ALG(\alpha_m)$ first sorts the jobs of R' in order of non-increasing processing time; ties are broken arbitrarily. Let J_r^i , $1 \leq i \leq |R'|$, be the i -th job in this sorted sequence and p_r^i be its processing time. For $i = 1, \dots, m$, $ALG(\alpha_m)$ forms jobs pairs consisting of the i -th largest and the $(2m + 1 - i)$ -th largest jobs provided that the processing time of the latter job is sufficiently high. A pairing strategy combining the i -th largest and the $(2m + 1 - i)$ -th largest jobs was also used by Graham [13]. Formally, $ALG(\alpha_m)$ builds sets P_1, \dots, P_m that contain up to two jobs. Initially, all these sets are empty. In a first step J_r^i is assigned to P_i , for any i with $1 \leq i \leq \min\{m, |R'|\}$. In a second step J_r^{2m+1-i} is added to P_i provided that $p_r^{2m+1-i} > p_r^i/2$, i.e. the processing time of J_r^{2m+1-i} must be greater than half times that of J_r^i . This second step is executed for any i such that $1 \leq i \leq m$ and $2m + 1 - i \leq |R'|$. For any set P_i , $1 \leq i \leq m$, let π_i be the total summed processing time of the jobs in P_i . $ALG(\alpha_m)$ now renumbers the sets in order of non-increasing π_i values such that $\pi_1 \geq \dots \geq \pi_m$. Then, for $i = 1, \dots, m$, it takes the set P_i and assigns the jobs of P_i to a machine with the smallest current load. If P_i contains two jobs, then both are placed on the same machine. Finally, if $R \setminus (P_1 \cup \dots \cup P_m) \neq \emptyset$, then $ALG(\alpha_m)$ takes care of the remaining jobs. These jobs may be scheduled in an arbitrary order. Each job of $R \setminus (P_1 \cup \dots \cup P_m)$ is scheduled on a machine having

Table 1 The values of α_m and μ_m , for small m

m	2	3	4	5	6	7	8	9	10	11
α_m	$\frac{4}{3}$	$\frac{15}{11}$	$\frac{11}{8}$	$\frac{125}{89}$	$\frac{137}{97}$	$\frac{273}{193}$	$\frac{586}{411}$	$\frac{1863}{1303}$	$\frac{5029}{3517}$	$\frac{58091}{40451}$
\approx		1.3636	1.375	1.4045	1.4124	1.4145	1.4258	1.4298	1.4299	1.4360
μ_m	10	9	9	8	8	8	8	8	8	7

the smallest current load. This concludes the description of $ALG(\alpha_m)$. A summary in pseudo-code is given in Fig. 1.

Theorem 1 $ALG(\alpha_m)$ is α_m -competitive and uses at most $(\lceil(2-\alpha_m)/(\alpha_m-1)\rceil+4)m$ job migrations.

As we shall see in the analysis of $ALG(\alpha_m)$ in the job migration phase the algorithm has to remove at most $\mu_m = \lceil(2-\alpha_m)/(\alpha_m-1)\rceil + 4$ jobs from each machine. Table 1 depicts the competitive ratios α_m (exactly and approximately) and the migration numbers μ_m , for small values of m . We point out that α_m is a rational number, for any $m \geq 2$.

2.2 Analysis of the Algorithm

We first show that the assignment operations in the job arrival phase are well defined. A corresponding statement was shown by Englert et al. [8]. The following proof is more involved because we have to take care of large jobs in the current schedule.

Lemma 3 At any time t there exists a machine M_j satisfying $\ell_s(j, t) \leq \beta(j)L_t^*$.

Proof Suppose that there exists a time t , $1 \leq t \leq n$, such that $\ell_s(j, t) > \beta(j)L_t^*$ holds for all M_j , $1 \leq j \leq m$. We will derive a contradiction.

As argued before, among the jobs J_1, \dots, J_t , at most $2m$ can be large at time t . Hence each of the jobs that is large at time t is represented by a positive entry in the sequence $\hat{p}_t^1, \dots, \hat{p}_t^{2m}$. Conversely, every positive entry in this sequence corresponds to a job that is large at time t and resides on one of the m machines or is equal to J_t if J_t is large at time t . Hence if J_t is large at time t , $\sum_{j=1}^m \ell(j, t) + p_t = \sum_{j=1}^m \ell_s(j, t) + \sum_{i=1}^{2m} \hat{p}_t^i$. If J_t is small at time t , then $\sum_{j=1}^m \ell(j, t) + p_t \geq \sum_{j=1}^m \ell(j, t) = \sum_{j=1}^m \ell_s(j, t) + \sum_{i=1}^{2m} \hat{p}_t^i$. In either case

$$\begin{aligned} \sum_{j=1}^m \ell(j, t) + p_t &\geq \sum_{j=1}^m \ell_s(j, t) + \sum_{i=1}^{2m} \hat{p}_t^i > \sum_{j=1}^m \beta(j)L_t^* + \sum_{i=1}^{2m} \hat{p}_t^i \\ &= m(\alpha_m - 1)L_t^* \sum_{j=1}^{\lfloor m/\alpha_m \rfloor} 1/(m-j) \\ &\quad + (m - \lfloor m/\alpha_m \rfloor)\alpha_m L_t^* + \sum_{i=1}^{2m} \hat{p}_t^i. \end{aligned}$$

Taking into account that $m - \lfloor m/\alpha \rfloor = \lceil (1 - 1/\alpha_m)m \rceil$ and that $f_m(\alpha_m) = 1$, we obtain

$$\begin{aligned} \sum_{j=1}^m \ell(j, t) + p_t &> mL_t^*((\alpha_m - 1)(H_{m-1} - H_{\lceil (1-1/\alpha_m)m \rceil - 1}) \\ &+ \lceil (1 - 1/\alpha_m)m \rceil \alpha_m/m) + \sum_{i=1}^{2m} \hat{p}_t^i \\ &= mL_t^* f_m(\alpha_m) + \sum_{i=1}^{2m} \hat{p}_t^i = m \left(1/m \sum_{i=1}^t p_i - 1/m \sum_{i=1}^{2m} \hat{p}_t^i \right) + \sum_{i=1}^{2m} \hat{p}_t^i = \sum_{i=1}^t p_i. \end{aligned}$$

This contradicts the fact that $\sum_{j=1}^m \ell(j, t) + p_t$ is equal to the total processing time $\sum_{i=1}^t p_i$ of J_1, \dots, J_t . □

We next analyze the job migration phase.

Lemma 4 *In the job removal step $ALG(\alpha_m)$ removes at most $\lceil (2 - \alpha_m)/(\alpha_m - 1)^2 \rceil + 4$ jobs from each of the machines.*

Proof Consider any M_j , with $1 \leq j \leq m$. We show that it suffices to remove at most $\lceil (2 - \alpha_m)/(\alpha_m - 1)^2 \rceil + 4$ so that M_j 's resulting load is upper bounded by $\max\{\beta(j)L^*, (\alpha_m - 1)L\}$. Since $ALG(\alpha_m)$ always removes the largest jobs the lemma follows.

Let time $n + 1$ be the time when the entire job sequence σ is scheduled and the job migration phase with the removal step starts. A job J_i , with $1 \leq i \leq n$, is *small at time* $n + 1$ if $p_i \leq (\alpha_m - 1)L$; otherwise it is *large at time* $n + 1$. Since $L = L_n$ any job that is small (large) at time $n + 1$ is also small (large) at time n . Let $\ell(j, n + 1)$ be the load of M_j at time $n + 1$. Similarly, $\ell_s(j, n + 1)$ is M_j 's load consisting of the jobs that are small at time $n + 1$. Throughout the proof let $k := \lceil (2 - \alpha_m)/(\alpha_m - 1)^2 \rceil$.

First assume $\ell_s(j, n + 1) \leq \beta(j)L^*$. If at time $n + 1$ machine M_j does not contain any jobs that are large at time $n + 1$, then $\ell(j, n + 1) = \ell_s(j, n + 1) \leq \beta(j)L^*$. In this case no job has to be removed and we are done. If M_j does contain jobs that are large at time $n + 1$, then it suffices to remove these jobs. Let time l be the last time when a job J_l that is large at time $n + 1$ was assigned to M_j . Since $L_l \leq L$, J_l was also large at time l and hence it was assigned to a least loaded machine. This implies that prior to the assignment of J_l , M_j has a load of at most $p_l^+/m \leq L_l \leq L$. Hence it could contain at most $1/(\alpha_m - 1)$ jobs that are large at time $n + 1$ because any such job has a processing time greater than $(\alpha_m - 1)L$. Hence at most $1/(\alpha_m - 1) + 1$ jobs have to be removed from M_j , and the latter expression is upper bounded by $k + 4$.

Next assume $\ell_s(j, n + 1) > \beta(j)L^*$. If $\ell_s(j, n) \leq \beta(j)L^* = \beta(j)L_n^*$, then J_n was assigned to M_j . In this case it suffices to remove J_n and, as in the previous case, at most $1/(\alpha_m - 1) + 1$ jobs that are large at time $n + 1$. Again $1/(\alpha_m - 1) + 2 \leq k + 4$.

In the remainder of this proof we consider the case that $\ell_s(j, n + 1) > \beta(j)L^*$ and $\ell_s(j, n) > \beta(j)L_n^*$. Let t^* be the earliest time such that $\ell_s(j, t) > \beta(j)L_t^*$ holds for

all times $t^* \leq t \leq n$. We have $t^* \geq 2$ because $\ell_s(j, 1) = 0 \leq \beta(j)L_1^*$. Hence time $t^* - 1$ exists. We partition the jobs residing on M_j at time $n + 1$ into three sets. Set T_1 is the set of jobs that were assigned to M_j at or before time $t^* - 1$ and are small at time $t^* - 1$. Set T_2 contains the jobs that were assigned to M_j at or before time $t^* - 1$ and are large at time $t^* - 1$. Finally T_3 is the set of jobs assigned to M_j at or after time t^* . We show a number of claims that we will use in the further proof.

- Claim 4.1** Each job in $T_2 \cup T_3$ is large at the time it is assigned to M_j .
- Claim 4.2** There holds $\sum_{J_i \in T_1 \setminus \{J_l\}} p_i \leq \beta(j)L_{t^*-1}^*$, where J_l is the job of T_1 that was assigned last to M_j .
- Claim 4.3** There holds $|T_2| \leq 3$.
- Claim 4.4** For any $J_l \in T_3$, M_j 's load immediately before the assignment of J_l is at most L_l .
- Claim 4.5** Let $J_l \in T_3$ be the last job assigned to M_j . If M_j contains at least k jobs, different from J_l , each having a processing time of at least $(\alpha_m - 1)^2L$, then it suffices to remove these k jobs and J_l such that M_j 's resulting load is upper bounded by $(\alpha_m - 1)L$.
- Claim 4.6** If there exists a $J_l \in T_3$ with $p_l < (\alpha_m - 1)^2L$, then M_j 's load immediately before the assignment of J_l is at most $(\alpha_m - 1)L$.

Proof of Claim 4.1. The jobs of T_2 are large at time $t^* - 1$ and hence at the time they were assigned to M_j . By the definition of t^* , $\ell_s(j, t) > \beta(j)L_t^*$ for any $t^* \leq t \leq n$. Hence at or after time t^* $ALG(\alpha_m)$ does not assign jobs that are small at their arrival time to this machine M_j .

Proof of Claim 4.2. All jobs of $T_1 \setminus \{J_l\}$ are small at time $t^* - 1$ and their total processing time is at most $\ell_s(j, t^* - 1)$. In fact, their total processing time is equal to $\ell_s(j, t^* - 1)$ if $l = t^* - 1$. By the definition of t^* , $\ell_s(j, t^* - 1) \leq \beta(j)L_{t^*-1}^*$.

Proof of Claim 4.3. We show that for any t , $1 \leq t \leq n$, when J_t has been placed on a machine, M_j can contain at most three jobs that are large at time t . The claim then follows by considering $t^* - 1$. Suppose that when J_t has been scheduled, M_j contained more than three jobs that are large as time t . Among these jobs let J_l be the one that was assigned last to M_j . Immediately before the assignment of J_l machine M_j had a load greater than L_l because the total processing time of three large jobs is greater than $3(\alpha_m - 1)L_t \geq 3(\alpha_m - 1)L_l \geq L_l$ since $\alpha_m \geq 4/3$, see Lemma 2. This contradicts the fact that J_l is placed on a least loaded machine, which has a load of at most $p_{l-1}^+ / m < p_l^+ / m \leq L_l$.

Proof of Claim 4.4. By Claim 4.1 J_l is large at time l and hence is assigned to a least loaded machine, which has a load of at most $p_l^+ / m \leq L_l$.

Proof of Claim 4.5. Claim 4.4 implies that immediately before the assignment of J_l machine M_j has a load of at most $L_l \leq L$. If M_j contains at least k jobs, different from J_l , with a processing time of at least $(\alpha_m - 1)^2L$, then the removal of these k jobs and J_l from M_j leads to a machine load of at most $L - k(\alpha_m - 1)^2L \leq L - \lceil (2 - \alpha_m) / (\alpha_m - 1)^2 \rceil (\alpha_m - 1)^2L \leq (\alpha_m - 1)L$, as desired.

Proof of Claim 4.6. By Claim 4.1 J_l is large at time l and hence $p_l > (\alpha_m - 1)L_l$. Since $p_l < (\alpha_m - 1)^2L$, it follows $L_l < (\alpha_m - 1)L$. By Claim 4.4, M_j 's load prior to the assignment of J_l is at most L_l and hence at most $(\alpha_m - 1)L$.

We now finish the proof of the lemma and distinguish two cases depending on the cardinality of $T_2 \cup T_3$.

Case 1: If $|T_2 \cup T_3| < k + 4$, then by Claim 4.2 it suffices to remove the jobs of $T_2 \cup T_3$ and the last job of T_1 assigned to M_j .

Case 2: Suppose $|T_2 \cup T_3| \geq k + 4$. By Claim 4.3, $|T_2| \leq 3$ and hence $|T_3| \geq k + 1$. Among the jobs of T_3 consider the last $k + 1$ ones assigned to M_j . If each of them has a processing time of at least $(\alpha_m - 1)^2L$, then Claim 4.5 ensures that it suffices to remove these $k + 1$ jobs. If one of them, say J_l , has a processing time smaller than $(\alpha_m - 1)^2L$, then by Claim 4.6 M_j 's load prior to the assignment of J_l is at most $(\alpha_m - 1)L$. Again it suffices to remove these $k + 1$ jobs from M_j . \square

After the job removal step each machine M_j , $1 \leq j \leq m$, has a load of at most $\max\{\beta(j)L^*, (\alpha_m - 1)L\}$. We first observe that this load is at most $\alpha_m L$. If $(\alpha_m - 1)L \geq \beta(j)L^*$, there is nothing to show. We evaluate $\beta(j)L^*$. If $j > \lfloor m/\alpha_m \rfloor$, then $\beta(j) = \alpha_m$ and $\beta(j)L^* = \alpha_m L^* \leq \alpha_m L$. If $j \leq \lfloor m/\alpha_m \rfloor$, then $\beta(j) = (\alpha_m - 1)m/(m - j) \leq (\alpha_m - 1)m/(m - \lfloor m/\alpha_m \rfloor) = (\alpha_m - 1)m/\lceil (1 - 1/\alpha_m)m \rceil \leq \alpha_m$ and thus $\beta(j)L^* \leq \alpha_m L$. Hence M_j 's load is upper bounded by $\alpha_m OPT$, where OPT denotes the value of the optimum makespan for the job sequence σ . The following lemma ensures that after the reassignment step, each machine still has a load of at most $\alpha_m OPT$.

Lemma 5 *After the reassignment step each machine M_j , $1 \leq j \leq m$, has a load of at most $\alpha_m OPT$.*

Proof We show that all scheduling operations in the reassignment step preserve a load of at most $\alpha_m OPT$ on each of the machines. We first consider the assignment of the sets P_1, \dots, P_m . Suppose that these sets are already sorted in order of non-increasing total processing times, i.e. $\pi_1 \geq \dots \geq \pi_m$. We first argue that π_1 and hence any π_i , $1 \leq i \leq m$, is upper bounded by OPT . If P_1 contains at most one job, there is nothing to show because OPT cannot be smaller than the processing time of any job in σ . Assume that P_1 contains two jobs. Then it consists of jobs $J_r^{i_1}$ and $J_r^{2m+1-i_1}$, for some i_1 with $1 \leq i_1 \leq m$. Since the two jobs are paired there holds $p_r^{2m+1-i_1} > p_r^{i_1}/2$ and hence $p_r^{2m+1-i_1} > \pi_1/3$. Let OPT' denote the optimum makespan for the job sequence $J_r^1, \dots, J_r^{2m+1-i_1}$. Since $J_r^{i_1}$ and $J_r^{2m+1-i_1}$ are paired, jobs J_r^i and J_r^{2m+1-i} are also paired, for any $i_1 < i \leq m$, because $p_r^{2m+1-i} \geq p_r^{2m+1-i_1} > p_r^{i_1}/2 \geq p_r^i/2$. Hence the sets P_1, \dots, P_m contain all the jobs $J_r^1, \dots, J_r^{2m+1-i_1}$, which implies $\pi_1 \geq OPT'$ and $p_r^{2m+1-i_1} > OPT'/3$. It follows $p_r^i > OPT'/3$, for all i with $1 \leq i \leq 2m + 1 - i_1$. Graham [13] showed that given a sequence of up to $2m$ jobs, each having a processing time greater than a third times the optimum makespan, an optimal schedule is obtained by repeatedly pairing the i -th largest and $(2m + 1 - i)$ -th largest jobs of the sequence. This is exactly the assignment computed by $ALG(\alpha_m)$ for $J_r^1, \dots, J_r^{2m+1-i_1}$. We conclude $\pi_1 = OPT'$ and $\pi_1 \leq OPT$.

A final observation is that each job of R' that is not contained in $P_1 \cup \dots \cup P_m$ has a processing time of at most $OPT/3$. A job in $R' \setminus (P_1 \cup \dots \cup P_m)$ is equal to a

job $J_r^{2m+1-i_0}$, with $1 \leq i_0 \leq m$. Since $J_r^{2m+1-i_0}$ is not paired with $J_r^{i_0}$, there holds $p_r^{2m+1-i_0} \leq p_r^{i_0}/2$. Assume that $p_r^{2m+1-i_0} > OPT/3$. Then $p_r^{2m+1-i_0}$ is greater than a third times the optimum makespan for the jobs $J_r^1, \dots, J_r^{2m+1-i_0}$. Using again the results by Graham [13], we obtain that an optimal schedule for the latter job sequence is obtained by repeatedly pairing J_r^i with J_r^{2m+1-i} . However, since $p_r^{2m+1-i_0} \leq p_r^{i_0}/2$, the processing time $p_r^{2m+1-i_0}$ is at most a third times the resulting optimum makespan for $J_r^1, \dots, J_r^{2m+1-i_0}$. Hence $p_r^{2m+1-i_0}$ is at most a third times OPT , which is a contradiction.

Next we compare the processing time of the jobs of $P_1 \cup \dots \cup P_m$ to $\sum_{i=1}^{2m} \hat{p}_n^i$. Set R' contains the jobs of R that are large at time $n + 1$. There exist at most $2m$ jobs that are large at time $n + 1$ and hence the processing time of each job in R' is represented by a positive entry in the sequence $\hat{p}_n^1, \dots, \hat{p}_n^{2m}$. It follows that the total processing time of the jobs in R' and hence the total processing time of the jobs in $P_1 \cup \dots \cup P_m$ is at most $\sum_{i=1}^{2m} \hat{p}_n^i$. Recall that $\pi_1 \geq \dots \geq \pi_m$. Then, for any j with $1 \leq j \leq m$, the product $j\pi_j$ is upper bounded by the total processing time of $P_1 \cup \dots \cup P_m$ and hence $j\pi_j \leq \sum_{i=1}^{2m} \hat{p}_n^i$.

Now consider the assignment of the sets P_1, \dots, P_m to the machines. Each set is assigned to a least loaded machine. Hence when P_j , $1 \leq j \leq m$, is scheduled, it is assigned to a machine whose current load is at most $\max\{\beta(j)L^*, (\alpha_m - 1)L\}$. If the load is at most $(\alpha_m - 1)L$, then the machine's load after the assignment is at most $(\alpha_m - 1)L + \pi_j \leq (\alpha_m - 1)L + OPT \leq \alpha_m OPT$. If the current load is only upper bounded by $\beta(j)L^*$, then we distinguish two cases.

If $j \leq \lfloor m/\alpha_m \rfloor$, then $j \leq m/\alpha_m$, which is equivalent to $m/(m - j) \leq \alpha_m/(\alpha_m - 1)$. The resulting machine load is at most

$$\begin{aligned} \beta(j)L^* + \pi_j &= (\alpha_m - 1) \frac{m}{m - j} \left(\frac{1}{m} \sum_{i=1}^n p_i - \frac{1}{m} \sum_{i=1}^{2m} \hat{p}_i^j \right) \\ &+ \pi_j \leq (\alpha_m - 1) \frac{1}{m - j} (mL - j\pi_j) + \pi_j. \end{aligned}$$

The last inequality follows because, as argued above, $j\pi_j \leq \sum_{i=1}^{2m} \hat{p}_i^j$. It follows that the machine load is upper bounded by

$$(\alpha_m - 1) \frac{1}{m - j} (mL - m\pi_j) + \alpha_m \pi_j \leq \alpha_m (L - \pi_j) + \alpha_m \pi_j = \alpha_m L.$$

The last inequality holds because $m/(m - j) \leq \alpha_m/(\alpha_m - 1)$, as mentioned above.

If $j > \lfloor m/\alpha_m \rfloor$, then $j \geq m/\alpha_m$ because j is integral. In this case the machine load is upper bounded by

$$\beta(j)L^* + \pi_j = \alpha_m \left(\sum_{i=1}^n p_i - \sum_{i=1}^{2m} \hat{p}_i^j \right) / m + \pi_j \leq \alpha_m \left(\sum_{i=1}^n p_i - j\pi_j \right) / m + \pi_j \leq \alpha_m L,$$

because $j\alpha_m \geq m$.

Finally we consider the jobs $R \setminus (P_1 \cup \dots \cup P_m)$. Each job of $R \setminus R'$ has a processing time of at most $(\alpha_m - 1)L$. As argued above, each job of $R' \setminus (P_1 \cup \dots \cup P_m)$ has a processing time of at most $OPT/3$, which is upper bounded by $(\alpha_m - 1)OPT$ since $\alpha_m \geq 4/3$. Hence each job of $R \setminus (P_1 \cup \dots \cup P_m)$ has a processing time of at most $(\alpha_m - 1)OPT$. Each of these jobs is scheduled on a least loaded machine and thus after the assignment the corresponding machine has a load of at most $OPT + (\alpha_m - 1)OPT \leq \alpha_m OPT$. \square

The proof of Theorem 1 is complete.

3 A Lower Bound

We present a lower bound showing that $ALG(\alpha_m)$ is optimal.

Theorem 2 *Let $m \geq 2$. No deterministic online algorithm can achieve a competitive ratio smaller than α_m if $o(n)$ job migrations are allowed.*

Proof Let A be any deterministic online algorithm that is allowed to use up to $g(n)$ job migrations on a job sequence of length n . Suppose that A achieves a competitive ratio smaller than α_m . We will derive a contradiction.

Choose an $\epsilon > 0$ such that A has a competitive ratio strictly smaller than $\alpha_m - \epsilon$. Let $\epsilon' = \epsilon/3$. Since $g(n) = o(n)$ there exists an n_0 such that $g(n)/n \leq \epsilon'/(2m)$, for all $n \geq n_0$. Hence there exists an n_0 such that $g(n+m)/(n+m) \leq \epsilon'/(2m)$, for all $n \geq \max\{m, n_0\}$. Let n' , with $n' \geq \max\{m, n_0\}$, be the smallest integer multiple of m . We have $g(n'+m)/n' \leq \epsilon'/m$ because $n'+m \leq 2n'$. An adversary constructs a job sequence consisting of $n' + m$ jobs. Let $p_1 = m/n'$. By our choice of n' , there holds $p_1 \leq \epsilon'/g(n'+m)$. The following adversarial sequence is similar to that used by Englert et al. [8]. However, here we have to ensure that in migrating $o(n)$ jobs, an online algorithm cannot benefit much.

First the adversary presents n' jobs of processing time p_1 . We will refer to them as p_1 -jobs. If after the assignment of these jobs A has a machine M_j , $1 \leq j \leq m$, whose load is at least α_m , then the adversary presents m jobs of processing time $p_2 = \epsilon'/m$. Using job migration, A can remove at most $g(n'+m)$ p_1 -jobs from M_j . Since $g(n'+m)p_1 \leq \epsilon'$, after job migration M_j still has a load of at least $\alpha_m - \epsilon$. On the other hand the optimal makespan is $1 + \epsilon'/m$. In an optimal assignment each machine contains n'/m p_1 -jobs and one p_2 -job. The ratio $(\alpha_m - \epsilon')/(1 + \epsilon'/m)$ is at least $\alpha_m - \epsilon$ by our choice of ϵ' and the fact that $\alpha_m \leq 2$, see Lemma 1. We obtain a contradiction.

In the following we study the case that after the assignment of the p_1 -jobs each machine in A 's schedule has a load strictly smaller than α_m . We number the machines in order of non-decreasing load such that $\ell(1) \leq \dots \leq \ell(m)$. Here $\ell(j)$ denotes the load of M_j after the p_1 -jobs have arrived, $1 \leq j \leq m$. For $j = 1, \dots, m-1$, define $\beta(j) = (\alpha_m - 1)m/(m-j)$. We remark that these values are identical to those used by $ALG(\alpha_m)$ for $j \leq \lfloor m/\alpha_m \rfloor$. For larger j , the values are different. We first argue that there must exist a machine M_j , $1 \leq j \leq m-1$, in A 's schedule whose load is at least $\beta(j)$. Suppose that each machine M_j , $1 \leq j \leq m-1$, had a load strictly

smaller than $\beta(j)$. By Lemma 1, $\alpha_m > 1$ and hence $\lceil(1 - 1/\alpha_m)m\rceil \geq 1$. Consider the $\lceil(1 - 1/\alpha_m)m\rceil$ machines with the highest load in A 's schedule. Each of these machines has a load strictly smaller than α_m . The remaining machines have a load strictly smaller than $\beta(j) = (\alpha_m - 1)m/(m - j)$, for $j = 1, \dots, m - \lceil(1 - 1/\alpha_m)m\rceil$. We conclude that after the arrival of the p_1 -jobs the total load on the machines is strictly smaller than

$$\begin{aligned} & (\alpha_m - 1)m \sum_{j=1}^{m - \lceil(1 - 1/\alpha_m)m\rceil} \frac{1}{m - j} + \lceil(1 - 1/\alpha_m)m\rceil \alpha_m \\ &= m((\alpha_m - 1)(H_{m-1} - H_{\lceil(1 - 1/\alpha_m)m\rceil - 1}) + \lceil(1 - 1/\alpha_m)m\rceil \alpha_m/m) \\ &= m f_m(\alpha_m) = m. \end{aligned}$$

The last equation holds because $f_m(\alpha_m) = 1$, by the choice of α_m . We obtain a contradiction to the fact that after the arrival of the p_1 -jobs a total load of exactly m resides on the machines.

Let M_{j_0} , with $1 \leq j_0 \leq m - 1$, be a machine whose load is at least $\beta(j_0)$. Since A 's machines are numbered in order of non-decreasing load there exist at most $j_0 - 1$ machines having a smaller load than $\beta(j_0)$. The adversary presents j_0 jobs of processing time $p_2 = m/(m - j_0)$. Using job migration A can remove at most $g(n' + m)$ p_1 -jobs from any of the machines, thereby reducing the load by at most ϵ' . Hence in A 's final schedule there exists a machine having a load of at least $\beta(j_0) + m/(m - j_0) - \epsilon'$. This holds true if the p_2 -jobs reside on different machines. If there exists a machine containing two p_2 -jobs, then its load is at least $2m/(m - j_0) \geq (\alpha_m - 1)m/(m - j_0) + m/(m - j_0) = \beta(j_0) + m/(m - j_0)$ as desired. The inequality holds because $\alpha_m \leq 2$, by Lemma 1. Hence A 's makespan is at least $\beta(j_0) + m/(m - j_0) - \epsilon'$.

The optimum makespan for the job sequence is upper bounded by $m/(m - j_0) + \epsilon'$. In an optimal schedule the j_0 p_2 -jobs are assigned to different machines. The n' p_1 -jobs are distributed evenly among the remaining $m - j_0$ machines. If n' is an integer multiple of $m - j_0$, then the load on each of these $m - j_0$ machines is exactly $n' p_1/(m - j_0) = m/(m - j_0)$, which is exactly equal to the processing time of a p_2 -job. If n' is not divisible by $m - j_0$, then the maximum load on any of these $m - j_0$ machines cannot be higher than $m/(m - j_0) + p_1 \leq m/(m - j_0) + \epsilon'/g(n' + m) \leq m/(m - j_0) + \epsilon'$.

Dividing the lower bound on A 's makespan by the upper bound on the optimum makespan we obtain $(\alpha_m m/(m - j_0) - \epsilon')/(m/(m - j_0) + \epsilon') \geq (\alpha_m - \epsilon')/(1 + \epsilon') \geq \alpha_m - \epsilon$. The last inequality holds because $\epsilon' = \epsilon/3$ and $\alpha_m \leq 2$, see Lemma 1. We obtain a contradiction to the assumption that A 's competitiveness is strictly smaller than $\alpha_m - \epsilon$. □

4 Algorithms Using Fewer Migrations

We present a family of algorithms $ALG(c)$ that uses a smaller number of job migrations. We first describe the family and then analyze its performance.

4.1 Description of $ALG(c)$

$ALG(c)$ is defined for any constant c with $5/3 \leq c \leq 2$, where c is the targeted competitive ratio. An important feature of $ALG(c)$ is that it partitions the machines M_1, \dots, M_m into two sets $A = \{M_1, \dots, M_{\lfloor m/2 \rfloor}\}$ and $B = \{M_{\lfloor m/2 \rfloor + 1}, \dots, M_m\}$ of roughly equal size. In a job arrival phase the jobs are preferably assigned to machines in A , provided that their load is not too high. In the job migration phase, jobs are mostly migrated from machines of A (preferably to machines in B) and this policy will allow us to achieve a smaller number of migrations. Setting $c = 5/3$ we obtain an algorithm $ALG(5/3)$ that is $5/3$ -competitive using $4m$ migrations. For $c = 1.75$ the resulting algorithm $ALG(1.75)$ is 1.75 -competitive and uses at most $2.5m$ migrations. In the following let $5/3 \leq c \leq 2$.

Algorithm $ALG(c)$: job arrival phase At any time t $ALG(c)$ maintains a lower bound L_t on the optimum makespan, which is defined as $L_t = \max\{\frac{1}{m}p_t^+, p_t^1, 2p_t^{m+1}\}$. Here we use the same notation as in Sect. 2. Recall that p_t^1 and p_t^{m+1} are the processing times of the largest and $(m + 1)$ -st largest jobs in J_1, \dots, J_t , respectively. A job J_i , with $i \leq t$, is *small at time t* if $p_i \leq (2c - 3)L_t$; otherwise the job is *large at time t* . For any machine M_j and any time t , $\ell(j, t)$ is M_j 's load immediately before J_t is assigned and $\ell_s(j, t)$ is its load consisting of the jobs that are small at time t .

Any job J_t , $1 \leq t \leq n$, is processed as follows. If J_t is small at its arrival time t , then $ALG(c)$ checks if there is a machine in A whose load value $\ell_s(j, t)$ is at most $(c - 1)L_t$. If this is the case, then among the machines in A with this property, J_t is assigned to one having the smallest $\ell_s(j, t)$ value. If there is no such machine in A , then J_t is assigned to a least loaded machine in B . If J_t is large at time t , then $ALG(c)$ checks if there is machine in A whose load value $\ell(j, t)$ is at most $(3 - c)L_t$. If this is the case, then J_t is scheduled on a least loaded machine in A . Otherwise J_t is assigned to a least loaded machine in B . At the end of the phase let $L = L_n$.

Job migration phase At any time during the phase let $\ell(j)$ denote the current load of M_j , $1 \leq j \leq m$. We first describe the job removal step. For any machine $M_j \in B$, $ALG(c)$ removes the largest job from that machine. Furthermore, while there exists a machine $M_j \in A$ whose current load exceeds $(c - 1)L$, $ALG(c)$ removes the largest job from the machine. Let R be the set of all removed jobs. In the job reassignment step $ALG(c)$ first sorts the jobs in order of non-increasing processing times. For any i , $1 \leq i \leq |R|$, let J_r^i be the i -th largest job in this sequence, and let p_r^i be the corresponding processing time. For $i = 1, \dots, |R|$, J_r^i is scheduled as follows. If there exists a machine $M_j \in B$ such that $\ell(j) + p_r^i \leq cL$, i.e. J_r^i can be placed on M_j without exceeding a makespan of cL , then J_r^i is assigned to this machine. Otherwise the job is scheduled on a least loaded machine in A . A pseudo-code description of $ALG(c)$ is given in Fig. 2.

Theorem 3 $ALG(c)$ is c -competitive, for any constant c with $5/3 \leq c \leq 2$.

The proof of the above theorem is presented in Sect. 4.2.1. In order to obtain good upper bounds on the number of job migrations, we focus on specific values of c . First, set $c = 5/3$. In $ALG(5/3)$ a job J_t is small if $p_t \leq 1/3 \cdot L_t$. In the arrival phase a small job is assigned to a machine in A if there exists a machine in this set whose load

Algorithm ALG(c): Let $5/3 \leq c \leq 2$.

Job arrival phase. Each $J_t, 1 \leq t \leq n$, is scheduled as follows.

- J_t is small at time t : Let $A' = \{M_j \in A \mid \ell_s(j, t) \leq (c - 1)L_t\}$. If $A' \neq \emptyset$, then assign J_t to a machine $M_j \in A'$ having the smallest $\ell_s(j, t)$ value. Otherwise assign J_t to a least loaded machine $M_j \in B$.
- J_t is large at time t : If there is an $M_j \in A$ with $\ell(j, t) \leq (3 - c)L_t$, then assign J_t to a least loaded machine in A . Otherwise assign J_t to a least loaded machine in B .

Job migration phase.

- Job removal: Set $R := \emptyset$. For any $M_j \in B$, remove the largest job from M_j and add it to R . While there exists an $M_j \in A$ with $\ell(j) > (c - 1)L$, remove the largest job from M_j and add it to R .
- Job reassignment: Sort the jobs of R in order of non-increasing processing time. For $i = 1, \dots, |R|$, schedule J_i^* as follows. If there is an $M_j \in B$ with $\ell(j) + p_i^* \leq cL$, then assign J_i^* to M_j . Otherwise assign it to a least loaded machine in A .

Fig. 2 The algorithm $ALG(c)$

consisting of jobs that are currently small is at most $2/3 \cdot L_t$. A large job is assigned to a machine in A if there exists a machine in this set whose load is at most $4/3L_t$.

Theorem 4 $ALG(5/3)$ is $\frac{5}{3}$ -competitive and uses at most $4m$ job migrations.

In fact, for any c with $5/3 \leq c \leq 2$, $ALG(c)$ uses at most $4m$ job migrations. Finally, let $c = 1.75$. In $ALG(1.75)$ a job J_t is small if $p_t \leq 0.5 \cdot L_t$. In the arrival phase a small job is assigned to a machine in A if there is a machine in this set whose load consisting of jobs that are currently small is no more than $0.75L_t$. A large job is assigned to a machine in A if there exists a machine in this set whose load is at most $1.25L_t$.

Theorem 5 $ALG(1.75)$ is 1.75-competitive and uses at most $2.5m$ job migrations.

Again, for any c with $1.75 \leq c \leq 2$, $ALG(c)$ uses at most $2.5m$ job migrations. The proofs of Theorems 4 and 5 are contained in Sect. 4.2.2.

4.2 Analysis of $ALG(c)$

In this section we analyze $ALG(c)$, for any c with $5/3 \leq c \leq 2$, and prove Theorems 3, 4 and 5. We first determine the competitive ratio of $ALG(c)$ and then bound the number of job migrations performed for $c = 5/3$ and $c = 1.75$.

4.2.1 Analysis of the Competitive Ratio

We start by showing two lemmas that will allow us to bound load on machines in B . The lemmas will be applied depending on the final loads of small jobs residing on machines in A . In particular, for any time t , Lemma 6 bounds the load of any machine in B when ignoring the last job assigned to that machine. Again, let time $n + 1$ be the time when the entire job sequence $\sigma = J_1, \dots, J_n$ has been scheduled and the migration phase starts. A job $J_i, 1 \leq i \leq n$, is *small at time $n + 1$* if $p_i \leq (2c - 3)L = (2c - 3)L_n$; otherwise the job is *large at time $n + 1$* . For any $M_j, 1 \leq j \leq m$, let $\ell(j, n + 1)$ be its load at time $n + 1$ and let $\ell_s(j, n + 1)$ be the load consisting of the jobs that are small at time $n + 1$. Let $L_{n+1} := L$.

Lemma 6 For any t , $1 \leq t \leq n + 1$, and any $M_j \in B$, there holds $\ell(j, t) - p_l \leq (3 - c)L_{t-1}$, where J_l with $l < t$ is the last job assigned to M_j .

Proof By the definition of $ALG(c)$, when J_l is assigned to M_j , all machines of A have a load greater than $(c - 1)L_l$ and M_j is a least loaded machine in B . Hence M_j 's load at time l is at most $(3 - c)L_l$ since otherwise the total load on the m machines would be greater than $\lfloor m/2 \rfloor (c - 1)L_l + \lceil m/2 \rceil (3 - c)L_l \geq mL_l \geq \sum_{i=1}^l p_i$, which is a contradiction. Hence $\ell(j, t) = \ell(j, l) + p_l \leq (3 - c)L_l + p_l \leq (3 - c)L_{t-1} + p_l$. \square

Lemma 7 Suppose that there exists a machine $M_{j^*} \in A$ with $\ell_s(j^*, n + 1) < (2 - c)L$. Then, for any $B \in M_j$, $\ell(j, n + 1) - p_l \leq (c - 1)L$, where J_l is the last job assigned to M_j .

Proof Consider any $M_j \in B$ and let J_l be the last job assigned to it. First assume that J_l is large at time l . By the definition of $ALG(c)$, at time l all machines of A have a load greater than $(3 - c)L_l$. Moreover, M_j is a least loaded machine in B at time l . We argue that a least loaded machine in B has a load of at most $(c - 1)L_l$. If this were not the case, then immediately after the assignment of J_l the total load on the m machines would be greater than $\lfloor m/2 \rfloor (3 - c)L_l + \lceil m/2 \rceil (c - 1)L_l + p_l \geq (m/2 - 1/2)(3 - c)L_l + (m/2 + 1/2)(c - 1)L_l + (2c - 3)L_l = mL_l + (3c - 5)L_l$. The inequality holds because $3 - c \geq c - 1$. Since $c \geq 5/3$ it follows $\lfloor m/2 \rfloor (3 - c)L_l + \lceil m/2 \rceil (c - 1)L_l + p_l \geq mL_l \geq \sum_{i=1}^l p_i$, which is a contradiction. Hence $\ell(j, n + 1) = \ell(j, l) + p_l \leq (c - 1)L_l + p_l \leq (c - 1)L + p_l$.

Next assume that J_l is small at time l . This implies $\ell_s(j, l) > (c - 1)L_l$, for all $M_j \in A$. In particular, $\ell_s(j^*, l) > (c - 1)L_l$. Since $\ell_s(j^*, l) \leq \ell_s(j^*, n + 1) < (2 - c)L$ it follows $L_l < (2 - c)/(c - 1) \cdot L$. By Lemma 6, $\ell(j, l + 1) \leq (3 - c)L_l + p_l$ and we conclude $\ell(j, n + 1) = \ell(j, l + 1) \leq (3 - c)L_l + p_l \leq (3 - c)(2 - c)/(c - 1) \cdot L + p_l \leq (c - 1)L + p_l$. The last inequality holds because $(3 - c)(2 - c)/(c - 1) \leq c - 1$ holds since $c \geq 5/3$. \square

We next analyze the job migration phase assuming that the job removal step has already taken place, i.e. each machine of A has a load of at most $(c - 1)L$ and the largest job was removed from each machine of B . We show that, given such a machine configuration, each job of R can be assigned to a machine so that a load bound of cL is preserved. For the analysis of the reassignment step we study two cases depending on whether or not at time $n + 1$ all machines $M_j \in A$ have a load $\ell_s(j, n + 1) \geq (2 - c)L$.

Lemma 8 If $\ell_s(j, n + 1) \geq (2 - c)L$, for all $M_j \in A$, then in the reassignment step all jobs of R are scheduled so that the resulting load on any of the machines is at most cL .

Proof By assumption, at the end of the job arrival phase $\ell_s(j, n + 1) \geq (2 - c)L$, for all $M_j \in A$. We first show that this property is maintained throughout the job removal step. Suppose that a job J_i that is small at time $n + 1$ is removed from a machine $M_j \in A$. Since $ALG(c)$ always removes the largest jobs from a machine, M_j currently contains no jobs that are large at time $n + 1$. Hence M_j 's current load $\ell(j)$ is equal to its current load $\ell_s(j)$ consisting of jobs that are small at time $n + 1$. Since a job removal needs to be

performed, $\ell_s(j) = \ell(j) > (c - 1)L$. Since $p_i \leq (2c - 3)L$, the removal of J_i leads to a load consisting of small jobs of at least $\ell_s(j) - p_i > (c - 1)L - (2c - 3)L = (2 - c)L$.

After the job removal step each machine $M_j \in A$ has a load of at most $(c - 1)L$. By Lemma 6 each machine of B has a load of at most $(3 - c)L < cL$ after $ALG(c)$ has removed the largest job from any of these machines. We show that each $J_k \in R$ can be scheduled on a machine such that the resulting load is at most cL . Consider any $J_k \in R$. There holds $p_k \leq L$. Suppose that J_k cannot be feasibly scheduled on any of the machines. Let $\ell(j)$ denote M_j 's load immediately before the assignment of J_k , $1 \leq j \leq m$. If J_k cannot be placed on a machine in A , then each machine $M_j \in A$ must have a load greater than $(c - 1)L$: If $\ell(j) \leq (c - 1)L$, then $\ell(j) + p_k \leq cL$ and the assignment of J_k to M_j would be feasible. Hence since the start of the reassignment step each machine $M_j \in A$ must have received at least one job J_{i_j} and its current load is $\ell(j) \geq (2 - c)L + p_{i_j}$. When J_{i_j} was reassigned, it could not be scheduled on any machine in B without exceeding a load of cL . This implies, in particular, that $\ell(\lfloor m/2 \rfloor + j) + p_{i_j} > cL$. Recall that the machines of A are numbered $1, \dots, \lfloor m/2 \rfloor$ and those of B are numbered $\lfloor m/2 \rfloor + 1, \dots, m$. Finally, since J_k cannot be placed on a machine in B , we have $\ell(m) + p_k > cL$.

It follows that when J_k has to be scheduled the total processing time of the jobs is at least

$$\sum_{j=1}^m \ell(j) + p_k \geq \lfloor m/2 \rfloor (2 - c)L + \sum_{j=1}^{\lfloor m/2 \rfloor} p_{i_j} + \sum_{j=\lfloor m/2 \rfloor + 1}^m \ell(j) + p_k.$$

If m is even, then $\sum_{j=\lfloor m/2 \rfloor + 1}^m \ell(j) = \sum_{j=1}^{m/2} \ell(m/2 + j)$. In this case we have

$$\begin{aligned} \sum_{j=1}^m \ell(j) + p_k &\geq m/2 \cdot (2 - c)L \\ &+ \sum_{j=1}^{m/2} (\ell(m/2 + j) + p_{i_j}) + p_k > m/2 \cdot (2 - c)L + m/2 \cdot cL = mL. \end{aligned}$$

If m is odd, then $\sum_{j=\lfloor m/2 \rfloor + 1}^m \ell(j) = \sum_{j=1}^{\lfloor m/2 \rfloor} \ell(\lfloor m/2 \rfloor + j) + \ell(m)$ and

$$\begin{aligned} \sum_{j=1}^m \ell(j) + p_k &\geq \lfloor m/2 \rfloor \cdot (2 - c)L + \sum_{j=1}^{\lfloor m/2 \rfloor} (\ell(\lfloor m/2 \rfloor + j) + p_{i_j}) + \ell(m) + p_k \\ &> \lfloor m/2 \rfloor \cdot (2 - c)L + \lfloor m/2 \rfloor \cdot cL + cL \\ &= (m/2 - 1/2)2L + cL > mL. \end{aligned}$$

In both cases with obtain $\sum_{i=1}^n p_i \geq \sum_{j=1}^m \ell(j) + p_k > mL$, which contradicts the definition of L . □

Lemma 9 *If $\ell_s(j^*, n + 1) < (2 - c)L$, for some $M_{j^*} \in A$, then in the reassignment step all jobs of R are scheduled so that the resulting load on any of the machines is at most cL .*

Proof In the removal step $ALG(c)$ removes the largest job from each machine $M_j \in B$. Hence, if $\ell_s(j^*, n + 1) < (2 - c)L$ for some $M_j \in A$, then by Lemma 7 each machine of B has a load of at most $(c - 1)L$ after the removal step. Moreover, each machine of A has a load of at most $(c - 1)L$ after the job removal.

Hence when the reassignment step starts, all machines have a load of at most $(c - 1)L$. By the definition of L each job has a processing time of at most L . Hence in the reassignment step the first m jobs can be scheduled without exceeding a load of cL on any of the machines. $ALG(c)$ sorts the jobs of R in order of non-increasing processing times. Thus when m jobs of R have been scheduled, each of the remaining jobs has a processing time of at most $1/2L$. This holds true because by the definition of L there cannot exist $m + 1$ jobs of processing time greater than $1/2L$. Each job of processing time at most $1/2L$ can be scheduled on a least loaded machine without exceeding a load of cL since $L + 1/2L < cL$. Hence every remaining job can be scheduled on a machine of B and A . \square

Lemmas 8 and 9 imply Theorem 3.

4.2.2 Analysis of the Job Migrations

It remains to evaluate the number of job removals in the job migration phase. $ALG(c)$ removes one job from each machine in B . Hence, in the following we analyze the number of jobs removed by $ALG(c)$ from any machine in A . To this end we consider the final machine loads consisting of small jobs.

Lemma 10 *Let $M_j \in A$. If $\ell_s(j, n + 1) \leq (c - 1)L$ or if $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) \leq (c - 1)L_n$, then $ALG(c)$ removes less than $(3 - c)/(2c - 3) + 2$ jobs from M_j .*

Proof We show that it suffices to remove less than $(3 - c)/(2c - 3) + 2$ jobs from M_j such that the resulting load is upper bounded by $(c - 1)L$. The lemma then follows because in each removal operation $ALG(c)$ removes the largest job.

First assume that $\ell_s(j, n + 1) \leq (c - 1)L$. In this case it suffices to remove all jobs that are large at time $n + 1$. Each such job has a processing time greater than $(2c - 3)L$ and was large at the time it was assigned to M_j . Consider the last time when such a job was assigned to M_j . At that time M_j had a load of at most $(3 - c)L$ and hence contained less than $(3 - c)/(2c - 3)$ jobs of processing time greater than $(2c - 3)L$. Thus at time $n + 1$ machine M_j contains less than $(3 - c)/(2c - 3) + 1$ of these large jobs.

Next assume $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) \leq (c - 1)L_n$. The latter inequality implies that J_n is assigned to M_j because $L = L_n$. Hence it suffices to remove J_n and, as shown in the last paragraph, less than $(3 - c)/(2c - 3) + 1$ additional jobs of processing time greater than $(2c - 3)L_n = (2c - 3)L$. \square

In the following we concentrate on a machine $M_j \in A$ such that $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) > (c - 1)L_n$. Let t^* be the earliest time such that $\ell_s(j, t) > (c - 1)L_t$ holds for all times $t \geq t^*$. We have $t^* > 1$ because $\ell_s(j, 0) = 0$. We partition the jobs that reside on M_j at time $n + 1$ into three sets. Set T_1 (set T_2) contains those jobs that were assigned to M_j at or before time $t^* - 1$ are small (large) at time $t^* - 1$. Set T_3 contains the remaining jobs, which have arrived at or after time t^* .

Claim 1 Each job of $T_2 \cup T_3$ is large at the time it is assigned to M_j .

Claim 2 There holds $\sum_{J_l \in T_1 \setminus \{J_l\}} p_i \leq (c - 1)L_{t^*-1}$, where J_l is the job of T_1 that was assigned last to M_j .

Claim 3 $|T_2|$ is smaller than $(3 - c)/(2c - 3) + 1$.

Claim 4 For any $J_l \in T_3$, M_j 's load immediately before the assignment of J_l is at most $(3 - c)L_l$.

Claim 5 Let $J_l \in T_3$ be the last job assigned to M_j . If M_j contains at least $\lceil 12(2 - c) \rceil$ jobs, different from J_l , each having a processing time of at least $1/6L$, then it suffices to remove these $\lceil 12(2 - c) \rceil$ jobs and J_l such that M_j 's resulting load is upper bounded by $(c - 1)L$.

Proof of Claim 1 The jobs of T_2 are large at time $t^* - 1$ and hence at the time they were assigned to M_j . By the definition of t^* , $\ell_s(j, t) > (c - 1)L_t$, for any $t^* \leq t \leq n$, and hence $ALG(c)$ does not assign small jobs to M_j . □

Proof of Claim 2 By the choice of t^* , all jobs of $T_1 \setminus \{J_l\}$ are small at time $t^* - 1$ and their total processing time is at most $\ell_s(j, t^* - 1) \leq (c - 1)L_{t^*-1}$. □

Proof of Claim 3 Each job of T_2 has a processing time greater than $(2c - 3)L_{t^*-1}$. Consider the last time l when a job $J_l \in T_2$ was assigned to M_j . Before the assignment, M_j had a load of at most $(3 - c)L_{t^*-1}$ and hence contained less than $(3 - c)/(2c - 3)$ jobs of processing time greater than $(2c - 3)L_{t^*-1}$. □

Proof of Claim 4 Consider any $J_l \in T_3$. By Claim 1 J_l is large at time l and hence M_j 's load prior to the assignment of J_l is at most $(3 - c)L_l$. □

Proof of Claim 5 By Claim 4 M_j 's load immediately before the assignment of J_l is at most $(3 - c)L_l$. Removing $\lceil 12(2 - c) \rceil$ jobs of processing time at least $1/6L$ each as well as J_l reduces M_j 's load to a value of at most $(3 - c)L_l - 12(2 - c)/6 \cdot L \leq (c - 1)L$. □

We are ready to analyze $ALG(5/3)$.

Lemma 11 In the removal step $ALG(5/3)$ removes at most seven jobs from each machine $M_j \in A$.

Proof Consider any $M_j \in A$. If $\ell_s(j, n + 1) \leq (c - 1)L$ or if $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) \leq (c - 1)L_n$, then by Lemma 10 $ALG(5/3)$ removes less than $(3 - c)/(2c - 3) + 2 = 6$ jobs from M_j . In the remainder of this proof we assume that $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) > (c - 1)L_n$. In this case Claims 1–5 hold. We need two additional statements.

Claim 6 *If there exists a $J_l \in T_3$ with $p_l < 1/6L$, then M_j 's load immediately before the assignment of J_l is at most $(c - 1)L = 2/3L$.*

Claim 7 *If there exists a $J_k \in T_2$ with $p_k < 1/6L$, then $\sum_{J_i \in T_1} p_i + p_k \leq (c - 1)L = 2/3L$.*

Proof of Claim 6 By Claim 1 J_l is large at time l and hence $p_l > (2c - 3)L_l = 1/3L_l$. Since $p_l < 1/6L$, we have $L_l < 1/2L$. By Claim 4, M_j 's load immediately before the assignment of J_l is at most $(3 - c)L_l = 4/3L_l$ and hence at most $2/3L$. \square

Proof of Claim 7 Job J_k is large at time $t^* - 1$ and hence $p_k > (2c - 3)L_{t^*-1} = 1/3L_{t^*-1}$. Since $p_k < 1/6L$ it follows $L_{t^*-1} < 1/2L$. By Claim 2, we have $\sum_{J_i \in T_1} p_i \leq (c - 1)L_{t^*-1} + p_l$, where J_l is the last job of T_1 assigned to M_j . Since p_l is small at time $t^* - 1$ we have $p_l \leq (2c - 3)L_{t^*-1} = 1/3L_{t^*-1} < 1/6L$. In summary $\sum_{J_i \in T_1} p_i + p_k \leq 1/3L + 1/6L + 1/6L = 2/3L$. \square

We distinguish two cases.

Case 1: Suppose that $|T_2 \cup T_3| \leq 4$. By Claim 2 it suffices to remove the jobs of $T_2 \cup T_3$ and the last job of T_1 assigned to M_j .

Case 2: Assume $|T_2 \cup T_3| \geq 5$. Then by Claim 3 $|T_2|$ is smaller than $(3 - c)/(2c - 3) + 1 = 5$. Thus $|T_2| \leq 4$ and $T_3 \neq \emptyset$. Let J_l be the last job of T_3 assigned to M_j . If $T_2 \cup T_3 \setminus \{J_l\}$ contains at least $\lceil 12(2 - c) \rceil = 4$ jobs of processing time at least $1/6L$, then by Claim 5 it suffices to remove these four jobs and J_l . So suppose that this is not the case. Then $T_2 \cup T_3 \setminus \{J_l\}$ must contain a job of processing time smaller than $1/6L$.

Assume there exists a job in $T_3 \setminus \{J_l\}$ with this property. Then let $J_{l'}$ be the last job assigned to M_j having a processing time smaller than $1/6L$. By Claim 6, immediately before the assignment of $J_{l'}$ machine M_j has a load of at most $2/3L$. Therefore it suffices to remove $J_{l'}$ and the jobs of T_3 subsequently scheduled on M_j . In addition to J_l , this sequence consists of at most three jobs $J_k \neq J_l$, because $T_3 \setminus \{J_l\}$ contains less than four jobs of processing time at least $1/6L$.

Finally consider the case that all jobs of $T_3 \setminus \{J_l\}$ have a processing time of at least $1/6L$ and there is a job $J_{l'} \in T_2$ having a processing time smaller than $1/6L$. By Claim 7 it suffices to remove $T_2 \setminus \{J_{l'}\} \cup T_3$. By Claim 3 we have $|T_2 \setminus \{J_{l'}\}| \leq 3$. Since $T_3 \setminus \{J_l\}$ contains less than four jobs, each having a processing time of at least $1/6L$, we have $|T_3| \leq 4$. We conclude that at most seven jobs have to be removed. \square

Lemma 11 ensures that in the job removal step $ALG(5/3)$ removes at most 7 jobs from any machine in A . For any machine in B , one job is removed. Hence the total number of migrations is at most $7\lceil m/2 \rceil + \lceil m/2 \rceil \leq 4m$. Combined with Theorem 3, this concludes the proof of Theorem 4. We next turn to the algorithm $ALG(1.75)$.

Lemma 12 *In the job removal step $ALG(1.75)$ removes at most four jobs from each machine $M_j \in A$.*

Proof Let M_j be any machine in A . If $\ell_s(j, n + 1) \leq (c - 1)L$ or if $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) \leq (c - 1)L_n$, then by Lemma 10 $ALG(1.75)$ removes less than $(3 - c)/(2c - 3) + 2 = 4.5$ jobs from M_j . Therefore, we focus on the case that $\ell_s(j, n + 1) > (c - 1)L$ and $\ell_s(j, n) > (c - 1)L_n$. Again, Claims 1–5 hold and we need two additional claims.

Claim 8 *If there exists a $J_l \in T_3$ with $p_l < 1/6L$, then M_j 's load immediately after the assignment of J_l is at most $(c - 1)L = 0.75L$.*

Claim 9 *If $T'_2 \subseteq T_2$ is a subset with $1 \leq |T_2| \leq 2$ and $p_i \leq 1/6L$, for all $J_i \in T_2$, then $\sum_{J_i \in T_1} p_i + \sum_{J_i \in T'_2} p_i \leq (c - 1)L = 0.75L$.*

Proof of Claim 8 By Claim 1 J_l is large at time l and hence $p_l > (2c - 3)L_l = 0.5L_l$. Since $p_l < 1/6L$, we have $L_l < 1/3L$. Using Claim 4 we obtain that M_j 's load immediately after the assignment of J_l is at most $(3 - c)L_l + p_l = 1.25L_l + p_l \leq 5/12L + 1/6L < 0.75L$. \square

Proof of Claim 9 Any job $J_i \in T'_2$ is large at time $t^* - 1$ and hence $p_i > (2c - 3)L_{t^*-1} = 0.5L_{t^*-1}$. Since $p_i < 1/6L$ it follows $L_{t^*-1} < 1/3L$. By Claim 2, we have $\sum_{J_i \in T_1} p_i \leq 0.75L_{t^*-1} + p_l \leq 0.25L + 1/6L$, where J_l is the last job of T_1 assigned to M_j . Thus $\sum_{J_i \in T_1} p_i + \sum_{J_i \in T'_2} p_i \leq 0.25L + 3 \cdot 1/6L \leq 0.75L$. \square

We finish the proof of the lemma using a case distinction on the size of T_3 . By Claim 3, T_2 contains no more than three jobs. Moreover, $\lceil 12(2 - c) \rceil = 3$, which is a quantity used in Claim 5.

- $|T_3| = 0$: Then by Claim 2 it suffices to remove T_2 and the last job of T_1 assigned to M_j .
- $|T_3| = 1$: We may assume that the only job $J_l \in T_3$ has a processing time of at least $1/6L$ since otherwise by Claim 8 no job has to be removed. Moreover, we may assume that $|T_2| = 3$ since otherwise, by Claim 2 it suffices to remove $T_2 \cup T_3$ and the last job of T_1 assigned to M_j . If all the jobs of T_2 have a processing time of at least $1/6L$, then Claim 5 ensures that it suffices to remove $T_2 \cup T_3$. If one job in T_2 has a processing time of at most $1/6L$, then Claim 9 ensures that it suffices to remove the other two jobs of T_2 and T_3 .
- $|T_3| = 2$: We assume that both jobs in T_3 have a processing time of at least $1/6L$ since otherwise, by Claim 8, we can just remove one job of T_3 and T_2 . If $|T_2| = 1$, then by Claim 2 it suffices to remove $T_2 \cup T_3$ and the last job of T_1 assigned to M_j . It remains to consider the case $|T_2| \geq 2$. If none of the jobs in T_2 has a processing time smaller than $1/6L$, then Claim 5 applies. If one of the jobs has a processing time smaller than $1/6L$, then Claim 9 applies and it suffices to remove the at most two other jobs of T_2 and the jobs of T_3 .
- $|T_3| = 3$: Again we assume that all jobs in T_3 have a processing time of at least $1/6L$ since otherwise the desired statement follows from Claim 8. Moreover, we assume $|T_2| > 0$; otherwise we can apply again Claim 2. If there is one job in T_2 having a processing time of at least $1/6L$, the desired number of job removals follows from Claim 5. If this is not the case, then Claim 9 ensures that it suffices to remove the last job of T_2 assigned to M_j as well as T_3 .
- $|T_3| \geq 4$: If four jobs in T_3 have a processing time of at least $1/6L$, then by Claim 5 it is sufficient to remove these four jobs. If at most three jobs have a processing time of at least $1/6L$, then let $J_l \in T_3$ be last jobs assigned to M_j having a processing time smaller than $1/6L$. By Claim 8 it suffices to remove the jobs of T_3 subsequently assigned to M_j , and there exist at most three of these.

This concludes the proof. □

Recall that $ALG(1.75)$ migrates $\lceil m/2 \rceil$ jobs from machines in B . Hence, using the above Lemma 12, we obtain that the total number of migrations is at most $4\lceil m/2 \rceil + \lceil m/2 \rceil \leq 2.5m$. Combined with Theorem 3, this finishes the proof of Theorem 5.

Appendix

Proof of Lemma 1 Fix $m \geq 2$. We first evaluate $f_m(2)$ and $f_m(1 + 1/(3m))$. For $\alpha = 2$, we have $\lceil (1 - 1/\alpha)m \rceil \geq m/2$. Hence $\lceil (1 - 1/\alpha)m \rceil \alpha/m \geq 1$ and $f_m(2) \geq 1$. For $\alpha = 1 + 1/(3m)$, there holds $\lceil (1 - 1/\alpha)m \rceil = 1$. Thus $f_m(1 + 1/(3m)) = 1/(3m)H_{m-1} + 1/m + 1/(3m^2) < 1/3 + 1/2 + 1/12 < 1$. It remains to show that $f_m(\alpha)$ is continuous and strictly increasing. To this end we show that, for any $\alpha > 1$ and small $\epsilon > 0$, $f_m(\alpha) - f_m(\alpha - \epsilon)$ and $f_m(\alpha + \epsilon) - f_m(\alpha)$ converge to 0 as $\epsilon \rightarrow 0$. Moreover $f_m(\alpha + \epsilon) - f_m(\alpha)$ is strictly positive.

First consider an $\alpha > 1$ such that $(1 - 1/\alpha)m \notin \mathbb{N}$. In this case we choose $\epsilon > 0$ such that $\lceil (1 - 1/(\alpha - \epsilon))m \rceil = \lceil (1 - 1/(\alpha + \epsilon))m \rceil = \lceil (1 - 1/\alpha)m \rceil$. We have

$$\begin{aligned} f_m(\alpha) &= (\alpha - 1)(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil \alpha/m \\ f_m(\alpha - \epsilon) &= (\alpha - \epsilon - 1)(H_{m-1} - H_{\lceil (1-1/(\alpha-\epsilon))m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil (\alpha - \epsilon)/m \\ f_m(\alpha + \epsilon) &= (\alpha + \epsilon - 1)(H_{m-1} - H_{\lceil (1-1/(\alpha+\epsilon))m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil (\alpha + \epsilon)/m. \end{aligned}$$

Thus $f_m(\alpha) - f_m(\alpha - \epsilon) = f_m(\alpha + \epsilon) - f_m(\alpha) = \epsilon(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil \epsilon/m$. Hence $f_m(\alpha) - f_m(\alpha - \epsilon)$ and $f_m(\alpha + \epsilon) - f_m(\alpha)$ tend to 0 as $\epsilon \rightarrow 0$. Since $\alpha > 1$ there holds $\lceil (1 - 1/\alpha)m \rceil \geq 1$ and thus $f_m(\alpha + \epsilon) - f_m(\alpha) > 0$.

Next let $\alpha > 1$ such that $(1 - 1/\alpha)m \in \mathbb{N}$. In this case we choose $\epsilon > 0$ such that $\lceil (1 - 1/(\alpha - \epsilon))m \rceil = \lceil (1 - 1/\alpha)m \rceil$ and $\lceil (1 - 1/(\alpha + \epsilon))m \rceil = \lceil (1 - 1/\alpha)m \rceil + 1$. There holds

$$\begin{aligned} f_m(\alpha) &= (\alpha - 1)(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil \alpha/m \\ f_m(\alpha - \epsilon) &= (\alpha - \epsilon - 1)(H_{m-1} - H_{\lceil (1-1/(\alpha-\epsilon))m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil (\alpha - \epsilon)/m \\ f_m(\alpha + \epsilon) &= (\alpha + \epsilon - 1)(H_{m-1} - H_{\lceil (1-1/(\alpha+\epsilon))m \rceil}) + (\lceil (1 - 1/\alpha)m \rceil + 1)(\alpha + \epsilon)/m. \end{aligned}$$

As above $f_m(\alpha) - f_m(\alpha - \epsilon) = \epsilon(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + \lceil (1 - 1/\alpha)m \rceil \epsilon/m$ and the latter expression tends to 0 as $\epsilon \rightarrow 0$. Taking into account that $(1 - 1/\alpha)m \in \mathbb{N}$ we obtain

$$\begin{aligned} f_m(\alpha + \epsilon) - f_m(\alpha) &= -(\alpha - 1) \cdot 1/((1 - 1/\alpha)m) + \epsilon(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil}) \\ &\quad + (\lceil (1 - 1/\alpha)m \rceil + 1)\epsilon/m + \alpha/m \\ &= \epsilon(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil}) + (\lceil (1 - 1/\alpha)m \rceil + 1)\epsilon/m. \end{aligned}$$

Again, $f_m(\alpha + \epsilon) - f_m(\alpha)$ is strictly positive and tends to 0 as $\epsilon \rightarrow 0$. □

Proof of Lemma 2 We first prove that $(\alpha_m)_{m \geq 2}$ is non-decreasing. A first observation is that $\alpha_m \leq m$ because $f_m(m) \geq 1$. We will show that, for any $m \geq 3$ and $1 < \alpha \leq m$,

there holds $f_{m-1}(\alpha) \geq f_m(\alpha)$. This implies $1 = f_{m-1}(\alpha_{m-1}) \geq f_m(\alpha_{m-1})$. By Lemma 1, f_m is strictly increasing and thus $\alpha_m \geq \alpha_{m-1}$. Consider a fixed α with $1 < \alpha \leq m$. We study two cases depending on whether or not $\lceil(1 - 1/\alpha)(m - 1)\rceil = \lceil(1 - 1/\alpha)m\rceil$.

If $\lceil(1 - 1/\alpha)(m - 1)\rceil = \lceil(1 - 1/\alpha)m\rceil$, then

$$\begin{aligned} f_m(\alpha) &= (\alpha - 1)(H_{m-1} - H_{\lceil(1-1/\alpha)m\rceil-1}) + \lceil(1 - 1/\alpha)m\rceil\alpha/m \\ f_{m-1}(\alpha) &= (\alpha - 1)(H_{m-2} - H_{\lceil(1-1/\alpha)m\rceil-1}) + \lceil(1 - 1/\alpha)m\rceil\alpha/(m - 1). \end{aligned}$$

We obtain $f_{m-1}(\alpha) - f_m(\alpha) = -(\alpha - 1)/(m - 1) + \lceil(1 - 1/\alpha)m\rceil\alpha/(m(m - 1)) \geq -(\alpha - 1)/(m - 1) + (\alpha - 1)/(m - 1) = 0$ and thus $f_{m-1}(\alpha) \geq f_m(\alpha)$.

If $\lceil(1 - 1/\alpha)(m - 1)\rceil < \lceil(1 - 1/\alpha)m\rceil$, then $\lceil(1 - 1/\alpha)(m - 1)\rceil = \lceil(1 - 1/\alpha)m\rceil - 1$ and

$$\begin{aligned} f_m(\alpha) &= (\alpha - 1)(H_{m-1} - H_{\lceil(1-1/\alpha)m\rceil-1}) + \lceil(1 - 1/\alpha)m\rceil\alpha/m \\ f_{m-1}(\alpha) &= (\alpha - 1)(H_{m-2} - H_{\lceil(1-1/\alpha)m\rceil-2}) + (\lceil(1 - 1/\alpha)m\rceil - 1)\alpha/(m - 1). \end{aligned}$$

Since $\alpha > 1$ there holds $\lceil(1 - 1/\alpha)(m - 1)\rceil \geq 1$. Hence in our case $\lceil(1 - 1/\alpha)m\rceil \geq 2$ and $\lceil(1 - 1/\alpha)m\rceil - 1 > 0$. We obtain

$$f_{m-1}(\alpha) - f_m(\alpha) = -\frac{\alpha-1}{m-1} + \frac{\alpha-1}{\lceil(1-1/\alpha)m\rceil-1} + \lceil(1 - 1/\alpha)m\rceil \frac{\alpha}{m(m-1)} - \frac{\alpha}{m-1}.$$

Choose x , with $0 \leq x < 1$, such that $\lceil(1 - 1/\alpha)m\rceil = (1 - 1/\alpha)m + x$. Then

$$\begin{aligned} f_{m-1}(\alpha) - f_m(\alpha) &= -\frac{\alpha-1}{m-1} + \frac{\alpha-1}{(1-1/\alpha)m+x-1} + (1 - 1/\alpha)m \frac{\alpha}{m(m-1)} + \frac{\alpha x}{m(m-1)} - \frac{\alpha}{m-1} \\ &= \frac{\alpha-1}{(1-1/\alpha)m+x-1} + \frac{\alpha x}{m(m-1)} - \frac{\alpha}{m-1} \end{aligned}$$

In order to establish $f_{m-1}(\alpha) - f_m(\alpha) \geq 0$ it suffices to show

$$\frac{\alpha-1}{(1-1/\alpha)m+x-1} \geq \frac{\alpha(m-x)}{m(m-1)}.$$

This is equivalent to $(\alpha - 1)m(m - 1) \geq (m - x)((\alpha - 1)m + \alpha x - \alpha)$. Standard algebraic manipulation yield that this is equivalent to $m \geq mx - \alpha x^2 + \alpha x$. Let $g(x) = mx - \alpha x^2 + \alpha x$, for any real number x . This function is increasing for any $x < (m + \alpha)/(2\alpha)$. Since $\alpha \leq m$, the function is increasing for any $x < 1$. As $g(0) = 0$ and $g(1) = m$, it follows that $m \geq mx - \alpha x^2 + \alpha x$ holds for all $0 \leq x < 1$. We conclude $f_{m-1}(\alpha) - f_m(\alpha) \geq 0$.

It is easy to verify that $f_2(4/3) = 1$. We show that $\lim_{m \rightarrow \infty} \alpha_m$ is upper bounded by $W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2))$. Cesáro [5] proved

$$0 < H_m - \frac{1}{2} \ln(m(m + 1)) - \gamma < \frac{1}{6m(m + 1)}, \tag{1}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Using this inequality we find, for any c with $0 < c \leq 1$ and $\lceil cm \rceil - 2 > 0$,

$$\begin{aligned} H_{m-1} - H_{\lceil cm \rceil - 2} &> \frac{1}{2} \ln((m-1)m) + \gamma - \frac{1}{2} \ln((\lceil cm \rceil - 2)(\lceil cm \rceil - 1)) \\ &\quad - \gamma - \frac{1}{6(\lceil cm \rceil - 2)(\lceil cm \rceil - 1)} \\ &\geq \frac{1}{2} (\ln(m-1) + \ln m - \ln(cm-1) - \ln(cm)) - \frac{1}{2(\lceil cm \rceil - 1)} \\ &= \frac{1}{2} (\ln(m-1) + \ln m - \ln(c(m-1/c)) - \ln(cm)) - \frac{1}{2(\lceil cm \rceil - 1)} \\ &= \frac{1}{2} (\ln(m-1) - \ln(m-1/c) - 2 \ln(c)) - \frac{1}{2(\lceil cm \rceil - 1)} \\ &\geq \frac{1}{2} (2 \ln(1/c)) - \frac{1}{2(\lceil cm \rceil - 1)} \\ &\geq \ln(1/c) - \frac{1}{2(cm-1)}, \end{aligned}$$

where the second to last inequality holds since $\ln(m-1/c) \leq \ln(m-1)$. for $0 < c \leq 1$ and sufficiently large m . We obtain

$$\begin{aligned} f_m(\alpha) &= (\alpha - 1)(H_{m-1} - H_{\lceil (1-1/\alpha)m \rceil - 1}) + (\lceil (1-1/\alpha)m \rceil) \frac{\alpha}{m} \\ &> (\alpha - 1) \left(\ln\left(\frac{\alpha}{\alpha-1}\right) - \frac{1}{2((1-1/\alpha)m-1)} - \frac{1}{\lceil (1-1/\alpha)m \rceil - 1} \right) \\ &\quad + (\lceil (1-1/\alpha)m \rceil) \frac{\alpha}{m} \\ &\geq (\alpha - 1) \left(\ln\left(\frac{\alpha}{\alpha-1}\right) - \frac{1}{(1-1/\alpha)m-1} \right) + \alpha - 1 =: F(m). \end{aligned}$$

Obviously, $\lim_{m \rightarrow \infty} F(m) = (\alpha - 1) \ln(\frac{\alpha}{\alpha-1}) + \alpha - 1$. We show that $(\alpha - 1) \ln(\frac{\alpha}{\alpha-1}) + \alpha - 1 = 1$, for $\alpha = \frac{1}{1-\delta}$, where $\delta = -1/W_{-1}(-1/e^2)$.

Equation $(\alpha - 1) \ln(\frac{\alpha}{\alpha-1}) + \alpha - 1 = 1$ is equivalent to $\ln(\frac{\alpha}{\alpha-1}) + 1 = \frac{1}{\alpha-1}$, which in turn is equivalent to

$$\frac{\alpha}{\alpha - 1} \cdot e = e^{\frac{1}{\alpha-1}}.$$

Substituting $x = 1/(\alpha - 1)$, which is equivalent to $\alpha = 1/x + 1$, we find that the above is equivalent to $xe + e = e^x$. Applying the Lambert W function we find that $x = -W_{-1}(-1/e^2) - 1$ is a solution of the former equality. Substituting we conclude that in fact $\alpha = W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2))$ satisfies the equality. Using the same techniques we can show that $\lim_{m \rightarrow \infty} \alpha_m$ is lower bounded by $W_{-1}(-1/e^2)/(1 + W_{-1}(-1/e^2))$. In the calculations, (1) yields that $H_{m-1} - H_{\lceil cm \rceil} < \ln(1/c) + 1/(2m)$. □

References

1. Aggarwal, G., Motwani, R., Zhu, A.: The load rebalancing problem. *J. Algorithms* **60**(1), 42–59 (2006)
2. Albers, S.: Better bounds for online scheduling. *SIAM J. Comput.* **29**, 459–473 (1999)
3. Bartal, Y., Karloff, H., Rabani, Y.: A better lower bound for on-line scheduling. *Inf. Process. Lett.* **50**, 113–116 (1994)
4. Bartal, Y., Fiat, A., Karloff, H., Vohra, R.: New algorithms for an ancient scheduling problem. *J. Comput. Syst. Sci.* **51**, 359–366 (1995)
5. Cesàro, E.: Sur la série harmonique. *Nouvelles Ann. de Math.* 3e Sér. **4**, 295–296 (1885)
6. Chen, B., van Vliet, A., Woeginger, G.J.: A lower bound for randomized on-line scheduling algorithms. *Inf. Process. Lett.* **51**, 219–222 (1994)
7. Chen, B., van Vliet, A., Woeginger, G.J.: A optimal algorithm for preemptive online scheduling. *Oper. Res. Lett.* **18**, 127–131 (1995)
8. Englert, M., Özmen, D., Westermann, M.: The power of reordering for online minimum makespan scheduling. *SIAM J. Comput.* **43**(3), 1220–1237 (2014)
9. Faigle, U., Kern, W., Turan, G.: On the performance of on-line algorithms for partition problems. *Acta Cybern.* **9**, 107–119 (1989)
10. Fleischer, R., Wahl, M.: Online scheduling revisited. *J. Sched.* **3**, 343–353 (2000)
11. Galambos, G., Woeginger, G.: An on-line scheduling heuristic with better worst case ratio than Graham’s list scheduling. *SIAM J. Comput.* **22**, 349–355 (1993)
12. Graham, R.L.: Bounds for certain multi-processing anomalies. *Bell Syst. Tech. J.* **45**, 1563–1581 (1966)
13. Graham, R.L.: Bounds on multiprocessing timing anomalies. *SIAM J. Appl. Math.* **17**(2), 416–429 (1969)
14. Gormley, T., Reingold, N., Torng, E., Westbrook, J.: Generating adversaries for request-answer games. In: *Proceedings of 11th ACM-SIAM Symposium on Discrete Algorithms*, pp. 564–565 (2000)
15. Hochbaum, D.S., Shmoys, D.B.: Using dual approximation algorithms for scheduling problems: theoretical and practical results. *J. ACM* **34**, 144–162 (1987)
16. Karger, D.R., Phillips, S.J., Torng, E.: A better algorithm for an ancient scheduling problem. *J. Algorithms* **20**, 400–430 (1996)
17. Min, X., Liu, J., Wang, Y.: Optimal semi-online algorithms for scheduling problems with reassignment on two identical machines. *Inf. Process. Lett.* **111**(9), 423–428 (2011)
18. Rudin, III, J.F.: Improved bounds for the on-line scheduling problem. Ph.D. Thesis. The University of Texas at Dallas, May 2001
19. Rudin III, J.F., Chandrasekaran, R.: Improved bounds for the online scheduling problem. *SIAM J. Comput.* **32**, 717–735 (2003)
20. Sanders, P., Sivadasan, N., Skutella, M.: Online scheduling with bounded migration. *Math. Oper. Res.* **34**(2), 481–498 (2009)
21. Sgall, J.: A lower bound for randomized on-line multiprocessor scheduling. *Inf. Process. Lett.* **63**, 51–55 (1997)
22. Tan, Z., Yu, S.: Online scheduling with reassignment. *Oper. Res. Lett.* **36**(2), 250–254 (2008)
23. Sleator, D.D., Tarjan, R.E.: Amortized efficiency of list update and paging rules. *Commun. ACM* **28**, 202–208 (1985)