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A Primal-Dual Approximation Algorithm for the Facility Location Problem with Submodular Penalties

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Abstract We consider the facility location problem with submodular penalties (FLPSP), introduced by Hayrapetyan et al. (Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 933–942, 2005), who presented a 2.50-approximation algorithm that is *non-combinatorial* because this algorithm has to solve the LP-relaxation of an integer program with exponential number of variables. The only known polynomial algorithm for this exponential LP is via the ellipsoid algorithm as the corresponding separation problem for its dual program can be solved in polynomial time. By exploring the properties of the submodular function, we offer a primal-dual 3-approximation *combinatorial* algorithm for this problem.

Keywords Facility location problem · Approximation algorithm · Submodular function

1 Introduction

Facility location problems have been extensively investigated since the early 1960s. These problems are usually *NP*-hard, implying that no polynomial-time exact algorithm is expected to exist unless $P = NP$. Consequently, most of the research has

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been focusing on designing approximation algorithms with good performance. Generally speaking, exisiting approaches for designing approximation algorithms for facility location problems can be divided into three categories, namely, *LP-rounding techniques*, *local search heuristics*, and *primal-dual method*. The first technique can result in non-combinatorial algorithms, while the last two techniques are usually combinatorial in nature. These approaches can also be mixed or combined with other techniques to yield better approximation algorithms.

One of the most basic facility location problems is the *metric uncapacitated facility location problem* (UFLP): Given a set of facility sites and a set of clients, we want to decide which facilities to open and how to assign the clients to the open facilities such that the total cost of opening facilities and assigning clients to the open facilities is minimized. Many approximation algorithms exist for this problem. The algorithms by Shmoys et al. [16], Chudak and Shmoys [7], and Sviridenko [17] are all based on LP-rounding technique. The algorithm of Korupolu et al. [14] is based on local search heuristic technique. The algorithm of Jain and Vazirani [12] is based on primal-dual technique. Moreover, these approaches can be mixed or combined with other skills such as greedy augmentation, cost scaling and dual-fitting [13, 15] to yield better approximation algorithms. So far, the best approximation ratio for the UFLP is 1.50 due to Byrka and Aardal [3]. Their algorithm is based on LP-rounding approach integrated with greedy and dual-fitting. In addition, Guha and Khuller [10] proved that the best approximation ratio is at least 1.463 unless $P = NP$.

Many variants of the basic UFLP have appeared in the literature (e.g., [1, 2, 8, 18, 21–23] and the references therein). We are particularly interested in one of the variants, namely, the *facility location problem with penalties* (FLPWP), which is the same as the UFLP except that not all clients are required to be connected to a facility and any unconnected client j incurs a penalty cost p_j. The objective is to minimize the total opening, connecting and penalizing cost. This problem was first studied by Charikar et al. [5], who gave a 3-approximation primal-dual (and hence combinatorial) algorithm. Later, Xu and Xu improved the ratio to $(2 + 2/e)$ and 1.8526 using LP-rounding and primal-dual with greedy adding technique respectively [19, 20].

The main focus of this work is an extension of the above problem, called the *facility location problem with submodular penalties* (FLPSP), first introduced by Hayrapetyan et al. [11]. The FLPSP extends the above FLPWP by assuming that the penalty cost is a monotone increasing submodular function $h(\cdot)$ defined on clients set D. A function $f(\cdot)$ is *submodular* if it is defined on a finite set V, satisfying

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y), \quad X, Y \subseteq V.$$

Function f is *monotone increasing* if $f(X) \leq f(Y)$ for any $X \subseteq Y$, where $X, Y \subseteq V$. They presented an $(1 + \gamma)$-approximation algorithm where γ is the approximation factor of an LP-based approximation algorithm for the corresponding UFLP, resulting in a 2.50-approximation algorithm for the FLPSP as the best known ratio of LP-based approximation algorithms for the UFLP is 1.50 [3]. However, their algorithm is non-combinatorial because it has to solve the LP-relaxation of an integer program with exponential number of variables. The only known algorithm for this exponential LP is via the ellipsoid algorithm as the corresponding separation problem for its dual program can be solved in polynomial time. Later, Chudak and Nagano [6] offered a more efficient (still non-combinatorial) approximation algorithm with slightly worse approximation ratio of $(1+\varepsilon)(1+\gamma)$. Their algorithm is based on an equivalent compact convex relaxation for the FLPSP. They proposed efficient non-combinatorial algorithms to solve this convex relaxation by using non-smooth convex optimization technique. We continue this line of research to offer a primal-dual 3-approximation *combinatorial* algorithm for the FLPSP by exploring the properties of submodular function.

We present our algorithm and its analysis in Sects. 2 and 3 respectively, followed by some concluding remarks in Sect. 4.

2 The Primal-Dual Algorithm

In the FLPSP, we are given a set of facilities F, and a set of clients D. Each facility has an open cost f_i, and a connection cost c_{ij} for assigning client j to facility i. There is also a monotone increasing submodular function $h(\cdot)$ defined on the clients set D, which serves as the penalty function for any set of clients $S \subseteq D$ that are not connected to any open facility. Moreover, we assume that $h(\cdot)$ is given by an oracle which returns $h(S)$ for any given $S \subseteq D$. Our objective is to choose a facility set $\bar{F} \subseteq F$ to open and a client set $\tilde{S} \subseteq D$ to be penalized such that all the clients in $D \backslash \tilde{S}$ are connected to the open facilities in $\bar{F}$, and the total cost of opening facilities, connecting clients to open facilities and penalizing unconnected clients is minimized. Finally, we assume that the connection cost c_{ij}'s satisfy symmetry and the triangle inequality, that is, the metric version.

This problem can be formulated as the following integer program:

$$\begin{aligned} \min\ & \sum_{i\in F, j\in D} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i + \sum_{S\subseteq D} h(S) z_S \\ \text{s.t.}\ & \sum_{i\in F} x_{ij} + \sum_{S\subseteq D: j\in S} z_S \ge 1, \quad \forall j \in D, \\ & x_{ij} \le y_i, \quad \forall i \in F, j \in D, \\ & x_{ij} \in \{0,1\}, \quad \forall i \in F, j \in D, \\ & y_i \in \{0,1\}, \quad \forall i \in F, \\ & z_S \in \{0,1\}, \quad \forall S \subseteq D. \end{aligned} \tag{2.1}$$

Variable y_i indicates whether facility i is open or not, x_{ij} indicates whether client j is connected to facility i or not, and z_S indicates whether the set $S \subseteq D$ is penalized or not. The first constraint states that a client is either connected to a facility or contained in some subset of D that is not connected. The second constraint states that if a client is connected to a facility, then this facility must be open. Relaxing the integrality constraints, we obtain the relaxation program (2.2) and its dual program (2.3)

$$\min \sum_{i\in F, j\in D} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i + \sum_{S\subseteq D} h(S) z_S$$

$$
\begin{aligned}
\text{s.t.}\ & \sum_{i\in F} x_{ij} + \sum_{S\subseteq D: j\in S} z_S \geq 1, \quad \forall j \in D,\\
& x_{ij} \leq y_i, \quad \forall i \in F, j \in D,\\
& x_{ij} \geq 0, \quad \forall i \in F, j \in D,\\
& y_i \geq 0, \quad \forall i \in F,\\
& z_S \geq 0, \quad \forall S \subseteq D.
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\max\ & \sum_{j\in D} \alpha_j\\
\text{s.t.}\ & \alpha_j \leq c_{ij} + \beta_{ij}, \quad \forall i \in F, j \in D,\\
& \sum_{j\in S} \alpha_j \leq h(S), \quad \forall S \subseteq D,\\
& \sum_{j\in D} \beta_{ij} \leq f_i, \quad \forall i \in F,\\
& \alpha_j \geq 0, \quad \forall j \in D,\\
& \beta_{ij} \geq 0, \quad \forall i \in F, j \in D.
\end{aligned} \tag{2.3}
$$

Intuitively, the first and the third constraints in (2.3) suggest that variable α_j being viewed as a budget that j is willing to pay for getting connected to a facility. Part of the budget pays for the connection cost and the rest pays for the facility open cost. The second constraints imply that any budget collection is less than the corresponding penalty collection cost. If some clients' budget collection equals to the corresponding penalty collection cost, these clients tend to pay the joint penalties.

Similar to the approach of Jain and Vazirani [12], we present a primal-dual algorithm for the FLPSP, which consists of two phases. In the first phase, we construct a dual feasible solution, leading to a primal feasible solution. But this solution may have redundancies, which needs to be purified in the second phase.

Algorithm 2.1 (The primal-dual algorithm)

Phase 1: Constructing a dual feasible solution. We introduce a notion of time t, and start the algorithm at time $t = 0$. Initially, all dual variables are set to 0. All facilities are *closed* and all clients are *unfrozen*. Let $\tilde{S}$ denote the set of penalized clients, and set $\tilde{S} := \emptyset$. We increase the dual variables α_j's for all unfrozen clients $j \in D$ uniformly at unit rate t. The algorithm declares an edge (i, j) *tight* if $\alpha_j = c_{ij}$. Henceforth, such an edge satisfies $\alpha_j = c_{ij} + \beta_{ij}$ as β_{ij} will be increased at the same rate as α_j. Keep increasing time t until there is no unfrozen client. As time increases, the following events may occur:

Event 1. Facility i is *temporarily* open if $\sum_{j\in D} \beta_{ij} = f_i$. In this case, freeze those unfrozen clients $j \in D$ with $\beta_{ij} > 0$ and connect them to facility i, which is called the *connecting witness* for j.

Event 2. If $\alpha_j = c_{ij}$ for temporarily open facility i and unfrozen client j, then freeze j and connect it to i, which is also called the connecting witness for j.
Event 3. If $\sum_{j\in S}\alpha_j = h(S)$ for some set S, then freeze those unfrozen clients in S and set $\tilde{S} := \tilde{S} \cup S$. Any client contained in $\tilde{S}$ is said to be a penalized client.
When all clients are frozen, the first phase terminates. If several events occur simultaneously, the algorithm executes them in an arbitrary order.

Phase 2: Opening facilities. Let F' be the set of temporarily open facilities. We choose set $\tilde{S}$ at the end of Phase 1 to be the set of unconnected clients. We say facilities i and i' ($i, i' \in F'$) are dependent if there exists some client $j \in D$ such that both $\beta_{ij} > 0$ and $\beta_{i'j} > 0$. We choose a maximal independent subset $\bar{F} \subseteq F'$ to open. Connect each client in $D \backslash \tilde{S}$ to an open facility that is closest to it.

Let *SOL* be the solution of Algorithm 2.1, whose cost consists of facility cost F_{SOL}, connection cost C_{SOL}, and penalty cost P_{SOL}.

3 Analysis

First, we will show that our algorithm is a well-defined polynomial time combinatorial algorithm. This claim follows from Lemmas 3.1 and 3.2 below.

Lemma 3.1 *Consider any time $\tilde{t}$ at which one of the three events of Phase* 1 *in Algorithm* 2.1 *occurs. Then we can find the next closest time t^* such that one of the three events occurs again in polynomial time.*

Proof Let $\tilde{D}$ be the set of frozen clients and $\tilde{F}$ be the set of temporarily open facilities at time $\tilde{t}$. Now we consider the following three possibilities.

Case 1. Event 1 occurs at the next closest time t_1^*. One can show that $t_1^* = \min_{i\in F\backslash\tilde{F}}\{t_{1i}^*\}$, where t_{1i}^* is the root of the following equation with respect to t

$$\sum_{j\in\tilde{D}}\beta_{ij} + \sum_{j\in D\backslash\tilde{D}}\max\{t - c_{ij},\ 0\} = f_i, \quad i \in F \setminus \tilde{F}.$$

Evidently t_1^* can be found in polynomial time by comparing t_{1i}^* for all $i \in F \setminus \tilde{F}$ as long as each t_{1i}^* can be found in polynomial time. For the latter, sort the costs $\{c_{ij} : j \in D \setminus \tilde{D}\}$ in a nondecreasing order, generating at most $|D \setminus \tilde{D}| + 1$ intervals along $[0, \infty)$. Then t_{1i}^* can be found by exhausting all these (polynomial number of) intervals.

Case 2. Event 2 occurs at the next closest time $t_2^* = \min_{i\in\tilde{F}, j\in D\backslash\tilde{D}}\{c_{ij}\}$.

Case 3. Event 3 occurs at the next closest time t_3^*. In Algorithm 2.1, we always maintain the following inequality from time $\tilde{t}$ to t_3^*

$$\sum_{j\in S\backslash\tilde{D}} t + \sum_{j\in S\cap\tilde{D}}\alpha_j \le h(S), \quad \forall S \subseteq D, \tag{3.1}$$

which implies that

$$t \le \frac{h(S) - \sum_{j\in S\cap\tilde{D}} \alpha_j}{|S\backslash\tilde{D}|}, \quad \forall S \subseteq D : S\backslash\tilde{D} \ne \emptyset. \tag{3.2}$$

Then we can calculate the time t_3^* by the following formula

$$t_3^* = \min_{S\subseteq D:S\backslash\tilde{D}\ne\emptyset} \frac{h(S) - \sum_{j\in S\cap\tilde{D}} \alpha_j}{|S\backslash\tilde{D}|}. \tag{3.3}$$

The last problem is the minimization of a ratio of a submodular function and a modular function, and it can be solved in polynomial time by a combinatorial algorithm [9].

Setting $t^* = \min\{t_1^*, t_2^*, t_3^*\}$ concludes the proof. □

Lemma 3.2 *At any time t of Phase 1 in Algorithm 2.1, the set $\tilde{S}$ always satisfies the following property*

$$\sum_{j\in\tilde{S}} \alpha_j(t) = h(\tilde{S}),$$

where $\alpha_j(t)$ is the value of the budget of client j at time t, and increases with time t until client j is frozen.

Proof Let S_1 and S_2 be two arbitrary sets such that

$$\sum_{j\in S_1} \alpha_j(t_1) = h(S_1)$$

at time t_1 and

$$\sum_{j\in S_2} \alpha_j(t_2) = h(S_2)$$

at time t_2 $(\ge t_1)$. Since client $j \in S_1$ is frozen at time t_1, we have $\alpha_j(t) = \alpha_j(t_1)$ for any $t \ge t_1$, $j \in S_1$.

To prove this lemma, we only need to show that

$$\sum_{j\in S_1\cup S_2} \alpha_j(t_2) = h(S_1 \cup S_2).$$

From Algorithm 2.1 and the submodularity of $h(\cdot)$, we have

$$\begin{aligned}\sum_{j\in S_1\cup S_2} \alpha_j(t_2) + \sum_{j\in S_1\cap S_2} \alpha_j(t_2) &= \sum_{j\in S_1} \alpha_j(t_1) + \sum_{j\in S_2} \alpha_j(t_2)\\ &= h(S_1) + h(S_2)\\ &\ge h(S_1\cup S_2) + h(S_1\cap S_2).\end{aligned} \tag{3.4}$$

Moreover, in Algorithm 2.1, we always maintain

$$\sum_{j\in S_1\cup S_2}\alpha_j(t_2)\le h(S_1\cup S_2),\qquad \sum_{j\in S_1\cap S_2}\alpha_j(t_2)\le h(S_1\cap S_2). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\sum_{j\in S_1\cup S_2}\alpha_j(t_2)+\sum_{j\in S_1\cap S_2}\alpha_j(t_2)=h(S_1\cup S_2)+h(S_1\cap S_2),$$

implying that

$$\sum_{j\in S_1\cup S_2}\alpha_j(t_2)=h(S_1\cup S_2).$$

□

Secondly, we will bound F_{SOL} and C_{SOL} in the next two lemmas. Let us denote the neighborhood of facility $i\in F$ as follows

$$N_i:=\{j\mid \beta_{ij}>0\}.$$

From the construction of $\bar{F}$, we have

$$N_i\cap N_{i'}=\emptyset,\quad \text{for all } i,\ i'\in\bar{F}. \tag{3.6}$$

It follows from Algorithm 2.1 that

Lemma 3.3

$$F_{SOL}=\sum_{i\in\bar{F}}f_i=\sum_{i\in\bar{F}}\sum_{j\in N_i}\beta_{ij}.$$

For any client $j\in D\setminus\tilde{S}$, let $i(j)$ be the connecting witness for j. Denote $D_{\bar{F}}:=\bigcup_{i\in\bar{F}}N_i$, $D_1:=\{j\mid i(j)\in\bar{F}, j\notin D_{\bar{F}}\cup\tilde{S}\}$, and $D_2:=D\setminus(D_{\bar{F}}\cup D_1\cup\tilde{S})$. The connection cost of SOL is bounded in the following lemma.

Lemma 3.4

$$C_{SOL}\le\sum_{i\in\bar{F}}\sum_{j\in N_i\setminus\tilde{S}}c_{ij}+\sum_{j\in D_1}\alpha_j+3\sum_{j\in D_2}\alpha_j.$$

Proof For any client $j\in D\setminus\tilde{S}$, consider the following three possibilities.

Case 1. $j\in D_{\bar{F}}\setminus\tilde{S}$. There exists $i\in\bar{F}$ such that $j\in N_i$ and $j\notin D_{\bar{F}}\setminus N_i$. Connect j to facility i with connection cost c_{ij}.

Case 2. $j\in D_1$. Since $i(j)\in\bar{F}$. Connect j to $i(j)$ with connection cost $c_{i(j)j}=\alpha_j$.

Case 3. $j\in D_2$. Since $i(j)\notin\bar{F}$, there exists a facility $i\in\bar{F}$ and a client j' such that $\beta_{i(j)j'}>0$ and $\beta_{ij'}>0$. Connect j to i. Let t_1 and t_2 be the times at which facilities $i(j)$ and i' are temporarily open respectively. Since $\beta_{i(j)j'}>0$ and $\beta_{ij'}>0$, we have $\alpha_{j'} \geq c_{i(j)j'}$ and $\alpha_{j'} \geq c_{ij'}$. Client j' is frozen earlier than $\min\{t_1, t_2\}$, implying that $\alpha_{j'} \leq \min\{t_1, t_2\}$. Since facility $i(j)$ is a connecting witness for j, we obtain that $\alpha_j \geq t_1$. Consequently, $\alpha_j \geq \alpha_{j'}$. From the triangle inequality, we have that

$$c_{ij} \leq c_{ij'} + c_{i(j)j'} + c_{i(j)j} \leq 3\alpha_j.$$

Summarizing the three cases above concludes the proof. □

Finally, we are ready to present our main result.

Theorem 3.1 *Algorithm 2.1 is a 3-approximation combinatorial algorithm for the FLPSP.*

Proof It follows from (3.6) and Lemmas 3.3–3.4 that the cost of *SOL* is at most

$$\begin{aligned}
\mathrm{cost}(SOL) &= F_{SOL} + C_{SOL} + P_{SOL} \\
&\leq \sum_{i\in\bar{F}}\sum_{j\in N_i}\beta_{ij} + \sum_{i\in\bar{F}}\sum_{j\in N_i\setminus\tilde{S}} c_{ij} + \sum_{j\in D_1}\alpha_j + 3\sum_{j\in D_2}\alpha_j + \sum_{j\in\tilde{S}}\alpha_j \\
&\leq \sum_{j\in D_{\bar{F}}\setminus\tilde{S}}\alpha_j + \sum_{j\in D_1}\alpha_j + 3\sum_{j\in D_2}\alpha_j + 2\sum_{j\in\tilde{S}}\alpha_j \\
&\leq 3\sum_{j\in D}\alpha_j.
\end{aligned}$$

□

4 Concluding Remarks

In this paper, we offer a primal-dual 3-approximation combinatorial algorithm for the FLPSP by exploring the properties of submodular functions. There are two interesting questions for future research.

Chariker and Guha [4] present a greedy augmentation procedure for the UFLP. They bound the cost of the solution after running the greedy augmentation procedure in terms of the cost of the initial solution and an arbitrary solution. It is natural to consider the following algorithm for the FLPSP (cf. [15, 20]).

Algorithm 4.1

1. Scale the opening costs of all facilities by a factor of δ (> 0) and then apply Algorithm 2.1 to obtain solution SOL_1.
2. Run the greedy augmentation procedure on SOL_1 and return solution SOL_2.

Currently we do not know how to bound the cost of SOL_1 in terms of those of the initial solution and an arbitrary solution of FLPSP. Therefore, how to analyze the performance guarantee of Algorithm 4.1 is an open question. One can obtain the following inequality from the proof of Theorem 3.1,

$$2F_{SOL} + C_{SOL} + P_{SOL} \leq 3 \sum_{j \in D} \alpha_j,$$

which may be useful to analyze the performance guarantee of Algorithm 4.1.

Another question is to further improve the approximation ratio for the FLPSP using other techniques such as dual-fitting (cf. [13]).

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