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On Covering Problems of Rado

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Abstract T. Rado conjectured in 1928 that if $\mathcal{F}$ is a finite set of axis-parallel squares in the plane, then there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise disjoint squares, such that $\mathcal{I}$ covers at least $1/4$ of the area covered by $\mathcal{F}$. He also showed that the greedy algorithm (repeatedly choose the largest square disjoint from those previously selected) finds an independent set of area at least $1/9$ of the area covered by $\mathcal{F}$. The analogous question for other shapes and many similar problems have been considered by R. Rado in his three papers (in Proc. Lond. Math. Soc. 51:232–264, 1949; 53:243–267, 1951; and J. Lond. Math. Soc. 42:127–130, 1968) on this subject. After 45 years, Ajtai (in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. 21:61–63, 1973) came up with a surprising example disproving T. Rado's conjecture. We revisit Rado's problem and present improved upper and lower bounds for squares, disks, convex bodies, centrally symmetric convex bodies, and others, as well as algorithmic solutions to these variants of the problem.

Keywords Discrete and computational geometry · Approximation algorithms

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1 Introduction

Rado's problem on selecting disjoint squares is a famous unsolved problem in geometry [5, Problem D6]: What is the largest number c such that, for any finite set $\mathcal{F}$ of axis-parallel squares in the plane, there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise disjoint squares with total area at least c times the union area of the squares in $\mathcal{F}$? T. Rado [14] observed that a greedy algorithm, which repeatedly selects the largest square disjoint from those previously selected, finds an independent subset $\mathcal{I}$ of disjoint squares with total area at least $1/9$ of the area of the union of all squares in $\mathcal{F}$. This lower bound has been improved by R. Rado [15] to $1/8.75$ in 1949, and further improved by Zalgaller [22] to $1/8.6$ in 1960. On the other hand, an upper bound of $1/4$ for the area ratio is obvious: take four unit squares sharing a common vertex, then only one of them may be selected.

T. Rado conjectured that, for any finite set of axis-parallel squares, at least $1/4$ of the union area can be covered by a subset of disjoint squares. For congruent squares, the conjecture was confirmed by Norlander [11], Sokolin [19], and Zalgaller [22]. For the general case, Ajtai [1] came up with an ingenious construction with several hundred squares and disproved T. Rado's conjecture in 1973! The problem of selecting disjoint squares has also been considered by R. Rado in a more general setting for various classes of convex bodies, in his three papers entitled "Some covering theorems" [15–17].

We introduce some definitions. Throughout the paper, the term "convex body" refers to a compact convex set with nonempty interior. For a convex body S in $\mathbb{R}^d$, denote by $|S|$ the Lebesgue measure of S, i.e., the length when $d = 1$, the area when $d = 2$, or the volume when $d \geq 3$. For a finite set $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$, denote by $|\mathcal{F}| = |\bigcup_{S \in \mathcal{F}} S|$ the Lebesgue measure of the union of the convex bodies in $\mathcal{F}$; when $d = 2$, we call $|\mathcal{F}|$ the *union area* of $\mathcal{F}$. For a convex body S in $\mathbb{R}^d$, define

$$F(S) = \inf_{\mathcal{F}} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{F}|},$$

where $\mathcal{F}$ ranges over all finite sets of convex bodies in $\mathbb{R}^d$ that are homothetic to S, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{F}$. Also define

$$f(S) = \inf_{\mathcal{F}_1} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{F}_1|},$$

where $\mathcal{F}_1$ ranges over all finite sets of convex bodies in $\mathbb{R}^d$ that are homothetic and *congruent* to S, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{F}_1$.

For the one-dimensional case, it is known that $f(S) = F(S) = 1/2$ for an interval S [2, 15]. The aforementioned results of Zalgaller [22] and Ajtai [1], respectively, give lower and upper bounds of $1/8.6 \leq F(S) \leq 1/4 - 1/1728$ for a square S. In Sect. 2, we present a very simple Algorithm A1 that improves the now 48 years old lower bound by Zalgaller [22]. Our algorithm is based on a novel idea that can be easily generalized to obtain improved lower bounds for hypercubes in *any* dimension $d \geq 2$.

Let λ_d be the unique solution in $[(5/2)^d, 3^d]$ to the equation

$$3^d - (\lambda^{1/d} - 2)^d/2 = \lambda. \tag{1}$$

As it will be shown later, $\lambda_d \leq (3 - (d + 2d \cdot 3^{d-1})^{-1})^d$. For $d = 2$, $\lambda_2 = (\sqrt{46} + 2)^2/9 = 8.5699\ldots$.

Theorem 1 *For a hypercube S in $\mathbb{R}^d$, $F(S) \geq 1/\lambda_d$. Moreover, given a set $\mathcal{F}$ of n axis-parallel hypercubes in $\mathbb{R}^d$, $d \geq 2$, an independent set $\mathcal{I} \subseteq \mathcal{F}$ such that $|\mathcal{I}|/|\mathcal{F}| \geq 1/\lambda_d$ can be computed by Algorithm* A1 *in $O(dn^2)$ time.*

In Sect. 3, we generalize the algorithmic idea of A1 to obtain an improved lower bound of $F(S) \geq 1/\lambda_2 > 1/8.5699$ for *any* centrally symmetric convex body S in the plane. The previous best lower bound of $F(S) \geq (1 + 1/200704)/9 = 1/8.999955\ldots$ was obtained by R. Rado in 1949 [15, Theorem 8].

Theorem 2 *For any centrally symmetric convex body S in the plane, $F(S) \geq 1/\lambda_2 > 1/8.5699$.*

In Sect. 4, we obtain an improved lower bound for the special case that the centrally symmetric convex body S is a disk. The previous best lower bound for disks was $F(S) > 1/8.4898$ [2].

Theorem 3 *For a disk S, $F(S) \geq 1/\lambda_{\text{disk}}$, where $\lambda_{\text{disk}} = 8.3539\ldots$. Moreover, given a set $\mathcal{F}$ of n disks in the plane, an independent set $\mathcal{I} \subseteq \mathcal{F}$ such that $|\mathcal{I}|/|\mathcal{F}| \geq 1/\lambda_{\text{disk}}$ can be computed by Algorithm* B1 *in $O(n^3)$ time.*

In Sect. 5, we present another Algorithm A2 that achieves an even better lower bound for squares. The Algorithm Z implicit in Zalgaller's lower bound [22] computes an independent set by repeatedly adding at most four disjoint squares at a time; our Algorithm A2 adds at most three squares at a time.

Theorem 4 *For a square S, $F(S) \geq 1/\lambda_{\text{square}}$, where $\lambda_{\text{square}} = 8.4797\ldots$. Moreover, given a set $\mathcal{F}$ of n axis-parallel squares in the plane, an independent set $\mathcal{I} \subseteq \mathcal{F}$ such that $|\mathcal{I}|/|\mathcal{F}| \geq 1/\lambda_{\text{square}}$ can be computed by Algorithm* A2 *in $O(n^2)$ time.*

In Sect. 6, we present an improved upper bound for squares. Our construction refines Ajtai's idea [1] and consists of an infinite number of squares tiling the plane.

Theorem 5 *For a square S, $F(S) \leq \frac{1}{4} - \frac{1}{384}$.*

We know much more about $f(S)$ than about $F(S)$. For example, $f(S) = 1/4$ for a square S; see [5]. R. Rado [15] showed that $f(S) = 1/6$ for a triangle S, $f(S) = 1/4$ for a centrally symmetric hexagon S, $f(S) \geq \frac{\pi}{8\sqrt{3}} > 1/4.4106$ for a disk S, $f(S) \geq 1/16$ for any convex body S in the plane, and $f(S) \geq 1/7$ for any centrally symmetric convex body S in the plane. In Sect. 7, we improve the lower bounds on $f(S)$ for the two latter cases.

Table 1

Convex body S in $\mathbb{R}^2$	Old bound		New bound	
Square	$F(S) < 1/4 - 1/1728$	[1]	$F(S) \leq 1/4 - 1/384$	Thm. 5
Square	$F(S) \geq 1/8.6$	[22]	$F(S) > 1/8.4797$	Thm. 4
Disk	$F(S) > 1/8.4898$	[2]	$F(S) > 1/8.3539$	Thm. 3
Centrally symmetric	$F(S) > 1/8.999955$	[15]	$F(S) > 1/8.5699$	Thm. 2
Centrally symmetric	$f(S) \geq 1/7$	[15]	$f(S) > 1/4.4810$	Thm. 6
Arbitrary	$f(S) \geq 1/16$	[15]	$f(S) \geq 1/6$	Thm. 6

Theorem 6 (i) *For any convex body S in the plane, $f(S) \geq 1/6$. This inequality cannot be improved.* (ii) *For any centrally symmetric convex body S in the plane, $f(S) \geq \delta(S)/4$ where $\delta(S)$ is the packing density of S; in particular, $f(S) > 1/4.4810$.*

We summarize our results for convex bodies in $\mathbb{R}^2$ in Table 1.

The covering problems of Rado are related to the classical optimization problem of computing a maximum weight independent set in geometric intersection graphs. It is well known that finding a maximum independent set in a set of axis-parallel unit squares is NP-hard [9]; therefore finding a maximum area independent set in a set of axis-parallel arbitrary-size squares is also NP-hard. On the other hand, maximum weight independent set in the intersection graphs of arbitrary-size squares (or, disks, fat objects, etc.) admits polynomial-time approximation schemes [4, 6, 10]. Our algorithms come with a different guarantee: while the previous approximation algorithms bound the weight of an independent set in terms of the maximum weight of any independent set, our algorithms bound the total area of an independent set in terms of the union area.

2 Lower Bounds for Squares and Hypercubes: Proof of Theorem 1

In this section we present Algorithm A1 that computes an independent set of squares meeting the requirements in Theorem 1. We first consider the planar case. Let $\mathcal{F}$ be a set of n axis-parallel squares. For each square S_q in $\mathcal{F}$, denote by T_q the smallest axis-parallel square that contains all squares in $\mathcal{F}$ that intersect S_q (T_q contains S_q but is not necessarily concentric with S_q). Denote by x_q the side length of S_q. Denote by y_q the side length of T_q. Put $z_q = y_q - x_q$.

Let $\lambda = \lambda_2$. Recall that $\lambda_2 = (\sqrt{46} + 2)^2/9 = 8.5699\ldots > (5/2)^2$. To construct an independent set $\mathcal{I}$, our Algorithm A1 initializes $\mathcal{I}$ to be empty, then repeats the following *selection round* until $\mathcal{F}$ is empty:

1. Find the largest square S_l in $\mathcal{F}$. Assume without loss of generality[1] that $x_l = 1$.
2. If $y_l \leq \sqrt{\lambda}$, add S_l to $\mathcal{I}$, delete from $\mathcal{F}$ the squares that intersect S_l, then stop. Otherwise, set $k \leftarrow l$ and continue with the next step.

[1] This assumption simplifies the analysis, and is not implemented in the algorithm. Our bounds are not affected by this assumption because they are area ratios.

3. Let S_i and S_j be two squares in $\mathcal{F}$ that intersect S_k and touch two opposite sides of T_k. (We will prove later that S_i and S_j exist, are disjoint, and are different from S_k.) If both z_i and z_j are at most z_k, add S_i and S_j to $\mathcal{I}$, delete from $\mathcal{F}$ the squares that intersect S_i or S_j, then stop. Otherwise, set $k \leftarrow i$ or j such that z_k increases, then repeat this step.

Intuitively, in each selection round, the algorithm selects either the largest square S_l with a small neighborhood in step 2, or two squares S_i and S_j each with a small neighborhood in step 3. There are at most n selection rounds. In each selection round, step 3 is repeated at most n times since z_k is strictly increasing. So the algorithm terminates in $O(n^2)$ steps. Later in this section we will describe an efficient implementation of the algorithm and analyze its running time in more detail.

We now prove that the algorithm achieves a lower bound of $1/\lambda$. Consider any selection round. If $y_l \le \sqrt{\lambda}$ in step 2, then

$$|S_l|/|T_l| = x_l^2/y_l^2 \ge 1/\lambda. \tag{2}$$

Now suppose that $y_l > \sqrt{\lambda}$. Then the algorithm proceeds to step 3. The two squares S_i and S_j clearly exist: even if no other squares in $\mathcal{F}$ intersect S_k, in which case $T_k = S_k$, we can still take $S_i = S_j = S_k$.

In every iteration of step 3, our choice of S_i and S_j implies that $x_i + x_j + x_k \ge y_k = x_k + z_k$. Therefore,

$$x_i + x_j \ge z_k. \tag{3}$$

Since both x_i and x_j are at most $x_l = 1$, it follows from (3) that $z_k \le 2$. The value z_k is strictly increasing; in the first iteration, it is equal to $z_l = y_l - x_l > \sqrt{\lambda} - 1$. Therefore, in every iteration,

$$\sqrt{\lambda} - 1 < z_k \le 2. \tag{4}$$

Either S_i or S_j becomes S_k for the next iteration. It then follows from (3) and (4) that, in every iteration,

$$\sqrt{\lambda} - 2 < x_k \le 1. \tag{5}$$

From (4) and (5) we have $y_k = x_k + z_k > (\sqrt{\lambda} - 2) + (\sqrt{\lambda} - 1) = 2\sqrt{\lambda} - 3 \ge 2(5/2) - 3 = 2$. Since $x_i + x_j \le 2$, it follows that, in every iteration,

$$y_k > x_i + x_j. \tag{6}$$

The strict inequality in (6) implies that S_i and S_j are disjoint, hence both S_i and S_j are different from S_k.

When the selection round ends, we have $z_i \le z_k$ and $z_j \le z_k$. Therefore,

$$|T_i| = y_i^2 = (x_i + z_i)^2 \le (x_i + z_k)^2, \qquad |T_j| = y_j^2 = (x_j + z_j)^2 \le (x_j + z_k)^2. \tag{7}$$

Since $S_k \subseteq T_i$ and $S_k \subseteq T_j$, we also have

$$|T_i \cap T_j| \ge |S_k| = x_k^2. \tag{8}$$

Then,

$$
\begin{aligned}
\frac{|T_i \cup T_j|}{|S_i|+|S_j|} &= \frac{|T_i|+|T_j|-|T_i \cap T_j|}{|S_i|+|S_j|} \\
&\le \frac{(x_i+z_k)^2+(x_j+z_k)^2}{x_i^2+x_j^2} - \frac{x_k^2}{x_i^2+x_j^2} \\
&= 1+2\frac{x_i+x_j}{x_i^2+x_j^2} z_k + \frac{2}{x_i^2+x_j^2} z_k^2 - \frac{x_k^2}{x_i^2+x_j^2} \\
&\le 1+2\frac{2}{x_i+x_j} z_k + \left(\frac{2}{x_i+x_j}\right)^2 z_k^2 - \frac{x_k^2}{x_i^2+x_j^2} \\
&= \left(1+\frac{2}{x_i+x_j} z_k\right)^2 - \frac{x_k^2}{x_i^2+x_j^2} \\
&\le (1+2)^2 - \frac{(\sqrt{\lambda}-2)^2}{2} \\
&= 9-(\sqrt{\lambda}-2)^2/2,
\end{aligned} \tag{9}
$$

where the last inequality follows from (3) and (5). Recall that $\lambda = \lambda_2$ is the solution to the equation $9-(\sqrt{\lambda}-2)^2/2 = \lambda$. So we have

$$\frac{|S_i|+|S_j|}{|T_i \cup T_j|} \ge 1/\lambda. \tag{10}$$

From the two inequalities (2) and (10), it follows by induction that $|\mathcal{I}|/|\mathcal{F}| \ge 1/\lambda_2 > 1/8.5699$. Note that this bound already improves the previous best bound of $1/8.6$ by Zalgaller [22].

Generalization to higher dimensions The algorithm can be easily generalized to any dimension $d \ge 2$ to achieve a bound of $1/\lambda_d$, where λ_d is the unique solution in $[(5/2)^d, 3^d]$ of (1). Set the threshold for y_l to $\lambda_d^{1/d}$ in step 2 of the selection round. Note that the inequality $x_i + x_j \ge z_k$ in (3) still holds, and (4) and (5) become

$$\lambda^{1/d} - 1 < z_k \le 2 \quad \text{and} \quad \lambda^{1/d} - 2 < x_k \le 1.$$

Since $\lambda_d \ge (5/2)^d$, the inequality $y_k > x_i + x_j$ in (6) also holds. Following the same chain of reasoning, (9) becomes

$$
\begin{aligned}
\frac{|T_i \cup T_j|}{|S_i|+|S_j|} &= \frac{|T_i|+|T_j|-|T_i \cap T_j|}{|S_i|+|S_j|} \\
&\le \frac{(x_i+z_k)^d+(x_j+z_k)^d}{x_i^d+x_j^d} - \frac{x_k^d}{x_i^d+x_j^d}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{t=0}^{d} \binom{d}{t} \frac{x_i^{d-t} + x_j^{d-t}}{x_i^d + x_j^d} z_k^t - \frac{x_k^d}{x_i^d + x_j^d} \\
&\le \sum_{t=0}^{d} \binom{d}{t} \frac{2}{x_i^t + x_j^t} z_k^t - \frac{x_k^d}{x_i^d + x_j^d} \\
&\le \sum_{t=0}^{d} \binom{d}{t} \frac{2^t}{(x_i + x_j)^t} z_k^t - \frac{x_k^d}{x_i^d + x_j^d} \\
&= \left(1 + \frac{2}{x_i + x_j} z_k\right)^d - \frac{x_k^d}{x_i^d + x_j^d} \\
&\le (1+2)^d - \frac{(\lambda^{1/d} - 2)^d}{2} \\
&= 3^d - (\lambda_d^{1/d} - 2)^d/2. \qquad (11)
\end{aligned}
$$

Recall that λ_d is the solution to the equation $3^d - (\lambda^{1/d} - 2)^d/2 = \lambda$. For any $d \ge 2$, this equation has a unique solution λ_d between $(5/2)^d$ and 3^d because the left side of the equation is larger than the right side when $\lambda = (5/2)^d$, is smaller than the right side when $\lambda = 3^d$, and is decreasing for $(5/2)^d \le \lambda \le 3^d$. A fairly accurate closed-form upper bound on λ_d can be derived as follows. Write $\lambda_d = (3-t)^d$, where $t \le 1/2$. We have

$$
\begin{aligned}
&3^d - (\lambda_d^{1/d} - 2)^d/2 = \lambda_d \\
&\implies \quad 3^d - \frac{(1-t)^d}{2} = (3-t)^d = 3^d\left(1 - \frac{t}{3}\right)^d \ge 3^d\left(1 - \frac{dt}{3}\right) \\
&\implies \quad \frac{dt}{3} \cdot 3^d \ge \frac{(1-t)^d}{2} \ge \frac{1-dt}{2} \\
&\implies \quad (d + 2d \cdot 3^{d-1})t \ge 1 \\
&\implies \quad t \ge \frac{1}{d + 2d \cdot 3^{d-1}}.
\end{aligned}
$$

Therefore,

$$
\lambda_d \le \left(3 - \frac{1}{d + 2d \cdot 3^{d-1}}\right)^d.
$$

Implementation A straightforward implementation of the algorithm takes $O(dn^3)$ time. We now discuss a faster implementation with $O(dn^2)$ running time. For efficiency, some preprocessing is done before the selection rounds. First, build $2d + 1$ sorted lists of the hypercubes: one list sorted by size, and $2d$ lists for the two opposite directions along each axis, sorted by coordinates. This takes $O(dn \log n)$ time. Next, for each hypercube, build $2d$ sorted lists of the hypercubes that intersect it, one for each of the two opposite directions along each axis. This takes $O(dn^2)$ time.

After the preprocessing, each step in a selection round, except the deletions of hypercubes from $\mathcal{F}$ and correspondingly from the sorted lists, takes only $O(d)$ time. Step 3 is repeated at most n times in each selection round. There are at most n selection rounds. So the total running time of the algorithm, except the deletions, is $O(dn^2)$. Perform the deletions in a "lazy" fashion: when deleting a hypercube from $\mathcal{F}$, simply mark it as removed; the actual removals of the hypercube from the lists of its neighbors take place during subsequent queries in these lists. Since each hypercube is removed at most once from each list, the total time spent on deletions is $O(dn^2)$.

3 Lower Bounds for Centrally Symmetric Convex Bodies in the Plane: Proof of Theorem 2

In this section we present Algorithm B1 for centrally symmetric convex bodies in the plane and prove Theorem 2. Note that Theorem 1 implies a bound of $F(S) \geq 1/\lambda_2 > 1/8.5699$ for a square S. Theorem 2 extends this bound for any centrally symmetric convex body S in the plane.

Let $\mathcal{F}$ be a set of n homothetic copies of S. For each convex body S_q in $\mathcal{F}$, define T_q as the convex hull of the union of the convex bodies in $\mathcal{F}$ that intersect S_q (T_q contains S_q). Define the *width of a convex body S along a line* (*or direction*) ℓ as the distance between the pair of supporting lines of S perpendicular to ℓ. For each line ℓ, denote by $w_q(\ell)$ the width of S_q along ℓ, and by $w'_q(\ell)$ the width of T_q along ℓ. Define $a_q = \max_\ell w'_q(\ell)/w_q(\ell)$, $x_q = \sqrt{|S_q|}$, $y_q = a_q \cdot x_q$, and $z_q = y_q - x_q$. Here our new definitions of x_q and y_q for centrally symmetric convex bodies are extended from those for squares in the previous section: for the definition of x_q, the square root of the area is exactly the side length when S is a square; for the definition of y_q, the maximum width ratio a_q is now taken over all directions rather than the only two directions along the sides of squares.

Let $\lambda = \lambda_2$. To construct an independent set $\mathcal{I}$, our Algorithm B1 initializes $\mathcal{I}$ to be empty, then repeats the following *selection round* until $\mathcal{F}$ is empty:

1. Find the largest convex body S_l in $\mathcal{F}$. Assume without loss of generality that $x_l = 1$.
2. If $y_l \leq \sqrt{\lambda}$, add S_l to $\mathcal{I}$, delete from $\mathcal{F}$ the convex bodies that intersect S_l, then stop. Otherwise, set $k \leftarrow l$ and continue with the next step.
3. Let ℓ be a line through the center of S_k such that $w'_k(\ell)/w_k(\ell) = a_k$. Among the convex bodies in $\mathcal{F}$ that intersect S_k, let S_i and S_j be any two convex bodies that are tangent, respectively, to the two supporting lines of T_k perpendicular to ℓ. If both z_i and z_j are at most z_k, add S_i and S_j to $\mathcal{I}$, delete from $\mathcal{F}$ the convex bodies that intersect S_i or S_j, then stop. Otherwise, set $k \leftarrow i$ or j such that z_k increases, then repeat this step.

The analysis remains largely the same as that for squares. The following lemma is analogous to (3).

Lemma 1 *In each iteration of step* 3 *of the selection round*, $x_i + x_j \geq z_k$.

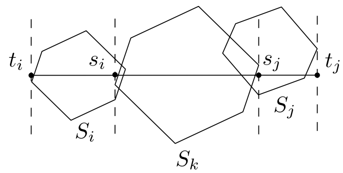


Fig. 1 Algorithm B1 for centrally symmetric convex bodies in the plane. The line ℓ is horizontal, and the four supporting lines are vertical in this example

Proof We refer to Fig. 1. The two supporting lines of S_k perpendicular to the line ℓ intersect ℓ at the two points s_i and s_j. The two supporting lines of T_k perpendicular to the line ℓ intersect ℓ at the two points t_i and t_j. The supporting line through t_i is tangent to S_i. The supporting line through s_i, which is tangent to S_k, must also intersect S_i because otherwise S_i would be disjoint from S_k. Now S_i intersects the two supporting lines through t_i and s_i. On the other hand, S_k is tangent to the two supporting lines through s_i and s_j. It follows by similarity that $x_i/x_k \geq |t_i s_i|/|s_i s_j|$. A symmetric argument also shows that S_j intersects the two supporting lines through s_j and t_j, and satisfies $x_j/x_k \geq |s_j t_j|/|s_i s_j|$. Therefore,

$$x_i + x_j \geq \frac{|t_i s_i| + |s_j t_j|}{|s_i s_j|} x_k = \frac{w'_k(\ell) - w_k(\ell)}{w_k(\ell)} x_k = (a_k - 1)x_k = y_k - x_k = z_k. \qquad \square$$

The inequality in the following lemma is analogous to the equality $|T_q| = y_q^2$ for squares, and maintains the overall inequalities in (2) and (7). Recall the concept of *Steiner symmetrization with respect to a point* [21, Exercise 6-9]: It is known that any convex body S is (can be viewed as) the intersection of infinitely many strips bounded by parallel supporting lines of S. The Steiner symmetrization of a convex body S with respect to a point o is the intersection of these strips translated to new positions such that each strip is symmetric with respect to o.

Lemma 2 *For each S_q in $\mathcal{F}$, $|T_q| \leq y_q^2$.*

Proof Let T_q^* be the Steiner symmetrization of T_q with respect to the center of S_q. Then $|T_q| \leq |T_q^*|$ since T_q is convex [21, Exercise 6-9]. Let S'_q be the concentric homothetic copy of S_q scaled by a_q. For each direction ℓ, we have (i) the width of T_q along ℓ is the same as the width of T_q^* along ℓ, i.e., $w_q^*(\ell) = w'_q(\ell)$; (ii) $w'_q(\ell) \leq a_q w_q(\ell)$ by the definition of a_q; (iii) $a_q w_q(\ell)$ equals the width of S'_q along ℓ. Hence for each direction ℓ, the width of T_q^* along ℓ is at most the width of S'_q along ℓ. Since both S'_q and T_q^* are symmetric with respect to o, it follows that S'_q contains T_q^*. Therefore $|T_q| \leq |T_q^*| \leq |S'_q| = (a_q \cdot x_q)^2 = y_q^2$. $\square$

We follow the same chain of reasoning from (2) to (10): the only difference is that here the first equalities in (2) and (7) are changed into inequalities because of Lemma 2; the strict inequality in (6) implies that S_i and S_j are contained in two disjoint parallel strips, hence they are disjoint from each other and different from S_k. Hence we obtain a bound of $F(S) \geq 1/\lambda_2 > 1/8.5699$ for any centrally symmetric convex body S in the plane. For the special case that S is a disk, we next derive a better bound of $F(S) \geq 1/\lambda_{\text{disk}} > 1/8.3539$ by a tighter analysis (Sect. 4).

4 A New Lower Bound for Disks: Proof of Theorem 3

In this section we prove Theorem 3 for a disk S. Let $\mathcal{F}$ be a set of n homothetic copies of a disk S. For each disk S_q in $\mathcal{F}$, define T_q as the convex hull of the union of the disks in $\mathcal{F}$ that intersect S_q, and define the width ratio a_q in the same way as in the previous section. For the convenience of analysis, define x_q as the diameter of S_q instead of $\sqrt{|S_q|}$. Define $y_q = a_q \cdot x_q$ and $z_q = y_q - x_q$ as before.

Let $\lambda = \lambda_{\text{disk}}$, where $\lambda_{\text{disk}} = 8.3539\ldots$ (the exact definition of λ_{disk} will be given later). To construct an independent set $\mathcal{I}$, we use the same Algorithm B1 that initializes $\mathcal{I}$ to be empty then repeats the following *selection round* until $\mathcal{F}$ is empty:

1. Find the largest disk S_l in $\mathcal{F}$. Assume without loss of generality that $x_l = 1$.
2. If $y_l \le \sqrt{\lambda}$, add S_l to $\mathcal{I}$, delete from $\mathcal{F}$ the disks that intersect S_l, then stop. Otherwise, set $k \leftarrow l$ and continue with the next step.
3. Let ℓ be a line through the center of S_k such that $w'_k(\ell)/w_k(\ell) = a_k$. Among the disks in $\mathcal{F}$ that intersect S_k, let S_i and S_j be any two disks that are tangent, respectively, to the two supporting lines of T_k perpendicular to ℓ. If both z_i and z_j are at most z_k, add S_i and S_j to $\mathcal{I}$, delete from $\mathcal{F}$ the disks that intersect S_i or S_j, then stop. Otherwise, set $k \leftarrow i$ or j such that z_k increases, then repeat this step.

It can be easily verified that our new definition of x_q does not change the inequalities in the chain of reasoning from (2) to (6) since the definition of the ratio $y_q/x_q = a_q$ remains the same. We next discuss the final iteration of step 3 of a selection round, for which we make a different analysis from (7) to (10). We will use some special properties of disks to obtain a tighter estimate for $|T_i \cup T_j|$. This is achieved, somewhat counter-intuitively, by "blowing up" both T_i and T_j.

Let R_i be the disk of radius $r_i = x_i/2 + z_k - 1$ that is concentric with S_i, and let T'_i be the convex hull of the union of T_i and R_i; see Fig. 2(a). Recall that x_i is the diameter of the disk S_i. Therefore, for each direction ℓ, we have

$$x_i = w_i(\ell).$$

It follows that

$$y_i = a_i \cdot x_i = \max_{\ell} \frac{w'_i(\ell)}{w_i(\ell)} x_i = \max_{\ell} w'_i(\ell).$$

Hence the maximum width of T_i along a line is $y_i = x_i + z_i \le x_i + z_k$. We now show that the maximum width of T'_i along a line is also at most $x_i + z_k$. Suppose the contrary. Then there must exist two parallel supporting lines of T'_i with a distance of more than $x_i + z_k$, one tangent to R_i and the other tangent to either R_i or T_i. But this is impossible because:

1. The distance from a line tangent to R_i to the center of S_i (the same as the center of S'_i) is exactly $x_i/2 + z_k - 1$, the radius of R_i.
2. The distance from a line tangent to T_i to the center of S_i is at most $x_i/2 + 1$, i.e., the radius of S_i plus the maximum diameter of a disk in $\mathcal{F}$ that intersects S_i.
3. $(x_i/2 + z_k - 1) + \max\{x_i/2 + z_k - 1, x_i/2 + 1\} \le (x_i/2 + z_k - 1) + (x_i/2 + 1) = x_i + z_k$.

Let T_i^* be the Steiner symmetrization of T_i' with respect to the center of S_i. Then $|T_i'| \le |T_i^*|$ since T_i' is convex [21, Exercise 6-9]. Let S_i' be the disk of diameter $x_i + z_k$ that is concentric with S_i. Then the same argument as in Lemma 2 shows that S_i' contains T_i^*. Therefore $|T_i'| \le |T_i^*| \le |S_i'| = (\pi/4)(x_i + z_k)^2$. We have proved the following inequality analogous to the first inequality in (7):

$$|T_i'| \le (\pi/4)(x_i + z_k)^2.$$

Similarly, let R_j be the disk of radius $r_j = x_j/2 + z_k - 1$ that is concentric with S_j, and let T_j' be the convex hull of the union of T_j and R_j. We have

$$|T_j'| \le (\pi/4)(x_j + z_k)^2.$$

The minimum radius of the two disks R_i and R_j satisfies

$$\min\{r_i, r_j\} = \min\{x_i, x_j\}/2 + z_k - 1 \ge 3(z_k - 1)/2 \ge 3\sqrt{\lambda}/2 - 3,$$

where the two inequalities follow from (3) and (4), respectively.

Since both S_i and S_j intersect S_k, the distance d_{ij} between the centers of the two disks R_i and R_j satisfies

$$d_{ij} \le x_i/2 + x_j/2 + x_k.$$

The intersection of R_i and R_j consists of a cap in R_i and a cap in R_j; refer to Fig. 2(b). The intersection is nonempty, since $h_{ij} \ge 0$, as shown below. The total height of the two caps is

$$\begin{aligned} h_{ij} &= r_i + r_j - d_{ij} \\ &\ge (x_i/2 + z_k - 1) + (x_j/2 + z_k - 1) - (x_i/2 + x_j/2 + x_k) \\ &= 2(z_k - 1) - x_k \end{aligned}$$

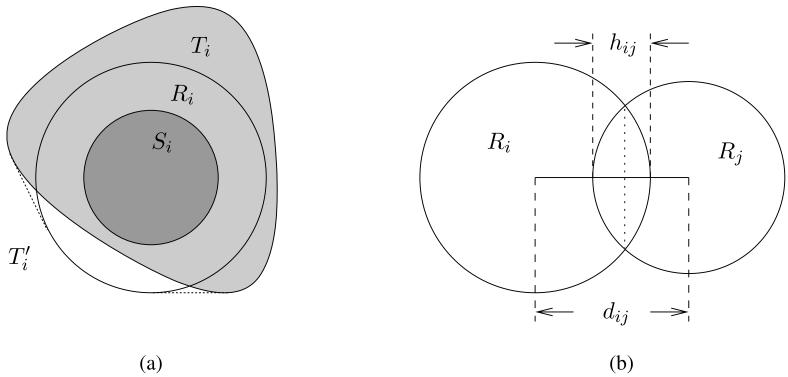


Fig. 2 (**a**) The disk S_i (*dark shaded*), the disk R_i (concentric with S_i), the convex hull T_i (*light shaded*), and the convex hull T_i' of the union of R_i and T_i. (**b**) The two disks R_i and R_j

$$\geq 2(\sqrt{\lambda} - 1 - 1) - 1$$
$$= 2(\sqrt{\lambda} - 5/2),$$

where the last inequality follows from (4) and (5).

Denote by $\mathrm{cap}(r, h)$ the area of a disk cap of height h and radius r. It is known [23] that

$$\mathrm{cap}(r, h) = r^2 \arccos(1 - h/r) - (r - h)\sqrt{r^2 - (r - h)^2}.$$

We now have the following inequality analogous to (8):

$$|T_i' \cap T_j'| \geq |R_i \cap R_j| \geq 2\,\mathrm{cap}\big(\min\{r_i, r_j\}, h_{ij}/2\big)$$
$$\geq 2\,\mathrm{cap}(3\sqrt{\lambda}/2 - 3, \sqrt{\lambda} - 5/2).$$

The following chain of inequalities is analogous to (9):

$$\frac{|T_i \cup T_j|}{|S_i| + |S_j|} \leq \frac{|T_i' \cup T_j'|}{|S_i| + |S_j|} = \frac{|T_i'| + |T_j'| - |T_i' \cap T_j'|}{|S_i| + |S_j|}$$
$$\leq \frac{(\pi/4)(x_i + z_k)^2 + (\pi/4)(x_j + z_k)^2}{(\pi/4)x_i^2 + (\pi/4)x_j^2} - \frac{2\,\mathrm{cap}(3\sqrt{\lambda}/2 - 3, \sqrt{\lambda} - 5/2)}{(\pi/4)x_i^2 + (\pi/4)x_j^2}$$
$$\leq 9 - \frac{\mathrm{cap}(3\sqrt{\lambda}/2 - 3, \sqrt{\lambda} - 5/2)}{\pi/4}.$$

Finally, let λ_{disk} be the solution to the equation

$$9 - \frac{\mathrm{cap}(3\sqrt{\lambda}/2 - 3, \sqrt{\lambda} - 5/2)}{\pi/4} = \lambda, \tag{12}$$

and we have a bound of $F(S) \geq 1/\lambda_{\mathrm{disk}}$. A calculation shows that $\lambda_{\mathrm{disk}} = 8.3539\ldots$.

Implementation We now show how to implement the algorithm B1 in $O(n^3)$ time. We perform some preprocessing before the selection rounds. For each disk S_q in $\mathcal{F}$, construct a circular list $\mathcal{F}_q$ of the other disks that intersect it; the disks in $\mathcal{F}_q$ are ordered by the directions of the vectors from the center of S_q to their centers. This can be done in $O(n^2)$ time by computing the arrangement of the lines $\{\ell_q \mid S_q \in \mathcal{F}\}$ dual to the disk centers $\{c_q \mid S_q \in \mathcal{F}\}$, where c_q denotes the center of S_q, since the circular order of the other disk centers around a disk center c_q corresponds to the linear order of intersections of the other dual lines with the dual line ℓ_q.

We next consider each selection round. The largest disk S_l can be found in $O(n)$ time. To select the two disks S_i and S_j in each iteration of step 3, first construct the convex hull of the disks in $\mathcal{F}_k$ using a variant of Graham scan, then apply the standard rotating calipers algorithm [13]. This can be done in $O(n)$ time since the list $\mathcal{F}_k$ is in circular order. To remove a disk from the circular lists, either in step 2 or in the last iteration of step 3, simply mark the disk "removed" and defer the actual removal until the convex hull construction of a later step. Step 3 is repeated at most n times in a selection round. There are at most n selection rounds. So the total running time of the algorithm is $O(n^3)$.

5 A New Lower Bound for Squares: Proof of Theorem 4

We present a simple greedy Algorithm A2 for axis-parallel squares and prove Theorem 4. Let $\mathcal{F}$ be a set of n axis-parallel squares. For each square $S_i = [x, x+l] \times [y, y+l]$ in $\mathcal{F}$, denote by x_i the side length l of S_i, and denote by S_i' the square $[x-1, x+l+1] \times [y-1, y+l+1]$, which contains all possible squares of side length at most 1 that intersect S_i. Note that S_i' is concentric with S_i. Given two axis-parallel squares S and T in the plane, we say that S is *tangent* to T if a side of S and a side of T are collinear and have non-empty intersection. Note that our usage of tangent in this section is not standard: S may intersect T in the interior and at the same time be tangent to T.

Let s be a real number to be chosen later, $3/4 < s < 1$. To construct an independent set $\mathcal{I}$, our Algorithm A2 initializes $\mathcal{I}$ to be empty, then repeats the following *selection round* until $\mathcal{F}$ is empty:

1. Let S_0 be the largest square in $\mathcal{F}$. Assume without loss of generality that S_0 is a unit square. Let $\mathcal{F}_0 \subseteq \mathcal{F} \setminus \{S_0\}$ be the set of squares of side length at least s that intersect S_0.
2. If $\mathcal{F}_0$ contains three disjoint squares S_1, S_2, and S_3, then add S_1, S_2, and S_3 to $\mathcal{I}$. Otherwise add S_0 to $\mathcal{I}$.
3. For each square S_i added to $\mathcal{I}$, remove from $\mathcal{F}$ the squares that intersect S_i.

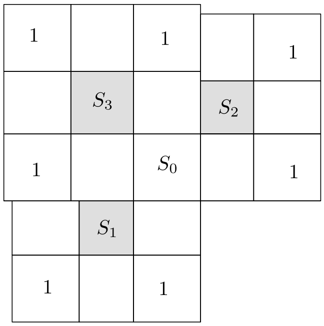
In a selection round, let $\mathcal{J}$ be the set of selected squares, and let $\mathcal{T}$ be the set of squares in $\mathcal{F}$ that intersect the selected squares. We prove the following two lemmas.

Lemma 3 *Suppose that the algorithm selects three disjoint squares S_1, S_2, and S_3, in a selection round. Then*

$$|\mathcal{T}|/|\mathcal{J}| \le (8 + 3s^2 + 10s)/(3s^2).$$

Proof We will show that the ratio of the area of the region $R = S_1' \cup S_2' \cup S_3'$ over the total area of the three squares S_1, S_2, and S_3 is maximized when each square intersects S_0 at a distinct corner as shown in Fig. 3 (possibly with a different correspondence

Fig. 3 Maximum area of $R = S_1' \cup S_2' \cup S_3'$



between the squares and the corners), and when the three squares have equal side lengths $x_1 = x_2 = x_3 = s$. The maximizing region is the union of 8 unit squares, 3 squares of side length s, and 10 rectangles of side lengths 1 and s.

We first prove that the area of the region $R = S'_1 \cup S'_2 \cup S'_3$ is maximized when each of the three squares S_1, S_2, and S_3 intersects S_0 at a distinct corner as shown in Fig. 3 (possibly with a different correspondence between the squares and the corners). We will use a sequence of axis-parallel translations such that, after each translation, (i) the area of R does not decrease, and (ii) the squares S_1, S_2, and S_3 are disjoint.

Suppose that $S_0 = [0, 1]^2$. Let B be the smallest axis-parallel rectangle that contains the three squares S_1, S_2, and S_3. Select a square $S_l \in \{S_1, S_2, S_3\}$ that is tangent to the left side of B. Translate S_l to the left for a distance of ϵ until its right side is tangent to the left side of S_0. The translation changes the region R by adding a rectangle of area $\epsilon(2 + x_l)$ outside S'_l and removing an area of at most $\epsilon(2 + x_l)$ inside S'_l, where $2 + x_l$ is the side length of S'_l. The area of R does not decrease. Similarly, select and translate a square S_r to the right, a square S_u up, and a square S_d down. One of the three squares S_1, S_2, and S_3 is selected at least twice. Assume without loss of generality that $S_l = S_d$. Thus $S_0 \cap S_l = \{(0, 0)\}$. We distinguish two cases:

- *Case 1*. Suppose that $S_r \neq S_u$. Since S_r and S_u are disjoint, one of them, say S_u, does not cover $(1, 1)$. Translate S_u to the left until $S_0 \cap S_u = \{(0, 1)\}$, see Fig. 4(a). Let y_1 be the distance between the upper sides of S'_u and S'_l. Let y_2 be the distance between the upper sides of S'_u and S'_r. The area of R does not decrease since $y_1 > y_2$: translating S_u for a small distance ϵ to the left increases the area of R by at least $\epsilon(y_1 - y_2)$. Then S_r can be translated up until $S_0 \cap S_r = \{(1, 1)\}$, or down until $S_0 \cap S_r = \{(1, 0)\}$, see Fig. 4(b).
- *Case 2*. Suppose that $S_r = S_u$. Then $S_r \cap S_0 = \{(1, 1)\}$. Let S_m be the third (middle) square, that is, $\{l, m, r\} = \{1, 2, 3\}$. If S_m is tangent to B, say on the right side, then we translate S_m to the right until S_m is tangent to S_0, see Fig. 5(a). This reduces to the case of different S_u and S_r (Case 1). Suppose that S_m is not tangent to B. Let y_1 be the distance between the upper sides of S'_m and S'_l. Let y_2 be the distance between the lower sides of S'_r and S'_m. If S_m intersects the line $y = 1$, then $y_1 \geq y_2$, and we translate S_m to the left. If S_m intersects the line $y = 0$, then $y_2 \geq y_1$, and

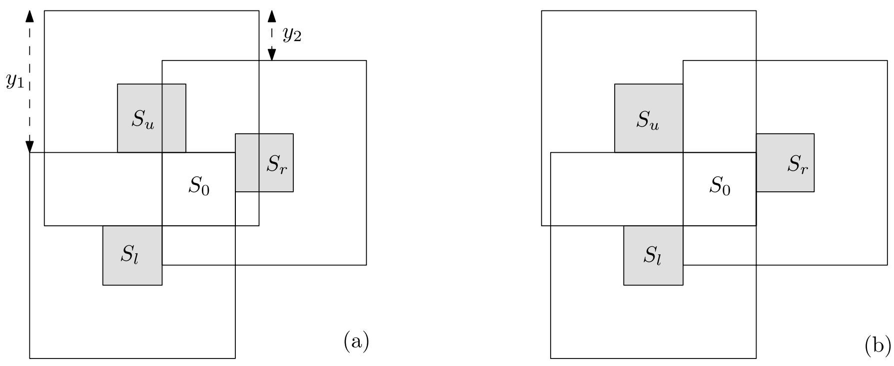


Fig. 4 (**a**) Translate S_u to the left. (**b**) Translate S_r up or down

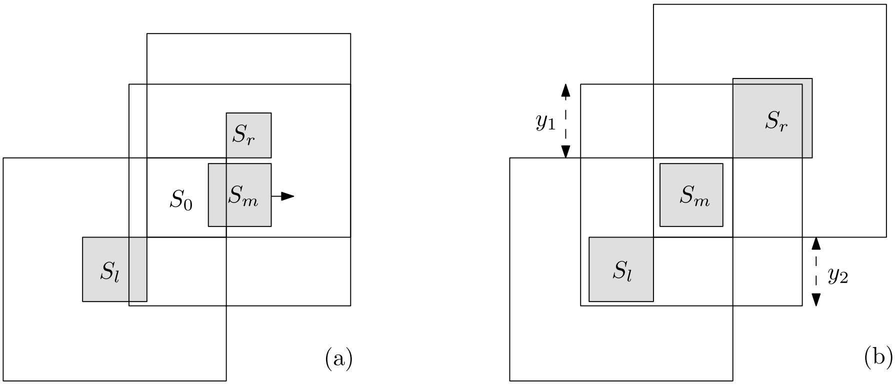


Fig. 5 (**a**) S_m is tangent to B. (**b**) S_m is not tangent to B

we translate S_m to the right. If S_m lies between the two lines $y = 0$ and $y = 1$, then we translate S_m to the left if $y_1 > y_2$, or to the right otherwise, see Fig. 5(b).

Stop the motion when S_m becomes tangent to S_0, and this reduces to the case of S_m tangent to B.

After the sequence of translations, each of the three squares S_1, S_2, and S_3 intersects S_0 at a distinct corner. Assume without loss of generality the correspondence between the squares and the corners as shown in Fig. 3. The region R is the union of (i) the three squares S_1, S_2, and S_3, (ii) 8 unit squares, and (iii) 10 rectangles. The area of R is $x_1^2 + x_2^2 + x_3^2 + 8 + c_1x_1 + c_2x_2 + c_3x_3$, where $c_1, c_2, c_3 \in \{2, 3, 4\}$ and $c_1 + c_2 + c_3 = 10$. There are four cases (the first case appears in Fig. 3):

1. If $x_3 \geq x_1$ and $x_3 \geq x_2$, then $c_1 = c_2 = 3$ and $c_3 = 4$.
2. If $x_3 \leq x_1$ and $x_3 \leq x_2$, then $c_1 = c_2 = 4$ and $c_3 = 2$.
3. If $x_1 \leq x_3 \leq x_2$, then $c_2 = 4$ and $c_1 = c_3 = 3$.
4. If $x_2 \leq x_3 \leq x_1$, then $c_1 = 4$ and $c_2 = c_3 = 3$.

Let $f : [3/4, 1]^3 \to \mathbb{R}$ be defined as follows:

$$f(x_1, x_2, x_3) = 1 + \frac{8 + c_1x_1 + c_2x_2 + c_3x_3}{x_1^2 + x_2^2 + x_3^2}.$$

Then we have $|\mathcal{T}|/|\mathcal{J}| \leq \max\{f(x_1, x_2, x_3) : (x_1, x_2, x_3) \in [s, 1]^3\}$. We show that $f(x_1, x_2, x_3)$ is a decreasing function of x_1 for $x_1 \in [3/4, 1]$ by taking the derivative f'_{x_1}. Consider the function

$$\begin{aligned} g(x_1) &= (x_1^2 + x_2^2 + x_3^2)^2 \cdot f'_{x_1} \\ &= c_1(x_1^2 + x_2^2 + x_3^2) - 2(8 + c_1x_1 + c_2x_2 + c_3x_3)x_1 \\ &= c_1(-x_1^2 + x_2^2 + x_3^2) - 2(8 + c_2x_2 + c_3x_3)x_1. \end{aligned}$$

Write $t = 3/4$. Since $g(x_1)$ is a quadratic function with the negative leading coefficient $-c_1 < 0$ and $g(0) > 0$, it suffices to show that $g(t) < 0$. Using $c_1 \leq 4$, $c_2, c_3 \geq 2$,

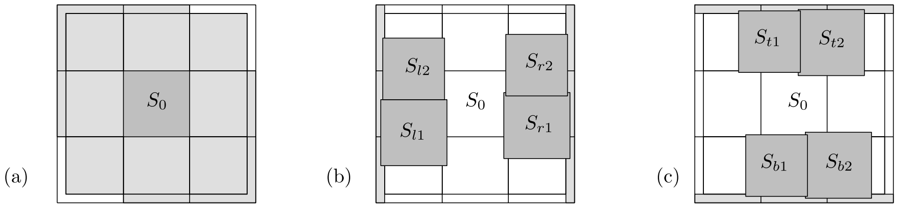


Fig. 6 (**a**) Maximum covered area in S'_0. (**b**) $\mathcal{F}_l$ and $\mathcal{F}_r$. (**c**) $\mathcal{F}_t$ and $\mathcal{F}_b$

and $t \le x_2, x_3 \le 1$ we have $-t^2 + x_2^2 + x_3^2 \ge t^2 > 0$ and $g(t) \le 4(-t^2+1+1) - 2(8+2t+2t)t = -43/4 < 0$. Similarly $f(x_1, x_2, x_3)$ is a decreasing function of x_2 and x_3 in $[3/4, 1]$. Recall that $x_1, x_2, x_3 \ge s$ as imposed by Algorithm A2, hence the function $f(\cdot)$ is maximized on the subdomain $[s, 1]^3$ when $x_1 = x_2 = x_3 = s$. □

Lemma 4 *Suppose that the algorithm selects one square*, S_0, *in a selection round. Then*

$$|\mathcal{T}|/|\mathcal{J}| \le 7 + 2s^2.$$

Proof We will show that the maximum covered area in S'_0 is the shaded area shown in Fig. 6(a), which contains 7 unit squares and 2 squares of side length s.

Suppose that $S_0 = [1, 2]^2$. Then $S'_0 = [0, 3]^2$. Let $a = 1 - s$. Let $\mathcal{F}_l \subseteq \mathcal{F}_0$ be the set of squares intersecting the rectangle $[0, a] \times [0, 3]$. If $\mathcal{F}_l$ is not empty, then let S_{l1} and S_{l2} be two squares in $\mathcal{F}_l$ containing points with the smallest and the largest y-coordinates y_{l1} and y_{l2}, respectively (S_{l1} and S_{l2} can coincide). Define the *left span* s_l as follows: if $\mathcal{F}_l$ is not empty, let $s_l = y_{l2} - y_{l1}$; otherwise, let $s_l = 0$. Similarly define the spans s_r, s_t and s_b for the three other sides, see Fig. 6(b) and (c). The covered area in S'_0 is at most $(1 + 2s)^2 + (s_l + s_r + s_t + s_b)a$.

Suppose that two spans are equal to zero. Then, for $s \le 1$, the covered area in S'_0 is at most

$$9 - 6a + a^2 = 4 + 4s + s^2 = 7 + 2s^2 + 1 - (s - 2)^2 \le 7 + 2s^2.$$

Suppose that only one span is equal to zero, say, $s_r = 0$, then $s_t, s_b \le 2$ (otherwise there would be three disjoint squares in $\mathcal{F}_t \cup \mathcal{F}_b$). Then, for $1/2 \le s \le 1$, the covered area in S'_0 is at most

$$(1 + 2s)^2 + 7a = 4s^2 - 3s + 8 = 7 + 2s^2 + (2s - 1)(s - 1) \le 7 + 2s^2.$$

Suppose that all spans are positive. Then each span is at most 2 by the above argument. If a span is at most $2s$, then the covered area in S'_0 is at most

$$(1 + 2s)^2 + (6 + 2s)a = 7 + 2s^2.$$

Suppose now that each span is larger than $2s$. Since $s_l > 2s$, it follows that either $y_{l2} < 1.5 - s$ or $y_{l1} > 1.5 + s$. Assume without loss of generality that $y_{l1} > 1.5 + s$.

The square S_{l1} is above the line $y = 1.5 + s - 1 = 1.5 - a$, and is disjoint from the squares in $\mathcal{F}_b$ because $1.5 - a > 1 + a$ for $a < 1/4$ ($s > 3/4$). The squares of $\mathcal{F}_r \cup \mathcal{F}_b$ pairwise intersect otherwise $\mathcal{F}_0$ would contain three disjoint squares: S_{l1}, one from $\mathcal{F}_r$, and one from $\mathcal{F}_b$. Then $x_{b1} \geq (2+s) - 2 = s$ and $x_{b2} > x_{b1} + 2s \geq 3s$. Since $3s > 3 - s = a + 2$ for $s \geq 3/4$, S_{b2} is disjoint from squares in $\mathcal{F}_l$. Therefore, symmetrically, the squares of $\mathcal{F}_l \cup \mathcal{F}_t$ pairwise intersect.

Consider the smallest axis-parallel rectangle R_{lt} that contains the squares in $\mathcal{F}_l \cup \mathcal{F}_t$. Let $(x, 3-y)$ be the left-top vertex of R_{lt}. Note that $0 \leq x, y \leq a$. Outside the square $[a, 3-a]^2$, the rectangle R_{lt} covers an area of

$$2(a-x) + 2(a-y) - (a-x)(a-y) = (4a - a^2) - (2-a)(x+y) - xy,$$

which is maximized when $x = y = 0$. A symmetrical argument applies to $\mathcal{F}_r \cup \mathcal{F}_b$. The maximum covered area in S'_0 is shown in Fig. 6(a). It is equal to $7 + 2s^2$. □

Balancing the two bounds in Lemmas 3 and 4, we obtain a quartic equation $3s^4 + 9s^2 - 5s - 4 = 0$, which has only one positive root $s_0 = 0.8601\ldots$. Choose $s = s_0$, and we have $(8 + 3s^2 + 10s)/(3s^2) = 7 + 2s^2 = \lambda_{\text{square}} = 8.4797\ldots$.

Implementation A straightforward implementation of the algorithm takes $O(n^3)$ time. The bottleneck is to decide whether $\mathcal{F}_0$ contains three disjoint squares in step 2 of a selection round; note that the size of $\mathcal{F}_0$ can be $\Omega(n)$. To reduce the running time to $O(n^2)$, we replace step 2 by the following step:

- Compute the four subsets $\mathcal{F}_l$, $\mathcal{F}_r$, $\mathcal{F}_t$, and $\mathcal{F}_b$, defined in the proof of Lemma 4, and their smallest axis-parallel enclosing rectangles. Find a set $\mathcal{F}'_0$ of at most 16 squares, four from each subset, tangent to the four sides of the corresponding rectangle. If $\mathcal{F}'_0$ contains three disjoint squares S_1, S_2, and S_3, then add S_1, S_2, and S_3 to $\mathcal{I}$. Otherwise add S_0 to $\mathcal{I}$.

Lemma 3 is unaffected by this modification. The proof for Lemma 4 remains valid after we substitute $\mathcal{F}_0$ by $\mathcal{F}'_0$, because (i) the spans for $\mathcal{F}_0$ and $\mathcal{F}'_0$ coincide, and (ii) the three disjoint squares used in the proof can be selected from $\mathcal{F}'_0$.

5.1 Comparison of Five Algorithms for Squares and Hypercubes

Besides Algorithm Z by Zalgaller [22] and our two Algorithms A1 and A2, we briefly review two Algorithms R1 and R2 by R. Rado [15]. Algorithm R1 is implicit in R. Rado's bound of $F(S) > 4/35$ for a square S. It repeatedly adds to the independent set either the largest square or two disjoint squares that intersect the largest square. Algorithm R1 can be easily generalized to any dimension $d \geq 2$ to achieve the following bound for a hypercube in $\mathbb{R}^d$:

$$F(S) \geq \min_{0 \leq x \leq 1} \max\left\{\frac{1}{(1+2x)^d}, \frac{2x^d}{2(x+2)^d - 1}\right\}. \tag{13}$$

Algorithm R2 is implicit in R. Rado's bound of $F(S) > 1/(3^d - 7^{-d})$ for a hypercube S in $\mathbb{R}^d$. It repeatedly adds to the independent set either the largest hypercube or 2^d

Table 2

d	R1	R2	Z	A1	A2
2	8.7436	8.9726	8.6	8.5699	8.4797
3	26.7478	26.9954		26.5260	
4	80.7493	80.9992		80.5091	
5	242.7498	242.9999		242.5031	

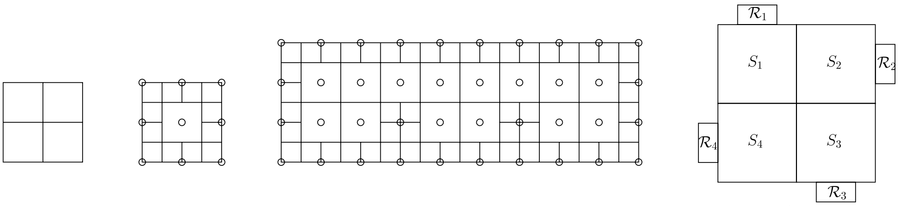


Fig. 7 (**a**) Starting point: a system of four congruent squares. (**b**) Ajtai's idea: an ambiguous system $\mathcal{Q}$ of 13 squares of sides 1 and 2. (**c**) Another ambiguous system $\mathcal{R}$ of 66 squares of sides 1 and 2. (**d**) Ajtai's construction shown schematically; $\mathcal{R}_i$, $i = 1, 2, 3, 4$, are rotated copies of $\mathcal{R}$

pairwise disjoint hypercubes that intersect the largest hypercube. The precise bound is

$$F(S) \geq \min_{0 \leq x < \frac{1}{2}} \max\left\{\frac{1}{3^d - x^d}, \frac{1}{5^d 2^{-d}(1-x)^{-d}}\right\}. \tag{14}$$

We believe that the ratio of Algorithm A1 is better than the ratios of both R1 and R2 for all $d \geq 2$ (a precise calculation is somewhat involved). We list some approximate numerical values of these bounds in Table 2.

6 A New Upper Bound for Squares: Proof of Theorem 5

We first describe briefly Ajtai's ingenious idea for the construction in [1]. The starting point is a system of 4 non-overlapping congruent squares shown in Fig. 7(a). Now slightly enlarge each square with respect to its center by a small $\varepsilon > 0$. All constructions we discuss will be obtained in the same way, by starting from a system of non-overlapping (i.e., interior disjoint) squares and then applying the above transformation; the effect is that any pair of touching squares results in a pair of squares intersecting in their interior. Finally by letting ε tend to zero, one recovers the same upper bound for systems of intersecting squares. Alternatively, one can consider the squares as closed sets, to start with, and use non-overlapping squares in the construction.

In the second step, consider a system $\mathcal{Q}$ of 13 squares of sides 1 and 2 as in Fig. 7(b). The system can also be viewed as four 2×2 squares A_1, A_2, A_3, and A_4, the vertices of which are drawn as circles. These 2×2 superimposed squares are not part of the system; they are only used in the analysis. The system $\mathcal{Q}$ has the nice property that any independent set can cover at most one quarter of (the area of) each A_i. Although $\mathcal{Q}$ by itself does not appear to be useful in reducing the conjectured

1/4 upper bound, Ajtai found a more elaborate system $\mathcal{R}$ that does so. The system $\mathcal{R}$ consists of 66 squares of sides 1 and 2 as in Fig. 7(c), whose union is a rectangle. His construction is shown schematically in Fig. 7(d); it consists of four large squares and four rotated copies of the system $\mathcal{R}$. A calculation shows that this construction yields an upper bound of $\frac{1}{4} - \frac{1}{1728}$, when the length of the rectangle equals the side length of the large squares. An obvious optimization uses eight copies of the system R bordering all eight outer sides of the four squares, and yields an improvement to $\frac{1}{4} - \frac{1}{1080}$.

Here we refine Ajtai's idea in several ways to obtain a better bound. We construct a *new* system $\mathcal{R}$ shown in Fig. 8(a). Our system, which serves the same purpose, has two desirable features: first, $\mathcal{R}$ is a *smaller* system (in a sense not meant to be precise) than that used in the previous construction; second, because of its symmetry, $\mathcal{R}$ permits a tiling (here we use this term in a broader sense, where the tile can have holes) of the plane, with adjacent blocks in the tiling *sharing* common parts of the system $\mathcal{R}$. The new system $\mathcal{R}$, shown in Fig. 8(a), consists of 48 unit squares and 16

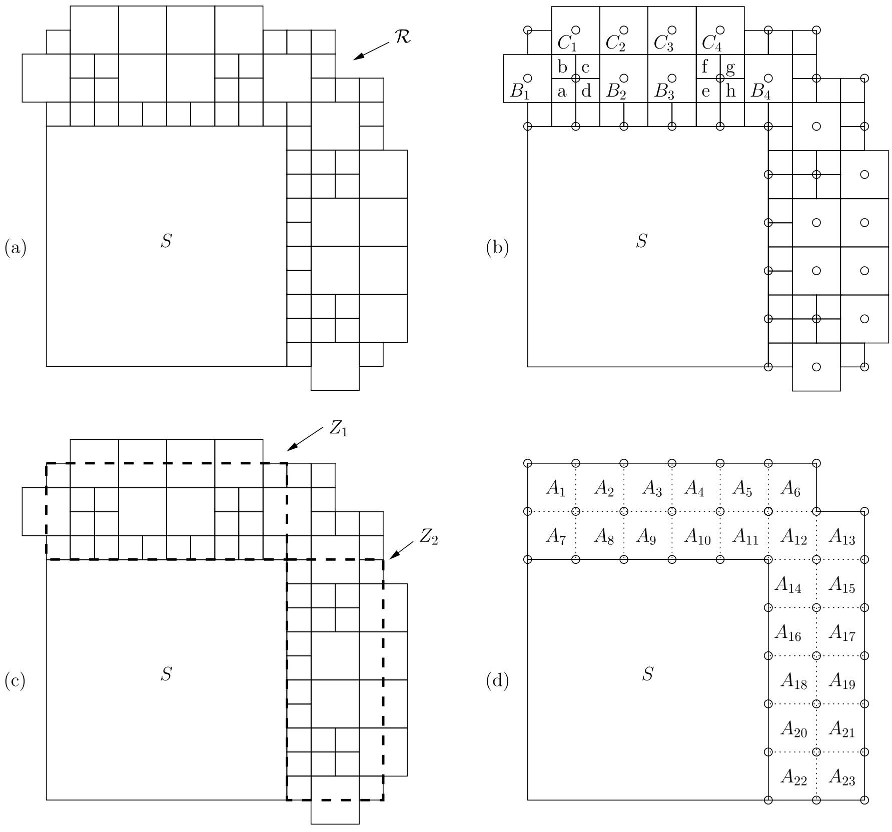


Fig. 8 (**a**) Preliminaries for the tiling: the new system $\mathcal{R}$ bordering two sides of a *large square* S. (**b**) The labeling of the squares used in the proof of the upper bound in Lemma 5. (**c**) *Two rectangles* Z_1 and Z_2 superimposed on $\mathcal{R}$. (**d**) A system of 23 squares of side 2, A_i, $i = 1, \ldots, 23$, superimposed on $\mathcal{R}$ (some of the squares in $\mathcal{R}$ are only partially covered by the squares A_i)

2×2 squares bordering two adjacent sides of a large 10×10 square S. By replicating copies of $\mathcal{R}$, rotated by 0°, 90°, 180°, and 270°, we construct a tiling of the plane, see Fig. 9. We say that a square A_i is *not covered* if 0% of its area is covered by $\mathcal{I}$.

Lemma 5 *Let $\mathcal{I}$ be an independent set of squares in the system $\mathcal{R} \cup \{S\}$ in Fig.* 8(a). *Let Z_1 be the* 10×4 *superimposed rectangle that borders S from above as in Fig.* 8(c). *Assume that $S \in \mathcal{I}$. Then $|\mathcal{I} \cap Z_1| \leq 9$.*

Proof Observe that $\mathcal{R}$ has the property that any independent set can cover at most one quarter of (the area of) each A_i, conform with Figs. 8(b) and 8(d). By the assumption, the 10 unit squares in the bottom row of squares A_7 through A_{11} cannot be in $\mathcal{I}$. It is enough to show that at least one of the squares A_i ($i \in \{1, 2, 3, 4, 5\} \cup \{7, 8, 9, 10, 11\}$ is not covered. Observe that either $B_2 \in \mathcal{I}$ or $B_3 \in \mathcal{I}$ (otherwise A_9 is not covered and we are done). Since $\mathcal{R} \cap Z_1$ admits a vertical symmetry axis, we can assume w.l.o.g. that $B_2 \in \mathcal{I}$. It follows that $e \in \mathcal{I}$ (otherwise A_{10} is not covered), and that $B_4 \in \mathcal{I}$ (otherwise A_{11} is not covered). But then A_4 is not covered, since $f, B_3, C_3, C_4 \notin \mathcal{I}$. This completes the proof. □

Obviously, the property in the lemma holds also for Z_2 in place of Z_1. We now move to the final step—the tiling—which completes our construction. Take four squares S_1, S_2, S_3, and S_4, each of side 10, and arrange them as in Fig. 9(b). Place four rotated copies of $\mathcal{R}$ bordering the outer 8 sides of $S_1 \cup S_2 \cup S_3 \cup S_4$ as in Fig. 9(b), and obtain a block (cell) of the tiling. Each block in the tiling contains 4 large 10×10 squares. Each large square has associated 23 2×2 squares that are shown in Fig. 8(d). We assign the (total of) 92 2×2 squares to the block that contains them. It is important to note that, although some of the 2×2 squares of $\mathcal{R}$ (the five squares B_1, C_1, C_2, C_3, C_4 and their symmetric counterparts) are shared between adjacent blocks in the tiling, the superimposed squares A_i used in the analysis are not shared, i.e., they are contained entirely in individual blocks.

Let $\mathcal{T}$ be the infinite set of squares as in Fig. 9(c), obtained by replicating the block in Fig. 9(b). Let $\mathcal{I}$ be an independent set of squares in $\mathcal{T}$. Fix any block σ in the tiling. Observe that at most one of the S_i can be in $\mathcal{I}$, so at most one quarter of the area of $S_1 \cup S_2 \cup S_3 \cup S_4$ is covered by $\mathcal{I}$. Similarly $\mathcal{I}$ covers at most one quarter of the area in each of the 92 2×2 squares assigned to σ. Observe that if one of the four large squares, say S_2, is selected in an independent set $\mathcal{I}$, it forces the 10 unit squares in both the bottom row of $\mathcal{R} \cap Z_1$ and the leftmost column of $\mathcal{R} \cap Z_2$ to be out of $\mathcal{I}$ in Fig. 8(c).

For the analysis, we can argue independently for each block. Fix any block σ in the tiling. The area covered by $\mathcal{T}$ in σ is

$$|\mathcal{T} \cap \sigma| = 4 \times 100 + 4 \times 92 = 768.$$

In the (easy) case that none of the S_i is in $\mathcal{I}$, the area covered by $\mathcal{I}$ in σ is

$$|\mathcal{I} \cap \sigma| \leq 4 \times 23 = 92, \quad \text{thus} \quad \frac{|\mathcal{I} \cap \sigma|}{|\mathcal{T} \cap \sigma|} \leq \frac{92}{768} = \frac{1}{4} - \frac{50}{384},$$

i.e., much smaller than required.

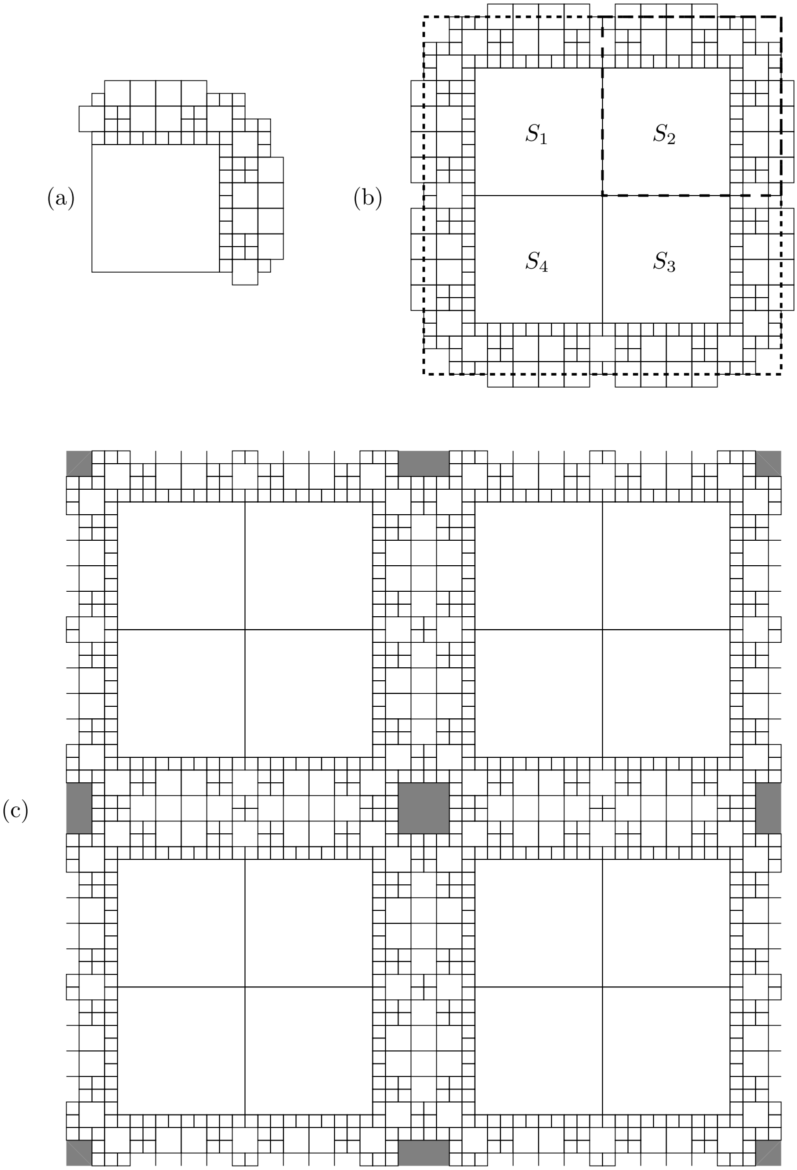


Fig. 9 (**a**) A *large square* of side 10 bordered by the system $\mathcal{R}$. (**b**) $S_1 \cup S_2 \cup S_3 \cup S_4$ bordered by 4 rotated copies of $\mathcal{R}$ (some squares are shared between adjacent copies). The block σ is the *large dashed square* containing $S_1 \cup S_2 \cup S_3 \cup S_4$. (**c**) Tiling of the plane with blocks composed of 4 *large squares* of side 10 bordered by 4 rotated copies of $\mathcal{R}$ (some squares are shared between adjacent blocks). The *shaded rectangles* in the figure represent holes in the tiling, and are not part of the square system

Assume now that one of the S_i, say S_2, belongs to $\mathcal{I}$. Observe that the 20 unit squares adjacent to top and right sides of S_2 do not belong to $\mathcal{I}$ (the same holds for the unit square in the corner, but this is irrelevant here). By Lemma 5,

$$|\mathcal{I} \cap \sigma| \le 100 + 4 \times 23 - 2 = 190, \quad \text{thus} \quad \frac{|\mathcal{I} \cap \sigma|}{|\mathcal{T} \cap \sigma|} \le \frac{190}{768} = \frac{1}{4} - \frac{1}{384},$$

as desired. Of course, one can get arbitrarily close to this bound, by using a suitably large (square) section of the tiling instead—since the boundary effects are negligible. This completes the proof of Theorem 5.

Remark Perhaps the above upper bound can be improved—the question is by how much? Ajtai wrote in his paper: "We now prove the conjecture is false for $d = 2$ (and thus for every $d > 2$ too)." While we also believe that the idea of his construction can be used to generate counterexamples in higher dimensions, the detailed arguments and the corresponding upper bounds still remain to be derived.

7 Lower Bounds for Convex Bodies in the Plane: Proof of Theorem 6

We prove Theorem 6 in this section. We first review some preliminaries. A lattice Λ is said to be *admissible* for a convex body S if at most one lattice point of Λ lies in the interior of S [15]. Denote by $|\Lambda|$ the area of a fundamental cell of Λ. Denote by $\Delta(S)$ the minimum fundamental cell area $|\Lambda|$ of a lattice Λ admissible for S. Consider a coordinate system with origin o. Define the *difference region* [12, pp. 38] of S as $S - S = \{s - s' \mid s, s' \in S\}$. Intuitively, $S - S$ is the union of all congruent homothetic copies of S that contain the origin o. $S - S$ is centrally symmetric, and is $\frac{1}{2}(S - S)$ scaled by 2, where $\frac{1}{2}(S - S)$ is the convex body obtained by Steiner symmetrization from S [21, Exercise 6-9] (we used this construction in Sect. 3). If S itself is centrally symmetric, then $S - S$ is a homothetic copy of S scaled by 2.

Arbitrary convex bodies We first prove a lower bound of $f(S) \ge 1/6$ for any convex body S in the plane. It is known by R. Rado's result [15, Theorem 7] that

$$f(S) \ge \frac{|S|}{\Delta(2S - 2S)} = \frac{|S|}{4\,\Delta(S - S)}.$$

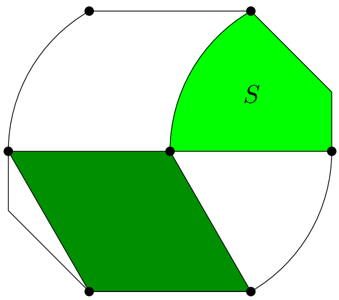
To prove that $f(S) \ge 1/6$, it suffices to show that

$$\frac{|S|}{\Delta(S - S)} \ge \frac{2}{3}.$$

We refer to Fig. 10. Using techniques from a classical lattice packing result by Fáry [7] (following [12, pp. 37–41]), one can show the following: First, $S - S$ contains an inscribed *affinely regular*[2] hexagon H, for any given direction $\vec{v}$ of one side of H. Second, any two vectors from the center of $S - S$ (also of H) to two non-opposite

[2] A convex hexagon is affinely regular if it is the image of regular hexagon under an affine transformation. Equivalently, a convex hexagon $p_1, \ldots, p_6$ is affinely regular if and only if (a) it is centrally symmetric, and (b) $\overrightarrow{p_2 p_1} + \overrightarrow{p_2 p_3} = \overrightarrow{p_3 p_4}$.

Fig. 10 Difference region $S - S$. Convex body S (with three straight sides and one curved side) *lightly shaded*. Center and vertices of inscribed hexagon H shown as *black dots*. Fundamental cell of lattice Λ (a parallelogram) *darkly shaded*



vertices of H form the basis of a lattice Λ. Third, with a suitable choice of $\vec{v}$, the lattice Λ satisfies

$$\frac{|S|}{|\Lambda|} \geq \frac{2}{3}.$$

Here $\delta_\Lambda(S) = |S|/|\Lambda|$ is the packing density of S in the lattice packing Λ, which is at least $2/3$ by Fáry's result [12, Theorem 4.1 and Exercise 4.1]. Since $S - S$ contains exactly one lattice point (the center) of Λ in its interior, Λ is admissible for $S - S$. It then follows by definition that

$$\frac{|S|}{\Delta(S - S)} \geq \frac{|S|}{|\Lambda|} \geq \frac{2}{3}.$$

The lower bound $f(S) \geq 1/6$ immediately follows. This bound cannot be improved, as R. Rado showed that $f(S) = 1/6$ for any triangle S [15, Theorem 10].

Centrally symmetric convex bodies We next prove a better lower bound for a centrally symmetric convex body S. Let T be a minimum-area convex hexagon that contains S. It is known that T is also centrally symmetric [12, Theorem 2.5]. The following results are also known:

1. $f(T) = 1/4$ [15, Theorem 10];
2. $f(S)/|S| \geq f(T)/|T|$ [15, Theorem 1];
3. $\delta(S) \leq |S|/|T|$, where $\delta(S)$ is the packing density of S [8] (see also [12, Corollary 3.4]);
4. $\delta(S) > 0.892656$ [20].

Therefore,

$$f(S) \geq f(T) \cdot |S|/|T| \geq \delta(S)/4 > 0.892656/4 > 1/4.4810.$$

This completes the proof of Theorem 6.

Remark Reinhardt [18] conjectured that $\delta(S) \geq 0.902414\ldots$ holds for any centrally symmetric convex body S in the plane, with equality only for the so-called smoothed octagon (see also [3, p. 11]). If this conjecture were to hold, the lower bound would be improved to $f(S) > 1/4.4325$. Compare this with the current best lower bound of $f(S) > \frac{\pi}{8\sqrt{3}} > 1/4.4106$ for a disk S [15, Theorem 10].

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