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Some properties of invariant measures of non symmetric dissipative stochastic systems

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Abstract. We consider transition semigroups generated by stochastic partial differential equations with dissipative nonlinear terms. We prove an integration by part formula and a Logarithmic Sobolev inequality for the invariant measure. No symmetry or reversibility assumptions are made. Furthemore we prove some compactness results on the transition semigroup and on the embedding of the Sobolev spaces based on the invariant measure. We use these results to derive asymptotic properties for a stochastic reaction–diffusion equation.

1. Introduction

The aim of this paper is to study asymptotic properties of a class of stochastic dissipative systems governed by a stochastic differential equation

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + \sqrt{C}dW(t), \quad t \geq 0, \\ X(0) = x \in H, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$, is the infinitesimal generator of a linear strongly continuous semigroup $\left(e^{tA}\right)_{t\geq 0}$ of contraction type, W is a cylindrical Wiener process on H, C is a symmetric positive linear operator on H bounded together with its inverse C^{-1}, and $F : D(F) \subset H \to H$ is a measurable mapping such that $F - \kappa$ is m-dissipative for some real number κ.

Under our assumptions, problem (1.1) has a unique mild solution $X(t, x)$. Then the corresponding transition semigroup $(P_t)_{t\geq 0}$, is defined on $C_b(H)$, the space of bounded and continuous functions on H, by

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$$P_t\phi(x) = \mathbb{E}[\phi(X(t,x))],\ \phi \in C_b(H),\ t \geq 0, \tag{1.2}$$

where $\mathbb{E}$ means expectation.

We shall also assume that there exists a Borel probability measure ν on H such that

$$\int_H P_t\phi(x)\nu(dx) = \int_H \phi(x)\nu(dx),\ \phi \in C_b(H).$$

The measure ν is said to be an *invariant measure* for the transition semigroup $(P_t)_{t\geq 0}$. If the invariant measure exists then $(P_t)_{t\geq 0}$ extends to a strongly continuous semigroup of contractions on $L^2(H,\nu)$. Its generator will be denoted by N_2 and its domain by $D(N_2)$. Several conditions ensuring existence of invariant measures are known, see e.g. [1], [16], [18], [23] and the references therein.

A first new result of the paper is that $D(N_2)$ is included in the Sobolev space $W^{1,2}(H,\nu)$ and that the following identity holds

$$\int_H \phi(x)N_2\phi(x)\nu(dx) = -\frac{1}{2}\int_H \left|C^{1/2}D\phi(x)\right|^2 \nu(dx), \quad \phi \in D(N_2). \tag{1.3}$$

This identity is quite natural. In fact, the infinitesimal generator N_2 of $(P_t)_{t\geq 0}$ is given formally by

$$N_2\phi(x) = \frac{1}{2}\,\mathrm{Tr}\left(CD^2\phi(x)\right) + \langle Ax + F(x), D\phi(x)\rangle\,, \tag{1.4}$$

where Tr represents the trace. Now, by a formal computation, we have

$$N_2\left(\phi^2\right) = 2\phi N_2\phi + \left|C^{1/2}D\phi\right|^2,$$

and, taking into account the invariance of ν, we find $\int_H N\left(\phi^2\right)d\nu = 0$, which yields (1.3).

The difficulty to prove (1.3) is to find a core Λ for N_2 such that $\phi^2 \in D(N_2)$ for any $\phi \in \Lambda$. We prove that this can be done in a very general setting and derive (1.3).

The identity (1.3) is the key point to prove that a Logarithmic Sobolev inequality holds, see section 4. As well known this implies exponential convergence to equilibrium for $(P_t)_{t\geq 0}$ and that N_2 has a gap in its spectrum (see [2]).

Note that we have no information in general on the bilinear form

$$a(\phi,\psi) = \int_H \psi(x)N_2\phi(x)\nu(dx),\ \phi,\psi \in D(N_2),$$

so that N_2 is not necessarily related to any Dirichlet form, see [22]. The mentionned results were known in this case, see [3] and also [18] ,[20], [21], [32], but they seem to be new in our general setting.

The second new result of the paper is that, under suitable assumptions, $(P_t)_{t\geq 0}$ is compact in $L^p(H,\nu)$ for any $p > 1$ and the embedding of the Sobolev space $W^{1,p}(H,\nu)$ into $L^p(H,\nu)$ is compact for any $p \geq 2$. To our knowledge, this is the first result of compactness for non trivial non gaussian measures in an infinite dimensional space.

To prove this result we need two tools. First we assume that a Logarithmic Sobolev inequality holds. Secondly that the invariant measure ν has a density with respect to the gaussian measure μ invariant with respect to the linear equation - i.e. with $F = 0$ in (1.1) - and that this density has a logarithmic derivative in $L^p(H, \nu)$ with $p > 1$. We note that this condition on the density of the invariant measure can be obtained by the results in [5] for instance.

We recall that in the gaussian case, necessary and sufficient conditions for compactness are known (see [9], [12], [26]). Results in the finite dimensional case have been obtained in [25] for non gaussian measures, and more recently in [7] under very weak assumptions. Such a compactness result is important since it allows to study other equations thanks to perturbation arguments. In this way, many properties on the invariant measures can be obtained. This approach has been used in many articles to treat perturbations of linear systems thanks to the previously known compactness result in the gaussian case.

In section 6 we apply the obtained results to a Reaction-Diffusion equation with a polynomial nonlinearity. We prove that the integration by parts formula (1.3) always holds. In the case of a general system of gradient type or of a strictly dissipative system not necessarily gradient, we obtain that a Logarithmic Sobolev inequality holds, that the transition semigroup is compact and that the above embedding of Sobolev spaces is compact. In the general case, we prove exponential convergence to equilibrium thanks a perturbation argument.

We note that the techniques used in this article are general and powerful. They can be applied to other systems generated by stochastic partial differential equations and could be used to get other results.

2. Assumptions and notations

We now set some notations, state our main assumptions, and recall some classical results concerning existence, uniqueness and regularity for problem (1.1).

Let H be a Hilbert space. We denote by $|\cdot|$ its norm and by $\langle\cdot,\cdot\rangle$ its inner product. Moreover $\mathcal{L}(H)$ (with norm $\|\cdot\|$) represents the Banach algebra of all linear bounded operators from H into H endowed with its usual norm, and $\mathcal{L}_1(H)$ is the subspace of all trace class operators.

If E is a Banach space we denote by $C_b(H; E)$ the Banach space of all continuous and bounded mappings from H into E, endowed with the sup norm $\|\cdot\|_0$. Moreover, for any $k \in \mathbb{N}$, $C_b^k(H; E)$ will represent the Banach space of all mappings from H into E which are continuous and bounded together with their Fréchet derivatives of order less or equal to k endowed with their natural norm denoted by $\|\cdot\|_k$. If $E = \mathbb{R}$, we set $C_b(H; \mathbb{R}) = C_b(H)$ and $C_b^k(H; \mathbb{R}) = C_b^k(H)$. Let $f : D(f) \subset H$ be a function such that $D(f)$ is another Banach space. If f is Gateau differentiable at a point x of $D(f)$, we write Df for its Gateau derivative.

We consider two linear operators A and C on H and we assume that the following holds.

Hypothesis 2.1. *(i) A generates a C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ on H and there exists $\omega > 0$ such that*

$$\left\| e^{tA} \right\| \leq e^{-\omega t}, \quad t \geq 0.$$

(ii) $C \in \mathcal{L}(H)$ is invertible, $C^{-1} \in \mathcal{L}(H)$ and for all $T > 0$ we have $Tr\,[Q_T] < \infty$, where

$$Q_T x = \int_0^T e^{sA} C e^{sA^*} x ds, \ x \in H,$$

and Tr stands for the trace.

We shall denote by W a cylindrical Wiener process on H on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By Hypothesis 2.1 it follows that the stochastic convolution

$$Z(t) = \int_0^t e^{(t-s)A} \sqrt{C} dW(s),$$

is a well defined Gaussian process and the law of $Z(t)$ has mean 0 and covariance operator Q_t. Moreover the equation

$$\begin{cases} dZ(t) = AZ(t)dt + \sqrt{C} dW(t), \\ Z(0) = x \in H, t \geq 0, \end{cases}$$

has a unique mild solution $Z(\cdot, x)$ that defines the so-called Ornstein–Uhlenbeck process,

$$Z(t, x) = e^{tA} x + \int_0^t e^{(t-s)A} \sqrt{C} dW(s), \ t \geq 0.$$

The corresponding transition semigroup,

$$R_t \phi(x) = \int_H \phi(e^{tA} x + y) \mathcal{N}(0, Q_t)(dy), \quad \phi \in C_b(H),$$

is not strongly continuous on $C_b(H)$ in general, it belongs to the class of π–continuous Markov semigroup, see [27]; roughly speaking it is continuous with respect to the pointwise bounded convergence. Its infinitesimal generator L can be defined through its resolvent by

$$R(\lambda, L)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} R_t \varphi(x) dt, \ x \in H, \ \lambda > 0, \ \ \varphi \in C_b(H).$$

We shall denote by $D(L)$ the domain of L. It can be shown, see [16], that $(R_t)_{t\geq 0}$ is strong Feller, $D(L) \subset C_b^1(H)$ and

$$\|R(\lambda, L)\| \leq \sqrt{\pi/\lambda}, \ \lambda > 0. \tag{2.1}$$

We also note that $(R_t)_{t\geq 0}$ admits a unique invariant measure μ; it is gaussian with covariance operator $Q : \mu = \mathcal{N}(0, Q)$, where

$$Qx = \int_0^\infty e^{tA} C e^{tA^*} x dt, \quad x \in H.$$

Before giving assumptions on F let us recall the notion of solution for (1.1). By a *mild* solution to (1.1) we mean an adapted stochastic process $X(\cdot) = X(\cdot, x)$ such that $X(0) = x \in D(F)$, $X(t, x) \in D(F)$ for all $t \in [0, T]$ and

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s, x))ds + \int_0^t e^{(t-s)A}\sqrt{C} dW(s), \ t \geq 0.$$

If $x \in H$ then we say that $X(\cdot, x)$ is a generalized solution to (1.1) if it is mean-square continuous on H and there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to x in H and such that

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \mathbb{E}\,|X(t, x_n) - X(t, x)|^2 = 0.$$

We assume that the domain $D(F)$ of F is a Banach space and $F : D(F) \to H$ is Gateaux differentiable for any $x \in D(F)$. Moreover, given a mild solution of (1.1), $X(t, x), \ t \in [0, T]$, we consider the equation

$$\begin{cases} \dfrac{d\eta^h(t)}{dt} = A\eta^h(t) + DF(X(t, x))\eta^h(t), \quad t \geq 0, \\ \\ \eta^h(0) = h \in H. \end{cases} \tag{2.2}$$

We define as above a mild or generalized solution of (2.2) and denote it by $\eta^h(\cdot, x)$.

We now state the assumptions on $F : D(F) \subset H \to H$.

Hypothesis 2.2. *(i) $D(F)$ is a Banach space and $F : D(F) \to H$ is Gateaux differentiable. There exists $\kappa > 0$ such that $F - \kappa$ is m–dissipative.*

(ii) Problems (1.1) and (2.2) have unique generalized solutions. Moreover $DX(t,x)$. $h = \eta^h(\cdot, x)$ for any $x \in D(F)$. Finally

$$X_\alpha(t, x) \to X(t, x), \ \eta^h_\alpha(t, x) \to \eta^h(t, x), \quad \text{a.s.}\ ,$$

where $X_\alpha(t, x)$ and η^h_α are the solutions to (1.1) and (2.2) with F replaced by the Yosida approximation :

$$F_\alpha(x) = \frac{1}{\alpha}(J_\alpha(x) - x) + \kappa I, \quad x \in H,$$

where $J_\alpha(x)$ is the unique solution of $J_\alpha(x) - \alpha(F(J_\alpha(x)) - \kappa J_\alpha(x)) = x$, and I is the identity mapping.

(iii) There exists an invariant measure ν for problem (1.1) and $\nu(D(F)) = 1$.

These assumptions hold for a large class of Reaction-Diffusion systems (see [8] and section 6).

Thanks to Hypothesis 2.2, we can define the transition semigroup $(P_t)_{t\geq 0}$ associated to equation (1.1)

$$P_t\phi(x) = \mathbb{E}(\phi(X(t,x))), \text{ for } \phi \in C_b(H), \ x \in H, \text{ and } t \geq 0.$$

As R_t the semigroup P_t belongs to the class of π–continuous Markov semigroups. Its infinitesimal generator N will be also defined through its resolvent by

$$R(\lambda, N)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)dt, \ x \in H, \ \lambda > 0, \quad \varphi \in C_b(H).$$

We shall denote by $D(N)$ the domain of N. Obviously, the operator N_2 is an extension of N. Moreover it is easy to check that we have

$$|\eta^h(t,x)| \leq e^{-(\omega-\kappa)t}|h|, \ x \in H, \ t \geq 0. \tag{2.3}$$

This easily implies that, for $\phi \in C_b^1(H)$ and $t \geq 0$, $P_t\phi \in C_b^1(H)$ and

$$|DP_t\phi(x)| \leq e^{-(\omega-\kappa)t} P_t(|D\phi|(x)). \tag{2.4}$$

Using the above definition of $R(\lambda, N)$, it is then not difficult to verify that P_t maps $R(\lambda, N)(C_b^1(H))$ into itself for any $\lambda > 0$ and $t \geq 0$. Moreover, $R(\lambda, N)$ $(C_b^1(H)) \subset C_b^1(H)$.

It is well known that P_t can be uniquely extended to a contraction semigroup, that we still denote by P_t, in $L^p(H, \nu)$, $\ p \geq 1$. Its infinitesimal generator will be denoted by N_p.

3. The integration by part formula

The aim of this section is to prove rigourously that (1.3) holds.

We will consider the linear span $\mathcal{E}_A(H)$ of real parts of all exponential functions $e^{i\langle h,x\rangle}$, $\ x \in H$, with $h \in D(A^*)$. Functions in $D(L)$ can be approximated by functions in $\mathcal{E}_A(H)$ in the following sense. Given a 4–sequence $(\phi_n)_{n\in\mathbb{N}^4}$ indexed by $n = (n_1, n_2, n_3.n_4) \in \mathbb{N}^4$, we say that it converges in a generalized sense to ϕ if

$$\lim_{n\to\infty} \phi_n(x) := \lim_{n_1\to\infty} \lim_{n_2\to\infty} \lim_{n_3\to\infty} \lim_{n_4\to\infty} \phi_n(x) = \phi(x), \quad x \in H.$$

It is proved in [13, Proposition 2.7] that for any $\phi \in D(L)$ there exists a 4–sequence $(\phi_n)_{n\in\mathbb{N}^4}$ in $\mathcal{E}_A(H)$ such that

$$\lim_{n\to\infty} \phi_n(x) = \phi(x), \quad \lim_{n\to\infty} D\phi_n(x) = D\phi(x), \ x \in H, \ \lim_{n\to\infty} L\phi_n(x) = L\phi(x) \tag{3.1}$$

and

$$\sup_{N\in\mathbb{N}^4, x\in H} \left\{ |\phi_n(x)| + |D\phi_n(x)| + \frac{|L\phi_n(x)|}{1+|x|^2} \right\} < +\infty. \tag{3.2}$$

We start with the case when F is bounded on H. In this case, taking into account (2.1), we see that $D(N) = D(L) \subset C_b^1(H)$ and we have

$$N\phi = L\phi + \langle F, D\phi\rangle \,,\ \phi \in D(L).$$

Lemma 3.1. *Assume that F is bounded. Let $f \in C_b(H)$, and $\phi = R(1, N)f$. Then $\phi^2 \in D(N)$, and the following identities hold,*

$$N\left(\phi^2\right) = 2\phi N\phi + \left|C^{1/2}D\phi\right|^2, \tag{3.3}$$

and

$$\phi^2 = R(2, N)\left(2\phi f - \left|C^{1/2}D\phi\right|^2\right). \tag{3.4}$$

Proof. Let $(\phi_n)_{n\in\mathbb{N}^4} \subset \mathcal{E}_A(H)$ be a 4–sequence such that (3.1) and (3.2) hold, and set $f_n = \phi_n - N\phi_n$. Then we have

$$f_n\phi_n = \phi_n^2 - N\phi_n\phi_n. \tag{3.5}$$

Since obviously

$$N(\phi_n^2) = 2\phi_n N\phi_n + |C^{1/2}D\phi_n|^2, \tag{3.6}$$

we have, by (3.5),

$$2\phi_n^2 - N(\phi_n^2) = 2f_n\phi_n - |C^{1/2}D\phi_n|^2. \tag{3.7}$$

Now, letting n tend to infinity in (3.6) and (3.7) gives (3.3) and (3.4) respectively. □

We now consider the case of an unbounded F. Note first that the approximating mappings $F_\alpha,\ \alpha > 0$, introduced in Hypothesis 2.2 are Lipschitz continuous on H. Therefore, we can choose a sequence $\left(F_{n,\alpha}\right)_{n\in\mathbb{N}}$ in $C_b^1(H; H)$ pointwise convergent to F_α on E and such that

$$\|F_{n,\alpha}\|_1 \le \|F_\alpha\|_1,\ n \in \mathbb{N},\ \alpha > 0.$$

Let us consider the problem

$$\begin{cases} dX(t) = \left(AX(t) + F_{n,\alpha}(X(t)\right)dt + \sqrt{C}dW(t), \\ X(0) = x \in H, t \ge 0, \end{cases} \tag{3.8}$$

whose solution is denoted by $X_{n,\alpha}(t, x)$. We set

$$P_t^{n,\alpha}\phi(x) = \mathbb{E}\left(\phi(X_{n,\alpha}(t, x))\right),\ n \in \mathbb{N},\ \alpha > 0,\ \phi \in C_b(H), \tag{3.9}$$

and denote by $N_{n,\alpha}$ the infinitesimal generator of $\left(P_t^{n,\alpha}\right)_{t\ge 0}$.

Lemma 3.2. *Assume that Hypothesis 2.1 and 2.2 hold and moreover that* $\phi \in R(1,N)(C_b^1(H))$. *Then* $\phi^2 \in D(N)$, *and*

$$N\left(\phi^2\right) = 2\phi N\phi + \left|C^{1/2}D\phi\right|^2. \tag{3.10}$$

Proof. Let $\phi = R(1,N)f$ with $f \in C_b^1(H)$, and set $\phi_{n,\alpha} = R(1,N_{n,\alpha})f$. Then by Lemma 3.1

$$\phi_{n,\alpha}^2 = R(2,N_\alpha)\left(2\phi_{n,\alpha}f - \left|C^{1/2}D\phi_{n,\alpha}\right|^2\right). \tag{3.11}$$

On the other hand it is not difficult to prove that for $t \geq 0,\ x \in\ D(F)$

$$X_{n,\alpha}(t,x) \to X(t,x), \quad DX_{n,\alpha}(t,x) \to DX(t,x),$$

$\mathbb{P}$–a.s.

Consequently

$$P_t^{n,\alpha}f(x) \to P_tf(x).$$

Also, since $DP_t^{n,\alpha}f(x) = \mathbb{E}\left(\left(DX_{n,\alpha}(t,x)\right)^* Df(X_{n,\alpha}(t,x))\right)$, noting that (2.3) holds also for the approximating differential and uniformly with respect to α, we have

$$DP_t^{n,\alpha}f(x) \to DP_tf(x).$$

It follows, using the definition of the resolvent,

$$\phi_{n,\alpha}(x) = R(1,N_\alpha)f(x) \to R(1,N)f(x),$$

$$D\phi_{n,\alpha}(x) = DR(1,N_\alpha)f(x) \to DR(1,N)f(x).$$

Therefore, letting $n \to \infty$ and $\alpha \to 0$ in (3.11), we find

$$\phi^2(x) = R(2,N)(2\phi f - \left|C^{1/2}D\phi\right|^2)(x), \quad x \in D(F).$$

Since, by Hypothesis 2.2, $\nu(D(F)) = 1$ we find that

$$2\phi^2 - N\phi^2 = 2\phi f - \left|C^{1/2}D\phi\right|^2 = 2\phi(\phi - N\phi) - \left|C^{1/2}D\phi\right|^2$$

and the conclusion follows. □

Proposition 3.3. *Assume that Hypotheses 2.1 and 2.2 hold. Then for any* $\phi \in R(1,N)(C_b^1(H))$ *we have*

$$\int_H \phi(x)N\phi(x)\nu(dx) = -\frac{1}{2}\int_H \left|C^{1/2}D\phi(x)\right|^2 \nu(dx). \tag{3.12}$$

Proof. Let $\phi \in R(1, N)(C_b^1(H))$. Then, since the measure ν is invariant for P_t, by Lemma 3.2

$$0 = \int_H N\left(\phi^2\right)(x)\nu(dx) = \int_H \left(2\phi(x)N\phi(x) + \left|C^{1/2}D\phi(x)\right|^2\right)\nu(dx),$$

and we obtain (3.12). □

From (3.12) it follows a useful identity for $u(t) = P_t\phi$.

Proposition 3.4. *Assume that Hypotheses 2.1 and 2.2 hold. Then for any* $\phi \in R(1, N)(C_b^1(H))$ *we have*

$$\int_H |P_t\phi|^2 d\nu + \int_0^t \int_H |C^{1/2}DP_s\phi|^2 ds d\nu = \int_H |\phi|^2 d\nu. \tag{3.13}$$

Proof. For any $\phi \in D(N)$ we have

$$\frac{d}{dt}\, P_t\phi = NP_t\phi,\ \ t \geq 0. \tag{3.14}$$

Multiplying both sides of (3.14) by $P_t\phi$, integrating in H with respect to ν, and taking into account (3.12), we find

$$\frac{d}{dt}\int_H |P_t\phi(x)|^2\nu(dx) = \int_H \left|C^{1/2}DP_t\phi(x)\right|^2 \nu(dx).$$

Now the conclusion follows, integrating in t. □

We are going now to prove (3.12) for any $\phi \in D(N_2)$. For this we need to introduce the Sobolev space $W^{1,2}(H, \nu)$, and first to prove closability of the derivative in $L^2(H, \nu)$.

Proposition 3.5. *Assume that Hypotheses 2.1, 2.2 hold. Then D is closable in* $L^2(H, \nu)$.

Proof. Let $(\phi_n)_{n\in\mathbb{N}} \subset R(1, N)C_b^1(H)$ be such that

$$\lim_{n\to\infty} \phi_n = 0 \quad \text{and} \quad \lim_{n\to\infty} D\phi_n = \psi \text{ in } L^2(H, \nu).$$

By assumption, for every ϕ_n and $t > 0$

$$DP_t\phi_n(x) = \mathbb{E}\left[X_x^*(t, x)D\phi_n\left(X(t, x)\right)\right] \tag{3.15}$$

and by (3.13)

$$\int_0^t \int_H |C^{1/2}DP_s\phi_n|^2 ds d\nu = \int_H |\phi_n|^2 d\nu - \int_H |P_t\phi_n|^2 d\nu \to 0,\ \text{as } n \to \infty.$$

Moreover

$$\int_0^t \int_H |C^{1/2} D P_s \phi_n|^2 ds d\nu$$

$$= \int_0^t \int_H \left| \mathbb{E}\left[\left(C^{-1/2} X_x(s,x) C^{1/2} \right)^* C^{1/2} D\phi_n\, (X(s,x)) \right] \right|^2 \nu(dx) ds,$$

and by (2.3)

$$\left| \left(\left| \int_0^t \int_H \mathbb{E}\left[\left(C^{-1/2} X_x(s,x) C^{1/2} \right)^* C^{1/2} D\phi_n\, (X(s,x)) \right] \right|^2 \nu(dx)\, ds \right)^{1/2} \right.$$

$$\left. - \left(\int_0^t \int_H \left| \mathbb{E}\left[\left(C^{-1/2} X_x(s,x) C^{1/2} \right)^* C^{1/2} \psi\, (X(s,x)) \right] \right|^2 \nu(dx)\, ds \right)^{1/2} \right|$$

$$\leq \left(\int_0^t \int_H e^{-2(\omega-\kappa)s} \mathbb{E}\left[\left| C^{1/2} D\phi_n (X(s,x)) - C^{1/2} \psi\, (X(s,x))) \right|^2 \right] \nu(dx)\, ds \right)^{1/2}$$

$$= \left(\int_0^t \int_H e^{-2(\omega-\kappa)s} \left| C^{1/2} D\phi_n\, (x) - \psi\, (x) \right|^2 \nu(dx)\, ds \right)^{1/2} \tag{3.16}$$

by the invariance of ν. By assumption, this goes to zero and we find that

$$\int_H \left| \mathbb{E}\left[(C^{-1/2} X_x(t,x) C^{1/2})^* C^{1/2} \psi(X(t,x)) \right] \right|^2 \nu(dx) = 0,$$

for a.e. t.

Let $h \in H$, then

$$\int_H \left| \mathbb{E}\langle C^{1/2} \psi(X(t,x)), C^{-1/2} X_x(t,x) C^{1/2} h \rangle \right|^2$$

$$= \int_H \left| \mathbb{E}\langle (C^{-1/2} X_x(t,x) C^{1/2})^* C^{1/2} \psi(X(t,x)), h \rangle \right|^2 \nu(dx) = 0,$$

for a.e. t, and

$$\int_H \left| \mathbb{E}\langle C^{1/2} \psi(X(t,x)), h \rangle \right| \nu(dx)$$

$$\leq \int_H \left| \mathbb{E}\langle C^{1/2} \psi(X(t,x)), C^{-1/2} X_x(t,x) C^{1/2} h - h \rangle \right| \nu(dx)$$

for a.e. t. By Cauchy-Schwarz inequality, (2.3), continuity of $X_x(t,x)$ with respect to t and dominated convergence we deduce that the right hand side goes to zero. However, the left hand side is equal to $\int_H |P_t(\langle C^{1/2}\psi, h\rangle)|\nu(dx)$ and since $\langle C^{1/2}\psi, h\rangle \in L^2(H,\nu)$ and P_t is strongly continuous on $L^1(H,\nu)$ we have

$$P_t(\langle C^{1/2}\psi, h\rangle) \to |\langle C^{1/2}\psi, h\rangle| \text{ as } t \to 0, \text{ in } L^1(H,\nu).$$

This implies

$$\int_H |P_t(\langle C^{1/2}\psi, h\rangle)|\nu(dx) \to \int_H |\langle C^{1/2}\psi, h\rangle|\nu(dx) \text{ as } t \to 0,$$

and

$$\int_H |\langle C^{1/2}\psi, h\rangle|\nu(dx) = 0.$$

Thus for any $h \in H$

$$\langle C^{1/2}\psi, h\rangle = 0, \ \nu - \text{a.s.}$$

and, since H is separable and $C^{1/2}$ invertible, the conclusion follows. □

We now define the spaces $W^{1,p}(H,\nu)$ for $p \geq 1$ as the completion of $C_b^1(H)$ with respect to the norm

$$\left(\int_H |\phi|^p d\nu + \int_H |D\phi|^p d\nu\right)^{1/p}.$$

By Proposition 3.5, D is closable for $p \geq 2$ so that

$$W^{1,p}(H,\nu) \subset L^p(H,\nu),\, p \geq 2.$$

For $\phi \in W^{1,p}(H,\nu)$, $p \geq 2$, we write $D\phi$ for its generalized derivative.

We are now in a position to prove the main result of this section.

Theorem 3.6. *Assume that Hypotheses 2.1 and 2.2 hold. Then $D(N_2) \subset W^{1,2}(H,\nu)$. Moreover for all $\phi \in D(N_2)$ we have*

$$\int_H \phi(x) N_2\phi(x)\nu(dx) = -\frac{1}{2}\int_H \left|C^{1/2} D\phi(x)\right|^2 \nu(dx). \tag{3.17}$$

Proof. Clearly $R(1,N)(C_b^1(H))$ is a core for $D(N_2)$, since $C_b^1(H)$ is dense in $L^2(H,\nu)$. Thus for any $\phi \in D(N_2)$ there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset R(1,N)$ $(C_b^1(H))$ such that

$$\phi_n \to \phi, \quad N_2\phi_n \to N_2\phi, \text{ in } L^2(H,\nu).$$

By Proposition 3.3 we have

$$\int_H \phi_n(x) N\phi_n(x)\nu(dx) = -\frac{1}{2}\int_H \left|C^{1/2} D\phi_n(x)\right|^2 \nu(dx).$$

This implies that $(\phi_n)_{n\in\mathbb{N}}$ is Cauchy in $W^{1,2}(H,\nu)$ and the conclusion follows. □

4. Logarithmic Sobolev inequality and spectral gap

We say that a Poincaré inequality holds if there exists a constant $\lambda > 0$ such that

$$\int_H \left|\phi(x) - \overline{\phi}\right|^2 \nu(dx) \le \lambda \int_H \left|C^{1/2} D\phi(x)\right|^2 \nu(dx), \quad \phi \in D(N_2), \tag{4.1}$$

where $\overline{\phi} = \int_H \phi(x)\nu(dx)$.

We show now that in this case there is a gap in the spectrum of N_2 and that the convergence of P_t to the equilibrium is exponential. We shall use the notation $\|\phi\|_\nu^2 = \int_H \phi^2 d\nu$.

We have in fact the well known result.

Proposition 4.1. *Assume, besides Hypothesis 2.1, 2.2 that (4.1) holds for some* $\lambda > 0$. *Then we have*

$$\sigma(N_2)\backslash\{0\} \subset \left\{z \in \mathbb{C} : \ \mathrm{Re}(z) \ < -\frac{1}{\lambda}\right\}.$$

Moreover

$$\int_H \left|P_t\phi(x) - \overline{\phi}\right|^2 \nu(dx) \le e^{-t/\lambda} \int_H |\phi(x)|^2 \nu(dx), \ \phi \in D(N_2). \tag{4.2}$$

Proof. The proof is an easy consequence of (3.17), see [2, Proposition 2.3]. We give here the simple proof for the reader's convenience. Let $L_0^2(H, \nu)$ be the closed subspace of $L^2(H, \nu)$ of all functions f such that $\overline{f} = 0$. Since $L_0^2(H, \nu)$ is invariant for $(P_t)_{t\ge 0}$ it is enough to show that

$$\langle N_2\phi, \phi\rangle_\nu \le -\frac{1}{\lambda} \ \|\phi\|_\nu^2, \ \phi \in L_0^2(H, \nu) \cap D(N_2). \tag{4.3}$$

If $\phi \in L_0^2(H, \nu)$ we have in fact, taking into account (3.17) and (4.1),

$$\langle N_2\phi, \phi\rangle_\nu = -\frac{1}{2} \int_H |C^{1/2} D\phi(x)|^2 \nu(dx) \le -\frac{1}{\lambda} \ \|\phi\|_\nu^2. \qquad \square$$

A sufficient condition in order that a Poincaré inequality holds is that N_p satisfies the Logarithmic Sobolev Inequality, that is if

$$\int_H |\phi(x)|^p \log |\phi(x)|^p \nu(dx) \le c(p)\left\langle -N_2\phi, \phi_p\right\rangle + \|\phi\|_{p,\nu}^p \log \|\phi\|_{p,\nu}^p \tag{4.4}$$

for $\phi \in D(N_p)$ and $p > 1$, where $\phi_p = \mathrm{sgn}\,\phi|\phi|^{p-1}$ and $c(p) < \infty$. If $p = 2$ then in view of Theorem 3.6 inequality (4.4) takes the form

$$\int_H |\phi(x)|^2 \log |\phi(x)|^2 \nu(dx) \le \frac{c(2)}{2} \left\|C^{1/2} D\phi\right\|_\nu^2 + \|\phi\|_\nu^2 \log \|\phi\|_\nu^2. \tag{4.5}$$

By the result of [29] inequality (4.5) implies the Poincaré inequality with the constant $\frac{c(2)}{4}$.

Also, similar arguments as in section 3 imply that for $p \geq 2$, (4.4) takes the form

$$\int_H |\phi(x)|^p \log |\phi(x)|^p \nu(dx)$$
$$\leq \frac{p-1}{2} c(p) \int_H |\phi(x)|^{p-2} |C^{1/2} D\phi(x)|^2 \nu(dx) + \|\phi\|_{p,\nu}^p \log \|\phi\|_{p,\nu}^p. \tag{4.6}$$

Moreover, since $(P_t)_{t\geq 0}$ is strongly continuous on $W^{1,p}(H,\nu)$ ([1]), for arbitrary $\phi \in W^{1,p}(H,\nu)$ we have a sequence in $D(N_p)$ converging to ϕ in $W^{1,p}(H,\nu)$. This easily imply that (4.6) holds for $\phi \in W^{1,p}(H,\nu)$.

Theorem 4.2. *Assume, besides Hypotheses 2.1 and 2.2, that* $\omega - \kappa > 0$. *Then (4.4) holds with*

$$c(p) = \frac{\omega - \kappa}{\|C\| \, \|C^{-1}\|} \frac{p^2}{p-1}.$$

Proof. In the proof we follow the method from [17] presented in the proof of Theorem 6.2.42.

We start with the case $p = 2$.

Step 1. Let $\phi \in D(N)$ and let $\phi \geq \delta > 0$. Then

$$\int_H N\phi \, \log \phi \, d\nu = -\frac{1}{2} \int_H \frac{1}{\phi} \, |C^{1/2} D\phi|^2 d\nu. \tag{4.7}$$

Let us first assume that $\phi \in C_b^2(H)$, $D^2\phi \in C_b(H; \mathcal{L}_1(H))$ and $A^* D\phi \in C_b(H; H)$ then we have

$$N\phi \, \log \phi = -N(\log \phi)\phi + N(\phi \log \phi) - \langle C^{1/2} D\phi, C^{1/2} D(\log \phi)\rangle. \tag{4.8}$$

Moreover, since

$$C^{1/2} D \log \phi = \frac{1}{\phi} C^{1/2} D\phi$$

and

$$(C^{1/2} D)^2 \log \phi = \frac{1}{\phi} (C^{1/2} D)^2 \phi - \frac{1}{\phi^2} C^{1/2} D\phi \otimes C^{1/2} D\phi,$$

we have

$$N \log \phi = \frac{1}{\phi} N\phi - \frac{1}{2\phi^2} \left| C^{1/2} D\phi \right|^2.$$

[1] This is proved in the next section.

Substituting in (4.8) we find

$$N\phi\ \log\phi = N(\phi\log\phi) - N\phi - \frac{1}{2\phi}\ |C^{1/2}D\phi|^2.$$

Now (4.7) follows integrating over H and taking into account the invariance of ν. The general case is then obtained by approximating ϕ by smoother functions satisfying the above assumptions.

Step 2 (Conclusion). Let $\phi \in R(1, N)(C_b^1(H))$ and set $u(t) = P_t(\phi^2)\log[P_t(\phi^2)]$, where $|\phi| \geq \delta > 0$. Then by Lemma 3.2 we have $\phi^2 \in D(N)$ and so u is differentiable with respect to t in $L^2(H, \nu)$ and it results

$$u'(t) = NP_t(\phi^2)\log[P_t(\phi^2)] + NP_t(\phi^2).$$

It follows

$$\begin{aligned} u(t) - u(0) &= P_t(\phi^2)\log[P_t(\phi^2)] - \phi^2\log(\phi^2) \\ &= \int_0^t NP_s(\phi^2)\log[P_s(\phi^2)]ds + \int_0^t NP_s(\phi^2)ds. \end{aligned}$$

Integrating with respect to ν, and taking into account (4.7), we find

$$\begin{aligned} &\int_H P_t(\phi^2)\log[P_t(\phi^2)]d\nu - \int_H \phi^2\log(\phi^2)d\nu \\ &= \int_0^t ds\int_H NP_s(\phi^2)\log[P_s(\phi^2)]d\nu \\ &= -\frac{1}{2}\int_0^t ds\int_H \frac{1}{P_s(\phi^2)}\ |C^{1/2}DP_s(\phi^2)|^2 d\nu. \end{aligned} \tag{4.9}$$

Now, recalling (2.4) and using the Hölder estimate, we find

$$\begin{aligned} |DP_s(\phi^2)|^2 &\leq e^{-2(\omega-\kappa)s}\left[P_s\left(|D(\phi^2)|\right)\right]^2 \\ &= 4e^{-2(\omega-\kappa)s}\left[P_s\left(|\phi||D\phi|\right)\right]^2 \\ &\leq 4e^{-2(\omega-\kappa)s}\left[P_s\left(\phi^2\right)\right]P_s\left(|D\phi|^2\right). \end{aligned}$$

Therefore, since C and C^{-1} are bounded,

$$\frac{1}{P_s(\phi^2)}\ |C^{1/2}DP_s(\phi^2)|^2 \leq 4\|C\|\ \|C^{-1}\|e^{-2(\omega-\kappa)s}P_s\left(|C^{1/2}D\phi|^2\right). \tag{4.10}$$

Substituting (4.10) into (4.9) gives

$$\int_H P_t(\phi^2)\log[P_t(\phi^2)]d\nu - \int_H \phi^2\log(\phi^2)d\nu$$

$$\geq -4\|C\|\,\|C^{-1}\|\int_0^t e^{-2(\omega-\kappa)s}ds\int_H |C^{1/2}D\phi|^2 d\nu$$

$$= -\frac{2\|C\|\,\|C^{-1}\|}{\omega-\kappa}\,(1-e^{-2(\omega-\kappa)t})\int_H |C^{1/2}D\phi|^2 d\nu. \tag{4.11}$$

Since we have assumed that $\omega-\kappa>0$, we know that $P_t\phi^2\to\overline{(\phi^2)}$ strongly (see [16]). Therefore, letting $t\to\infty$, we find

$$\overline{(\phi^2)}\log\left[\overline{(\phi^2)}\right] - \int_H \phi^2\log(\phi^2)d\nu \geq -\frac{2\|C\|\,\|C^{-1}\|}{\omega-\kappa}\int_H |C^{1/2}D\phi|^2 d\nu,$$

which proves (4.5) for $p=2$ and $|\phi|\geq\delta>0$ such that $\phi\in D(N)\cap C_b^1(H)$. Approximating any $\phi\in D(N_2)$ with functions ϕ_n such that $\phi_n\in D(N)\cap C_b^1(H)$ and $|\phi_n|\geq\delta>0$, we prove (4.5) for any $\phi\in D(N_2)$.

The case of general p can be treated similarly, taking $u(t)=P_t\phi^p\log(P_t\phi^p)$. It can also be obtained from the case $p=2$ using the result of [20]. □

The assumption $\omega-\kappa>0$ in Theorem 4.2 is rather restrictive. To relax this assumption in the application considered below, we will use the following well known result.

Theorem 4.3. *Let ν_1, ν_2 be two probability measures on H such that ν_1 satisfies (4.4) and $\nu_2(dx)=r(x)\nu_1(dx)$ with $m\leq r(x)\leq M$, m, M being positive constants. Then ν_2 satisfies also (4.4) with $c(p)$ replaced by $\frac{Mc(p)}{m}$.*

Proof. We use the same argument as in [17]. It is sufficient to consider a smooth ϕ. We rewrite (4.4) as

$$\int_H |\phi(x)|^p\log\frac{|\phi(x)|^p}{\|\phi\|^p_{p,\nu_1}}\,\nu_1(dx)\leq\frac{p-1}{2}c(p)\int_H |\phi(x)|^{p-2}|C^{1/2}D\phi(x)|^2\nu_1(dx)$$

and write

$$\int_H |\phi(x)|^p\log\frac{|\phi(x)|^p}{\|\phi\|^p_{p,\nu_2}}\,\nu_2(dx)$$

$$=\inf_{t>0}\left(\int_H(|\phi(x)|^p\log|\phi(x)|^p-|\phi(x)|^p\log t-|\phi(x)|^p+t)\nu_2(dx)\right)$$

$$\leq M\inf_{t>0}\left(\int_H(|\phi(x)|^p\log|\phi(x)|^p-|\phi(x)|^p\log t-|\phi(x)|^p+t)\nu_1(dx)\right)$$

$$= M \int_H |\phi(x)|^p \log \frac{|\phi(x)|^p}{\|\phi\|^p_{p,\nu_1}} \nu_1(dx)$$

$$\leq M \frac{p-1}{2} c(p) \int_H |\phi(x)|^{p-2} |C^{1/2} D\phi(x)|^2 \nu_1(dx)$$

$$\leq \frac{M}{m} \frac{p-1}{2} c(p) \int_H |\phi(x)|^{p-2} |C^{1/2} D\phi(x)|^2 \nu_2(dx) \qquad \square$$

Remark 4.4. As shown in [20], a Logarithmic Sobolev Inequality implies that the transition semigroup $(P_t)_{t\geq 0}$ is hypercontractive.

5. Compact embedding of Sobolev spaces based on ν

Our aim in this section is to derive a sufficient condition for the compactness of P_t in $L^p(H,\nu)$, $p > 1$, and of the embedding

$$W^{1,p}(H,\nu) \subset L^p(H,\nu), \;\; p \geq 2.$$

Our result is the following.

Theorem 5.1. *Assume that Hypotheses 2.1, 2.2 hold and*

(i) *ν satisfies a defective logarithmic Sobolev inequality:*
For any $\varphi \in W^{1,\overline{p}}(H,\nu)$

$$\int_H |\varphi(x)|^{\overline{p}} \log |\varphi(x)|^{\overline{p}} \nu(dx)$$

$$\leq c(\overline{p}) \int_H |\varphi(x)|^{\overline{p}-2} |C^{1/2} D\varphi(x)|^2 d\nu + \|\varphi\|^{\overline{p}}_{\overline{p}} \log \|\varphi\|^{\overline{p}}_{\overline{p}} + \lambda \|\varphi\|^{\overline{p}}_{\overline{p}},$$

with $\overline{p} \geq 2$, $\lambda \geq 0$ and $c(\overline{p}) > 0$.

(ii) *ν has a density with respect to μ, $\nu(dx) = \rho(x)\mu(dx)$, and $\log \rho \in W^{1,1+\delta}(H,\nu)$, with $\delta \geq 1$ and $\delta > \frac{1}{\overline{p}-1}$.*

Then the transition semigroup $(P_t)_{t\geq 0}$ is compact in $L^p(H,\nu)$ for any $p > 1$ and the embedding

$$W^{1,p}(H,\nu) \subset L^p(H,\nu),$$

is compact for any $p \geq 2$.

Proof. For $r > 0$ we define

$$B_r = \left\{ x \in H : \; \rho(x) < \frac{1}{r} \right\},$$

and we denotes by B_r^c the complementary set of B_r. Clearly $B_r \subset B_{\tilde{r}}$ if $r \geq \tilde{r}$ and

$$\nu(B_r) = \int_{B_r} \rho(x)\mu(dx) \leq \frac{1}{r},$$

so that $\nu\left(\bigcup_{r>0} B_r^c\right) = 1$.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,\overline{p}}(H,\nu)$ such that

$$|\varphi_n|_{W^{1,\overline{p}}(H,\nu)} \le 1,$$

and θ a smooth positive function with compact support such that

$$\theta(s)\begin{cases} = 1, & \text{if } s \le 1 \\ = 0, & \text{if } s \ge 2, \\ \le 1, & \text{otherwise.} \end{cases}$$

We set for $r > 1$

$$\theta_r(x) = \theta\left(-\frac{2\log\rho(x)}{\log r}\right),$$

and $\varphi_n^r(x) = \theta_r(x)\varphi_n(x)$. Then

$$\varphi_n^r(x) = 0 \text{ if } x \in B_r,$$

$$\varphi_n^r(x) = \varphi_n(x) \text{ if } x \in B^c_{\sqrt{r}}.$$

Let $\varepsilon = \frac{\delta(\overline{p}-1)-1}{1+\delta+\overline{p}}$, we are going to show that the sequence $(\varphi_n^r)_{n\in\mathbb{N}}$ is bounded in $W^{1,1+\varepsilon}(H,\mu)$. (The definition of $W^{1,p}(H,\mu)$ for any $p \ge 1$ is classical.) We have in fact

$$\begin{aligned}\int_H |\varphi_n^r(x)|^{1+\varepsilon}\mu(dx) &= \int_{B_r^c} |\varphi_n^r(x)|^{1+\varepsilon}\mu(dx) \\ &\le r\int_{B_r^c} |\varphi_n(x)|^{1+\varepsilon}\nu(dx) \\ &\le r\left(\int_H |\varphi_n(x)|^{\overline{p}}\nu(dx)\right)^{\frac{1+\varepsilon}{\overline{p}}} \le r.\end{aligned}$$

Moreover it can be easily shown that $\varphi_n^r \in W^{1,1+\varepsilon}(H,\mu)$ and that the following formula holds

$$D\varphi_n^r(x) = -\frac{2}{\log r}\,\theta'\left(\frac{-2\log\rho(x)}{\log r}\right) D\log\rho(x)\varphi_n(x) + \theta_r(x)D\varphi_n(x),$$

so that

$$\begin{aligned}&\int_H \left|D\varphi_n^r(x)\right|^{1+\varepsilon}\mu(dx) \\ &\le c_\varepsilon\Bigg[\frac{2^{1+\varepsilon}}{(\log r)^{1+\varepsilon}}\,\|\theta\|_1^{1+\varepsilon}\int_{B_r^c} |\varphi_n(x)|^{1+\varepsilon}|D\log\rho|^{1+\varepsilon}\mu(dx) \\ &\quad + \int_{B_r^c} |D\varphi_n(x)|^{1+\varepsilon}\mu(dx)\Bigg].\end{aligned}$$

We majorize the two terms in the right side as follows

$$\frac{2^{1+\varepsilon}}{(\log r)^{1+\varepsilon}}\ \|\theta\|_1^{1+\varepsilon} \int_{B_r^c} |\varphi_n(x)|^{1+\varepsilon} |D\log\rho|^{1+\varepsilon} \mu(dx)$$

$$\le \frac{2^{1+\varepsilon}}{(\log r)^{1+\varepsilon}}\ \|\theta\|_1^{1+\varepsilon} r \int_{B_r^c} |\varphi_n(x)|^{1+\varepsilon} |D\log\rho|^{1+\varepsilon} \nu(dx)$$

$$\le \frac{2^{1+\varepsilon}}{(\log r)^{1+\varepsilon}}\ \|\theta\|_1^{1+\varepsilon} r \left(\int_H |\varphi_n(x)|^{\overline{p}} \nu(dx)\right)^{\frac{1+\varepsilon}{\overline{p}}}$$

$$\times \left(\int_H |D\log\rho(x)|^{1+\delta} \nu(dx)\right)^{\frac{1+\varepsilon}{1+\delta}}$$

and

$$\int_{B_r^c} |D\varphi_n(x)|^{1+\varepsilon} \mu(dx) \le r \int_{B_r^c} |D\varphi_n(x)|^{1+\varepsilon} \nu(dx)$$

$$\le r \left(\int_H |D\varphi_n(x)|^{\overline{p}} \nu(dx)\right)^{\frac{1+\varepsilon}{\overline{p}}}.$$

This proves that for any $r > 0$, the sequence $(\varphi_n^r)_{n\in\mathbb{N}}$ is bounded in $W^{1,1+\varepsilon}(H,\mu)$. Since the embedding of $W^{1,1+\varepsilon}(H,\mu)$ in $L^{1+\varepsilon}(H,\mu)$ is compact, see [9], it has a convergent subsequence in $L^{1+\varepsilon}(H,\mu)$. Thus we can extract a subsequence which converges μ almost surely, and by (ii) ν almost surely on $B^c_{\sqrt{r}}$.

Using a diagonal extraction, we are able to construct a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ which converges ν–almost surely on any B_r^c, and thus on H.

We now use (i) to obtain that $(|\varphi_{n_k}|^{\overline{p}})_{k\in\mathbb{N}}$ is uniformly integrable and deduce that $(\varphi_{n_k})_{k\in\mathbb{N}}$ converges in $L^{\overline{p}}(H,\nu)$. The result is thus proved for $p = \overline{p}$.

Let $\varphi \in C_b^1(H)$, then we have (see [4], [19])

$$\langle DP_t\varphi(x), h\rangle = \frac{1}{t}\ \mathbb{E}\left(\varphi(X(t,x)\int_0^t \langle C^{-1/2}\eta_h(s,x), dW(s)\rangle\right),$$

and, by Hölder's inequality

$$|\langle DP_t\varphi(x), h\rangle| \le \frac{1}{t}\ \mathbb{E}\left(\varphi^{\overline{p}}(X(t,x))\right)^{\frac{1}{\overline{p}}} \mathbb{E}\left(\left(\int_0^t \langle C^{-1/2}\eta_h(s,x), dW(s)\rangle\right)^{\overline{q}}\right)^{\frac{1}{\overline{q}}},$$

where $\frac{1}{\overline{p}} + \frac{1}{\overline{q}} = 1$. We now use the Burkholder inequality to derive

$$|\langle DP_t\varphi(x), h\rangle| \le \frac{1}{t}\ \left(P_t|\varphi|^{\overline{p}}(x)\right)^{\frac{1}{\overline{p}}} \mathbb{E}\left(\left(\int_0^t |C^{-1/2}\eta_h(s,x)|^2 ds\right)^{\overline{q}/2}\right)^{\frac{1}{\overline{q}}}.$$

Finally the boundedness of $C^{-1/2}$ and (2.3) give

$$|\langle DP_t\varphi(x), h\rangle| \le \frac{1}{t}\ \left(P_t|\varphi|^{\overline{p}}(x)\right)^{\frac{1}{\overline{p}}} \|C^{-1/2}\| \left(\frac{1}{\omega-k}\ \left(1-e^{-2(\omega-k)t}\right)\right)^{1/2} |h|.$$

Therefore for $t > 0$

$$|DP_t\varphi(x)| \le g(t)\ \left(P_t|\varphi|^{\overline{p}}(x)\right)^{\frac{1}{\overline{p}}},$$

for some function $g(t)$. By integration we deduce

$$|DP_t\varphi|_{L^{\overline{p}}(H,\nu)} \le g(t)|\varphi|_{L^{\overline{p}}(H,\nu)},$$

thanks to the invariance of ν. By approximation this is true for any $\varphi \in L^{\overline{p}}(H,\nu)$. Since P_t is a contraction semigroup on $L^{\overline{p}}(H,\nu)$, we deduce

$$|P_t\varphi|_{W^{1,\overline{p}}(H,\nu)} \le (g(t)+1)|\varphi|_{L^{\overline{p}}(H,\nu)},$$

which clearly implies that P_t is compact on $L^{\overline{p}}(H,\nu)$.

Moreover P_t is bounded in $L^1(H,\nu)$ and on $L^\infty(H,\nu)$ and, by interpolation, we prove that P_t is compact on $L^p(H,\nu)$ for any $p \in (1,\infty)$.

Let us now take $p \ge 2$. In a similar way as in section 3, we can prove that for any $\varphi \in D(N_p)$

$$|P_t\varphi|^p_{L^p(H,\nu)} + \frac{p(p-1)}{2} \int_0^t \int_H |P_s\varphi|^{p-2}|C^{1/2}DP_s\varphi|^2 d\nu ds = |\varphi|^p_{L^p(H,\nu)}.$$

Let now assume only that $\varphi \in W^{1,p}(H,\nu)$. It is not difficult to see that $(P_t)_{t\ge0}$ is strongly continuous on $W^{1,p}(H,\nu)$ thus there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $D(N_p)$ which converges to φ in $W^{1,p}(H,\nu)$. Then for any $n \in \mathbb{N}$:

$$|P_t\varphi_n|^p_{L^p(H,\nu)} + \frac{p(p-1)}{2} \int_0^t \int_H |P_s\varphi_n|^{p-2}|C^{1/2}DP_s\varphi_n|^2 d\nu ds = |\varphi_n|^p_{L^p(H,\nu)}.$$

Moreover $P_t\varphi_n \to P_t\varphi$ in $L^p(H,\nu)$ so that $|P_t\varphi_n|_{L^p(H,\nu)} \to |P_t\varphi|_{L^p(H,\nu)}$.

It is easily checked that for any s

$$\int_H |P_s\varphi_n|^{p-2}|C^{1/2}DP_s\varphi_n|^2 d\nu \to \int_H |P_s\varphi|^{p-2}|C^{1/2}DP_s\varphi|^2 d\nu.$$

Since

$$\int_H |P_s\varphi_n|^{p-2}|C^{1/2}DP_s\varphi_n|^2 d\nu \le K_1|\varphi_n|^p_{W^{1,p}(H,\nu)},$$

with $K_1 \ge 0$, we deduce from the dominated convergence theorem that

$$\int_0^t \int_H |P_s\varphi_n|^{p-2}|C^{1/2}DP_s\varphi_n|^2 d\nu ds \to \int_0^t \int_H |P_s\varphi|^{p-2}|C^{1/2}DP_s\varphi|^2 d\nu ds.$$

Then the above identity holds for any $\varphi \in W^{1,p}(H,\nu)$.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(H,\nu)$. Since P_t is compact, for any t there exists a subsequence of $(P_t\varphi_n)_{n\in\mathbb{N}}$ which converges in $L^p(H,\nu)$. By diagonal extraction, we construct a subsequence $(\varphi_{n_h})_{h\in\mathbb{N}}$ such that $(P_{1/l}\varphi_{n_h})_{h\in\mathbb{N}}$ converges for any $l\in\mathbb{N}$. Let us write

$$
\begin{aligned}
&|\varphi_{n_h}-\varphi_{n_{\tilde h}}|^p_{L^p(H,\nu)}\\
&\quad =|P_{1/l}(\varphi_{n_h}-\varphi_{n_{\tilde h}})|^p_{L^p(H,\nu)}\\
&\qquad +\frac{p(p-1)}{2}\int_0^{1/l}\int_H |P_s(\varphi_{n_h}-\varphi_{n_{\tilde h}})|^{p-2}|C^{1/2}D(\varphi_{n_h}-\varphi_{n_{\tilde h}})|^2 d\nu ds\\
&\quad \le |P_{1/l}(\varphi_{n_h}-\varphi_{n_{\tilde h}})|^p_{L^p(H,\nu)}+\frac{K_1}{l}\sup_{n\in\mathbb{N}}|\varphi_n|^p_{W^{1,p}(H,\nu)}.
\end{aligned}
$$

This proves that (φ_{n_k}) is a Cauchy sequence in $L^p(H,\nu)$ and is thus convergent. □

6. Application to equations of reaction-diffusion type

6.1. Preliminaries

In this section we consider a specific example of an equation of Reaction-Diffusion type:

$$
\begin{cases} dX(t)=(AX(t)+F(X(t))dt+\sqrt{C}dW(t),\\[2mm] X(0)=x\in H, t\ge 0.\end{cases} \tag{6.1}
$$

We set $H=L^2(0,1)$ with inner product denoted by $\langle\cdot,\cdot\rangle$. Then we consider the realization A of an elliptic operator with Dirichlet boundary conditions:

$$
\begin{cases} Ax=\dfrac{\partial^2}{\partial\zeta^2}(ax)+b\dfrac{\partial}{\partial\zeta}x+cx,\ \ x\in\ D(A),\\ D(A)=H^2(0,1)\cap H^1_0(0,1)\end{cases}
$$

where $a\in C^1([0,1])$ and, b, c are given continuous function on $(0,1)$ such that there exist $\lambda_0>0$, $\lambda_1\in\mathbb{R}$, $\lambda_2\in\mathbb{R}$ satisfying

$$
\lambda_0\le a(\zeta),\quad b(\zeta)\le\lambda_1,\quad \lambda_2\le c(\zeta),\quad \zeta\in(0,1)
$$

and we assume[2]

$$
\omega=(\lambda_0\pi-\lambda_1)\pi+\lambda_2>0. \tag{6.2}
$$

[2] This assumption is not optimal. For instance if b is constant we can assume instead $\lambda_0\pi^2+\lambda_2>0$.

It is not difficult to check, see [16], that the stochastic convolution

$$Z(t) = \int_0^t e^{(t-s)A} \sqrt{C} dW(s),$$

is well defined and has continuous paths on $[0, 1] \times [0, \infty)$. Moreover, the corresponding Ornstein–Uhlenbeck transition semigroup is strong Feller. Finally for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\mathbb{E}\left(\sup_{t \geq 0} |Z(t)|^{2m}_{L^{2m}(0,1)}\right) \leq C_m. \tag{6.3}$$

We consider a nonlinear term F given by a polynomial of odd degree with strictly negative dominant coefficient :

$$F(x) = \sum_{k=0}^{2m-1} a_k x^k, \ \ x \in \mathbb{R},$$

with

$$a_{2m-1} < 0.$$

It is elementary to check that there exists $\kappa \in \mathbb{R}$ such that

$$(F(x) - F(y))(x - y) \leq \kappa |x - y|^2, \quad x, y \in \mathbb{R}.$$

Moreover, for any $r \in \mathbb{N}$ there exist positive numbers $c_{1,r}, \ c_{2,r}$ such that

$$F(x+z)x^{2r-1} \leq \frac{a_{2m-1}}{2}\, x^{2m+2r-2} + c_{1,r}|z|^{2m-2} + c_{2,r}, \quad x, z \in \mathbb{R}. \tag{6.4}$$

Finally W is a cylindrical Wiener process on $H = L^2(0, 1)$ and C is a bounded linear operator with bounded inverse on H.

Under these assumptions, (6.1) has a unique solution and Hypothesis 2.1, 2.2 are satisfied. Moreover, the invariant measure ν is unique. This last statement is proved by the strong Feller property and the irreducibilty of the transition semigroup.

We give however the proof of the existence of an invariant measure ν since it also gives as a byproduct, an estimate for $\int_H |F(x)|^2 \nu(dx)$ needed later. For this we need a lemma.

Lemma 6.1. *Let $X(\cdot, x)$ be the solution to equation (6.1). Then for any $r \in \mathbb{N}$ and for any $x \in L^{2r}(0, 1)$, there exists an increasing positive function g_x such that*

$$|X(t, x)|^{2r}_{L^{2r}(0,1)} \leq g_x(|Z(t)|_{L^{2r}(0,1)}), \ \ t \geq 0. \tag{6.5}$$

Proof. Setting $Y(t) = X(t,x) - Z(t)$ problem (6.1) reduces to

$$\begin{cases} Y' = AY + F(Y+Z), \\ \\ Y(0) = x. \end{cases}$$

Multiplying both sides by Y^{2r-1} and integrating on (0, 1) we obtain

$$\frac{1}{2r}\,\frac{d}{dt}\,|Y(t)|^{2r}_{L^{2r}(0,1)} + (2r-1)\int_0^1 a\left(\frac{\partial}{\partial\zeta}Y\right)\left(\frac{\partial}{\partial\zeta}Y\right)Y^{2r-2}d\zeta$$
$$+\int_0^1 b\left(\frac{\partial}{\partial\zeta}Y\right)\,Y^{2r-1}d\zeta + \int_0^1 cY^{2r}d\zeta + \int_0^1 F(Y+Z)Y^{2r-1}d\zeta = 0.$$

Now the conclusion follows taking into account (6.4), (6.2) and using standard arguments. □

Proposition 6.2. *There exists an invariant measure* ν *for (6.1) and we have*

$$\int_H |F(x)|^2\nu(dx) < \infty. \tag{6.6}$$

Proof. Let $x \in L^{4m}(0,1)$, and let $\epsilon < \frac{1}{4}$. Then we have

$$(-A)^{\varepsilon}X(t,x) = (-A)^{\varepsilon}e^{tA}x + \int_0^t (-A)^{\varepsilon}e^{(t-s)A}F(X(s,x))ds + (-A)^{\varepsilon}Z(t).$$

By Lemma 6.1 it follows that there exists a positive constant C such that

$$\mathbb{E}\left[\left|(-A)^{\varepsilon}X(t,x)\right|^2\right] \le C, \quad t \ge 0.$$

This implies tightness of the sequence $(\mathcal{L}(X(t,x)))$ of the laws of $X(t,x)$, and by the Krylov–Bogoliubov theorem the existence of an invariant measure ν.

It remains to prove (6.6). Note first that by (6.5) there exists $C_1(x) > 0$ such that

$$\mathbb{E}[|F(X(t,x)|^2] \le C_1(x),\ t \ge 0.$$

Let now x be fixed and let $t_n \to \infty$ be such that $(\mathcal{L}(X(t_n,x)))_{n\in\mathbb{N}}$ is weakly convergent to ν.

Then we have

$$\mathbb{E}|F(X(t_n,x)|^2] = \int_H |F(y)|^2\mathcal{L}(X(t_n,x))(dy) \to \int_H |F(y)|^2\nu(dy),$$

and the conclusion follows. □

6.2. The strictly dissipative case

All the results of section 3 hold without any further assumptions. To apply the other results, we need stonger assumptions. In this section, we assume

$$\omega - \kappa > 0 \tag{6.7}$$

then Theorem 4.2 applies and ν satisfies a Logarithmic Sobolev inequality.

We also assume that the Ornstein-Uhlenbeck semigroup $(R_t)_{t\geq 0}$ is symmetric. Since $W^{1,2}(H,\mu)$ is compactly embedded in $L^2(H,\mu)$, then Proposition 6.2 and the result in [5] imply that the invariant measure ν has a density ρ with respect to μ such that $\log\rho \in W^{1,2}(H,\nu)$.

We can therefore deduce from Theorem 5.1 that $(P_t)_{t\geq 0}$ is compact in $L^p(H,\nu)$ for any $p > 1$ and $W^{1,p}(H,\nu) \subset L^p(H,\nu)$ for $p \geq 2$ with compact embedding.

6.3. The general case

We now treat the general case - i.e. without assuming (6.7) - by pertubation. We however still assume that the Ornstein-Uhlenbeck semigroup is symmetric.

It is not difficult to see that there exists $r_1 \geq 0$ such that

$$F'(x) \leq 0, \text{ if } |x| \geq r_1.$$

Using elementary arguments, we then construct two C^1 functions F_1 and F_2 such that

$$F_1(x) = F(x), \ |x| \geq r_2,$$

$$F_1'(x) \leq 0, \ x \in \mathbb{R},$$

$$F = F_1 + F_2$$

where $r_2 \geq 0$. The function F_2 is compactly supported, it is also possible to construct it in such a way that

$$\int_{-r_2}^{r_2} F_2(x)dx = 0$$

so that it has a globally bounded antiderivative U_2.

The stochastic equation where F is replaced by F_1 satisfies all the assumptions of section 6.2. Let us denote by $(P_t^1)_{t\geq 0}$ the corresponding transition semigroup with generator N_1 and invariant measure ν_1.

Then, we can define

$$N\phi = N_1\phi + \langle F_2, D\phi\rangle$$

for $\phi \in D(N_1)$. Since F_2 is bounded and $(P_t^1)_{t\geq 0}$ is compact, we can apply the same arguments as in [9], [15] and prove that N generates a C^0 semigroup $(P_t)_{t\geq 0}$ in $L^2(H,\nu_1)$. This semigroup is the transition semigroup associated to (6.1). It has a unique invariant measure ν with a density $r \in L^p(H, \nu_1)$, for any $p > 1$, with respect to ν_1. Furthermore $(P_t)_{t\geq 0}$ is also compact in $L^p(H, \nu_1)$, for any $p > 1$.

Finally, applying the result in [11], we prove that there exist $\alpha > 0$ and $M \geq 1$ such that

$$\left\| P_t\phi - \int_H \phi(x)\nu(dx) \right\|_{p,\nu_1} \leq Me^{-\alpha t} \|\phi\|_{p,\nu_1}.$$

Remark 6.3. Note that we could generalize our results in many directions. Indeed, the compactness of $(P_t^1)_{t\geq 0}$ is a very powerful tool. For instance, as in [9], we could consider a perturbation which is only Borel and bounded.

6.4. The gradient case

In this section, we show that in the case of a system of gradient type we can considerably strengthen the above result.

We assume that $C = Id$ and that A is self-adjoint. Then (6.1) is a of gradient type and it is well known that in this case the invariant measure ν has a density ρ with respect to the gaussian measure μ which is given by

$$\rho(x) = Ke^{-\frac{1}{2}U(x)}$$

where U is an antiderivative of F and K is a constant.

We use again the decomposition introduced in section 6.3 and set $U_1 = U - U_2$ so that

$$\rho(x) = r(x)\rho_1(x)$$

where ρ_1 is the density of ν_1 with respect to μ and $r(x) = \widetilde{K}e^{-\frac{1}{2}U_2(x)}$. We clearly have

$$\widetilde{K}e^{-\frac{1}{2}\|U_2\|_\infty} \leq r(x) \leq \widetilde{K}e^{\frac{1}{2}\|U_2\|_\infty}$$

where $\|U_2\|_\infty = \sup_{x\in H} |U_2(x)|$.

We are in position to apply Theorem 4.3 and we obtain that ν satisfies the Logarithmic Sobolev Inequality (4.4) for any $p > 1$.

We argue as in section 6.2 and obtain exactly the same results as in the strictly dissipative case.

Remark 6.4. Again, as in section 6.3, we could consider perturbations by Borel and bounded functions.

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