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The genealogy of continuous-state branching processes with immigration

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Abstract. Recent works by J.F. Le Gall and Y. Le Jan [15] have extended the genealogical structure of Galton-Watson processes to continuous-state branching processes (CB). We are here interested in processes with immigration (CBI).

The height process H which contains all the information about this genealogical structure is defined as a simple local time functional of a strong Markov process $X^\star$, called the genealogy-coding process (GCP). We first show its existence using Itô's synthesis theorem. We then give a pathwise construction of $X^\star$ based on a Lévy process X with no negative jumps that does not drift to $+\infty$ and whose Laplace exponent coincides with the branching mechanism, and an independent subordinator Y whose Laplace exponent coincides with the immigration mechanism. We conclude the construction with proving that the local time process of H is a CBI-process.

As an application, we derive the analogue of the classical Ray–Knight–Williams theorem for a general Lévy process with no negative jumps.

1. Introduction

A continuous-state branching process (CB) is a strong Markov process Z with values in $[0, \infty]$, 0 and ∞ being absorbing states. It is characterized by its branching mechanism function ψ and enjoys the following branching property. The sum of two independent CB(ψ) starting respectively from x and y, is a CB(ψ) starting from $x + y$. CB-processes are the analogue of (Galton-Watson) discrete-branching processes (DB) in continuous time and continuous state-space. The very difference between DB and CB-processes is that the definition of a DB-process is based on a random tree (Z_n is the number of particles at the n-th generation), whereas that of a CB-process is intrinsic. In this direction, J.F. Le Gall and Y. Le Jan [15] have defined a continuous genealogical structure via a non-Markovian process called the height process. It is the continuous analogue of the process of successive heights in the finite discrete tree explored in the lexicographical order. The motivation for the study of the genealogical structure of CB-processes is to extend the construction of superprocesses with quadratic branching to more general branching mechanisms.

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In the special case of quadratic branching mechanism, a natural construction of the superprocess involves the path-valued process known as the Brownian snake, which loosely speaking combines quadratic branching and Brownian spatial motion. Extending these results to other branching mechanisms requires detailed information about the genealogical structure of the associated CB-process. For a deep understanding of this topic, see [16].

As in the discrete setting, the CB-process does not contain the information on the genealogy. The height process is therefore only defined in law. Nevertheless, a pathwise construction can be given from the paths of a spectrally positive (i.e. with no negative jumps) Lévy process X whose Laplace exponent coincides with the branching mechanism function ψ. Namely, for every $t \geq 0$, H_t is defined by

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s - \inf_{s \leq r \leq t} X_r < \varepsilon\}} ds. \tag{1}$$

Roughly speaking, as H_t is the height in the tree of particle t, the total 'time spent' by H at level $x \geq 0$ is the amount of population belonging to generation x. Indeed the main theorem of [15] states that the local time process of H as a function of the space variable $(Z_x, x \geq 0)$ is a CB(ψ).

Next consider a Galton-Watson tree and add independently from the tree at each generation n a random number Y_n of particles, where the Y_i's are i.i.d. Then the process that associates to every integer n the number of particles of the n-th generation of the modified tree is called a discrete-branching process with immigration (DBI). Adding at each generation $n-1$ a virtual father to the immigrating particles allows us to keep up with the tree structure. The aim of the present paper is to find out the continuous analogue of such a genealogy.

Indeed DBI-processes have a continuous analogue known as CBI-processes. These are strong Markov processes valued in $[0, \infty]$, where 0 is no longer absorbing. They are characterized by their branching mechanism function ψ and their immigration mechanism function ϕ. The sum of a CBI(ψ , ϕ) started at x and an independent CB(ψ) started at y is a CBI(ψ , ϕ) started at $x + y$. To give a pathwise construction of the height process, we now need more than the information contained in the paths of the spectrally positive Lévy process X. We thus have to show the existence of a strong Markov process $X^\star$ called the genealogy-coding process (GCP) satisfying the next assertion. Applying an analogue of the local time functional (1) to $X^\star$ gives rise to a newly distributed height process $H^\star$, whose local time process is a CBI(ψ, ϕ).

The GCP is defined by its excursion measure $N^\star$ away from 0. Let Y denote a subordinator with Laplace exponent ϕ, and X a Lévy process with Laplace exponent ψ independent from Y. The measure $N^\star$ is then defined in terms of the law of X killed upon reaching 0 and the Lévy measure of jumps of Y. The existence of a measure of probability $\mathbb{P}^\star$ with excursion measure $N^\star$ follows from Itô's synthesis theorem. A pathwise construction of $X^\star$ is also given by

$$X_t^\star = X_t + Y(Y^{-1}(-\inf_{s \leq t} X_s)).$$

This is completed by proving that the local time process of $H^\star$ is as expected a CBI(ψ, ϕ).

The last section is devoted to the extension of the Ray–Knight–Williams theorem to general spectrally positive Lévy processes. For simplicity, assume that X is a recurrent Lévy process with no negative jumps and Laplace exponent ψ. The existence of a law $\mathbb{P}^\uparrow$ of X conditioned to stay positive is well-known. Our theorem identifies the Lévy process conditioned to stay positive and the genealogy-coding process associated to a branching mechanism ψ, and an immigration mechanism ϕ, where

$$\phi(\lambda) = \frac{\psi(\lambda)}{\lambda}, \qquad \lambda \geq 0.$$

In the Brownian case, the height process under $\mathbb{P}^\uparrow$ is a Bessel process of dimension 3 (BES(3)). Our theorem thus reduces in this case to the Ray–Knight–Williams theorem which ensures that the local time process of a BES(3) is a CBI(ψ, ϕ), where $\psi(\lambda) = \lambda^2/2$, $\phi(\lambda) = \lambda/2$, that is a squared Bessel process of dimension 2 (see [25]).

The paper is organized as follows. In the next section, we set the main notations and recall some known facts about CBI-processes. We also give details in the discrete setting concerning the height process and the genealogy-coding walk (GCW). In section 3, we show the existence of the law of the GCP $X^\star$ and give a pathwise construction of $X^\star$. In section 4, we check that the height process derived from the GCP has the requested law, that is its local time process as a function of the space variable is a CBI-process. The last section deals with the extension of the Ray–Knight–Williams theorem.

2. Preliminaries

Consider a finite rooted tree, using the coding of Neveu. A vertex u of the tree which belongs to generation $n \in \mathbb{N}$ is denoted by a finite sequence of positive integers $u = (u_0, \dots, u_n)$ defined recursively as follows. For any $k = 0, \dots, n$, the unique ancestor $(u_0, \dots, u_{k-1})$ of u at generation $k-1$ (i.e. the root if $k = 0$) has a distinguishable offspring ordered from left to right. Then the ancestor of u at generation k belongs to this offspring, and u_k denotes its rank in this offspring. Explore this tree according to the lexicographical order associated to this coding (for example $1 < 11 < 12 < 121$). To the n-th visited particle, associate W_n the sum of the numbers of younger brothers of all its ancestors, including itself. Define the height process H_n as the number of generation of the n-th particle. It can be recovered from W by

$$H_n = \mathrm{card}\{j : 0 \leq j < n, W_j = \inf_{j \leq l \leq n} W_l\}.$$

We call H the exploration process, or height process. It is clear that this process contains the whole information about the genealogy of the tree.

Let us introduce probability measures on trees. First consider $f(s) = \sum_{k \geq 0} \nu(k) s^k$ a probability generating function, and the probability measure associated to Galton-Watson trees with offspring distribution ν. The key idea of [15] is that under this probability, W is a random walk on the integers with jump distribution $\tilde{\nu}(k) = \nu(k+1), k = -1, 0, 1, \dots$ killed at its hitting time of -1. The associated Galton-Watson process Z, or DB-process (discrete branching), is then equal to

$$Z_p = \sum_{n\geq 0} \mathbf{1}_{H_n = p}, \qquad p \geq 0.$$

It is clear that conversely one can start with a random walk with jump distribution $\tilde{\nu}$. The same method then applies to construct a Galton-Watson tree (thanks to the knowledge of H) and the associated DB-process.

As a second step, add some immigration. Let $g(s) = \sum_{k\geq 0} \mu(k)s^k$, be a probability generating function. We can still achieve the immigration procedure sticking to a tree-like structure. We define this tree by giving a virtual father to the immigrating particles. Start with $N + 1$ particles, and mark the rightmost one.

1. each generation contains one and only one marked particle. Give it k children with probability $\mu(k - 1)$, $k = 1, 2, \dots$ Give independently to the other particles an offspring with distribution ν.
2. at each generation, mark the rightmost particle.

The discrete-time branching process with branching mechanism ν and immigration mechanism μ, denoted by DBI(f, g), is the process that associates to each integer $n \geq 0$ the number Z_n of unmarked particles of the n-th generation. It is a Markov chain on the nonnegative integers with transition matrix (P_{ij}) given by

$$\mathbf{E}_i(s^{Z_1}) = \sum_{j\geq 0} P_{ij}s^j = (f(s))^i g(s), \qquad i \in \mathbb{N}.$$

In particular, a DBI(f, 1) is a DB(f) (a time-discrete branching process with branching mechanism f), and if $u_k(s)$ denotes the quantity $\mathbf{E}_1(s^{Z_k})$ when $g \equiv 1$ (u_k is known to be the k-th iterate of f), then for every DBI(f, g) starting from $i \in \mathbb{N}$,

$$\mathbf{E}_i(s^{Z_n}) = (u_n(s))^i \prod_{k=0}^{n-1} g(u_k(s)), \qquad s \in [0, 1]. \tag{2}$$

With this new probability measure on (marked) trees, we are able to state a more general result. First set some notations. Let ϵ denote a generic path with finite duration $V(\epsilon) \geq 1$, defined on $\{1, \dots, V(\epsilon)\}$. Now for a sequence $(\epsilon_i)_{i\geq 1}$ of finite paths, define recursively their concatenation $[\epsilon] = [\epsilon_i]_{i\geq 1}$ as follows. $[\epsilon]_0 = 0$, and

$$[\epsilon]_{V(\epsilon_1)+\cdots+V(\epsilon_n)+k} = \epsilon_{n+1}(k), \qquad n \geq 1, 1 \leq k \leq V(\epsilon_{n+1}).$$

In the following statement, we are still interested in the total number of younger unmarked brothers W_n of the ancestors of the n-th particle, and in its height in the tree, but this tree now is distributed according to the branching mechanism ν and immigration μ. We skip the proof for conciseness.

Proposition 1. *Recall that* $\tilde{\nu}(k) = \nu(k + 1)$, $k \geq -1$.

Set $W^\star \doteq [W^{(i)}]_{i\geq 1}$, *where the* $W^{(i)}$*'s are i.i.d. finite random paths such that each* $W^{(i)}$ *is a random walk with initial distribution* μ, *with step distribution* $\tilde{\nu}$, *and killed at* $T_0 + 1$, *where* $T_0 = \inf\{n \geq 1 : W_n = 0\}$. *The process* $W^\star$ *is called* *the GCW(f,g), or genealogy-coding walk associated to the probability generating functions f and g. Define*

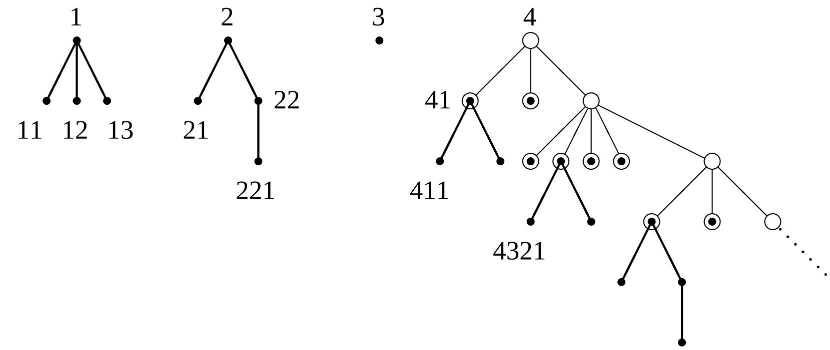
$$H_n^\star = card\{j : 0 \le j < n,\, W_j^\star = \inf_{j \le l \le n} W_l^\star\}, \qquad n \ge 0,$$

and

$$Z_p^\star = \sum_{n \ge 0} \mathbf{1}_{H_n^\star = p}, \qquad p \ge 0.$$

Then $(Z_p^\star - 1,\, p \ge 0)$ *is a DBI(f, g).*

We stress that by construction, $Z_p^\star$ is the total number of particles belonging to generation p. We thus have to remove the marked particle at each generation to recover the DBI-process.



Immigrating particles (⦿) are given a virtual father (○) and have independent descendant trees with the same branching mechanism as ordinary branching particles (•).

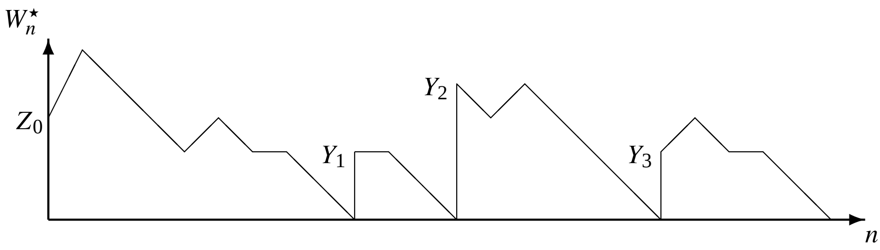


Fig. 1. Galton-Watson tree with immigration and associated GCP $W^\star$ ($W_n^\star$ is the total number of younger unmarked brothers of the ancestors of the n-th particle)

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In the tree semantics, $H_n^\star$ is the height in the tree of the n-th visited particle. If $\gamma_n = \sup\{j \le n : W_j^\star = 0\}$, then

$$H_n^\star = \mathrm{card}\{0 \le j < \gamma_n : W_j^\star = 0\} + \mathrm{card}\{\gamma_n \le j \le n : W_j^\star = \inf_{j \le l \le n} W_l^\star\}.$$

The first term is the number of times when the GCW visits the rightmost branch of the tree, that is the height of the last visited marked particle. Then the n-th particle is a descendant of this marked particle, and the second term is its height in the corresponding subtree.

Notice that two random walks (r.w.) underlie the discrete setting. The first is a so-called left-continuous r.w. W with step distribution $\tilde{\nu}$. The second is the renewal process Y with jump distribution μ that gives at time n the total number of immigrants up until the n-th generation.

In [15], the tree is linked to a LIFO queue. Every (positive) increment $W_n - W_{n-1}$ is viewed as some service required by a customer arrived at time n. The task is over as soon as W reaches W_{n-1}, and in the meanwhile, each new service is prioritary (Last In First Out). The genealogy is built up by saying that each customer is the son of who he interrupted. In the case with immigration, we suppose that extra service is required every time the queue is empty. These services are i.i.d. and independent of the rest of the queue (i.e. the increments of the renewal process Y).

We now deal with the continuous setting. It is well-known that DBI-processes have a continuous state-space time-continuous analogue, called CBI-processes. Let ψ be the Laplace exponent of a spectrally positive Lévy process, and ϕ that of a subordinator (ψ is convex and ϕ is concave). They are specified by the Lévy-Khinchin formula

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{r<1})\Lambda(dr), \qquad \lambda \ge 0, \tag{3}$$

where $\alpha \in \mathbb{R}$, $\beta \ge 0$ denotes the Gaussian coefficient, and the Lévy measure Λ is a measure on $(0, \infty)$ such that $\int_0^\infty (r^2 \wedge 1)\Lambda(dr) < \infty$. Similarly

$$\phi(\lambda) = \delta\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda r})\Gamma(dr), \qquad \lambda \ge 0, \tag{4}$$

where $\delta \ge 0$ is the drift coefficient and the Lévy measure Γ is a measure such that $\int_0^\infty (r \wedge 1)\Gamma(dr) < \infty$.

A CBI-process with branching mechanism ψ and immigration mechanism ϕ is denoted by CBI(ψ, ϕ). It is a Markov process Z taking values in $[0, \infty]$, whose transition kernels are characterized by their Laplace transform

$$\begin{aligned}\mathbf{E}_x(\mathrm{e}^{-\lambda Z_t}) &= \mathbf{E}(\mathrm{e}^{-\lambda Z_t} \mid Z_0 = x)\\ &= \exp[-x u_t(\lambda) - \int_0^t \phi(u_s(\lambda))ds], \qquad x \ge 0, t \ge 0,\end{aligned}$$

where $u_t(\lambda)$ is the unique nonnegative solution of the integral equation

$$v(t) + \int_0^t \psi(v(s))ds = \lambda, \qquad \lambda \ge 0, t \ge 0. \tag{5}$$

For existence and unicity of such a process Z, see [13, Theorem 1.1].

Notice that a spectrally positive Lévy process is the continuous analogue of a left-continuous r.w., and a subordinator that of a renewal process (the jumps of the subordinator embody the arrival of immigrants). Compare the preceding equations with (2), and see [19] for more details, and the references therein.

In particular, a CBI(ψ, 0) is a CB(ψ) that satisfies the following branching property. The sum of two independent CB(ψ) starting respectively from x and y, is a CB(ψ) starting from $x + y$. In that case ($\phi \equiv 0$), we give a brief account of the continuous analogue results given in [14] and [15].

Consider a spectrally positive Lévy process X with Laplace exponent ψ, and assume that $\psi'(0+) \geq 0$ (the process X does not drift to $+\infty$). Define the height process H_t as the local time at 0 at time t of $S^{(t)} - X^{(t)}$, where

$$X_s^{(t)} = X_t - X_{(t-s)-}, \qquad 0 \leq s \leq t,$$

and $S^{(t)}$ its associated supremum process $S_s^{(t)} = \sup_{0\leq r\leq s} X_r^{(t)}$. The normalization for this local time is such that

$$H_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{S_s^{(t)} - X_s^{(t)} < \varepsilon\}} ds = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s - \inf_{s\leq r\leq t} X_r < \varepsilon\}} ds. \tag{6}$$

Moreover, there is a lower semicontinuous version of the process $(H_t, t \geq 0)$ with values in $[0, \infty]$. Next let T_y denote the first hitting time of $(-\infty, y)$ by X $(y \leq 0)$. The main theorem of [15] states that the random measure $\mathcal{Z}_x$ on $\mathbb{R}_+$ defined by

$$< \mathcal{Z}_x, h >= \int_0^{T_{-x}} h(H_s) ds,$$

has a density $(Z_a, a \geq 0)$ w.r.t. Lebesgue measure, that is a CB(ψ) started at x (Z can be viewed as the local time process of H).

When some immigration is added, we should like to find out an analogue of the genealogy-coding process (W in the discrete setting, X in the continuous setting) and then check that the same kind of construction gives rise to a branching process with immigration.

We point out that there is an alternative way of defining the genealogy of a continuous tree. Though, this definition does not seem appropriate for the study of snakes and superprocesses. We could recall these results (see [6]), but it is as easy to give straight away the (more general) analogue in the case with immigration.

We first emphasize the role of the initial value of the CBI-process and write $Z_t = Z(t, a)$ for the value at time t of a CBI(ψ, ϕ) starting from $Z_0 = a \in [0, \infty)$. The additive property for branching processes implies that if $Z'(\cdot, b)$ is an independent CBI(ψ, 0) (that is a CB(ψ)) starting from b, then $Z(\cdot, a) + Z'(\cdot, b)$ has the same law as $Z(\cdot, a + b)$. Invoking Kolmogorov's theorem, we can thus construct a process $(Z(t, a), t \geq 0$ and $a \geq 0)$ such that $Z(\cdot, 0)$ is a CBI(ψ, ϕ) starting from 0, and for every $a, b \geq 0$, $Z(\cdot, a + b) - Z(\cdot, a)$ is independent from the family of processes $(Z(\cdot, c), 0 \leq c \leq a)$ and has the law of a CB(ψ) starting from b.

In particular, for each fixed $t \geq 0$, we will choose the right-continuous modification of the process $Z(t, \cdot)$, which is then a subordinator with Laplace exponent $u_t(\cdot)$ and initial value α_t a positive r.v. with Laplace exponent

$$\mathbf{E}(e^{-\lambda \alpha_t}) = \exp(-\int_0^t \phi(u_s(\lambda))\, ds), \qquad \lambda \geq 0, t \geq 0.$$

In the following statement, the positive real number $S^{(s,t)}(a)$ is to be interpreted as the total progeny at time t of the amount of population $[0, a]$ present at time s, with branching-immigrating mechanism (ψ, ϕ). The proof of this statement is easily adapted from [6, Proposition 1].

Proposition 2. *On some probability space, there exists a process $(S^{(s,t)}(a), 0 \leq s \leq t$ and $a \geq 0)$ such that*

(i) For every $0 \leq s \leq t$, $S^{(s,t)} = (S^{(s,t)}(a), a \geq 0)$ is a subordinator with Laplace exponent $u_{t-s}(\cdot)$ starting from a r.v. $S^{(s,t)}(0)$ distributed as α_{t-s}.

(ii) For every integer $p \geq 2$, and $0 \leq t_1 \leq \cdots \leq t_p$, the subordinators $S^{(t_1,t_2)}, \ldots, S^{(t_{p-1},t_p)}$ are independent and

$$S^{(t_1,t_p)}(a) = S^{(t_{p-1},t_p)} \circ \cdots \circ S^{(t_1,t_2)}(a), \qquad \forall a \geq 0 \qquad a.s.$$

Finally, the processes $(S^{(0,t)}(a), t \geq 0$ and $a \geq 0)$ and $(Z(t, a), t \geq 0$ and $a \geq 0)$ have the same finite-dimensional marginals.

As in [6], this proposition enables us to make the following consistent definition of genealogy. For every $a, b \geq 0$, and $0 \leq s \leq t$, we say that the individual a in population at time t has ancestor at time s the individual b if b is a jump time of $S^{(s,t)}$ and

$$S^{(s,t)}(b-) < a < S^{(s,t)}(b).$$

3. The genealogy-coding process $X^\star$

In this section, we introduce the continuous analogue of the genealogy-coding walk defined in the last section. We will call it the genealogy-coding process (GCP) $X^\star$. Roughly speaking, the GCP is a spectrally positive Lévy process reflected on the range of an independent subordinator Y. In particular when Y is deterministic, the GCP is the Lévy process X reflected at 0, that is, $X^\star = X - I$, where $I_t = \inf_{s \leq t} X_s$. In the next section we will see the link between its paths and CBI-processes, in a way similar to that of Proposition 1.

3.1. Itô's synthesis

In the last section, the sample-paths of the GCW(f, g) could be viewed as the concatenation of a sequence of independent excursions away from 0, which started with a jump of law μ (immigration mechanism), and then proceeded as the random walk with step distribution $\tilde{\nu}$ (branching mechanism) killed upon hitting 0. Informally, this suggests that we should define the GCP as the concatenation of a sequence of independent excursions which start with the jump of a certain subordinator Y (with

Laplace exponent ϕ for the immigration), and then evolve like a certain spectrally positive Lévy process X (with Laplace exponent ψ for the branching mechanism) killed upon reaching 0.

This fits the classical problem of recurrent extensions of Markov processes, which goes back to Feller and Itô [11]. More precisely, given a Markov process X killed upon hitting some given point x_0 for the first time, the program is to characterize all the recurrent Markov processes X' having the same law as X when killed at the first hitting time of x_0. The killed Markov process X is called the minimal process, and the possible X' are the extensions of the minimal process. In the Brownian case, the extensions are known as Feller Brownian motions (see [12, p.186]).

The definitive treatment of this question was done in [23],[24], and a survey can be found in [8]. See also [21] for an analytic counterpart.

The usual way of tackling the extension problem relies on excursion theory. Roughly speaking, it uses Itô's synthesis theorem which links i.i.d. excursions together in order to produce a Markov process. To apply this tool, we need to set some notations and recall some facts about excursion theory.

We use the canonical representation. Let $\mathcal{D} = \mathcal{D}([0,\infty), \mathbb{R})$ be the space of càdlàg functions, endowed with Skorohod's topology and the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $\mathcal{E}$ be the space of excursions ϵ in $\mathcal{D}$ (paths with finite lifetime $V(\epsilon)$). Now let M be a recurrent real-valued strong Markov process with càdlàg paths, such that 0 is regular, that is

$$\mathbb{P}_0(\inf\{s > 0 : M_s = 0\} = 0) = 1.$$

Then it is known that there exists a unique (up to a multiplicative constant) continuous increasing process L adapted to the natural filtration of M, called local time at level 0 of M, such that

1. The support of dL coincides with the closure of $\{t : M_t = 0\}$ a.s.
2. L is an additive functional of M.

The inverse local time τ

$$\tau_s = \inf\{t \geq 0 : L_t > s\}$$

is a subordinator whose jumps coincide exactly with the excursion intervals of M. Specifically, we can index the excursions of M away from 0 by the jump times of τ. For $t > 0$, set

$$e_t = \begin{cases} (M_{\tau_{t-}+s}, 0 \leq s < \tau_t - \tau_{t-}) & \text{if } \tau_{t-} < \tau_t \\ \Upsilon & \text{otherwise,} \end{cases}$$

where Υ is an additional isolated point. A fundamental theorem by Itô asserts that $(e_t, t \geq 0)$ is a Poisson point process in $\mathcal{E} \cup \{\Upsilon\}$. Its characteristic measure is called the excursion measure and is a (σ-finite) infinite measure on $\mathcal{E}$. This measure has the strong Markov property and its semigroup is that of M killed upon reaching 0.

Conversely, the program of Salisbury and Itô is to produce a Markov process with given excursion measure. In our particular case, we work by analogy with the discrete setting. Recall that the excursion of the GCW(f, g) starts with some immigration with law μ and then proceeds as the random walk of jump distribution $\tilde{\nu}$ until it hits 0. Now in the continuous setting, the immigration is driven by some subordinator Y and thus has two components, a linear immigration (due to the drift part of the subordinator), and an immigration by jumps (the jump part of the subordinator). We thus have to cope with two kinds of excursions, those starting with a jump of Y that behave as the Lévy process X until it reaches 0, and those starting at 0 (infinitesimal immigration). Intuitively, the latter should have in some sense the limiting distribution as $x \to 0+$ of X started at x and killed upon reaching 0. We will see that a good choice for this measure is the excursion measure of $X - I$ away from 0, where I stands for the infimum process of X. Indeed, since 0 is regular for $(-\infty, 0)$ w.r.t. X, it is regular for itself w.r.t. $X - I$, and it is known that $X - I$ is a strong Markov process (see Proposition VI.1 in [4]). We are therefore able to define this measure up to a multiplicative constant. We shall take the associated local time equal to $-I$, which forces the choice of the constant factor. We stress that the semigroup of this measure is that of X killed upon reaching 0.

Consider the family of measures $(N_r, r \geq 0)$ defined on $\mathcal{E}$ as follows

- N_0 stands for the excursion measure of $X - I$ away from 0 under $\mathbb{P}_0$.
- For $r > 0$, N_r denotes the law of X started at r and killed upon reaching 0.

Now we can lay out the problem in proper terms. Let $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$ be the first and second coordinate processes on $\Omega = \mathcal{D} \times \mathcal{D}$. Let $(\mathbb{P}_x)_{x>0}$ be a family of probability measures on Ω for which X is a spectrally positive (i.e. with no negative jumps) Lévy process starting from x, and Y an independent subordinator starting from 0. The Laplace exponents of X and Y are those specified in the Preliminaries by (3) and (4). The paths of X are assumed not to drift to $+\infty$ and to have infinite variation, that is $\psi'(0+) \geq 0$ and either $\int_0 r\Lambda(dr) = \infty$ or $\beta > 0$. Analogously to the discrete setting, we have to find out a Markov process $X^\star$ (the genealogy-coding process) whose excursion measure $N^\star$ away from 0 can be described as follows

$$N^\star = \int_0^\infty \Gamma(dr) N_r + \delta N_0,$$

where we remind that Γ and δ are defined in (4).

More precisely $N^\star$ is the sum of two disjoint σ-finite measures on $\mathcal{E}$. The first has support $\{\epsilon \in \mathcal{E} : \epsilon(0) > 0\}$ and satisfies

(i) For $x > 0$, conditional on $\{\epsilon(0) = x\}$, ϵ behaves as the process X started at x and killed upon reaching 0,

(ii) The σ-finite distribution of $\epsilon(0)$ is

$$N^\star(\epsilon(0) \in dr) = \Gamma(dr) \qquad r > 0.$$

The second has support $\{\epsilon \in \mathcal{E} : \epsilon(0) = 0\}$ and is equal to δN_0, where we recall that N_0 is the excursion measure of $X - I$ away from 0.

We prove thanks to Itô's synthesis theorem that there exists a unique Markovian family of probability measures $(\mathbb{P}_x^\star, x \geq 0)$ on $\mathcal{D}$ such that 0 is instantaneous ($\mathbb{E}_0^\star(\int_0^\infty \mathbf{1}_{\{X_t=0\}} dt) = 0$) and

(i) The excursion measure away from 0 under $\mathbb{P}_0^\star$ is $N^\star$.
(ii) For any nonnegative measurable F and G,

$$\mathbb{E}_x^\star(F(X_s, s \le T_0)G(X_{s+T_0}, s \ge 0)) = \mathbb{E}_x(F(X_s, s \le T_0))\mathbb{E}_0^\star(G(X_s, s \ge 0)),$$

where T_0 stands for the first hitting time of 0.

We call the canonical process under $\mathbb{P}^\star$ the genealogy-coding process associated to the Laplace exponents ψ and ϕ, abbreviated GCP(ψ, ϕ). We shall always assume in the sequel that Y is not a compound Poisson process, as otherwise our results would merely reduce to those of [15].

Let us start with the uniqueness result for the family $\mathbb{P}^\star$. Since 0 is instantaneous, it is known that

$$\mathbb{E}_0^\star(\int_0^\infty \mathrm{e}^{-t} g(X_t)dt) = N^\star(\int_0^V \mathrm{e}^{-t} g(\epsilon_t)dt),$$

where g is any bounded measurable function. The definition of $\mathbb{P}_x^\star$ then implies that

$$\mathbb{E}_x^\star(\int_0^\infty \mathrm{e}^{-t} g(X_t)dt) = \mathbb{E}_x(\int_0^{T_0} \mathrm{e}^{-t} g(X_t)dt) + \mathbb{E}_x(\mathrm{e}^{-T_0})N^\star(\int_0^V \mathrm{e}^{-t} g(\epsilon_t)dt),$$

hence the knowledge of $N^\star$ and that of $(\mathbb{P}_x, x \ge 0)$ determine that of the last quantities for every $x \ge 0$ and bounded measurable g. It follows from standard results that the semigroup of $(\mathbb{P}_x^\star, x \ge 0)$ is then uniquely determined.

As for the existence of the family $\mathbb{P}^\star$, we apply Itô's synthesis theorem to the excursion measure $N^\star$. We check that the hypotheses of Theorem V.2.10 p.145 in [8] hold in our special case. Specifically, required properties such as the Feller property of the minimal process and technical properties about $(\mathbb{E}_x(\mathrm{e}^{-\lambda T_0}), \lambda, x \ge 0)$ are easily verified. We need only show that $N^\star$ is compatible with the minimal semigroup. Namely, for all $s > 0$, $\Theta \in \mathcal{F}_s$, and bounded measurable F,

$$N^\star(F \circ \theta_s; \Theta \cap \{V > s\}) = N^\star(N_{\epsilon_s}(F); \Theta \cap \{V > s\}).$$

This follows from the fact that the measures $(N_r, r \ge 0)$ have the same semigroup as that of the killed Lévy process.

3.2. *Pathwise construction of the GCP*

Our aim is to give here a pathwise construction of a version of the GCP, still denoted $X^\star$, based on the paths of X and Y. As Y is a.s. increasing, we can define its right inverse Y^{-1} by

$$Y_s^{-1} = \inf\{t : Y_t > s\}, \qquad s \ge 0.$$

Set also

$$I_t = (\inf_{s \le t} X_s) \wedge 0, \qquad t \ge 0.$$

We can now state the

Theorem 3. *Define*

$$X_t^\star \doteq X_t + \inf(\mathcal{R} \cap (-I_t, \infty)) = X_t + Y(Y^{-1}(-I_t)),$$

where $\mathcal{R}$ stands for the range of Y. Under $\mathbb{P}_x$, the law of this process is equal to $\mathbb{P}_x^\star$. More precisely, we may (and will) define the local time at 0 of $X^\star$ by

$$L_t^\star = Y^{-1}(-I_t),$$

and then the excursion measure of $X^\star$ away from 0 is $N^\star$.

Proof. The first step is showing that $L^\star$ as defined in the theorem is a local time for $X^\star$. The second step uses the compensation formula for X and Y, to prove that the excursion process of $X^\star$ away from 0 is a Poisson point process with excursion measure $N^\star$. Then the theorem will follow from the uniqueness of the family $\mathbb{P}^\star$ (in particular, $X^\star$ as defined by this pathwise construction, is a Markov process).

First step. As X is spectrally positive, $-I$ is continuous increasing a.s., and so is Y^{-1} (for Y is not a compound Poisson process), thus by composition the same holds for $L^\star$.

We prove that $L^\star$ is an additive functional adapted to the natural filtration of $X^\star$. When $\Gamma = 0$, $L_t^\star = -I_t/\delta$ is known to be an adapted additive functional of $X - I = X^\star$.

When Γ is finite, the number of excursions of $X^\star$ away from 0 starting from positive values and occurring before time t is finite. It then suffices to proceed as in the previous case after discarding these excursions.

When Γ is infinite, a standard argument shows there is a sequence $\varepsilon_n \to 0$ such that with probability one for all t, $N_{\varepsilon_n}(t)/\Gamma(\varepsilon_n, \infty)$ converges to t, where

$$N_\varepsilon(t) = \text{Card}\{s \le t : \Delta Y_s > \varepsilon\}.$$

Hence $N_{\varepsilon_n}(Y^{-1}(-I_t))/\Gamma(\varepsilon_n, \infty)$ converges a.s. to $L_t^\star$. Next notice that $X_{s-}^\star = 0$ iff $X_s = I_s$ and $-I_s \in \mathcal{R}$. Since a dual ladder time (i.e. an increase time for $-I$) cannot be a jump time for X,

$$X_{s-}^\star = 0 \text{ and } \Delta X_s^\star > \varepsilon_n \Leftrightarrow X_s = I_s \text{ and } \Delta Y(Y^{-1}(-I_s)) > \varepsilon_n,$$

and it follows that $N_{\varepsilon_n}(Y^{-1}(-I_t))$ is exactly equal to

$$\text{Card}\{s \le t : X_{s-}^\star = 0, \Delta X_s^\star > \varepsilon_n\},$$

which is obviously measurable relative to $\sigma\{X_s^\star, s \le t\}$.

We get the fact that $L^\star$ is a local time by proving that for all $x \ge 0$,

$$\text{supp}(dL^\star) = \{t : X_{t-}^\star = 0\} \quad \mathbb{P}_x - \text{a.s.}$$

Take two rationals $a < b$. If for all $t \in (a, b)$ $X_{t-}^\star > 0$, then $(-I_b, \infty) \cap \mathcal{R} = (-I_a, \infty) \cap \mathcal{R}$, that is $(-I_b, -I_a) \cap \mathcal{R} = \emptyset$, hence $Y^{-1}(-I_a) = Y^{-1}(-I_b)$.

Consider

$$\zeta = \inf\{s \ge a : X_{s-}^\star = 0\} \in (a, \infty) \quad \text{a.s.}$$

Conditional on $\mathcal{R}$, ζ is a stopping time relative to the natural filtration of X. Now a spectrally positive Lévy process started at 0 immediately takes on negative values and so by the strong Markov property applied at ζ, ζ is a right increase time for $-I$. But $I_\zeta \in \mathcal{R}$ a.s., that is $-I_\zeta$ is a right increase time for Y^{-1}, hence ζ is a right increase time for $L^\star$. We just proved that if there exists $t \in (a, b)$ such that $X^\star_{t-} = 0$, then $L^\star_a < L^\star_b$ a.s. Then by continuity of $L^\star$, for all $x \geq 0$,

$$\mathbb{P}_x(\forall a < b, [\forall t \in (a, b)\ X^\star_{t-} > 0 \Leftrightarrow L^\star_a = L^\star_b]) = 1.$$

Second step. As previously mentioned, we shall prove that the excursion process of $X^\star$ is a Poisson point process with excursion measure $N^\star$. Specifically, set $\tau^\star$ the inverse local time

$$\tau^\star_s = \inf\{t : L^\star_t > s\} = T_{-Y_s}, \qquad s \geq 0.$$

We recall that since $\psi : [0, \infty) \to [0, \infty)$ is strictly increasing, it has an inverse ψ^{-1}, which is known to be the Laplace exponent of the subordinator $(T_{-x}, x \geq 0)$ (until the end of the proof, we write T_{-x} instead of $T_{(-\infty,-x)}$). Note then that $\tau^\star$ is the composition in the sense of Bochner of two independent subordinators with Laplace exponents ψ^{-1} and ϕ respectively, it is hence a subordinator with exponent $\phi \circ \psi^{-1}$. Then for $s \geq 0$ set as usual in $\mathcal{E}$

$$e^\star_s = \begin{cases} (X^\star_{\tau^\star_{s-}+t}, 0 \leq t < \tau^\star_s - \tau^\star_{s-}) & \text{if } \tau^\star_{s-} < \tau^\star_s \\ \Upsilon & \text{otherwise.} \end{cases}$$

We point out that as the set of jump times of Y is a.s. countable, with probability one, for all $s \geq 0$ $\Delta Y_s > 0 \Rightarrow \Delta T_{-Y_{s-}} = 0$, so that one can a.s. describe the set of jump times of $\tau^\star$ as

$$\{s : \Delta Y_s > 0\} \cup \{s : \Delta Y_s = 0, \Delta T_{-Y_s} \neq 0\}.$$

Define the filtration $(\mathcal{H}_t)_{t\geq 0}$ by saying that $\Theta \in \mathcal{H}_t$ if for every $r \geq 0$, $\Theta \cap \{T_{-Y_t} \leq r\}$ is in $\sigma\{(Y_s, s \leq t); (X_s, s \leq r)\}$. Then let F be a process predictable relative to $\mathcal{H}$, taking values in the nonnegative measurable functionals on $\mathcal{E} \cup \{\Upsilon\}$ and such that $F_t(\Upsilon) = 0$, for all t. We now apply the compensation formula to the process of jumps of Y and to the excursion process of $X - I$ away from 0. Taking predictable projections successively w.r.t. $\sigma\{(X_s, s \geq 0)\}$ and $\sigma\{(Y_s, s \geq 0)\}$, we get

$$\begin{aligned} \mathbb{E}\Big(\sum_{s\geq 0} F_s(e^\star_s)\Big) &= \mathbb{E}\Big(\sum_{s\geq 0, \Delta Y_s > 0} F_s(X^\star_{t+T_{-Y_{s-}}}, t \leq T_{-Y_s} - T_{-Y_{s-}})\Big) \\ &\quad + \mathbb{E}\Big(\sum_{s\geq 0, \Delta T_{-Y_s} > 0} F_s(X^\star_{t+T_{(-Y_s)-}}, t \leq T_{-Y_s} - T_{(-Y_s)-})\Big) \\ &= \mathbb{E}\int_0^\infty ds \int_0^\infty \Gamma(dr) F_s(X^\star_{t+T_{-Y_{s-}}}, t \leq T_{-Y_{s-}+r} - T_{-Y_{s-}}) \\ &\quad + \mathbb{E}\Big(\sum_{u\geq 0, \Delta T_{-u} > 0} \mathbf{1}_{\{u\in\mathcal{C}\}}\, F_{Y_u^{-1}}(X^\star_{t+T_{(-u)-}}, t \leq T_{-u} - T_{(-u)-})\Big), \end{aligned}$$

where $\mathcal{C} = \{Y_s; s \geq 0, \Delta Y_s = 0\}$,

$$= \int_0^\infty ds \int_0^\infty \Gamma(dr)\mathbb{E}_r(F_s(X_t, t \le T_0)) + \mathbb{E}\int_0^\infty du\, N_0(F_{Y_u^{-1}})\mathbf{1}_{u\in\mathcal{C}}$$
$$= \int_0^\infty ds \int_0^\infty N_r(F_s)\Gamma(dr) + \mathbb{E}\int_0^\infty N_0(F_s)\mathbf{1}_{\{\Delta Y_s=0\}}\, dY_s$$
$$= \int_0^\infty ds\, (\int_0^\infty N_r(F_s)\Gamma(dr) + \delta N_0(F_s))$$
$$= \int_0^\infty ds\, N^\star(F_s),$$

and the proof is complete. □

The next section is devoted to the continuous analogue of Proposition 1, that is to give a pathwise definition of the height process based on the GCP, and then check that it is distributed as the height process linked to a CBI(ψ, ϕ).

4. The height process $H^\star$

4.1. Definitions and genealogy-decoding

Consider the GCP(ψ, ϕ) $X^\star$ defined in the last section. This Markov process provides a handy way to code for the genealogy of the continuous analogue of a Galton-Watson tree with immigration. This continuous tree is well understood when considering the successive heights of its individuals. If $\phi \equiv 0$, we recall that this so-called height process H_t can be recovered from the paths of X by measuring the set $\mathcal{J}_t$ of times when X meets or crosses its future infimum on $[0, t]$, thanks to formula (6). When $\phi \not\equiv 0$, denote by g_t for each positive t, the last zero of $X^\star$ before time t, that is

$$g_t = \sup(\{s \le t : X^\star_{s-} = 0\} \cup \{0\}).$$

Then, for before time g_t the future infimum on $[0, t]$ is equal to 0 a.s., we set

$$\mathcal{K}_t = \{s < g_t : X^\star_{s-} = 0\},$$

$$\mathcal{L}_t = \{g_t \le s \le t : X^\star_{s-} \le \inf_{s\le r\le t} X^\star_r\},$$

which implies that $\mathcal{J}_t = \{s \le t : X^\star_{s-} \le \inf_{s\le r\le t} X^\star_r\}$ is the disjoint union of $\mathcal{K}_t$ and $\mathcal{L}_t$. Analogously to the discrete setting, the first set can be considered as the ‘time spent’ when the exploration process hits the rightmost branch of the tree, that is the height of the last visited immigrant. The measure of $\mathcal{K}_t$ is taken equal to the local time $L^\star_t$ at level 0 at time t of $X^\star$.

Next the (incomplete) excursion $(X^\star_s, g_t \le s \le t)$ explores the standard descendant tree starting from the last wave of immigration $\Delta X^\star_{g_t}$, and thus it reduces to the definition of the height process given in [14] when there is no immigration.

Indeed, conditional on $\{X^\star_{g_t}=0\}$, the finite path $(X^\star_s, g_t \le s \le t)$ is distributed as $(X_s - I_t, G_t \le s \le t)$ under $\mathbb{P}_0$, where

$$G_t = \sup\{s \le t : X_s = I_t\}.$$

And conditional on $\{X^\star_{g_t} = x, t - g_t = r\}$, the finite path $(X^\star_s, g_t \le s \le t)$ is distributed as $(X_s, 0 \le s \le r)$ under $\mathbb{P}_x(\cdot \mid T_0 > r)$. We will thus deal with the measure of $\mathcal{L}_t$ given by (6), provided the paths of $X^\star$ have infinite variation. In conclusion, we can thus make the following

Definition 4. *The height process $H^\star$ is defined as a functional of $X^\star$ by*

$$H^\star_t = L^\star_t + \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{g_t}^t \mathbf{1}_{\{X^\star_s - \inf_{s\le r\le t} X^\star_r < \varepsilon\}} ds.$$

Then $H^\star$ is a progressively measurable process and we can define the random measure $\mathcal{Z}^\star$ by

$$< \mathcal{Z}^\star, h >= \int_0^\infty h(H^\star_s) ds,$$

where h is any nonnegative measurable function with compact support. Moreover, Theorem 4.7 in [15] and the continuity of $L^\star$ entail that $H^\star$ is continuous a.s. if and only if $\int^\infty d\lambda/\psi(\lambda) < \infty$.

We can state the main theorem of this section. It shows that the local time process of $H^\star$, as a function of the space variable, is a CBI(ψ,ϕ).

Theorem 5. *Under $\mathbb{P}^\star_x$, the random measure $\mathcal{Z}^\star$ has a.s. a càdlàg density $(Z^\star_a, a \ge 0)$, w.r.t. Lebesgue measure on $\mathbb{R}_+$, and the process $Z^\star$ is a CBI(ψ,ϕ) starting from x.*

We start with stating a lemma on CBI-processes, which proof is moved to the appendix.

Lemma 6. *If h denotes any nonnegative bounded measurable function with compact support on $\mathbb{R}_+$, and if Z denotes a CBI(ψ,ϕ) started at x, then*

$$\mathbf{E}_x(\exp[-\int_0^\infty h(a) Z_a da]) = \exp[-xw(0) - \int_0^\infty \phi(w(s))ds],$$

where $(w(t), t \ge 0)$ is the unique nonnegative solution of the integral equation

$$v(t) + \int_t^\infty \psi(v(s))ds = \int_t^\infty h(s)ds, \qquad t \ge 0. \tag{E_h}$$

We are now able to provide the

Proof of Theorem 5. We have to establish the equality

$$\mathbb{E}^\star_x(\exp[-\int_0^\infty h(H^\star_s)ds]) = \exp[-xw(0) - \int_0^\infty \phi(w(s))ds],$$

where w is the nonnegative solution of (E_h).

The choice we made for the definition of $H^\star$ allows us to use the following two expressions computed in [14, p.141], where V stands for the lifetime of the generic path

$$N_0(1-\exp(-\int_0^V h(H_u+t)du)) = w(t), \qquad t \geq 0,$$

and

$$N_y(\exp[-\int_0^V h(H_u+t)du]) = \exp(-yw(t)) \qquad t \geq 0, y > 0.$$

We decompose the genealogy-coding process $X^\star$ into its excursions away from 0. Thanks to the two foregoing equations, and using the excursion measure $N^\star$ that defined the process $X^\star$ in Section 3, we get

$$\begin{aligned}
&\mathbb{E}^\star_x(\exp(-\int_0^\infty h(H^\star_s)ds)) \\
&= \mathbb{E}^\star_x(\exp(-\int_0^{T_0} h(H^\star_s)ds)\mathbb{E}^\star_0(\exp-\sum_{s\geq 0}\int_{\tau^\star_{s-}}^{\tau^\star_s} h(H^\star_u)du) \\
&= \mathbb{E}_x(\exp(-\int_0^{T_0} h(H^\star_s)ds))\exp\left(-\int_0^\infty ds\, N^\star(1-\exp(-\int_0^V h(s+H_u)du))\right) \\
&= N_x(\exp(-\int_0^V h(H_u)du)) \,.\, \exp\left(-\int_0^\infty ds\left[\int_0^\infty \Gamma(dr)\right.\right. \\
&\quad \times N_r(1-\exp(-\int_0^V h(s+H_u)du)) \\
&\qquad\qquad \left.\left. +\delta N_0(1-\exp(-\int_0^V h(s+H_u)du))\right]\right) \\
&= \exp(-xw(0))\exp-\int_0^\infty ds\,[\int_0^\infty \Gamma(dr)(1-\mathrm{e}^{-rw(s)})+\delta w(s)] \\
&= \exp[-xw(0)-\int_0^\infty \phi(w(s))ds],
\end{aligned}$$

which is the expected expression. □

Remark. In the case when X is a standard Brownian motion, and $Y = (\delta t, t \geq 0)$, with δ a positive real number, Theorem 5 reduces to a theorem by M. Yor and J.F. Le Gall [17]. Writing $\psi(\lambda) = \lambda^2/2$, $\phi(\lambda) = \delta\lambda$, and referring to Theorem 3, the GCP(ψ, ϕ) is the Feller Brownian motion $X^\star = (X_t - I_t, t \geq 0)$ with local time at time t at level 0 equal to $-I_t/\delta$.

For $t \geq 0$, consider the finite path $\beta = (X^\star_t - X^\star_{t-s}, 0 \leq s \leq t - g_t)$, with g_t the last zero of $X^\star$ before time t, and let $\overline{\beta}$ be its supremum process. By definition of the height process $H^\star$ of $X^\star$,

$$H^\star_t = \frac{-I_t}{\delta} + \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{t-g_t} \mathbf{1}_{\{\overline{\beta}_s-\beta_s<\varepsilon\}}ds.$$

But β is a Brownian path started at 0 and killed upon reaching the positive real number $X_t - I_t$, hence

$$H_t^\star = \frac{-I_t}{\delta} + 2(X_t - I_t).$$

By Lévy's equivalence theorem, $H^\star$ is distributed as $2 \mid X \mid + L^0/\delta$, where L^0 denotes the local time at level 0 of the Brownian motion X, and a theorem by J.F. Le Gall and M. Yor states that

$$(L_\infty^a(2 \mid X \mid + L^0/\delta), a \geq 0) \overset{\mathcal{L}}{=} 4^{-1}\mathrm{BESQ}(4\delta),$$

where the notation in the r.h.s. refers to a squared Bessel process of dimension 4δ starting from 0.

In agreement with Theorem 5, a BESQ(4δ) is a CBI(ψ, ϕ) up to a multiplicative factor 4, with $\psi(\lambda) = \lambda^2/2$, $\phi(\lambda) = \delta\lambda$.

5. An extension of a Ray–Knight–Williams theorem

We show how Lévy processes that drift to $+\infty$ and Lévy processes conditioned to stay positive code the genealogy of certain CBI-processes. In the Brownian case, the genealogy related to the Bessel process of dimension 3 (BES(3)), which is the Brownian motion conditioned to stay positive, is that of a squared Bessel process of dimension 2 ((BESQ(2)), that is a CBI-process with branching mechanism $\lambda \mapsto \lambda^2/2$, and immigration mechanism $\lambda \mapsto \lambda/2$. This result is known as the Ray–Knight–Williams theorem (see [25]).

5.1. Main result

We stick to the framework described in Section 3. Specifically, $\mathbb{P}_x$ denotes the law of a spectrally positive Lévy process X started at $x \in \mathbb{R}$, with Laplace exponent the convex function

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{r<1})\Lambda(dr), \qquad \lambda \geq 0.$$

Denote by ξ the largest root of ψ. If $\xi > 0$, ψ has exactly two roots (0 and ξ), otherwise it has a unique root $\xi = 0$. According as $\psi'(0+) < 0$, $\psi'(0+) = 0$, or $\psi'(0+) > 0$, the paths of X a.s. drift to $+\infty$, oscillate, or drift to $-\infty$, and the associated branching mechanism is supercritical, critical, or subcritical. We assume throughout the rest of this section that $\psi'(0+) \leq 0$, and again that the paths of X have a.s. infinite variation.

We introduce briefly two laws connected with $\mathbb{P}$.

1. The probability measure $\mathbb{P}_x^\uparrow$ is the law of the Lévy process started at $x > 0$ and conditioned to stay positive. When X drifts to $+\infty$, the conditioning is taken in the usual sense, since X stays positive with positive probability. When X oscillates, as X reaches 0 continuously, the process $(X_t \mathbf{1}_{\{t<T_0\}}, t \geq 0)$ is a

martingale (T_0 denotes the first hitting time of 0 by X). Then $\mathbb{P}_x^\uparrow$ is defined by local absolute continuity w.r.t. $\mathbb{P}_x$ with density $x^{-1}X_t\mathbf{1}_{\{t<T_0\}}$ on $\mathcal{F}_t$ ($t \geq 0$). It is known that the probability measures $\mathbb{P}_x^\uparrow$ converge weakly as $x \to 0+$ to a Markovian law $\mathbb{P}_0^\uparrow$. For more details, see [1], [9],[10].

2. The law $\mathbb{P}^\natural$ is that of the spectrally positive Lévy process with Laplace exponent $\psi^\natural : \lambda \mapsto \psi(\lambda + \xi)$. When $\xi > 0$, the path of X a.s. drifts to $+\infty$, and if I_∞ denotes its overall infimum (Lemma VII.7(i) in [4]), then

$$\lim_{x\to\infty} \mathbb{P}(\Lambda \mid I_\infty < -x) = \mathbb{P}^\natural(\Lambda), \qquad \Lambda \in \mathcal{F}_t, t > 0.$$

This process is thus called the Lévy process conditioned to drift to $-\infty$, and for every $x \in \mathbb{R}$, $\mathbb{P}_x^\natural$ is defined by local absolute continuity w.r.t. $\mathbb{P}_x$ with density $\exp(-\xi(X_t - x))$ on $\mathcal{F}_t$ ($t \geq 0$). In the sequel, it will be implicit that the superscript $\natural$ refers to $\mathbb{P}^\natural$. For more details, see [1].

Recall that the definition (1) of the functional H makes sense for any Lévy process with no negative jumps. By local absolute continuity, H is still well defined under $\mathbb{P}$, and $\mathbb{P}_x^\uparrow$ for every $x > 0$. We shall see in Lemma 8 that the same holds under $\mathbb{P}_0^\uparrow$. In [15], the main result asserts that, provided X does not drift to $+\infty$ under $\mathbb{P}_x$, the occupation measure $\mathcal{Z}_x$ of H defined for any nonnegative h by

$$< \mathcal{Z}_x, h >= \int_0^{T_{-x}} h(H_s)ds,$$

has a density $(Z_a, a \geq 0)$ w.r.t. Lebesgue measure, which is a CB-process started at x. We now state the analogue under $\mathbb{P}^\uparrow$ and under $\mathbb{P}$ when X drifts to $+\infty$.

Theorem 7. *Define*

$$\phi(\lambda) = \frac{\psi^\natural(\lambda)}{\lambda + \xi} = \frac{\psi(\lambda + \xi)}{\lambda + \xi}, \qquad \lambda \geq 0.$$

Both $\mathbb{P}$-a.s. and $\mathbb{P}^\uparrow$-a.s., the occupation measure of H has a density w.r.t. Lebesgue measure. We denote by $(Z_a, a \geq 0)$ the càdlàg version of this density.

(i) Let $x \geq 0$. Under $\mathbb{P}_x^\uparrow$, Z is a CBI($\psi^\natural$, ϕ) with initial distribution μ_x, where

1. *The measure μ_0 is the Dirac mass at 0.*
2. *For $x > 0$ and $\xi = 0$, μ_x is the uniform distribution on $(0, x)$.*
3. *For $x > 0$ and $\xi > 0$,*

$$\mu_x(dy) = \frac{\xi e^{-\xi y}}{1 - e^{-\xi x}} dy, \qquad 0 < y < x.$$

(ii) Assume the branching mechanism is supercritical ($\psi'(0+) < 0$). Then under $\mathbb{P}$, Z is a CBI($\psi^\natural$, ϕ) with initial distribution the exponential distribution with parameter ξ.

The proof uses the following lemma, which shows the connection with the GCP. When X drifts to $+\infty$, we denote by $\underline{\underline{X}}$ its future infimum

$$\underline{\underline{X}}_t = \inf_{s \geq t} X_s, \qquad t \geq 0.$$

Lemma 8. *Under $\mathbb{P}_0^{\uparrow}$, we may (and will) define the local time $\underline{\underline{L}}$ at 0 for $X - \underline{\underline{X}}$ by*

$$\underline{\underline{L}}_t \overset{P}{=} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s - \underline{\underline{X}}_s < \varepsilon\}} ds, \qquad t \geq 0. \tag{7}$$

Then with this normalization of local time,

(i) The process $X - \underline{\underline{X}}$ is a version of the GCP($\psi^{\natural}, \phi$).

(ii) The functional $\overline{H}$ of X is well defined by (1) and is equal to the height process $H^{\star}$ of the GCP $X - \underline{\underline{X}}$.

Before proving the theorem, we establish the link with the Ray–Knight–Williams theorem. In the Brownian case, the law $\mathbb{P}^{\uparrow}$ is that of the Bessel process of dimension 3 (BES(3)). Invoking Pitman's theorem (see [20]), the bivariate process $(X, \underline{\underline{X}})$ has the same law as $(2S - B, S)$, where B stands for a standard Brownian motion, and S for its supremum process. Hence by Lévy's equivalence theorem, $2\underline{\underline{X}}$ is a local time at 0 for $X - \underline{\underline{X}}$. It is then easily checked referring to the remark ending last section that

$$H_t^{\star} = 2\underline{\underline{X}}_t + 2(X - \underline{\underline{X}})_t = 2X_t, \qquad t \geq 0,$$

and by Lemma 8(ii), the process H is again (up to a factor 4) a BES(3). Then the Ray–Knight–Williams theorem states that the local time process of a BES(3) is a squared Bessel process of dimension 2 starting from 0 (BESQ(2)), which is (up to a factor 4) a CBI(ψ, ϕ), with $\psi(\lambda) = \lambda^2/2$, and $\phi(\lambda) = \lambda/2$ (see [25, Theorem 65 p.38]).

Proof of Theorem 7.

(i) When $x = 0$, the statement follows readily from Lemma 8 and Theorem 5.

Let $x > 0$. We have the following absolute continuity relationship (see [10])

$$\mathbb{P}_x^{\uparrow}(\Theta) = \mathbb{E}_x\left(\frac{h(X_t)}{h(x)}, \Theta, t < T_0\right), \qquad \Theta \in \mathcal{F}_t, x > 0, \tag{8}$$

where $h(y) = y$ when $\xi = 0$, and $h(y) = \xi^{-1}(1 - \mathrm{e}^{-\xi y})$ when $\xi > 0$. For any $0 \leq y \leq x$, (8) yields

$$\mathbb{P}_x^{\uparrow}(I_\infty \leq y) = \frac{h(y)}{h(x)} \mathbb{P}_x(T_y < \infty) = \frac{h(y)}{h(x)} \mathrm{e}^{-\xi(x-y)}. \tag{9}$$

Hence, with the notation in Theorem 7,

$$\mathbb{P}_x^{\uparrow}(x - I_\infty \in dy) = \mu_x(dy).$$

Now a theorem by L. Chaumont ([9, Théorème 2]) states that under $\mathbb{P}_x^{\uparrow}$, conditional on $I_\infty = y$, the pre-minimum process and the post-minimum process are independent with respective laws $N_{x-y}^{\natural}$ and $\mathbb{P}_0^{\uparrow}$. Hence conditional on $I_\infty = y$, $X - \underline{\underline{X}}$ is the juxtaposition of the killed Lévy process under $\mathbb{P}^{\natural}_{x-y}$ and an independent GCP($\psi^{\natural}$, ϕ) started at 0, it is thus distributed as a GCP($\psi^{\natural}$, ϕ) with initial distribution μ_x. The equality between its height process $H^{\star}$ and the functional H is again straightforward from Lemma 8 and an application of Theorem 5 completes the proof.

(ii) We work under $\mathbb{P}_0$. When $\xi > 0$, we know that $-I_\infty$ is exponential with parameter ξ. The key is a result of P.W. Millar [18] and J. Bertoin [1, Théorème 2] which states that conditional on $-I_\infty = y$, the pre-minimum process and the post-minimum process are independent with respective laws $N^{\natural}_y$ and $\mathbb{P}^{\uparrow}_0$. We conclude as previously. □

5.2. *Proof of Lemma 8*

We first set some definitions and state a general result describing the excursion measure of $X - S$ away from 0 under $\mathbb{P}$.

Recall that since X is a Lévy process with paths of infinite variation, the processes $X - I$ and $X - S$ (I denotes the infimum process, and S the supremum process) are strong Markov processes for which 0 is a regular point. Therefore, one can associate to each an excursion measure away from 0, denoted by N_0 and $\overline{n}$ respectively, for the following normalization of local time. It is well-known that the process $-I$ provides a local time at 0 for $X - I$. The local time L at 0 for $X - S$ is defined (see [14, p.133]) by

$$L_t \overset{P}{=} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{S_s - X_s < \varepsilon\}} ds, \qquad t \geq 0. \tag{10}$$

We start with the following lemma concerning $\overline{n}$, and next use it in the proof of Lemma 8. Its proof is moved to the appendix.

Lemma 9. *We denote the reversed generic excursion by $\hat{\epsilon} = (-\epsilon_{(V-t)-}, 0 \leq t \leq V)$. Then under $\overline{n}$,*

(i) $\overline{n}(\hat{\epsilon}_0 \in dr) = \Lambda(r, \infty) e^{-\xi r} dr, r > 0$.

(ii) For $r > 0$, conditional on $\{\hat{\epsilon}_0 = r\}$, $\hat{\epsilon}$ is distributed as X started at r and killed upon reaching 0 under $\mathbb{P}^{\natural}$.

(iii) $\hat{\overline{n}}(\cdot, \epsilon_0 = 0)$ is proportional to the excursion measure ${N_0}^{\natural}$ of $X - I$ away from 0 under $\mathbb{P}^{\natural}$. More precisely, $\hat{\overline{n}}(\cdot, \epsilon_0 = 0) = \beta {N_0}^{\natural}$.
In other words, $\hat{\overline{n}} = N^{\star}$, where we set

$$N^{\star} = \int_0^\infty dr\, e^{-\xi r} \Lambda(r, \infty) N^{\natural}_r + \beta {N_0}^{\natural}.$$

We point out that when the Gaussian component β of the Laplace exponent of X vanishes, (i) and (ii) are known results by L.C.G. Rogers [22, Theorem 1] and J. Bertoin [2, Corollary 1], respectively. The previous lemma allows us to prove Lemma 8.

Proof of Lemma 8. For every $t \geq 0$, define $\underline{g}_t = \sup\{s < t : X_s = S_s\}$, $\underline{d}_t = \inf\{s > t : X_s = S_s\}$, and introduce the process

$$\mathcal{R}(X-S)_t = \begin{cases} (S-X)_{(\underline{d}_t+\underline{g}_t-t)-} & \text{if } \underline{d}_t > \underline{g}_t \\ 0 & \text{if } \underline{d}_t = \underline{g}_t, \end{cases}$$

obtained by reversing each excursion of $S-X$. When X drifts to $+\infty$ under $\mathbb{P}$, Lemme 4 in [1] states that under $\mathbb{P}_0$ the process $(\mathcal{R}(X-S)_t, t \geq 0)$ has the same law as $X - \underline{\underline{X}}$ under $\mathbb{P}_0^{\uparrow}$. Let us give a short argument to prove that this still holds when X oscillates. Let T be an independent exponential r.v. with parameter $\varepsilon > 0$, and $\rho = \rho(T)$ the first time when X reaches its future infimum on $[0, T]$

$$\rho = \inf\{s \leq T : X_s = \inf_{s \leq r \leq T} X_r\}.$$

The same arguments as those developed in the proof of [1, Lemme 4] show that the processes $(\mathcal{R}(X-S)_t, 0 \leq t < \underline{g}_T)$ and $((X-\underline{\underline{X}})_{\rho+t}, 0 \leq t \leq T-\rho)$ have the same law. The result then follows from [3, Corollary 3.2] according to which the laws of $(X_{\rho+t}, 0 \leq t \leq T-\rho)$ converge to $\mathbb{P}_0^{\uparrow}$ as $\varepsilon \downarrow 0$. Hence under $\mathbb{P}_0$, the process $(\mathcal{R}(X-S)_t, t \geq 0)$ has the same law as $X - \underline{\underline{X}}$ under $\mathbb{P}_0^{\uparrow}$. As a consequence, $\{t : X_t - \underline{\underline{X}}_t = 0\}$ is distributed under $\mathbb{P}_0$ as $\{t : X_t = S_t\}$ under $\mathbb{P}_0$. Recall that $\{t : X_t = S_t\}$ is a regenerative set with local time L defined by (10), that is

$$L_t = L_{\underline{g}_t} = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^{\underline{g}_t} \mathbf{1}_{\{S_s - X_s < \varepsilon\}} ds = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^{\underline{g}_t} \mathbf{1}_{\{\mathcal{R}(S-X)_s < \varepsilon\}} ds, \qquad t \geq 0.$$

Hence $\{t : X_t - \underline{\underline{X}}_t = 0\}$ is a regenerative set and the functional $\underline{\underline{L}}$ defined by (7) is its local time. Furthermore, it follows also from this identity in law that the associated excursion measure $\underline{\underline{n}}$ of $X - \underline{\underline{X}}$ away from 0 satisfies

$$\underline{\underline{n}} = \hat{\hat{n}},$$

where $\hat{\hat{n}}$ still denotes the image of the excursion measure of $S - X$ away from 0 (normalized by (10)) by the time-reversal map. Hence referring to Lemma 9, $\underline{\underline{n}} = N^\star$, with

$$N^\star = \int_0^\infty dr\, \mathrm{e}^{-\xi r} \Lambda(r, \infty) N_r^\natural + \beta N_0{}^\natural.$$

Now notice that after elementary calculation

$$\phi(\lambda) = \beta\lambda + \int_0^\infty dr\, \mathrm{e}^{-\xi r} \Lambda(r, \infty)(1 - \mathrm{e}^{-\lambda r}), \qquad \lambda \geq 0, \qquad (11)$$

which ensures that ϕ is the Laplace exponent of a subordinator, and that $N^\star$ is the excursion measure of the GCP$(\psi^\natural, \phi)$. The zeros of $X - \underline{\underline{X}}$ are instantaneous, hence the uniqueness of $\mathbb{P}_0^\star$ yields that $X - \underline{\underline{X}}$ is a GCP$(\psi^\natural, \phi)$ started at 0.

It thus only remains to show that if $H^\star$ denotes its height process as in Definition 4, then $H^\star$ is equal to the r.h.s. in (1). For every positive t, split the interval $[0, t]$ into $[0, \underline{\underline{g}}_t) \cup [\underline{\underline{g}}_t, t]$, where

$$\underline{\underline{g}}_t = \sup\{s \leq t : X_s = \underline{\underline{X}}_s\}.$$

Then by definition of $H^\star$ and $\underline{\underline{L}}$,

$$H_t^\star = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\underline{\underline{g}}_t} \mathbf{1}_{\{X_s-\underline{\underline{X}}_s<\varepsilon\}}ds + \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\underline{\underline{g}}_t}^t \mathbf{1}_{\{(X-\underline{\underline{X}})_s-\inf_{s\le r\le t}(X-\underline{\underline{X}})_r<\varepsilon\}}ds.$$

Note that for any $s\le t$, $\underline{\underline{X}}_s=\min(\underline{\underline{X}}_t,\inf_{s\le r\le t}X_r)$. Then for any $s\in[0,\underline{\underline{g}}_t)$, $\underline{\underline{X}}_s=\inf_{s\le r\le t}X_r$, and for any $s\in[\underline{\underline{g}}_t,t]$, $\underline{\underline{X}}_s=\underline{\underline{X}}_t$. Replacing in the previous equality yields

$$H_t^\star = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\underline{\underline{g}}_t} \mathbf{1}_{\{X_s-\inf_{s\le r\le t}X_r<\varepsilon\}}ds + \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\underline{\underline{g}}_t}^t \mathbf{1}_{\{X_s-\underline{\underline{X}}_t-\inf_{s\le r\le t}(X_r-\underline{\underline{X}}_t)<\varepsilon\}}ds,$$

which entails the existence of

$$H_t = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{X_s-\inf_{s\le r\le t}X_r<\varepsilon\}}ds,$$

and the identity $H=H^\star$. □

Remark. An easy way of building a GCP is to erase the negative excursions of X under $\mathbb{P}$. We consider here that $\psi'(0+)=0$ (critical case). Set

$$A_t^+ = \int_0^t \mathbf{1}_{\{X_s>0\}}ds, \qquad t\ge 0,$$

$$A_t^- = \int_0^t \mathbf{1}_{\{X_s<0\}}ds, \qquad t\ge 0,$$

and α^+, α^- their respective right-inverses. Referring to the remark p.1470 in [2], the excursion ϵ of X away from 0 and $\hat\epsilon$ are equally distributed. Hence the excursion ϵ of $X\circ\alpha^+$ is distributed as the reversed excursion $\hat\epsilon$ of $X\circ\alpha^-$. But [2, Lemma 2] entails that $X\circ\alpha^-$ and $X-S$ have the same law. In conclusion, the excursion measure of $X\circ\alpha^+$ away from 0 is equal to $\hat{\hat{n}}$, and $X\circ\alpha^+$ is thus a GCP(ψ,ϕ), where

$$\phi(\lambda)=\frac{\psi(\lambda)}{\lambda}, \qquad \lambda>0.$$

6. Appendix

6.1. Proof of Lemma 6

Every solution v of (E_h) is continuous and has its support included in that of h. Hence the range of v is compact and ψ is Lipschitz on this compact set. The uniqueness of the solution then follows from Gronwall's lemma.

Remember that $t\mapsto u_t(\lambda)$ is the unique nonnegative solution of

$$v(t)+\int_0^t \psi(v(s))ds=\lambda, \qquad \lambda\ge 0,\ \ t\ge 0,$$

and that

$$\mathbf{E}_x(\mathrm{e}^{-\lambda Z_t}) = \exp[-xu_t(\lambda) - \int_0^t \phi(u_s(\lambda))ds].$$

As a consequence, for any $t_1 \geq 0$, $\lambda_1 \geq 0$, $t \mapsto w_t(\lambda_1) = \mathbf{1}_{[0,t_1]}(t)u_{t_1-t}(\lambda_1)$ is the unique nonnegative solution of

$$v(t) + \int_t^\infty \psi(v(s))ds = \lambda_1 \mathbf{1}_{[0,t_1]}(t), \qquad t \geq 0, \qquad (E'(\lambda_1, t_1))$$

and furthermore

$$\mathbf{E}_x(\mathrm{e}^{-\lambda_1 Z_{t_1}}) = \exp[-xw_0(\lambda_1) - \int_0^{t_1} \phi(w_s(\lambda_1))ds].$$

More generally, we define for $0 \leq t_1 < \cdots < t_n$, and $\lambda_1, \ldots, \lambda_n \geq 0$ the integral equation

$$v(t) + \int_t^\infty \psi(v(s))ds = \sum_{j=1}^n \lambda_j \mathbf{1}_{[0,t_j]}(t), \qquad t \geq 0. \quad (E'(\lambda_1, t_1, \ldots, \lambda_n, t_n))$$

We argue by induction on n to show that the solution w of $(E'(\lambda_1, t_1, \ldots, \lambda_n, t_n))$ satisfies

$$\mathbf{E}_x(\exp - \sum_{j=1}^n \lambda_j Z_{t_j}) = \exp[-xw(0) - \int_0^{t_n} \phi(w(s))ds].$$

The first step was just proved in the preceding lines. Now let $n \geq 2$, and assume that the result holds up to the order $n-1$. By the Markov property at t_1,

$$\begin{aligned}\mathbf{E}_x(\exp - \sum_{j=1}^n \lambda_j Z_{t_j}) &= \mathbf{E}_x(\mathrm{e}^{-\lambda_1 Z_{t_1}} \mathbf{E}_{Z_{t_1}}(\exp - \sum_{j=2}^n \lambda_j Z_{t_j - t_1})) \\ &= \mathbf{E}_x(\mathrm{e}^{-\lambda_1 Z_{t_1}} \exp[-Z_{t_1}\tilde{w}(0) - \int_0^{t_n - t_1} \phi(\tilde{w}(s))ds]),\end{aligned}$$

where $\tilde{w}$ is the nonnegative solution of $(E'(\lambda_2, t_2 - t_1, \ldots, \lambda_n, t_n - t_1))$.

Thanks to the first step ($n = 1$),

$$\mathbf{E}_x(\exp - \sum_{j=1}^n \lambda_j Z_{t_j}) = \exp[-x\bar{w}(0) - \int_0^{t_1} \phi(\bar{w}(s))ds - \int_0^{t_n - t_1} \phi(\tilde{w}(s))ds],$$

where $\bar{w}$ is the nonnegative solution of $(E'(\lambda_1 + \tilde{w}(0), t_1))$. Hence

$$w(t) = \mathbf{1}_{[0,t_1]}(t)\bar{w}(t) + \mathbf{1}_{(t_1,\infty)}(t)\tilde{w}(t - t_1)$$

is the nonnegative solution of $(E'(\lambda_1, t_1, \ldots, \lambda_n, t_n))$ and satisfies

$$\mathbf{E}_x(\exp - \sum_{j=1}^{n} \lambda_j Z_{t_j}) = \exp[-x\bar{w}(0) - \int_0^{t_1} \phi(\bar{w}(s))ds - \int_{t_1}^{t_n} \phi(\tilde{w}(s - t_1))ds]$$
$$= \exp[-xw(0) - \int_0^{t_n} \phi(w(s))ds].$$

Now go back to the general case with h some nonnegative bounded measurable function with compact support. The mapping $t \mapsto \int_t^\infty h(s)ds$ is a continuous decreasing function that we may approximate by a pointwise increasing sequence of step functions φ_n

$$\varphi_n(t) = \sum_{j=1}^{n} \lambda_j^n \mathbf{1}_{[0,t_j^n]}(t) \uparrow \int_t^\infty h(s)ds, \qquad t \geq 0.$$

Then it is clear that the associated differential equation $(E'(\lambda_1^n, t_1^n, \ldots, \lambda_n^n, t_n^n))$ has a unique nonnegative solution w_n satisfying

$$0 \leq w_n(t) \leq \int_t^\infty h(s)ds, \qquad t \geq 0.$$

In particular, the $(w_n, n \geq 0)$ are uniformly bounded and have a common compact support. Applying Gronwall's lemma to the increments $w_{n+p}(t) - w_n(t)$ for each $t \geq 0$, we deduce that the sequence $(w_n(t), n \geq 0)$ has a limit, say $w(t)$, as $n \to \infty$. It follows from the dominated convergence theorem that $\int_t^\infty \psi(w_n(s))ds \to \int_t^\infty \psi(w(s))ds$, that $\int_t^\infty \phi(w_n(s))ds \to \int_t^\infty \phi(w(s))ds$, and that

$$\mathbf{E}_x(\exp - \sum_{j=1}^{n} \lambda_j^n Z_{t_j^n}) \to \mathbf{E}_x(\exp - \int_0^\infty h(a)Z_a da), \text{ as } n \to \infty.$$

Hence w satisfies (E_h), which provides a proof for the existence of solutions, moreover

$$\mathbf{E}_x(\exp[- \int_0^\infty h(a)Z_a da]) = \exp[-xw(0) - \int_0^\infty \phi(w(s))ds],$$

and the proof is complete. □

6.2. Proof of Lemma 9

We first give some further details about $\mathbb{P}$, and state two preliminary lemmas.

The scale function is defined as the unique continuous function $W : [0, \infty) \to [0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-qx} W(x)dx = \frac{1}{\psi(q)}, \qquad q > \xi.$$

It satisfies for any $0 \le x \le y$

$$\mathbb{P}_x(T_0 < T_{[y,+\infty)}) = \frac{W(y-x)}{W(y)}. \tag{12}$$

We introduce also the positive increasing mappings $W^{(\lambda)}$ on $(0,\infty)$ specified by their Laplace transforms

$$\int_0^\infty e^{-qx} dW^{(\lambda)}(x) = \frac{q}{\psi(q)-\lambda}, \qquad \psi(q) > \lambda \ge 0.$$

In particular, $W^{(0)} = W$. Since when $\beta > 0$, $\psi(q) \sim \beta q^2$ as $q \to \infty$, it follows from a Tauberian theorem that for any $\lambda \ge 0$,

$$W^{(\lambda)}(x) \sim \beta^{-1} x, \qquad \text{as } x \to 0^+. \tag{13}$$

We stress that $\overline{n}$ is normalized by (10). Set L^{-1} the right-inverse of L, and ψ^{-1} the inverse of $\psi_{|[\xi,\infty)}$ (ψ is strictly increasing on $[\xi,\infty)$ with $\psi(\xi)=0$). Referring to [14, p.133] and [7], $((L_t^{-1}, S_{L_t^{-1}}), t \ge 0)$ is a bivariate subordinator with Laplace exponent κ satisfying

$$\kappa(\lambda,0) = \frac{\lambda}{\psi^{-1}(\lambda)}, \qquad \kappa(0,\lambda) = \frac{\psi(\lambda)}{\lambda-\xi}, \qquad \lambda \ge 0.$$

Lemma 10. *Let $m(\epsilon)$ stand for the supremum of the generic excursion ϵ. Then*

$$\overline{n}(m \ge x) = \frac{1}{W(x)}, \qquad x > 0.$$

Proof. For every $0 < x \le y$, it follows from the strong Markov property applied at T_{-x} under $\overline{n}$ that

$$\frac{\hat{\overline{n}}(m \ge y)}{\hat{\overline{n}}(m \ge x)} = \mathbb{P}_{-x}(T_{-y} < T_{[0,\infty)}).$$

Thanks to (12), there is some positive constant K such that

$$\overline{n}(m \ge x) = \frac{K}{W(x)}, \qquad x > 0.$$

In order to compute K, we recall that for every $\lambda > 0$, if $\mathbf{e}_\lambda$ is an independent exponential r.v. with parameter λ, then

$$\overline{n}(V > \mathbf{e}_\lambda) = \kappa(\lambda, 0) = \frac{\lambda}{\psi^{-1}(\lambda)}. \tag{14}$$

But on the other hand,

$$\begin{aligned}\overline{n}(V > \mathbf{e}_\lambda) &= \lim_{\varepsilon \downarrow 0} \overline{n}(m \ge \varepsilon)\mathbb{P}_{-\varepsilon}(T_{[0,\infty)} > \mathbf{e}_\lambda)\\ &= \lim_{\varepsilon \downarrow 0} \frac{K}{W(\varepsilon)} \mathbb{P}_0(S_{\mathbf{e}_\lambda} < \varepsilon).\end{aligned}$$

Referring for example to [5, p.158],

$$\mathbb{P}_0(S_{\mathbf{e}_\lambda} \in dx) = \frac{\lambda}{\psi^{-1}(\lambda)} W^{(\lambda)}(dx) - \lambda W^{(\lambda)}(x)dx.$$

Hence thanks to (13)

$$\overline{n}(V > \mathbf{e}_\lambda) = \lim_{\varepsilon \downarrow 0} \frac{K}{W^{(0)}(\varepsilon)} \frac{\lambda}{\psi^{-1}(\lambda)} W^{(\lambda)}(\varepsilon) = K \frac{\lambda}{\psi^{-1}(\lambda)},$$

and we conclude from (14) that $K = 1$. □

Lemma 11. *Let σ_ε stand for the last passage time below ε*

$$\sigma_\varepsilon \doteq \sup\{s \geq 0 : X_s < \varepsilon\}, \qquad \varepsilon > 0.$$

Then for any $t > 0$ and any bounded $\mathcal{F}_t$-measurable Θ,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}_0^\uparrow(\Theta, t < \sigma_\varepsilon) = N_0{}^\natural(\Theta, t < V).$$

Proof. Thanks to (9), we know that for any $0 \leq \varepsilon \leq x$,

$$\mathbb{P}_x^\uparrow(I_\infty \leq \varepsilon) = \frac{h(\varepsilon)}{h(x)} \mathrm{e}^{-\xi(x-\varepsilon)}.$$

Since $h(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0+$,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}_0^\uparrow(\Theta, t < \sigma_\varepsilon) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}_0^\uparrow(\Theta, I_\infty \circ \theta_t \leq \varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}_0^\uparrow(\Theta, \frac{h(\varepsilon)}{h(X_t)} \mathrm{e}^{-\xi(X_t - \varepsilon)}) \\
&= N_0(\Theta, t < V, \mathrm{e}^{-\xi X_t}),
\end{aligned}$$

the last equality stemming from the following absolute continuity relationship (see [10])

$$\mathbb{P}_0^\uparrow(\Theta) = N_0(h(X_t), \Theta, t < V), \qquad \Theta \in \mathcal{F}_t. \tag{15}$$

According to Lemma VII.7(ii) in [4], the law of X killed upon reaching $-x$ ($x > 0$), is the same under $\mathbb{P}_0^\natural$ as under $\mathbb{P}_0(\cdot \mid T_{-x} < \infty)$, and it is easy to deduce that $N_0{}^\natural = N_0(\cdot, V < \infty)$. Hence we conclude as follows thanks to the Markov property under N_0

$$\begin{aligned}
N_0{}^\natural(\Theta, t < V) &= N_0(\Theta, t < V, \mathbb{P}_{X_t}(I_\infty < 0)) \\
&= N_0(\Theta, t < V, \mathrm{e}^{-\xi X_t}),
\end{aligned}$$

and the proof is complete. □

We now are able to give the

Proof of Lemma 9.

(i) According to Theorem 1 in [22], if $\mathbf{e}_\lambda$ stands for some independent exponential r.v. with parameter $\lambda > 0$, then

$$\overline{n}(\hat{\epsilon}_0 \in dr, V < \mathbf{e}_\lambda) = \Lambda(r,\infty)\frac{\kappa(\lambda,0)}{\lambda}\mathbb{P}_0(-I_{\mathbf{e}_\lambda} \in dr), \qquad r > 0.$$

But $-I_{\mathbf{e}_\lambda}$ has an exponential distribution with parameter $\psi^{-1}(\lambda)$ under $\mathbb{P}_0$, thus recalling that $\kappa(\lambda,0) = \lambda/\psi^{-1}(\lambda)$, and $\psi^{-1}(0) = \xi$, letting $\lambda \to 0+$ yields

$$\overline{n}(\hat{\epsilon}_0 \in dr) = \Lambda(r,\infty)\mathrm{e}^{-\xi r}dr, \qquad r > 0.$$

(ii) When $\beta = 0$, $\hat{\overline{n}}(\epsilon_0 = 0) = 0$ and Corollary 1 in [2] asserts that for any positive r, under $\hat{\overline{n}}(\cdot \mid \epsilon_0 = r)$, ϵ has the law of X killed upon reaching 0 under $\mathbb{P}_r(\cdot \mid T_0 < \infty)$. The result follows once again from Lemma VII.7(ii) in [4], that is $\hat{\overline{n}}(\cdot \mid \epsilon_0 = r) = N_r^\natural$. When $\beta > 0$, $\hat{\overline{n}}(\epsilon_0 = 0) = \infty$ but the arguments developed in the prooves of Lemma 1 and Corollary 1 in [2] still apply. Hence we have

$$\hat{\overline{n}}(\cdot, \epsilon_0 \neq 0) = \int_{(0,\infty)} \hat{\overline{n}}(\epsilon_0 \in dr) N_r^\natural.$$

(iii) We have to prove that when $\beta > 0$, $\nu = \beta N_0{}^\natural$, where we wrote

$$\nu = \hat{\overline{n}}(\cdot, \epsilon_0 = 0).$$

According to Theorem 4.1 in [10], the law of $(-X_{(T_{[0,\infty)}-t)-}, t \le T_{[0,\infty)})$ under $\mathbb{P}_{-\varepsilon}(\cdot \mid X_{T_{[0,\infty)}} = 0)$ is $\mathbb{P}_0^\uparrow \circ \mathrm{k}_{\sigma_\varepsilon}$, where k stands for the killing operator. Hence thanks to Lemma 10,

$$\begin{aligned}
\nu(\Theta, t < V) &= \lim_{\varepsilon\downarrow 0} \hat{\overline{n}}(\Theta, t < V, \epsilon_0 = 0, m \ge \varepsilon)\\
&= \lim_{\varepsilon\downarrow 0} \hat{\overline{n}}(m \ge \varepsilon)\mathbb{P}_0^\uparrow \circ \mathrm{k}_{\sigma_\varepsilon}(\Theta, t < V)\mathbb{P}_{-\varepsilon}(X_{T_{[0,\infty)}} = 0)\\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{W(\varepsilon)}\mathbb{P}_0^\uparrow(\Theta, t < \sigma_\varepsilon)\mathbb{P}_0(X_{T_{[\varepsilon,\infty)}} = \varepsilon).
\end{aligned}$$

Now

$$\mathbb{P}_0(X_{T_{[\varepsilon,\infty)}} = \varepsilon) = \mathbb{P}_0(\exists t : S_{L_t^{-1}} = \varepsilon),$$

and $(S_{L_t^{-1}}, t \ge 0)$, is a subordinator with Laplace exponent $\lambda \mapsto \kappa(0,\lambda) = \psi(\lambda)/(\lambda-\xi)$. Setting $\pi(dr) = \int_0^\infty dy\, \mathrm{e}^{-\xi y}\Lambda(y+dr)$, an easy calculation provides the identity

$$\frac{\psi(\lambda)}{\lambda - \xi} = \beta\lambda + \int_0^\infty \pi(dr)\,(1 - \mathrm{e}^{-\lambda r}), \qquad \lambda > 0.$$

It is known that (such) a subordinator with positive drift hits a fixed point ε with positive probability $v(\varepsilon)$, and that $\lim_{\varepsilon\downarrow 0} v(\varepsilon) = 1$ (see Theorem III.5 in [4]). Hence we conclude thanks to Lemma 11 and the estimate (13). □

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