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Asymptotically exact nonparametric hypothesis testing in sup-norm and at a fixed point

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Abstract. For the signal in Gaussian white noise model we consider the problem of testing the hypothesis $H_0 : \ f \equiv 0,$ (the signal f is zero) against the nonparametric alternative $H_1 : \ f \in \Lambda_\varepsilon$ where Λ_ε is a set of functions on $\mathbf{R}^1$ of the form $\Lambda_\varepsilon = \{f : \ f \in \mathcal{F}, \varphi(f) \geq C\psi_\varepsilon\}$. Here $\mathcal{F}$ is a Hölder or Sobolev class of functions, $\varphi(f)$ is either the sup-norm of f or the value of f at a fixed point, $C > 0$ is a constant, ψ_ε is the minimax rate of testing and $\varepsilon \to 0$ is the asymptotic parameter of the model. We find exact separation constants $C^* > 0$ such that a test with the given summarized asymptotic errors of first and second type is possible for $C > C^*$ and is not possible for $C < C^*$. We propose asymptotically minimax test statistics.

1. Introduction

Consider the stochastic process $Y(t)$ defined on $[0, 1]$ and satisfying the stochastic differential equation

$$dY(t) = f(t)dt + \varepsilon dW(t) \ , \tag{1}$$

where $W(t)$ is the standard Wiener process on $[0, 1]$, f is an unknown real-valued function and $0 < \varepsilon < 1$.

Suppose that f is defined on the whole real line $\mathbf{R}^1$. Given the observation $\{Y(t), \ 0 \leq t \leq 1\}$, consider the problem of testing the simple hypothesis

$$H_0 : \quad f(t) = 0, \quad \forall\, t \in [0, 1] \ ,$$

against the nonparametric alternative

$$H_1 : f \in \Lambda_\varepsilon \ .$$

Here Λ_ε is a set of functions on $\mathbf{R}^1$ of the form

$$\Lambda_\varepsilon = \ \{f : \ f \in \mathcal{F}, \ \varphi(f) \geq C\psi_\varepsilon\} \ ,$$

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where $\mathscr{F}$ is some functional class, $\varphi(f)$ is a functional on $\mathscr{F}$ (usually a distance between f and the zero function), $C > 0$ is a constant and ψ_ε is some positive function of ε tending to 0 as ε tends to 0.

In words, the set of alternatives Λ_ε is the set of functions separated from 0 by the φ-distance $C\psi_\varepsilon$ and belonging to some functional class $\mathscr{F}$. The set Λ_ε is defined by four parameters: $\mathscr{F}$, $\varphi(\cdot)$, C and ψ_ε. However, it can be shown that, given $\mathscr{F}$ and $\varphi(\cdot)$, the constant C and the function ψ_ε cannot be chosen in an arbitrary way, if one wants to have a statistically meaningful setup. This fact was first noticed by Yuri Ingster who indicated the optimal choice of C and ψ_ε for different classes $\mathscr{F}$ and functionals $\varphi(\cdot)$ (see the survey of Ingster (1993)). In such a choice only the product value $C\psi_\varepsilon$ is important. (Introducing the two parameters C and ψ_ε is conventional and follows the tradition to consider separately the "rate" ψ_ε and the "constant" C.) It turns out that, if $C\psi_\varepsilon$ is too small, then it is not possible to test the hypothesis H_0 against H_1 with the given summarized errors of the first and second type. On the other hand, if $C\psi_\varepsilon$ is very large, such a testing is possible. The problem is to find the smallest threshold value $C\psi_\varepsilon$ for which such a testing is still possible, and to indicate the corresponding test. Let us give the precise definitions.

Let T_ε be a test statistic, i.e. an arbitrary function with zero-one values, being measurable w.r.t $\{Y(t),\ 0 \le t \le 1\}$. The value $T_\varepsilon = 0$ means that H_0 is accepted, and $T_\varepsilon = 1$ means that H_0 is rejected.

We measure the error of the test T_ε by the summarized probability of errors of first and second type

$$r_\varepsilon(C, \psi_\varepsilon, T_\varepsilon) = \mathbf{P}_0(T_\varepsilon = 1) + \sup_{f \in \Lambda_\varepsilon} \mathbf{P}_f(T_\varepsilon = 0) \ ,$$

where $\mathbf{P}_f$ is the probability measure generated by $\{Y(t),\ 0 \le t \le 1\}$, when the signal in (1) is f. The error r_ε depends on C and ψ_ε, since Λ_ε does.

Fix a number $0 < \gamma < 1$. The function ψ_ε is called **minimax rate of testing** (MRT) if the following two conditions hold:

there exists a constant $C_* > 0$ such that for every $C < C_*$ we have

$$\liminf_{\varepsilon \to 0} \inf_{T_\varepsilon} r_\varepsilon(C, \psi_\varepsilon, T_\varepsilon) \ge \gamma \ , \tag{2}$$

where $\inf_{T_\varepsilon}$ denotes infimum over all test statistics,

there exist a constant C^* and a test statistic T_ε^* such that

$$\limsup_{\varepsilon \to 0} r_\varepsilon(C, \psi_\varepsilon, T_\varepsilon^*) \le \gamma \tag{3}$$

for each $C > C^*$.

Thus the MRT ψ_ε is such that a meaningful test of H_0 is impossible if the distance between the null hypothesis and the alternative is smaller than $C_*\psi_\varepsilon$, and that such a test is possible if this distance is greater than $C^*\psi_\varepsilon$. The MRT ψ_ε is not unique: it is defined up to an arbitrary positive scaling factor. In the following we fix natural scaling factors for MRT (corresponding to the "minimal writing length" of the expression for ψ_ε), and thus avoid the non-uniqueness. Clearly, $C^* \ge C_*$ and an interesting question is, whether in some cases $C^* = C_*$? If (2) and (3) are satisfied with $C^* = C_*$, then C^* is called **exact separation constant** (ESC) and T^*_ε is called **asymptotically minimax test statistic**.

The study of MRT was initiated by Ingster (1982) who found such rates for the case where $\mathcal{F}$ is an ellipsoid in $\mathbf{L}_2[0,1]$ and $\varphi(\cdot)$ is the $\mathbf{L}_2[0,1]$ norm. Ermakov (1990) obtained the ESC and the asymptotically minimax test for this setup. Ingster (1987, 1993) derived the MRT for the setup with Sobolev classes $\mathcal{F}$ and $\mathbf{L}_p[0,1]$ norms $\varphi(\cdot),\ 2 \le p \le \infty$. For the most complete survey on this subject as well as for the similar problem of testing hypotheses on probability densities see Ingster (1993). Except the cited example of ellipsoids in $\mathbf{L}_2$ with the $\mathbf{L}_2$-norm $\varphi(\cdot)$, the ESC are obtained for a number of problems, where the class $\mathcal{F}$ and the norm $\varphi(\cdot)$ are defined in a coordinate form (ellipsoids in $\mathbf{l}_p$: Ingster (1990, 1993), Suslina (1993); coordinate Besov bodies: Ingster and Suslina (1995)). For the classes $\mathcal{F}$ defined in functional form (such as usual Hölder or Sobolev classes), and when $\varphi(\cdot)$ is the $\mathbf{L}_p$ norm, much less is known about the exact asymptotics. To our knowledge, such asymptotics is available only for the case of Hölder classes with smoothness parameter less than 1 and the $\mathbf{L}_\infty$-norm $\varphi(\cdot)$ (Lepski (1993)).

Here we consider the problem of nonparametric hypothesis testing where $\mathcal{F}$ is a Hölder or Sobolev class of functions. Let $\beta > 0,\ L > 0$, and $1 \le p \le \infty$ be given. Denote

$$\mathcal{P} = \{(\beta, p) : \text{either } p = \infty, \beta > 0, \text{or } 1 \le p < \infty, \beta \in \{1, 2, \ldots\}, \beta p > 1\} \ .$$

Consider the class $\mathcal{F} = \mathcal{F}(\beta, L, p),\ (\beta, p) \in \mathcal{P},\ L > 0$, defined as follows.

If $1 \le p < \infty,\ \beta \in \{1, 2, ...\},\ \beta p > 1$, then

$$\mathcal{F}(\beta, L, p) = \left\{ f :\ f^{(\beta-1)} \text{ is absolutely continuous and } \|f^{(\beta)}\|_p \le L \right\} \ .$$

If $p = \infty, \beta > 0$, then

$$\mathcal{F}(\beta, L, p) = \left\{ f :\ |f^{(l)}(x) - f^{(l)}(x')| \le L|x - x'|^{\beta - l}, \forall\, x, x' \in \mathbf{R}^1 \right\} \ ,$$

where $l = \lfloor \beta \rfloor$ is the maximal integer that is strictly less than β.

Here and later

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}, \quad 1 \le p < \infty \ ,$$

and $f^{(k)}$ denotes the k-th derivative of f.

In words, for $1 \le p < \infty$ the classes $\mathcal{F}(\beta, L, p)$ are Sobolev classes, and for $p = \infty$ they represent the Hölder classes. Everywhere in the sequel it is tacitly assumed that β belongs to the set of integers $\{1, 2, ...\}$, whenever $p < \infty$.

In this paper we find ESC and construct the asymptotically minimax tests for two different problems.

In the first problem $\varphi(f)$ is the supremum norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| \ ,$$

and we study the following set Λ_ε of alternatives:

$$\Lambda_\varepsilon^\infty(C) = \left\{ f : f \in \mathcal{F}(\beta, L, p), \ \|f\|_\infty \geq C(\varepsilon^2 \ln(1/\varepsilon))^{\alpha/(2\alpha+1)} \right\} ,$$

where

$$\alpha = \beta - 1/p .$$

It is known from Ingster (1987) that if $p = \infty$ the MRT for this problem is

$$\psi_{\varepsilon,\infty} = (\varepsilon^2 \ln(1/\varepsilon))^{\beta/(2\beta+1)} .$$

We show that the MRT for $1 \leq p < \infty$ is

$$\psi_{\varepsilon,\infty} = (\varepsilon^2 \ln(1/\varepsilon))^{\alpha/(2\alpha+1)} ,$$

where $\alpha = \beta - 1/p$, and we find the ESC.

In the second problem $\varphi(\cdot)$ is the pseudo-distance at a fixed point $t_0 \in (0, 1)$: $\varphi(f) = f(t_0)$, and we consider the set of alternatives

$$\Lambda_\varepsilon^0(C) = \left\{ f : f \in \mathcal{F}(\beta, L, p), \ f(t_0) \geq C\varepsilon^{2\alpha/(2\alpha+1)} \right\} .$$

We show that the MRT for this problem is

$$\psi_{\varepsilon,0} = \varepsilon^{2\alpha/(2\alpha+1)} ,$$

and we find the ESC.

In the case $p = \infty$, $0 < \beta \leq 1$, the explicit expressions for ESC and for asymptotically minimax tests are obtained by Lepski (1993). Here we give the solution for the case $(\beta, p) \in \mathcal{P}$, and thus extend the result of Lepski (1993) to the whole scale of classes $\mathcal{F}(\beta, L, p)$. The solution (constants and tests) is expressed in terms of certain optimization problems related to optimal recovery (see Donoho (1994a, b), Korostelev (1996), Lepski and Spokoiny (1997), Leonov (1997)). For some interesting examples the solutions are given explicitly.

2. Main results

Introduce the semi-norm

$$\rho_{\beta,p}(f) = \begin{cases} \|f^{(\beta)}\|_p & \text{if } 1 \leq p < \infty, \\ & \quad \beta \in \{1, 2, \ldots\}, \ \beta p > 1, \\ \sup\limits_{x,x' \in \mathbf{R}^1, x \neq x'} \dfrac{|f^{(l)}(x) - f^{(l)}(x')|}{|x - x'|^{\beta - l}} & \text{if } p = \infty, \ \beta > 0 . \end{cases}$$

Then

$$\mathcal{F}(\beta, L, p) = \left\{ f : \rho_{\beta,p}(f) \leq L \right\} .$$

Consider the following optimization problem

$$\begin{aligned} &\max g(0), \ \ subject\ to \\ &\|g\|_2 \le 1, \\ &\rho_{\beta,p}(g) \le 1 \ . \end{aligned} \tag{4}$$

It is well-known that (4) has a solution (see e.g. Arestov (1989)). Denote this solution g^* (clearly, g^* depends on β, p, but we drop this dependence in the notation, assuming here and later that a pair (β, p) is fixed). Some examples of solutions g^* are given in Section 5. The constant $g^*(0)$ is the value of the problem (4).

The relation between the problems of the type (4) and nonparametric estimation has been pointed out by Donoho and Low (1992), Donoho (1994a, b). They showed that for estimation of f in (1) at a fixed point or in sup-norm the linear minimax estimators on the class $\mathcal{F}(\beta, L, p)$ can be found in the form of kernel estimators, where the kernels are properly renormalized solutions of (4). An important point to understand these results is the fact that (up to a normalization) g^* is at the same time the solution of the optimal recovery problem, i.e. the problem of minimax reconstruction of f in the deterministic model associated to (1): $Y(t) = f(t) + \varepsilon Z(t)$ where Z is a non-random "noise" function with $\|Z\|_2 \le 1$ (see Micchelli and Rivlin (1977) and earlier results cited in Arestov (1989)). This analogy is particularly useful for the case of sup-norm loss, where the stochastic term of the estimation error is asymptotically degenerate, and the reduction to the deterministic model is natural. For the loss at a fixed point, however, there is no such a transparent heuristics.

Our results go further and may be interpreted as establishing similar link between optimal recovery (or, equivalently, the problem (4)) and nonparametric testing. Our tests are based on estimators, and the above "degeneracy heuristics" works perfectly, at least in the case of the sup-norm. Furthermore, we show that for testing the situation is more favorable than for estimation: the tests based on optimal recovery attain not only the rate but also the exact minimax constant. Recall that in estimation problems the optimal recovery argument does not give constants, except for the unique example of estimation in sup-norm on the Hölder class $\mathcal{F}(\beta, L, \infty)$ (Korostelev (1993), Donoho (1994a)). This has probably not only technical but intrinsic origin, and linear minimax methods do not attain optimal constants in most of estimation problems on the classes $\mathcal{F}(\beta, L, p)$. However, as we show below, the tests based on linear minimax estimators attain optimal constants in testing.

Let us start with the first problem: testing in the supremum norm.

Theorem 1. *Let the set of alternatives be* $\Lambda_\varepsilon = \Lambda_\varepsilon^\infty(C)$ *and let* $(\beta, p) \in \mathcal{P}$, $\psi_\varepsilon = (\varepsilon^2 \ln(1/\varepsilon))^{\alpha/(2\alpha+1)}$, *where* $\alpha = \beta - 1/p$.

Then the ESC has the form

$$C_\infty^* = \ g^*(0) L^{\frac{1}{2\alpha+1}} \left(\frac{4}{2\alpha+1}\right)^{\frac{\alpha}{2\alpha+1}} \ .$$

Moreover, for each $C < C_\infty^*$ *we have*

$$\liminf_{\varepsilon\to 0} \inf_{T_\varepsilon} \left(\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f\in\Lambda_\varepsilon^\infty(C)} \mathbf{P}_f(T_\varepsilon = 0) \right) \ge 1 \ . \tag{5}$$

Remark 1. *Note that the constant* C^*_∞ *does not depend on* γ *and also, in view of (5), the inequality (2) is satisfied for all* $0 < \gamma < 1$, *provided* $\Lambda_\varepsilon = \Lambda^\infty_\varepsilon(C),\ C < C^*_\infty$. *This reflects the fact that, for testing in sup-norm, the limiting distribution of the asymptotically minimax test statistics* T^*_ε *is degenerate.*

The asymptotically minimax test statistic T^*_ε for the case of sup-norm will be denoted T^∞_ε. To define T^∞_ε consider the following restricted analog of the optimization problem (4):

$$\begin{aligned} &\max g(0),\ \textit{subject to} \\ &\int_{-D}^{D} g^2(x)dx \le 1\ , \\ &\rho_{\beta,p}(g) \le 1\ , \end{aligned} \tag{6}$$

where $D > 0$ is a constant. Denote $A = [-D, D]$. As shown in Section 4 below, all the solutions $g^*_A(x)$ of the problem (6) are such that $g^*_A(x)I\{x \in A\}$, their restriction to A, is unique, and $0 < \int_{-D}^{D} g^*_A < \infty$. (Here and later $I\{\cdot\}$ denotes the indicator function.)

Set

$$K_A(x) = \frac{g^*_A(x)I\{x \in A\}}{\int_{-D}^{D} g^*_A(u)du}\ ,$$

and consider the kernel estimator

$$f_\varepsilon(t) = \frac{1}{h_\varepsilon}\int_0^1 K_A\left(\frac{u-t}{h_\varepsilon}\right)dY(u),\ t \in (0,1)\ .$$

Note that the restricted optimization problem (6) is introduced here in place of (4) for technical reasons. In fact, there are no results guaranteeing that g^*, the solution of (4), is integrable and satisfies $\int g^* \neq 0$ in general case. If we knew this, we could replace K_A by $K^* = g^*/\int g^*$ everywhere. In some particular examples where g^* is given explicitly (see Section 5) this is possible, and we do not make a restriction to $[-D, D]$.

Denote $f_{\varepsilon,\infty}(t) = f_\varepsilon(t)$, if $h_\varepsilon = h_{\varepsilon,\infty}$, where

$$h_{\varepsilon,\infty} = L^{-\frac{2}{2\alpha+1}}\left(\frac{4}{2\alpha+1}\right)^{\frac{1}{2\alpha+1}}(\varepsilon^2\ln(1/\varepsilon))^{\frac{1}{2\alpha+1}},\quad \alpha = \beta - 1/p\ .$$

For boundary modification of f_ε, introduce the one-sided kernels $K_- : [0,1] \to \mathbf{R}^1$ and $K_+ : [-1,0] \to \mathbf{R}^1$ defined as

$$K_-(u) = \sum_{j=0}^{l} p_j(0)p_j(u),\ K_+(u) = K_-(-u)\ ,$$

where $l = \lfloor\beta\rfloor$, and $p_0, .., p_l$ are the first $l+1$ orthonormal Legendre polynomials on $[0,1]$. It is easy to see that

$$\operatorname{supp}(K_-) = [0,1], \quad \operatorname{supp}(K_+) = [-1,0],$$
$$\int_0^1 K_-(u)u^j du = \int_{-1}^0 K_+(u)u^j du = 0, \quad j = 1,\dots,l, \tag{7}$$
$$\int_0^1 K_-(u)du = \int_{-1}^0 K_+(u)du = 1 \ .$$

The choice of these particular kernels K_- and K_+ for boundary correction is not crucial. Higher order boundary kernels, as well as other boundary correction procedures ensuring proper rates can be used. The contribution of boundary terms is of smaller order than the main term in the asymptotics of the test statistic.

Set $s_\varepsilon = \varepsilon^{2/(2\alpha+1)}$,

$$f^*_{\varepsilon,\infty}(t) = \begin{cases} f_{\varepsilon,\infty}(t), & t \in [Dh_{\varepsilon,\infty}, 1 - Dh_{\varepsilon,\infty}], \\ \dfrac{1}{s_\varepsilon}\displaystyle\int_0^1 K_-\left(\frac{u-t}{s_\varepsilon}\right)dY(u), & t \in [0, Dh_{\varepsilon,\infty}), \\ \dfrac{1}{s_\varepsilon}\displaystyle\int_0^1 K_+\left(\frac{u-t}{s_\varepsilon}\right)dY(u), & t \in (1 - Dh_{\varepsilon,\infty}, 1] \ . \end{cases} \tag{8}$$

Now, the test statistic T_ε^∞ is defined by

$$T_\varepsilon^\infty = \begin{cases} 0, & \|f^*_{\varepsilon,\infty}\|_\infty < (1+\delta_\varepsilon)Q_\infty(\varepsilon^2 \ln(1/\varepsilon))^{\frac{\alpha}{2\alpha+1}}, \\ 1, & \text{otherwise} \ . \end{cases} \tag{9}$$

Here $Q_\infty = \|K_A\|_2 L^{\frac{1}{2\alpha+1}} \left(\frac{4}{2\alpha+1}\right)^{\frac{\alpha}{2\alpha+1}}$, $\delta_\varepsilon = 1/\sqrt{\ln\ln(1/\varepsilon)}$.

Theorem 2. *Let* $(\beta, p) \in \mathcal{P}$ *and* $\psi_\varepsilon = (\varepsilon^2 \ln(1/\varepsilon))^{\frac{\alpha}{2\alpha+1}}$, $\alpha = \beta - 1/p$. *Assume that* $D = D_\varepsilon \to \infty$ *and* $D_\varepsilon = \mathcal{O}\left((\ln(1/\varepsilon))^a\right)$, *as* $\varepsilon \to 0$, *for some* $a > 0$.

Then the statistic T_ε^∞ *defined in (9) satisfies*

$$\limsup_{\varepsilon\to 0}\left(\mathbf{P}_0(T_\varepsilon^\infty = 1) + \sup_{f\in\Lambda_\varepsilon^\infty(C)} \mathbf{P}_f(T_\varepsilon^\infty = 0)\right) = 0 \ , \tag{10}$$

for each $C > C^*_\infty$.

Theorem 2 shows that for the test statistic T_ε^∞ the upper bound (3) on the error holds with any γ. This is related to the fact that for the sup-norm, appearing in (9), the asymptotic distribution is degenerate. Note also that relations (5) and (10) together yield the first statement of Theorem 1. Thus we get the following:

Corollary 1. *Under the conditions of Theorem 2 the test statistic* T_ε^∞ *defined in (9) is asymptotically minimax for testing in sup-norm, for any* $0 < \gamma < 1$.

The case $0 < \beta \leq 1$, $p = \infty$ was considered by Lepski (1993). The ESC obtained by Lepski (1993) is the corresponding particular case of the constant C^*_∞ defined in Theorem 1.

Consider now the second problem: testing at a fixed point t_0. The asymptotic behavior of the minimax test for this problem is not degenerate, and the test depends on the chosen level γ.

Denote q_γ the $(1-\gamma)$-quantile of the standard normal distribution: $\Phi(q_\gamma) = 1-\gamma$, where $\Phi(\cdot)$ is the c.d.f. of the normal $\mathcal{N}(0,1)$ random variable. Define

$$h_{\varepsilon,0} = L^{-\frac{2}{2\alpha+1}} (2q_{\gamma/2})^{\frac{2}{2\alpha+1}} \varepsilon^{\frac{2}{2\alpha+1}} \quad ,$$

and consider the test statistic

$$T_\varepsilon^0 = \begin{cases} 0 & \text{if } f_{\varepsilon,0}(t_0) < Q_0 \varepsilon^{\frac{2\alpha}{2\alpha+1}}, \\ 1 & \text{if } f_{\varepsilon,0}(t_0) \ge Q_0 \varepsilon^{\frac{2\alpha}{2\alpha+1}}, \end{cases}$$

where $f_{\varepsilon,0}(t)$ stands for the estimator $f_\varepsilon(t)$, with $h_\varepsilon = h_{\varepsilon,0}$, and

$$Q_0 = (L/2)^{\frac{1}{2\alpha+1}} \left(q_{\gamma/2}\right)^{\frac{2\alpha}{2\alpha+1}} \|K_A\|_2 \ .$$

Theorem 3. *Let the set of alternatives be $\Lambda_\varepsilon = \Lambda_\varepsilon^0(C)$, and let $(\beta, p) \in \mathcal{P}$ and $\psi_\varepsilon = \varepsilon^{\frac{2\alpha}{2\alpha+1}}$, where $\alpha = \beta - 1/p$.*

Then the ESC has the form

$$C_0^* = g^*(0) L^{\frac{1}{2\alpha+1}} \left(2q_{\gamma/2}\right)^{\frac{2\alpha}{2\alpha+1}} \quad .$$

Moreover,

$$\liminf_{\varepsilon\to 0} \inf_{T_\varepsilon} \left(\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f\in\Lambda_\varepsilon^0(C_0^*)} \mathbf{P}_f(T_\varepsilon = 0) \right) \ge \gamma \ , \tag{11}$$

and, if $D_\varepsilon \to \infty$, $D_\varepsilon h_{\varepsilon,0} \to 0$, as $\varepsilon \to 0$, the test statistic T_ε^0 is asymptotically minimax:

$$\limsup_{\varepsilon\to 0} \left(\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f\in\Lambda_\varepsilon^0(C)} \mathbf{P}_f(T_\varepsilon = 0) \right) \le \gamma \ , \tag{12}$$

for all $C > C_0^*$.

Observe that (11) implies

$$\liminf_{\varepsilon\to 0} \inf_{T_\varepsilon} \left(\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f\in\Lambda_\varepsilon^0(C)} \mathbf{P}_f(T_\varepsilon = 0) \right) \ge \gamma \ ,$$

for any $C < C_0^*$, since in this case $\Lambda_\varepsilon^0(C) \supset \Lambda_\varepsilon^0(C_0^*)$, and thus (2) is satisfied.

Remark 2. *The results of this section can be easily extended to the case of the simple hypothesis $H_0 : f \equiv c$ where c is a given constant, or $H_0 : f(t) = f_0(t)$ where $f_0(t)$ is a given function that is smoother than every $f \in \mathcal{F}$ (if $\mathcal{F} = \mathcal{F}(\beta, L, p)$, it suffices that $f_0 \in \mathcal{F}(\beta', L, p)$ for some $\beta' > \beta$).*

The results similar to the Theorems 1, 2 and 3 can be obtained in another setup of hypothesis testing, where one fixes an upper bound $\gamma_1 \in (0, 1)$ on the error of the first type and tries to minimize the second type error under this constraint. Then it is relevant to look for a test T_ε^* that is minimax not among all tests, but among all tests T_ε of asymptotical level γ_1, i.e. such that

$$\limsup_{\varepsilon\to 0} \mathbf{P}_0(T_\varepsilon = 1) \le \gamma_1 \ .$$

Denote $\Gamma(\gamma_1)$ the class of all such tests T_ε. For this setup $C^* > 0$ is called the ESC and $T_\varepsilon^* \in \Gamma(\gamma_1)$ is called asymptotically minimax test statistic of the level γ_1 if there exists $\gamma_2 \in (0, 1)$ such that simultaneously the following two relations hold:

$$\liminf_{\varepsilon\to 0} \inf_{T_\varepsilon\in\Gamma(\gamma_1)} \sup_{f\in\Lambda_\varepsilon(C)} \mathbf{P}_f(T_\varepsilon = 0) \ge \gamma_2 \ , \tag{13}$$

for all $C < C^*$, and

$$\limsup_{\varepsilon\to 0} \sup_{f\in\Lambda_\varepsilon(C)} \mathbf{P}_f(T_\varepsilon^* = 0) \le \gamma_2 \ , \tag{14}$$

for all $C > C^*$. The value γ_2 is then the minimal asymptotical second type error.

Let us apply this definition to the two problems considered here.

For the problem of testing in supremum norm, where $\Lambda_\varepsilon(C) = \Lambda_\varepsilon^\infty(C)$, it follows from Theorems 1 and 2 that the relations (13) and (14) are satisfied with $C^* = C_\infty^*,\ T_\varepsilon^* = T_\varepsilon^\infty$ and $\gamma_2 = 1 - \gamma_1$. This is a consequence of degenerate character of the result for supremum norm.

For the second problem (testing at a fixed point) the answer is quite different. It follows from (11) and from the proof of (12) that, under assumptions of Theorem 3, the relations (13) and (14) hold with $C^* = C_0^*(2\gamma_1),\ T_\varepsilon^* = T_\varepsilon^0(2\gamma_1)$, and $\gamma_1 = \gamma_2$, where $C_0^*(2\gamma_1)$ and $T_\varepsilon^0(2\gamma_1)$ are defined as C_0^* and T_ε^0 above, with $\gamma = 2\gamma_1$.

We end this section by a comparison of the results on exact minimax testing and estimation. A standard connection between estimation and testing would suggest to reject the null hypothesis H_0 in favor of the alternative $\Lambda_\varepsilon = \{f : f \in \mathcal{F},\ \varphi(f) \ge C\psi_\varepsilon\}$ if $\varphi(\hat{f}_\varepsilon) \ge Q(\varepsilon)$ where $\hat{f}_\varepsilon$ is a good estimator of $f \in \mathcal{F}$ and $Q(\varepsilon)$ is a properly chosen threshold. Ingster (1990, 1993) has shown that, as concerns the rates of testing, this connection does not work in nonparametric situation if $\varphi(f) = \|f\|_q$, except for the case of sup-norm ($q = \infty$). This is exactly the case considered in Theorems 1 and 2, and the critical region of the optimal test procedure (9) is of the form $\varphi(f_{\varepsilon,\infty}^*) \ge Q(\varepsilon)$. Note that here $f_{\varepsilon,\infty}^*$ is a good estimator not only in rate, but also in constant, as soon as we consider the unique case where constants are available for estimation (Hölder classes, i.e. $p = \infty$, see Korostelev (1993), Donoho (1994a)). Thus, the above mentioned connection between estimation and testing works perfectly (even in constants) for this particular case $\mathcal{F} = \mathcal{F}(\beta, L, \infty), \varphi(f) = \|f\|_\infty$. For other combinations of $\varphi(f)$ and $\mathcal{F}$ considered in this paper such a comparison is not possible since the exact constants for the associated estimation problems are not available.

3. Preliminary lemmas

In this section we give some auxiliary results about the properties of solutions to the problems of the type (6), the asymptotic behavior of suprema of Gaussian processes and the boundary kernels.

3.1. Properties of solutions

Let $\rho(g)$ be a convex functional defined on the space of functions $g : \mathbf{R}^1 \to \mathbf{R}^1$ and satisfying the following two conditions.

(i) The functional $\rho(\cdot)$ is convex, nonnegative and symmetric, i.e. $\rho(-g) = \rho(g)$, for all g such that $\rho(g) < \infty$.

(ii) The functional $\rho(\cdot)$ is **renormalizable** with exponent $\alpha > 0$, i.e. for every g, such that $\rho(g) < \infty$, one has

$$\rho(ag(bt) + c) = ab^{\alpha}\rho(g(t)), \quad \forall\, a \geq 0,\ b > 0,\ c \in \mathbf{R}^1 \ .$$

The notion of renormalization was introduced in the context of nonparametric estimation by Donoho and Low (1992). Note, that the functionals $\rho = \rho_{\beta,p}$ satisfy (i) and they are renormalizable with exponent $\alpha = \beta - 1/p$. In particular, for the Hölder case ($p = \infty$) we have $\alpha = \beta$.

Let us study the following optimization problem

$$\begin{aligned} &\max g(0),\ \textit{subject to} \\ &\int_{-D}^{D} g^2(x)dx \leq 1, \\ &\rho(g) \leq 1 \ . \end{aligned} \tag{15}$$

This is a generalization of (6).

The next lemma can be deduced from the results on optimal recovery (see e.g. Gabushin (1970), Micchelli and Rivlin (1977), Arestov (1989)). Some versions of it, in a more particular context, appeared recently in statistical literature (Donoho (1994), Korostelev (1996), Lepski and Spokoiny (1997)). For convenience, we state the lemma in the form adapted to our purposes. We give a simple self-contained proof, which does not refer to the theory of optimal recovery.

Denote

$$G_A = \left\{ g : \mathbf{R}^1 \to \mathbf{R}^1 :\ \int_{-D}^{D} g^2(x)dx \leq 1,\ \rho(g) \leq 1 \right\},\ A = [-D, D] \ .$$

Lemma 1. *Let the conditions (i) and (ii) hold and let $|g(0)| < \infty$ for each $g \in G_A$. Then the following properties are valid.*

(L1) The problem (15) has a solution g_A^*.

(L2) Any solution of (15) is attained on the boundary of G_A, i.e.

$$\int_{-D}^{D} (g_A^*(x))^2 dx = 1 \ . \tag{16}$$

(L3) The restriction to $[-D, D]$ *of any solution* g_A^* *is unique a.e. with respect to the Lebesgue measure on* $[-D, D]$.

(L4)

$$g_A^*(0) > 0 \ . \tag{17}$$

(L5) For each f*, such that* $\rho(f) \le 1$ *and* $f(0) = g_A^*(0)$*, we have*

$$\int_{-D}^{D} f g_A^* \ge \int_{-D}^{D} (g_A^*)^2 = 1 \ , \tag{18}$$

and

$$g_A^*(0) \int_{-D}^{D} g_A^* \ge 1 \ . \tag{19}$$

(L6)

$$g_A^*(0) = \|K_A\|_2 + B_A(\rho) \ , \tag{20}$$

where

$$K_A(u) = \frac{g_A^*(u)}{\int_{-D}^{D} g_A^*} I\{u \in [-D, D]\} \ ,$$

and

$$B_A(\rho) = \sup_{g:\ \rho(g) \le 1} \left| \int_{-\infty}^{\infty} K_A(u)(g(u) - g(0))du \right| \ .$$

Proof of Lemma 1.

(L1). Consider g as an element of the weighted $\mathbf{L}_2$-space, with the indicator weight $I\{x \in A\}$. This space is reflexive, and the set G_A is bounded in this space. Moreover, since the functionals $\int_{-D}^{D} g^2$ and $\rho(g)$ are convex, the set G_A is weakly closed (Vainberg (1972), p. 111–112). Similarly, since the functional $g(0)$ is convex and such that $|g(0)| < \infty$, $\forall g \in G_A$, it is weakly upper semi-continuous. Finally, by the generalized first Weierstrass theorem (Vainberg (1972), Theorem 9.2) any weakly upper semi-continuous functional attains its maximum on a bounded weakly closed subset of a reflexive Banach space.

(L2). Assume that there exists a solution g_A^* of (15) such that

$$\int_{-D}^{D} (g_A^*)^2 < 1 \ .$$

Then

$$\kappa = \left[\int_{-D}^{D} (g_A^*)^2 \right]^{-\frac{\alpha}{2\alpha+1}} > 1 \ .$$

Denote $f^*(t) = \kappa g_A^*(\kappa^{-\frac{1}{\alpha}} t)$. Then, by the renormalization property (ii),

$$\rho(f^*) = \rho(g_A^*) \le 1 \ ,$$

and $f^*(0) = \kappa g_A^*(0) > g_A^*(0)$. Moreover,

$$\int_{-D}^{D} (f^*)^2 = \kappa \int_{-D}^{D} (g_A^*(\kappa^{-\frac{1}{\alpha}} t))^2 dt$$
$$= \kappa^{2+\frac{1}{\alpha}} \int_{-D\kappa^{-\frac{1}{\alpha}}}^{D\kappa^{-\frac{1}{\alpha}}} (g_A^*)^2 \le \kappa^{2+\frac{1}{\alpha}} \int_{-D}^{D} (g_A^*)^2 = 1 \ .$$

Thus, $f^* \in G_A$ and $f^*(0) > g_A^*(0)$, which contradicts the assumption that $g_A^*(0) = \max_{g \in G_A} g(0)$. The contradiction proves (16).

(L3). Assume that g_1 and g_2 are two different solutions of the problem (15). Consider the function $f = (g_1 + g_2)/2$. By convexity of ρ and of the $\mathbf{L}_2[-D, D]$-norm, we have $f \in G_A$. Also, $f(0) = (g_1(0) + g_2(0))/2 = g_A^*(0)$. Thus, f is a solution of (15). Now

$$\int_{-D}^{D} f^2 = \int_{-D}^{D} \left[\frac{g_1^2 + g_2^2}{2} - \frac{(g_1 - g_2)^2}{2} \right] = 1 - \int_{-D}^{D} \frac{(g_1 - g_2)^2}{2} \ , \tag{21}$$

where we used the fact that (by (16))

$$\int_{-D}^{D} g_1^2 = \int_{-D}^{D} g_2^2 = 1 \ .$$

If $g_1 \ne g_2$ on a set of positive Lebesgue measure in $[-D, D]$ then $\int_{-D}^{D} (g_1 - g_2)^2 > 0$, and (21) implies $\int_{-D}^{D} f^2 < 1$. Since f is a solution to (15), this contradicts $(L2)$.

(L4). Assume that $g_A^*(0) < 0$. Then for the function $f = -g_A^*$ we have $f(0) = -g_A^*(0) > 0 > g_A^*(0)$, and $f \in G_A$. This contradicts the fact that g_A^* is a solution to (15).

(L5). Denote $\mathcal{F}_0 = \{f : \rho(f) \le 1,\ f(0) = g_A^*(0)\}$. Note that

$$\int_{-D}^{D} f^2 \ge \int_{-D}^{D} (g_A^*)^2 = 1, \quad \forall f \in \mathcal{F}_0 \ . \tag{22}$$

In fact, if $\int_{-D}^{D} f^2 < 1$, then $f \in G_A$ and $f(0) = g_A^*(0)$, therefore f is a solution of (15) which is not on the boundary of G_A. This contradicts $(L2)$ and proves (22).

In particular, (22) yields that g_A^* is a minimizer of the convex functional $\int_{-D}^{D} f^2$ on the convex set $\mathcal{F}_0$. Thus, the directional derivatives of this functional at g_A^* are non-negative:

$$\lim_{t \downarrow 0} \frac{1}{t} \left[\int_{-D}^{D} (g_A^* + t(f - g_A^*))^2 - \int_{-D}^{D} (g_A^*)^2 \right] \ge 0, \quad \forall f \in \mathcal{F}_0 \ . \tag{23}$$

The inequality (23) is equivalent to (18). Finally, note that the function $f(t) \equiv g_A^*(0),\ \forall t \in \mathbf{R}^1$, belongs to $\mathcal{F}_0$. In fact, by the renormalization property (ii),

$$\rho(f) = \rho(g_A^*(0)) = \rho(0 \cdot g(t) + g_A^*(0)) = 0 \cdot \rho(g(t)) = 0 \ ,$$

where g is any function such that $\rho(g) < \infty$. By putting in (18) $f(t) \equiv g_A^*(0)$, we obtain (19).

(L6). Introduce the scalar product

$$(f, g) = \int_{-D}^{D} f(x)g(x)dx \quad .$$

By definition of K_A,

$$B_A(\rho) = \sup_{g:\rho(g)\leq 1} |(K_A, g - g(0))|$$
$$= \sup_{g\in\mathcal{G}} |(K_A, g)| = \left(\int_{-D}^{D} g_A^*\right)^{-1} \sup_{g\in\mathcal{G}} |(g_A^*, g)| \quad , \qquad (24)$$

where

$$\mathcal{G} = \{g : \mathbf{R}^1 \to \mathbf{R}^1 : \ \rho(g) \leq 1, \ g(0) = 0\} \quad .$$

For any $g \in\mathcal{G}$ the function $f(t) = g_A^*(0) - g(t)$ belongs to $\mathcal{F}_0$, since $f(0) = g_A^*(0)$ and $\rho(f) = \rho(-g) \leq 1$ (here we used (i) and (ii)). Therefore, for this function f we can apply (18), which yields

$$(g_A^*, g) \leq g_A^*(0) \int_{-D}^{D} g_A^* - 1, \quad \forall\, g \in\mathcal{G} \quad . \qquad (25)$$

The right side of (25) is nonnegative, in view of (19). Moreover, for the function $g_-(t) = g_A^*(0) - g_A^*(t) \in\mathcal{G}$ we have, using (16),

$$(g_A^*, g_-) = g_A^*(0) \int_{-D}^{D} g_A^* - 1 \quad ,$$

that is the equality in (25) is attained on $g = g_-$. Hence

$$\sup_{g\in\mathcal{G}} |(g_A^*, g)| = \sup_{g\in\mathcal{G}} (g_A^*, g) = g_A^*(0) \int_{-D}^{D} g_A^* - 1 \quad . \qquad (26)$$

Applying (26), (24), (16) and the definition of K_A, we get

$$B_A(\rho) = g_A^*(0) - \left(\int_{-D}^{D} g_A^*\right)^{-1} = g_A^*(0) - \|K_A\|_2 \quad .$$

The lemma is proved.

Remark 3. *The conditions of Lemma 1 are satisfied for $\rho = \rho_{\beta,p}$, $(\beta, p) \in \mathcal{P}$, and hence Lemma 1 holds for $\rho_{\beta,p}$. Moreover, as in this case all functions $g \in G_A$ are continuous, one can drop the words "almost everywhere" in (L3): the restriction of each solution g_A^* to $[-D, D]$ is unique.*

Next, we consider the analog of the problem (4) with a support constraint:

$$\begin{aligned} &\max g(0), \ \ subject\ to \\ &\|g\|_2 \le 1, \\ &\rho_{\beta,p}(g) \le 1, \\ &supp(g) \subseteq [-d, d] \ \ , \end{aligned} \tag{27}$$

where $d > 0$ is a fixed number. For the same reason as before, the problem (27) has a solution that will be denoted $\bar{g}_d(\cdot)$. The following lemma states that the values of the problems (6) and (27) approach the value of the problem (4) as the sizes of supports d and D tend to ∞.

Lemma 2. *If* $(\beta, p) \in \mathcal{P}$*, then*

$$g^*(0) \ge \bar{g}_d(0), \ \ \bar{g}_d(0) \to g^*(0), \quad as\ d \to \infty \ \ ; \tag{28}$$

$$g^*(0) \le g_A^*(0), \ \ g_A^*(0) \to g^*(0), \quad as\ D \to \infty \ \ . \tag{29}$$

This lemma is established by the methods of Donoho and Low (1992, Theorem 3), see also Donoho (1994).

Next, we need to characterize the smoothness properties of the kernel K_A in $\mathbf{L}_2(\mathbf{R}^1)$.

Lemma 3. *If* $(\beta, p) \in \mathcal{P}$*, then there exists a constant* $c_1 > 0$ *such that*

$$\int_{-\infty}^{\infty} (K_A(t+u) - K_A(u))^2 du \le c_1 |t|, \quad \forall\, t \in \mathbf{R}^1 \ \ . \tag{30}$$

Proof of Lemma 3. By definition, $K_A(x) = g_A^*(x) / \int_{-D}^{D} g_A^*$ for $x \in (-D, D)$, K_A possibly has jumps at points $-D, D$ and $K_A(x) = 0, \ x \notin [-D, D]$. Let $t > 0$ (the case $t < 0$ is quite analogous). Suppose that $t \le 2D$, since for $t > 2D$ the supports of $K_A(t + \cdot)$ and $K_A(\cdot)$ are non-overlapping and (30) is trivial. We have for $0 < t \le 2D$,

$$\begin{aligned} &\int_{-\infty}^{\infty} (K_A(t+u) - K_A(u))^2 du \\ &\quad = \left(\int_{(-D-t,-D]} + \int_{(-D,D-t]} + \int_{(D-t,D]} \right) (K_A(t+u) - K_A(u))^2 du \ \ . \end{aligned} \tag{31}$$

Here

$$\left(\int_{(-D-t,-D]} + \int_{(D-t,D]} \right) (K_A(t+u) - K_A(u))^2 du \le 8 K_{max}^2 t \ \ , \tag{32}$$

where $K_{max} = \max_x |K_A(x)|$. To estimate the second integral in the right hand side of (31) use the fact that $K_A(t+u) - K_A(u) = (g_A^*(t+u) - g_A^*(u))/\int_{-D}^D g_A^*$ for $u \in (-D, D-t)$. Hence,

$$\int_{(-D,D-t]} (K_A(t+u) - K_A(u))^2 du$$
$$\leq \left(\int_{-D}^{D} g_A^*\right)^{-2} \int_{-\infty}^{\infty} (g_A^*(t+u) - g_A^*(u))^2 du \ , \qquad (33)$$

Note that $g_A^* \in \mathcal{F}(\beta, L, p)$, $(\beta, p) \in \mathcal{P}$. For $(\beta, p) \in \mathcal{P}$ we have $\beta p > 1$, and thus, by embedding theorems for Sobolev and Besov spaces (see e.g. Triebel (1992)) one gets $g_A^* \in B_{\infty,\infty}^{\beta-1/p}(\mathbf{R}^1) \subset B_{2,\infty}^{\beta-1/p+1/2}(\mathbf{R}^1) \subset B_{2,\infty}^{1/2}(\mathbf{R}^1)$, where $B_{r,q}^s(\mathbf{R}^1)$ denotes the Besov space of functions on $\mathbf{R}^1$. This entails that the last integral in (33) does not exceed $c_2|t|$ where c_2 is a constant. This remark, together with (31)–(33), proves (30).

Finally, the following property will be used later in the proofs.

Lemma 4. *For $(\beta, p) \in \mathcal{P}$ and any $h > 0$, $D > 0$ such that $Dh < \frac{1}{2}$ we have*

$$\sup_{t\in[Dh,1-Dh]} \sup_{f\in\mathcal{F}(\beta,L,p)} \left| \frac{1}{h}\int_0^1 K_A\left(\frac{u-t}{h}\right) f(u)du - f(t)\right| \leq Lh^{\alpha} B_A(\rho_{\beta,p}) \ ,$$

where $\alpha = \beta - 1/p$.

Proof of Lemma 4. Since $supp(K_A) \subseteq [-D, D]$, we have

$$\sup_{t\in[Dh,1-Dh]} \left| \frac{1}{h}\int_0^1 K_A\left(\frac{u-t}{h}\right) f(u)du - f(t)\right|$$
$$= \sup_{t\in[Dh,1-Dh]} \left| \int_{-t/h}^{(1-t)/h} K_A(w) f(t+wh)dw - f(t)\right|$$
$$= \sup_{t\in[Dh,1-Dh]} \left| \int_{-\infty}^{\infty} K_A(w) f(t+wh)dw - f(t)\right|$$
$$\leq \sup_{t\in\mathbf{R}^1} \left| \int_{-\infty}^{\infty} K_A(w)(f(t+wh) - f(t))dw\right| \ . \qquad (34)$$

For any fixed $t \in \mathbf{R}^1$ and any $f \in \mathcal{F}(\beta, L, p)$ denote $f_1(u) = f(t+u)$. Clearly, $\rho_{\beta,p}(f_1) = \rho_{\beta,p}(f) \leq L$. For $w \in \mathbf{R}^1$ consider the function

$$g(w) = \frac{f(t+wh)}{Lh^{\alpha}} \ ,$$

If $f \in \mathcal{F}(\beta, L, p)$, then by the renormalization property (ii)

$$\rho_{\beta,p}(g) = \rho_{\beta,p}\left(\frac{f(t+wh)}{Lh^{\alpha}}\right) = \frac{1}{L}\rho_{\beta,p}(f_1) \leq 1 \ .$$

Thus, for every $t \in \mathbf{R}^1$,

$$\sup_{f \in \mathcal{F}(\beta,L,p)} \left| \int_{-\infty}^{\infty} K_A(w)(f(t+wh) - f(t))dw \right|$$
$$\leq \; Lh^{\alpha} \sup_{g:\rho_{\beta,p}(g) \leq 1} \left| \int_{-\infty}^{\infty} K_A(w)(g(w) - g(0))dw \right| = \; Lh^{\alpha} B_A(\rho_{\beta,p}) \; .$$

This, together with (34), proves the lemma.

3.2. Supremum of a Gaussian process

Let $T > 0$. On the interval $[0, T]$ define the random process

$$X(t) = \frac{1}{\sqrt{h}} \int_0^1 K\left(\frac{u-t}{h}\right) dW(u) \; ,$$

where $h > 0$, $W(\cdot)$ is the standard Wiener process on $[0, 1]$, and $K : \mathbf{R}^1 \to \mathbf{R}^1$ is a function such that $\|K\|_2 < \infty$.

Lemma 5. *Let for some $c_1 > 0$,*

$$\int_{-\infty}^{\infty} (K(t+u) - K(u))^2 du \leq c_1 |t| \; , \tag{35}$$

for $\forall\, t \in \mathbf{R}^1$. Then

$$\mathbf{P}\left\{ \sup_{t \in [0,T]} |X(t)| \geq \left(1 + \delta\left(\frac{T}{h}\right)\right) \|K\|_2 \sqrt{2 \ln\left(\frac{T}{h}\right)} \right\} \to 0 \; ,$$

as $h/T \to 0$, for any positive function $\delta(x)$ defined for $x > 0$ and such that

$$\frac{\delta(x) \ln x}{\ln \ln x} \to \infty \; ,$$

as $x \to \infty$.

We omit the proof. Results close to Lemma 5 are well known in the literature on the extrema of Gaussian processes: Pickands (1969), Konakov and Piterbarg (1983, 1984), Leadbetter, Lindgren and Rootzén (1986, Theorem 12.2.9). A direct proof of Lemma 5 can be obtained following the lines of Adler (1990, p. 119–120).

3.3. Boundary kernels

Lemma 6. *Let either $S = [0, D_\varepsilon h_{\varepsilon,\infty}]$, $K = K_-$, or $S = [1 - D_\varepsilon h_{\varepsilon,\infty}, 1]$, $K = K_+$, where $D_\varepsilon \to \infty$ as $\varepsilon \to 0$, and $D_\varepsilon = \mathcal{O}\left((\ln \frac{1}{\varepsilon})^a\right)$, for some $a > 0$.*

Then there exists a constant $c_* > 0$ such that

$$\sup_{f \in \mathcal{F}(\beta,L,p)} \sup_{t \in S} \left| \frac{1}{s_\varepsilon} \int_0^1 K\left(\frac{u-t}{s_\varepsilon}\right) f(u)du - f(t) \right| \leq c_* \varepsilon^{\frac{2\alpha}{2\alpha+1}} \; , \tag{36}$$

and

$$\mathbf{P}\left(\sup_{t\in S}\left|\frac{\varepsilon}{s_\varepsilon}\int_0^1 K\left(\frac{u-t}{s_\varepsilon}\right)dW(u)\right|\geq c_*\varepsilon^{\frac{2\alpha}{2\alpha+1}}\sqrt{\ln\ln\left(\frac{1}{\varepsilon}\right)}\right)\to 0\ , \qquad (37)$$

as $\varepsilon\to 0$. *Here* $\alpha=\beta-1/p$.

Proof of Lemma 6 . It suffices to consider the case $S=[0,D_\varepsilon h_{\varepsilon,\infty}]$, $K=K_-$. Let us prove (36). For any $f\in\mathcal{F}(\beta,L,p)$, any $t\in[0,D_\varepsilon h_{\varepsilon,\infty}]$, and $l=\lfloor\beta\rfloor$, we have

$$\begin{aligned}
&\left|\frac{1}{s_\varepsilon}\int_0^1 K_-\left(\frac{u-t}{s_\varepsilon}\right)f(u)du-f(t)\right|\\
&=\left|\int_0^1 K_-(w)\left(f(t+ws_\varepsilon)\right)dw-f(t)\right|\\
&=\left|\int_0^1 K_-(w)\left(f^{'}(t)ws_\varepsilon+\frac{f^{''}(t)}{2}(ws_\varepsilon)^2+\cdots+\frac{f^{(l)}(t+\theta w h_{1,\varepsilon})}{l!}(ws_\varepsilon)^l\right)dw\right|\\
&=s_\varepsilon^l\left|\int_0^1 K_-(w)w^l f^{(l)}(t+\theta ws_\varepsilon)dw\right|\\
&=\ s_\varepsilon^l\left|\int_0^1 K_-(w)w^l(f^{(l)}(t)-f^{(l)}(t+\theta ws_\varepsilon))dw\right|\ , \qquad (38)
\end{aligned}$$

where $0\leq\theta\leq 1$, and we used (7). If $p=\infty$, the last expression is bounded as follows

$$\left|\int_0^1 K_-(w)w^l(f^{(l)}(t)-f^{(l)}(t+\theta ws_{1,\varepsilon}))dw\right|\leq Ls_\varepsilon^{\beta-l}\int_0^1|K_-(w)w^\beta|dw\ , \qquad (39)$$

where the fact that $f\in\mathcal{F}(\beta,L,\infty)$ was used. Combining (38) and (39) we get (36) for $p=\infty$. If $1\leq p<\infty$, then β is an integer, $l=\beta-1$ and instead of (39) we obtain the following estimate

$$\begin{aligned}
&\left|\int_0^1 K_-(w)w^l(f^{(l)}(t)-f^{(l)}(t+\theta ws_\varepsilon))dw\right|\\
&\leq\int_0^1|K_-(w)w^l|\int_t^{t+ws_\varepsilon}|f^{(\beta)}(\tau)|d\tau dw\\
&\leq\left(\int_0^1|K_-(w)w^l|^q dw\right)^{\frac{1}{q}}\left(\int_0^1\left[\int_t^{t+ws_\varepsilon}|f^{(\beta)}(\tau)|d\tau\right]^p dw\right)^{\frac{1}{p}}\\
&\leq c_6\left(\int_0^1(ws_\varepsilon)^{p-1}\int_t^{t+ws_\varepsilon}|f^{(\beta)}(\tau)|^p d\tau dw\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq c_6 s_\varepsilon^{1-\frac{1}{p}} \|f^{(\beta)}\|_p \left[\int_0^1 w^{p-1} dw\right]^{\frac{1}{p}} \leq c_7 L s_\varepsilon^{1-\frac{1}{p}} \ , \tag{40}$$

where $q = p/(p-1)$, $c_6 > 0$, $c_7 > 0$ depend only on K_- and p, l. Combining (38) and (40) we obtain (36) for $p < \infty$.

Let us prove (37) for $K = K_-,\ S = [0, D_\varepsilon h_{\varepsilon,\infty}]$. Note that $K = K_-$ satisfies the condition (35): in fact, K_- is a polynomial of order l on the interval $[0, 1]$ and possibly has jumps at the endpoints 0 and 1. Using the same argument as in (31)–(33), we get (35). Thus, we can apply Lemma 5 to prove (37). In our case $T = D_\varepsilon h_{\varepsilon,\infty},\ h = s_\varepsilon$, and

$$\frac{T}{h} = \frac{D_\varepsilon h_{\varepsilon,\infty}}{s_\varepsilon} = c_8 D_\varepsilon \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2\alpha+1}} \to \infty \ ,$$

as $\varepsilon \to 0$, since $D_\varepsilon \to \infty$. Here $c_8 > 0$ is a constant. Also

$$\|K\|_2 \sqrt{2\ln\left(\frac{T}{h}\right)} = \|K\|_2 \sqrt{2\left(\ln D_\varepsilon + \frac{1}{2\alpha+1}\ln\ln\left(\frac{1}{\varepsilon}\right) + \ln c_8\right)}$$
$$\leq c_9 \sqrt{\ln\ln\left(\frac{1}{\varepsilon}\right)} \ ,$$

in view of the condition $D_\varepsilon = \mathbb{O}\left((\ln(\frac{1}{\varepsilon}))^a\right),\ \varepsilon \to 0$. Here $c_9 > 0$ is a constant. This and Lemma 5 yield (37). Lemma 6 is proved.

4. Proofs of Theorems 1–3

To prove Theorems 1 and 2, it suffices to show (5) and (10) under the conditions of these theorems respectively.

Proof of (5). Fix β, p and a positive constant $C < C^*_\infty$. Write for convenience $C = (1-\delta)C^*_\infty,\ 0 < \delta < 1$. Let $d > 0$ be such that $\bar{g}_d(0) > g^*(0)(1-\delta/2)$, where $\bar{g}_d$ is the solution of the problem (27) and g^* is the solution of (4). Such a value d exists in view of (28). Denote

$$h(C) = \left(\frac{C}{L\bar{g}_d(0)}\right)^{\frac{1}{\alpha}} \left(\varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2\alpha+1}} ,$$
$$M = \left\lfloor \frac{1}{2dh(C)} \right\rfloor - 1,\ x_k = (2k-1)dh(C), \quad k = 1, \ldots, M,$$
$$G(x) = L h^\alpha(C) \bar{g}_d\left(\frac{x}{h(C)}\right), \quad \alpha = \beta - 1/p,$$
$$f_k(x) = G(x - x_k), \quad k = 1, \ldots, M \ .$$

Since the functional $\rho_{\beta,p}$ is renormalizable with exponent α, we get

$$\rho_{\beta,p}(f_k) = \rho_{\beta,p}(G) = L\rho_p(\bar{g}_d) \leq L \ .$$

Hence, $f_k \in \mathscr{F}(\beta, L, p),\ k = 1, ..., M$. Moreover,

$$\|f_k\|_\infty \geq |f_k(x_k)| = Lh^\alpha(C)\bar{g}_d(0) = C\left(\varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{\alpha}{2\alpha+1}} \quad .$$

Thus,

$$f_k \in \Lambda_\varepsilon^\infty(C), \quad k = 1, ..., M \quad . \tag{41}$$

Note also that $\sigma^2 = \|f_k\|_2^2$ does not depend on k, and

$$\begin{aligned}
\sigma^2 =&\ L^2 h^{2\alpha}(C) \int \bar{g}_d^2\left(\frac{x}{h(C)}\right) dx \\
=&\ L^2 h^{2\alpha+1}(C)\|\bar{g}_d\|_2^2 \leq L^2\left(\frac{C}{L\bar{g}_d(0)}\right)^{\frac{2\alpha+1}{\alpha}} \varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) \\
=&\ L^2\left(\frac{(1-\delta)C_\infty^*}{L\bar{g}_d(0)}\right)^{\frac{2\alpha+1}{\alpha}} \varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) \\
=&\ \frac{4}{2\alpha+1}(1-\delta)^{\frac{2\alpha+1}{\alpha}}\left(\frac{g^*(0)}{\bar{g}_d(0)}\right)^{\frac{2\alpha+1}{\alpha}} \varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) \\
\leq&\ \frac{4}{2\alpha+1}\left(1-\frac{\delta}{2}\right)^{\frac{2\alpha+1}{\alpha}} \varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) \leq \frac{4}{2\alpha+1}\left(1-\frac{\delta}{2}\right)\varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) \quad ,
\end{aligned} \tag{42}$$

where we used the inequalities $\|\bar{g}_d\|_2^2 \leq 1$ and $g^*(0)/\bar{g}_d(0) < (1-\delta/2)^{-1}$. In view of (41), we have, for an arbitrary test statistic T_ε,

$$\begin{aligned}
\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f \in \Lambda_\varepsilon^\infty(C)} \mathbf{P}_f(T_\varepsilon = 0) &\geq \mathbf{P}_0(T_\varepsilon = 1) + \frac{1}{M}\sum_{k=1}^M \mathbf{P}_k(T_\varepsilon = 0) \\
&\geq \mathbf{E}_0\left(I(T_\varepsilon = 1) + I(T_\varepsilon = 0)Z\right) \\
&\geq (1-\delta)\mathbf{P}_0(Z \geq 1-\delta) \quad ,
\end{aligned} \tag{43}$$

where $\mathbf{P}_k = \mathbf{P}_{f_k}$, for brevity, and

$$Z = \frac{1}{M}\sum_{k=1}^M \frac{\mathbf{dP}_k}{\mathbf{dP}_0} \quad .$$

Now

$$\begin{aligned}
\mathbf{P}_0(|Z-1| \geq \delta) &= P\left(\left|\frac{1}{M}\sum_{k=1}^M \xi_k - 1\right| \geq \delta\right) \\
&= P\left(\left|\frac{1}{M}\sum_{k=1}^M (\xi_k - \mathbf{E}(\xi_k))\right| \geq \delta\right) \quad .
\end{aligned} \tag{44}$$

where

$$\xi_k = \exp\left(\varepsilon^{-1}\sigma\zeta_k - \frac{\varepsilon^{-2}\sigma^2}{2}\right) ,$$

and $\zeta_k,\ k = 1, ..., M$, are normal $\mathcal{N}(0, 1)$ random variables that are independent since the supports of f_k and f_j are non-overlapping for $k \neq j$. Obviously, $\mathbf{E}(\xi_k) = 1$. Using (44) and the Bahr-Esseen inequality for the moments of sums of independent random variables (Petrov (1995, p. 82)) we find that for all $0 < \nu < 1$

$$\mathbf{P}_0(|Z - 1| \geq \delta) \leq \frac{c_{10}\mathbf{E}|\xi_k|^{1+\nu}}{\delta^{1+\nu}M^{\nu}} , \tag{45}$$

where $c_{10} > 0$ is a constant that depends on ν only. Direct calculation gives

$$\mathbf{E}|\xi_k|^{1+\nu} = \exp\left(\frac{\varepsilon^{-2}\sigma^2}{2}(\nu + \nu^2)\right) \leq \exp\left(\frac{2(\nu + \nu^2)}{2\alpha + 1}\left(1 - \frac{\delta}{2}\right)\ln\frac{1}{\varepsilon}\right) ,$$

where we applied (42). Using this and (45) and choosing $\nu = \delta/4$ we obtain

$$\mathbf{P}_0(|Z - 1| \geq \delta) \leq c_{11}\varepsilon^q\left(\ln\frac{1}{\varepsilon}\right)^{\frac{\nu}{2\alpha+1}} ,$$

where $c_{11} > 0$ and

$$q = \frac{\delta^2}{8(2\alpha + 1)}\left(1 + \frac{\delta}{2}\right) > 0 .$$

Thus

$$\mathbf{P}_0(|Z - 1| \geq \delta) \to 0, \quad as\ \varepsilon \to 0 . \tag{46}$$

Since the probability in (46) does not depend on T_ε, we deduce (5) from (43) and (46).

Proof of (10). Here we put $\psi_\varepsilon = \left(\varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{\alpha}{2\alpha+1}}$, $\alpha = \beta - 1/p$, $\delta_\varepsilon = \frac{1}{\sqrt{\ln\ln\frac{1}{\varepsilon}}}$.

We have

$$\begin{aligned}
\mathbf{P}_0(T_\varepsilon^\infty = 1) &= \mathbf{P}_0\left(\|f^*_{\varepsilon,\infty}\|_\infty \geq (1 + \delta_\varepsilon)Q_\infty\psi_\varepsilon\right) \\
&\leq \mathbf{P}\left(\sup_{t\in[0,Dh_{\varepsilon,\infty}]}\left|\frac{\varepsilon}{s_\varepsilon}\int_0^1 K_-\left(\frac{u-t}{s_\varepsilon}\right)dW(u)\right| \geq \frac{\delta_\varepsilon}{4}Q_\infty\psi_\varepsilon\right) \\
&\quad + \mathbf{P}\left(\sup_{t\in[1-Dh_{\varepsilon,\infty},1]}\left|\frac{\varepsilon}{s_\varepsilon}\int_0^1 K_+\left(\frac{u-t}{s_\varepsilon}\right)dW(u)\right| \geq \frac{\delta_\varepsilon}{4}Q_\infty\psi_\varepsilon\right) \\
&\quad + \mathbf{P}\left(\sup_{t\in[Dh_{\varepsilon,\infty},1-Dh_{\varepsilon,\infty}]}\left|\frac{\varepsilon}{h_{\varepsilon,\infty}}\int_0^1 K_A\left(\frac{u-t}{h_{\varepsilon,\infty}}\right)dW(u)\right|\right. \\
&\quad \left.\geq \left(1 - \frac{\delta_\varepsilon}{2}\right)(1 + \delta_\varepsilon)Q_\infty\psi_\varepsilon\right) .
\end{aligned} \tag{47}$$

Note that $\frac{\delta_\varepsilon}{4} Q_\infty \psi_\varepsilon \ \geq \ c_* \varepsilon^{\frac{2\alpha}{2\alpha+1}} \sqrt{\ln\ln\left(\frac{1}{\varepsilon}\right)}$ for ε small enough, where c_* is the constant in Lemma 6. On the other hand, for ε small enough,

$$\begin{aligned}
&\left(1-\frac{\delta_\varepsilon}{2}\right)(1+\delta_\varepsilon)\, Q_\infty \psi_\varepsilon \\
&\geq \left(1+\frac{\delta_\varepsilon}{4}\right) Q_\infty \psi_\varepsilon \\
&\geq \left(1+\frac{\delta_\varepsilon}{4}\right) \|K_A\|_2 \frac{\varepsilon}{\sqrt{h_{\varepsilon,\infty}}} \sqrt{2\ln\left(\frac{1}{h_{\varepsilon,\infty}}\right)} \left(1+\mathbb{O}\left(\sqrt{\frac{\ln\ln(1/\varepsilon)}{\ln(1/\varepsilon)}}\right)\right) \\
&\geq \left(1+\frac{\delta_\varepsilon}{8}\right) \|K_A\|_2 \frac{\varepsilon}{\sqrt{h_{\varepsilon,\infty}}} \sqrt{2\ln\left(\frac{T}{h_{\varepsilon,\infty}}\right)}\ ,
\end{aligned}$$

where $T = 1 - 2Dh_{\varepsilon,\infty}$. Using these remarks and evaluating the last three probabilities in (47) by means of Lemma 5 and (37), we find

$$\mathbf{P}_0(T_\varepsilon^\infty = 1) = \ o(1), \quad as\ \varepsilon \to 0\ . \tag{48}$$

Let us show that for $C = C_\infty^*(1+\delta),\ \forall\, \delta > 0$,

$$\sup_{f\in\Lambda_\varepsilon^\infty(C)} \mathbf{P}_f(T_\varepsilon^\infty = 0) = \ o(1), \quad as\ \varepsilon \to 0\ . \tag{49}$$

Together (48) and (49) yield (10). To prove (49) fix $f \in \Lambda_\varepsilon^\infty(C)$, and denote $\bar{t}$ the minimal number in [0, 1], such that

$$|f(\bar{t})| = \max_{t\in[0,1]} |f(t)|\ .$$

Introduce the random variable

$$\xi = \mathbf{E}_f(f_{\varepsilon,\infty}^*(\bar{t})) - f_{\varepsilon,\infty}^*(\bar{t})\ .$$

Using the fact that $f \in \Lambda_\varepsilon^\infty(C)$, we get

$$\begin{aligned}
\|f_{\varepsilon,\infty}^*\|_\infty &\geq |f_{\varepsilon,\infty}^*(\bar{t})| \geq |\mathbf{E}_f(f_{\varepsilon,\infty}^*(\bar{t}))| - |\xi| \\
&\geq C\psi_\varepsilon - |f(\bar{t})| + |\mathbf{E}_f(f_{\varepsilon,\infty}^*(\bar{t}))| - |\xi| \\
&\geq C\psi_\varepsilon - |f(\bar{t}) - \mathbf{E}_f(f_{\varepsilon,\infty}^*(\bar{t}))| - |\xi| \\
&\geq C\psi_\varepsilon - \sup_{t\in[0,1]} |\mathbf{E}_f(f_{\varepsilon,\infty}^*(t)) - f(t)| - |\xi|\ .
\end{aligned} \tag{50}$$

Now, Lemma 4 and (36) yield

$$\sup_{t\in[0,1]} |\mathbf{E}_f(f_{\varepsilon,\infty}^*(t)) - f(t)| \leq Lh_{\varepsilon,\infty}^\alpha B_A(\rho_{\beta,p}) + 2c_*\varepsilon^{\frac{2\alpha}{2\alpha+1}}\ .$$

This and (50) imply

$$\|f_{\varepsilon,\infty}^*\|_\infty \geq C\psi_\varepsilon - |\xi| - Lh_{\varepsilon,\infty}^\alpha B_A(\rho_{\beta,p}) - 2c_*\varepsilon^{\frac{2\alpha}{2\alpha+1}}\ .$$

Therefore,

$$\mathbf{P}_f(T_\varepsilon^\infty = 0) = \mathbf{P}_f(\|f^*_{\varepsilon,\infty}\|_\infty \le (1+\delta_\varepsilon)Q_\infty\psi_\varepsilon) \le \mathbf{P}_f(|\xi| \ge \Delta_\varepsilon) \ , \quad (51)$$

where

$$\Delta_\varepsilon = C\psi_\varepsilon - Lh^\alpha_{\varepsilon,\infty}B_A(\rho_{\beta,p}) - 2c_*\varepsilon^{\frac{2\alpha}{2\alpha+1}} - (1+\delta_\varepsilon)Q_\infty\psi_\varepsilon \ .$$

Using (20), (29), the definition of Q_∞, C^*_∞ and the formula $C = (1+\delta)C^*_\infty$, one obtains

$$\begin{aligned}
\Delta_\varepsilon &= C\psi_\varepsilon - Lh^\alpha_{\varepsilon,\infty}B_A(\rho_{\beta,p}) - Q_\infty\psi_\varepsilon + o(\psi_\varepsilon)\\
&= \delta C^*_\infty\psi_\varepsilon + \psi_\varepsilon L^{\frac{1}{2\alpha+1}}\left(\frac{4}{2\alpha+1}\right)^{\frac{\alpha}{2\alpha+1}}\left[g^*(0) - B_A(\rho_{\beta,p}) - \|K_A\|_2\right] + o(\psi_\varepsilon)\\
&= \delta C^*_\infty\psi_\varepsilon + \psi_\varepsilon L^{\frac{1}{2\alpha+1}}\left(\frac{4}{2\alpha+1}\right)^{\frac{\alpha}{2\alpha+1}}\left[g^*(0) - g_A(0)\right] + o(\psi_\varepsilon)\\
&= (\delta + o(1))C^*_\infty\psi_\varepsilon, \quad as\ \varepsilon \to 0 \ .
\end{aligned}$$

Note that ξ is a Gaussian zero-mean random variable, and its variance does not depend of f. It is easy to see that

$$\mathbf{E}(\xi^2) = \mathcal{O}\left(\frac{\varepsilon^2}{s_\varepsilon}\right) = \mathcal{O}\left(\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right), \quad as\ \varepsilon \to 0 \ .$$

Thus,

$$\sup_{f\in\Lambda^\infty_\varepsilon(C)} \mathbf{P}_f(|\xi| \ge \Delta_\varepsilon) \to 0, \quad as\ \varepsilon \to 0 \ . \quad (52)$$

Together (51) and (52), prove (49). Therefore, (10) follows.

Proof of Theorem 3. We prove in turn (11) and (12).

Proof of (11). Fix β and p. Let $d > 0$ be large enough, as in the proof of (5). Denote

$$\tau = \left(\frac{C^*_0}{L\bar{g}_d(0)}\right)^{\frac{1}{\alpha}}, \quad h = \tau\varepsilon^{\frac{2}{2\alpha+1}},$$
$$f_1(x) = Lh^\alpha\bar{g}_d\left(\frac{x-t_0}{h}\right), \quad \alpha = \beta - 1/p \ .$$

Clearly, $f_1 \in \mathcal{F}(\beta, L, p)$ and

$$f_1(t_0) = Lh^\alpha\bar{g}_d(0) = C^*_0\varepsilon^{\frac{2\alpha}{2\alpha+1}} \ .$$

Hence, $f_1 \in \Lambda^0_\varepsilon(C^*_0)$, and for any test statistic T_ε we get

$$\begin{aligned}
\mathbf{P}_0(T_\varepsilon = 1) + \sup_{f\in\Lambda^0_\varepsilon(C^*_0)} \mathbf{P}_f(T_\varepsilon = 0) &\ge \mathbf{P}_0(T_\varepsilon = 1) + \mathbf{P}_1(T_\varepsilon = 0)\\
&\ge \mathbf{P}_0(T^b_\varepsilon = 1) + \mathbf{P}_1(T^b_\varepsilon = 0) \ , \quad (53)
\end{aligned}$$

where $\mathbf{P}_1 = \mathbf{P}_{f_1}$, and T_ε^b is the Bayesian decision rule, i.e.

$$T_\varepsilon^b = \begin{cases} 1 & \text{if } \frac{\mathbf{dP}_1}{\mathbf{dP}_0} \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} \mathbf{P}_0(T_\varepsilon^b = 1) &= \mathbf{P}_0\left(\ln \frac{\mathbf{dP}_1}{\mathbf{dP}_0} \geq 0\right) \\ &= \mathbf{P}\left(\varepsilon^{-1}\sigma_0\xi_0 - \frac{1}{2}\varepsilon^{-2}\sigma_0^2 \geq 0\right) = \mathbf{P}\left(\xi_0 \geq \frac{1}{2}\varepsilon^{-1}\sigma_0\right) \quad , \quad (54) \end{aligned}$$

where $\xi_0 \sim \mathcal{N}(0,1)$ and $\sigma_0 > 0$ is defined by

$$\begin{aligned} {\sigma_0}^2 &= \int_0^1 f_1^2(x)dx = L^2h^{2\alpha}\int_0^1 \bar{g}_d^2\left(\frac{x-t_0}{h}\right)dx \\ &= L^2h^{2\alpha+1}\|\bar{g}_d\|_2^2 \leq L^2h^{2\alpha+1} = L^2\tau^{2\alpha+1}\varepsilon^2 \quad , \quad (55) \end{aligned}$$

if $\varepsilon > 0$ is so small that $(t_0 - dh, t_0 + dh) \subset [0,1]$ (observe that $supp\left[\bar{g}_d\left(\frac{x-t_0}{h}\right)\right] \subseteq (t_0 - dh, t_0 + dh)$). From (54) and (55) we find that, for ε small enough,

$$\mathbf{P}_0(T_\varepsilon^b = 1) \geq \mathbf{P}\left(\xi_0 \geq \frac{L}{2}\tau^{\frac{2\alpha+1}{2}}\right) \quad . \quad (56)$$

Quite similarly, for ε small enough,

$$\begin{aligned} \mathbf{P}_1(T_\varepsilon^b = 0) &= \mathbf{P}_1\left(\ln \frac{\mathbf{dP}_1}{\mathbf{dP}_0} < 0\right) \\ &= \mathbf{P}\left(\varepsilon^{-1}\sigma_0\xi_0 + \frac{1}{2}\varepsilon^{-2}{\sigma_0}^2 < 0\right) \geq \mathbf{P}\left(\xi_0 < -\frac{L}{2}\tau^{\frac{2\alpha+1}{2}}\right) \quad . \quad (57) \end{aligned}$$

It follows from (56) and (57) that

$$\begin{aligned} \liminf_{\varepsilon\to 0}\left[\mathbf{P}_0(T_\varepsilon^b = 1) + \mathbf{P}_1(T_\varepsilon^b = 0)\right] &\geq \mathbf{P}\left(|\xi_0| > \frac{L}{2}\tau^{\frac{2\alpha+1}{2}}\right) \\ &= \mathbf{P}\left(|\xi_0| > q_{\frac{\gamma}{2}}\left(\frac{\bar{g}_d(0)}{g^*(0)}\right)^{\frac{2\alpha}{2\alpha+1}}\right) \\ &= 2\left(1 - \Phi\left(q_{\frac{\gamma}{2}}\left(\frac{\bar{g}_d(0)}{g^*(0)}\right)^{\frac{2\alpha}{2\alpha+1}}\right)\right) \quad , \end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal c.d.f. Using Lemma 2, we get that the last expression tends to $2(1 - \Phi\left(q_{\frac{\gamma}{2}}\right)) = \gamma$, as $d \to \infty$. This and (53) prove (11).

Proof of (12). Assume that $\varepsilon > 0$ is small enough to have $(t_0 - Dh_{\varepsilon,0}, t_0 + Dh_{\varepsilon,0}) \subset (0,1)$. Then $f_{\varepsilon,0}(t_0)$ is a normal random variable with mean

$$\mathbf{E}_f\left(f_{\varepsilon,0}(t_0)\right) = \frac{1}{h_{\varepsilon,0}}\int_{-\infty}^{\infty} K_A\left(\frac{u-t_0}{h_{\varepsilon,0}}\right) f(u)du \quad , \quad (58)$$

and variance

$$\mathbf{Var}_f(f_{\varepsilon,0}(t_0)) = \frac{1}{h_{\varepsilon,0}^2}\mathbf{E}\left[\left(\varepsilon\int_0^1 K_A\left(\frac{u-t_0}{h_{\varepsilon,0}}\right)dW(u)\right)^2\right]$$
$$= \varepsilon^2 h_{\varepsilon,0}^{-1}\|K_A\|_2^2 \ . \tag{59}$$

Applying (58) and (59) in the case $f \equiv 0$, we get

$$\mathbf{P}_0(T_\varepsilon^0 = 1) = \mathbf{P}_0\left(f_{\varepsilon,0}(t_0) \geq Q_0\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)$$
$$= \mathbf{P}_0\left(\frac{f_{\varepsilon,0}(t_0) - \mathbf{E}_0(f_{\varepsilon,0}(t_0))}{\sqrt{\mathbf{Var}_0(f_{\varepsilon,0}(t_0))}} \geq \frac{Q_0\varepsilon^{\frac{2\alpha}{2\alpha+1}}}{\varepsilon\|K_A\|_2}\sqrt{h_{\varepsilon,0}}\right)$$
$$= \mathbf{P}\left(\xi_0 \geq q_{\frac{\gamma}{2}}\right) = \frac{\gamma}{2} \ , \tag{60}$$

where $\xi_0 \sim \mathcal{N}(0,1)$. Next, in view of Lemma 4, we get

$$|\mathbf{E}_f(f_{\varepsilon,0}(t_0)) - f(t_0)| \leq L h_{\varepsilon,0}^\alpha B_A(\rho_{\beta,p}) \ ,$$

for any $f \in \mathcal{F}(\beta, L, p)$. Using this and (59), we find that, for all $f \in \Lambda_\varepsilon^0(C)$,

$$\mathbf{P}_f(T_\varepsilon^0 = 0)$$
$$= \mathbf{P}_f\left(f_{\varepsilon,0}(t_0) < Q_0\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)$$
$$= \mathbf{P}_f\left([f_{\varepsilon,0}(t_0) - \mathbf{E}_f(f_{\varepsilon,0}(t_0))] + [\mathbf{E}_f(f_{\varepsilon,0}(t_0)) - f(t_0)] < Q_0\varepsilon^{\frac{2\alpha}{2\alpha+1}} - f(t_0)\right)$$
$$\leq \mathbf{P}_f\left(f_{\varepsilon,0}(t_0) - \mathbf{E}_f(f_{\varepsilon,0}(t_0)) < L h_{\varepsilon,0}^\alpha B_A(\rho_{\beta,p}) + (Q_0 - C)\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)$$
$$= \mathbf{P}_f\left(\frac{f_{\varepsilon,0}(t_0) - \mathbf{E}_f(f_{\varepsilon,0}(t_0))}{\sqrt{\mathbf{Var}_f(f_{\varepsilon,0}(t_0))}} < \frac{\sqrt{h_{\varepsilon,0}}\left(L h_{\varepsilon,0}^\alpha B_A(\rho_{\beta,p}) + (Q_0 - C)\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)}{\varepsilon\|K_A\|_2}\right)$$
$$= \mathbf{P}\left(\xi_0 < \varepsilon^{-1}\sqrt{h_{\varepsilon,0}}\|K_A\|_2^{-1}\left(L h_{\varepsilon,0}^\alpha B_A(\rho_{\beta,p}) + (Q_0 - C)\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)\right) \ . \tag{61}$$

Since $C > C_0^*$, write $C = (1+\delta)C_0^*,\ \delta > 0$. Note that, in view of (20),

$$\varepsilon^{-1}\sqrt{h_{\varepsilon,0}}\left(L h_{\varepsilon,0}^\alpha B_A(\rho_{\beta,p}) + (Q_0 - (1+\delta)C_0^*)\varepsilon^{\frac{2\alpha}{2\alpha+1}}\right)$$
$$= 2q_{\frac{\gamma}{2}}\|K_A\|_2^{-1}\left(B_A(\rho_{\beta,p}) + \frac{\|K_A\|_2}{2} - (1+\delta)g^*(0)\right)$$
$$= 2q_{\frac{\gamma}{2}}\|K_A\|_2^{-1}\left(B_A(\rho_{\beta,p}) - g_A^*(0) + \frac{\|K_A\|_2}{2} + g_A^*(0) - (1+\delta)g^*(0)\right)$$
$$= -q_{\frac{\gamma}{2}} + 2q_{\frac{\gamma}{2}}\|K_A\|_2^{-1}(g_A^*(0) - (1+\delta)g^*(0)) \leq -q_{\frac{\gamma}{2}} \ , \tag{62}$$

if ε is small enough to have $g_A^*(0) < (1+\delta)g^*(0)$ (the last inequality is satisfied for ε small enough, since $D_\varepsilon \to \infty$ and (29) holds). Combining (61) and (62) we obtain

$$\limsup_{\varepsilon\to 0} \sup_{f\in\Lambda_\varepsilon^0(C)} \mathbf{P}_f(T_\varepsilon^0 = 0) \le \mathbf{P}(\xi_0 < -q_{\frac{\gamma}{2}}) = \frac{\gamma}{2} \ . \tag{63}$$

Finally, (12) follows from (60) and (63).

5. Examples and extensions

In this section we study special cases where the asymptotically minimax test statistics and the ESC can be written explicitly, the definition of test statistics $T_\varepsilon^\infty,\ T_\varepsilon^0$ can be simplified, and the results of Theorems 1 to 3 sharpened. We also discuss an extension to the Besov classes of functions.

5.1. Compactly supported solutions

Assume that the pair (β, p) is such that the solution g^* of the optimization problem (4) is compactly supported. Then there exists $D_0 > 0$ such that $g_A^* = g^*$ for $A = [-D_0, D_0]$, and for any $A = [-D, D],\ D > D_0$. Thus the condition $D \to \infty$, as $\varepsilon \to 0$ is no longer useful, and one may replace g_A^* everywhere by $g^*,\ K_A(x)$ by

$$K^*(x) = \frac{g^*(x)}{\int g^*(u)du} \ ,$$

and $B_A(\rho_{\beta,p})$ by

$$B^*(\rho_{\beta,p}) = \sup_{g\in\mathscr{F}(\beta,1,p)} \left| \int K^*(u)(g(u) - g(0))du \right| \ .$$

All the results of Sections 2, 3 and 4 remain valid with these changes. Moreover, inspection of the proof of (12) shows that, in the case of compactly supported solutions, the test statistic T_ε^0, defined with

$$Q_0 = \left(\frac{L}{2}\right)^{\frac{1}{2\alpha+1}} (q_{\frac{\gamma}{2}})^{\frac{2\alpha}{2\alpha+1}} \|K^*\|_2$$

satisfies the stronger relation

$$\limsup_{\varepsilon\to 0} \left(\mathbf{P}_0(T_\varepsilon^0 = 1) + \sup_{f\in\Lambda_\varepsilon^\infty(C_0^*)} \mathbf{P}_f(T_\varepsilon^0 = 0) \right) \le \gamma \ ,$$

i.e. $C = C_0^*$ is included in (12). (In fact, since $g_A^* = g^*$, (62) holds for $\delta = 0$ as well.)

Consider some examples.

Example 1. *Hölder classes* ($p = \infty$) *with* $0 < \beta \le 1$.

The solution g^* is easy to find explicitly (see Korostelev (1993)) and g^* is compactly supported. The exact formulas are

$$g^*(0) = \left(\frac{(2\beta+1)(\beta+1)}{4\beta^2} \right)^{\frac{\beta}{2\beta+1}} ,$$
$$g^*(t) = (g^*(0) - |t|^\beta)_+,$$
$$K^*(t) = \frac{\beta+1}{2\beta}(g^*(0))^{-\frac{\beta+1}{\beta}} g^*(t), \; \|K^*\|_2 = \frac{\beta+1}{2\beta}(g^*(0))^{-\frac{\beta+1}{\beta}} ,$$
$$C^*_\infty = g^*(0) L^{\frac{1}{2\beta+1}} \left(\frac{4}{2\beta+1} \right)^{\frac{\beta}{2\beta+1}} , C^*_0 = g^*(0) L^{\frac{1}{2\beta+1}} \left(2q_{\frac{\gamma}{2}} \right)^{\frac{2\beta}{2\beta+1}} ,$$
$$Q_\infty = \frac{\beta+1}{2\beta} \left(\frac{4}{2\beta+1} \right)^{\frac{\beta}{2\beta+1}} (g^*(0))^{-\frac{\beta+1}{\beta}} L^{\frac{1}{2\beta+1}},$$
$$Q_0 = \frac{\beta+1}{2\beta} 2^{-\frac{1}{2\beta+1}} (g^*(0))^{-\frac{\beta+1}{\beta}} L^{\frac{1}{2\beta+1}} \left(q_{\frac{\gamma}{2}} \right)^{\frac{2\beta}{2\beta+1}} \quad .$$

The above expressions for ESC C^*_∞ and C^*_0 are due to Lepski (1993).

Example 2. *Hölder classes* ($p = \infty$) *with* $\beta > 1$.

Leonov (1997) shows that the solution g^* of (4) is compactly supported for all $\beta > 1$. Note that for $\beta = 2$ this fact was first proved by Fuller (1960) whose study of the problem (4) was motivated by applications in optimal control. Then it was rediscovered by Gabushin (1968) in a slightly more general framework. The case $\beta = 2$ is, to our knowledge, the only example among Hölder classes with $\beta > 1$ where the explicit solution of (4) is currently available.

The solution for $\beta = 2$ is expressed as follows (Fuller (1960), see also Leonov (1997)). Let

$$q = \frac{1}{16} \left(3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}} \right)^2 \approx 0.0586,$$
$$\theta = \frac{2(23q^2 - 14q + 23)\sqrt{1+q}}{30(1 - q^{\frac{5}{2}})} \approx 1.528 \quad .$$

The constant θ is the value of the problem dual to (4). The solution $g^*(\cdot)$ of (4) is

$$g^*(t) = \theta^{-\frac{2}{5}} \phi \left(\theta^{\frac{1}{5}} t \right), \; g^*(0) = \theta^{-\frac{2}{5}} \quad ,$$

where $\phi(\cdot)$, the solution of the dual problem, is a symmetric compactly supported function with infinitely many local extrema, defined for $t \geq 0$ by

$$\phi(t) = \sum_{j=0}^{\infty} \left[(-1)^j q^j + \frac{1}{2}(-1)^{j+1}(t - t_{2j})^2 \right] I\{t \in [t_{2j-1}, t_{2j+1})\} \quad ,$$

where $t_{-1} = t_0 = 0,\ t_1 = \sqrt{1+q},\ t_{2j+1} = t_{2j} + q^{\frac{j}{2}}\sqrt{1+q},\ t_{2j} = 2\sqrt{1+q}\sum_{i=0}^{j-1} q^{\frac{i}{2}}$. The support of ϕ is contained in the interval $(-D_\phi, D_\phi)$, where

$$D_\phi = \lim_{j\to\infty} t_{2j} = 2\sqrt{1+q}(1-\sqrt{q})^{-1} \approx 2.71 \ .$$

Denote

$$I_\phi = \int \phi(t)dt = 2(1-q)\sqrt{1+q}/(1+q^{3/2}) \approx 1.91 \ .$$

Then

$$K^*(t) = I_\phi^{-1}\theta^{\frac{1}{5}}\phi(\theta^{\frac{1}{5}}t),\ \|K^*\|_2 = I_\phi^{-1} \ ,$$

and the exact constants for the problem of hypothesis testing are

$$C_\infty^* = \left(\frac{4}{5\theta}\right)^{\frac{2}{5}} L^{\frac{1}{5}}, \quad C_0^* = \left(\frac{4}{\theta}\right)^{\frac{2}{5}} L^{\frac{1}{5}}(q_{\frac{\gamma}{2}})^{\frac{4}{5}},$$
$$Q_\infty = \left(\frac{4}{5}\right)^{\frac{2}{5}} L^{\frac{1}{5}} I_\phi^{-1}, \quad Q_0 = \left(\frac{1}{2}\right)^{\frac{1}{5}} L^{\frac{1}{5}}(q_{\frac{\gamma}{2}})^{\frac{4}{5}} I_\phi^{-1} \ .$$

Example 3. *Sobolev classes with* $\beta = 1,\ p > 2$.

The solution g^* of (4) was obtained by Sz.-Nagy (1941). It has a compact support:

$$g^*(t) = g^*(0)(1-|bt|)_+^{\frac{p}{p-2}} \ ,$$

where

$$g^*(0) = \left(\frac{3p-2}{2p}\right)^{\frac{p}{3p-2}}\ ,\ \ b = \frac{2p-4}{3p-2}(g^*(0))^2 \ .$$

Other exact values for this example are

$$K^*(t) = \frac{2p-2}{3p-2}(g^*(0))^2(1-|bt|)_+^{\frac{p}{p-2}},$$
$$\|K^*\|_2 = 2(p-1)(3p-2)^{-\frac{2p-2}{3p-2}}(2p)^{-\frac{p}{3p-2}},$$
$$C_\infty^* = 2^{\frac{p-1}{3p-2}}\left(\frac{3p-2}{2p}\right) L^{\frac{p}{3p-2}},\ \ C_0 = g^*(0)(2q_{\frac{\gamma}{2}})^{\frac{2p-2}{3p-2}} L^{\frac{p}{3p-2}},$$
$$Q_\infty = \|K^*\|_2\left(\frac{4p}{3p-2}\right)^{\frac{p-1}{3p-2}} L^{\frac{p}{3p-2}},\ \ Q_0 = 2^{-\frac{p}{3p-2}}\|K^*\|_2\left(q_{\frac{\gamma}{2}}\right)^{\frac{2p-2}{3p-2}} L^{\frac{p}{3p-2}} \ .$$

Note that the rates of convergence in this example are $\psi_\varepsilon = \varepsilon^{\frac{2p-2}{3p-2}}$ for testing at a fixed point, and $\psi_\varepsilon = \left(\varepsilon^2 \ln \frac{1}{\varepsilon}\right)^{\frac{p-1}{3p-2}}$ for testing in sup-norm.

5.2. Non-compactly supported solutions

If the solution of (4) is not compactly supported the definition of asymptotically minimax test statistics T_ε^∞ and T_ε^0 depends on the solution of the restricted problem (15), that, in general, should be obtained numerically. The application of our results in this situation, therefore, requires more work. There exists, however one important particular case, where the exact solution g^* of (4) is known explicitly, is not compactly supported , but the results of this paper hold, with K_A in the definition of $T_\varepsilon^\infty,\ T_\varepsilon^0$ replaced by $K^* = g^*/\int g^*$. This particular case is $p = 2$. Let us discuss it in more detail.

We add the condition that the functions f are uniformly bounded. That is, instead of the class $\mathscr{F}(\beta, L, 2)$ we consider

$$\mathscr{F}_1(\beta, L, 2) = \{f \in \mathscr{F}(\beta, L, 2)\ :\ \|f\|_\infty \le L_1\}\ ,$$

where $L_1 > 0$ is a given constant. This constant may not be known to a statistician, since the construction of the tests does not depend on L_1. The sets of alternatives are defined, respectively, as

$$\Lambda_{\varepsilon,1}^\infty(C) = \left\{f \in \Lambda_\varepsilon^\infty(C) :\ \|f\|_\infty \le L_1\right\},$$
$$\Lambda_{\varepsilon,1}^0(C) = \left\{f \in \Lambda_\varepsilon^0(C) :\ \|f\|_\infty \le L_1\right\}\ .$$

As shown by Taikov (1968) the solution of the problem (4) for $p = 2$ is

$$g^*(t) = \frac{\beta}{\pi}\left(\sin\frac{\pi}{2\beta}\right)^{\frac{1}{2}}(2\beta-1)^{-\frac{2\beta-1}{4\beta}} K_0\left((2\beta-1)^{\frac{1}{2\beta}} t\right)\ , \tag{64}$$

where

$$K_0(x) = \int_{-\infty}^{\infty}\frac{e^{iux}}{1+|u|^{2\beta}}du = 2\int_0^\infty\frac{\cos(ux)}{1+|u|^{2\beta}}du\ . \tag{65}$$

The function $K_0(\cdot)$ can be calculated explicitly, for example, if $\beta = 1$ then

$$K_0(x) = \pi e^{-|x|}\ ,$$

and if $\beta = 2$, we get the Silverman (1984) kernel

$$K_0(x) = \frac{\pi}{2}e^{-\frac{|x|}{\sqrt{2}}}\cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{4}\right)\ .$$

General formula for K_0 is given in Gradshteyn and Ryzhik (1980, formula 3.738). It implies, in particular, that for every β there exist positive constants $a_1 = a_1(\beta)$ and $a_2 = a_2(\beta)$ such that

$$|K_0(u)| \le a_1\exp(-a_2|u|), \quad \forall u \in \mathbf{R}^1\ . \tag{66}$$

Define

$$
\begin{aligned}
g^*(0) &= (2\beta-1)^{-\frac{2\beta-1}{4\beta}}\left(\sin\frac{\pi}{2\beta}\right)^{-\frac{1}{2}}, \\
K^*(x) &= (2\pi)^{-1}(2\beta-1)^{\frac{1}{2\beta}}K_0\left((2\beta-1)^{\frac{1}{2\beta}}x\right), \\
C^*_\infty &= g^*(0)\left(\frac{2}{\beta}\right)^{\frac{2\beta-1}{4\beta}}L^{\frac{1}{2\beta}}, \quad C^*_0 = g^*(0)\left(2q_{\frac{\gamma}{2}}\right)^{\frac{2\beta-1}{2\beta}}L^{\frac{1}{2\beta}}, \qquad (67)\\
Q_\infty &= \frac{1}{2\beta}\left(\frac{2}{\beta}\right)^{\frac{2\beta-1}{4\beta}}L^{\frac{1}{2\beta}}(2\beta-1)^{\frac{2\beta+1}{4\beta}}\left(\sin\frac{\pi}{2\beta}\right)^{-\frac{1}{2}}, \\
Q_0 &= \frac{1}{2\beta}2^{-\frac{1}{2\beta}}(2\beta-1)^{\frac{2\beta+1}{4\beta}}L^{\frac{1}{2\beta}}\left(\sin\frac{\pi}{2\beta}\right)^{-\frac{1}{2}}\left(q_{\frac{\gamma}{2}}\right)^{\frac{2\beta-1}{2\beta}},
\end{aligned}
$$

where K_0 is defined in (65). Note that $g^*(0)$ is the value of the function (64) at $t=0$, and $K^* = g^*/\int g^*$, where g^* is the function (64). The constants of the type (67) are obtained for the corresponding minimax estimation problem by Tsybakov (1998).

Let $T^\infty_{\varepsilon,1},\ T^0_{\varepsilon,1}$ be the test statistics defined in the same way as $T^\infty_\varepsilon,\ T^0_\varepsilon$, but with replacing K_A by K^*, and such that $K^*,\ Q_\infty$ and Q_0 are defined by (67). Note that, in the construction of $T^\infty_{\varepsilon,1}$, we still need the parameter D_ε, but only for boundary correction: it defines the length of boundary intervals (cf. (8)).

Theorem 4. **I.** *Let* $p=2,\ \beta\in\{1,2...,\},\ \psi_\varepsilon=\left(\varepsilon^2\ln\frac{1}{\varepsilon}\right)^{\frac{2\beta-1}{4\beta}}$. *Let* $D_\varepsilon=\left(\ln\frac{1}{\varepsilon}\right)^a$, *for some* $a>1$. *Then*

$$
\limsup_{\varepsilon\to 0}\left(\mathbf{P}_0(T^\infty_{\varepsilon,1}=1)+\sup_{f\in\Lambda^\infty_{\varepsilon,1}(C)}\mathbf{P}_f(T^\infty_{\varepsilon,1}=0)\right)=0, \quad \forall C>C^*_\infty\ ,
$$

where C^*_∞ *is defined in (67). Moreover,*

$$
\liminf_{\varepsilon\to 0}\inf_{T_\varepsilon}\left(\mathbf{P}_0(T_\varepsilon=1)+\sup_{f\in\Lambda^\infty_{\varepsilon,1}(C)}\mathbf{P}_f(T_\varepsilon=0)\right)\geq 1, \quad \forall C<C^*_\infty\ .
$$

II. *Let* $p=2,\ \beta\in\{1,2...,\},\ \psi_\varepsilon=\varepsilon^{\frac{2\beta-1}{4\beta}},\ 0<\gamma<1$. *Then for* C^*_0, *defined in (67),*

$$
\limsup_{\varepsilon\to 0}\left(\mathbf{P}_0(T^0_{\varepsilon,1}=1)+\sup_{f\in\Lambda^0_{\varepsilon,1}(C^*_0)}\mathbf{P}_f(T^0_{\varepsilon,1}=0)\right)\leq\gamma\ .
$$

Moreover,

$$
\liminf_{\varepsilon\to 0}\inf_{T_\varepsilon}\left(\mathbf{P}_0(T_\varepsilon=1)+\sup_{f\in\Lambda^0_{\varepsilon,1}(C^*_0)}\mathbf{P}_f(T_\varepsilon=0)\right)\geq\gamma\ .
$$

We omit the proof of this theorem, because it follows the same lines as the proofs of Theorems 1 to 3. It is only needed to check that conditions of Lemmas 1 to 4 are fulfilled for the kernel $K^*(\cdot)$, defined in (67). This is established by direct calculations, since one has the explicit expression for $K^*(\cdot)$.

5.3. Extension to the Besov classes of alternatives

Consider the semi-norm

$$\rho^B_{\beta,p}(f) = \sup_{h>0} \frac{\|\Delta^l_h f\|_p}{|h|^\beta} \ ,$$

where $l = \lfloor \beta \rfloor,\ \beta > 1/p,\ 1 \le p \le \infty$, and $\Delta^l_h f$ is the l-th difference of a function f, with step h. This semi-norm is related to the Besov space $B^\beta_{p\infty}(\mathbf{R}^1)$ (see e.g. Triebel (1992)). It is easy to check that the functional $\rho^B_{\beta,p}$ satisfies the assumptions of Section 3, namely it is convex, nonnegative, symmetric, and renormalizable with exponent $\alpha = \beta - 1/p$. Hence, Lemma 1 holds for $\rho = \rho^B_{\beta,p}$. Quite similarly, one shows that Lemmas 2, 3 and 4 hold, with $\rho_{\beta,p}$ replaced by $\rho^B_{\beta,p}$. Thus, Theorems 1, 2 and 3 remain valid, with the following changes: $\Lambda^\infty_\varepsilon(C),\ \Lambda^0_\varepsilon(C)$ should be replaced by the Besov sets of alternatives

$$\Lambda^\infty_{\varepsilon,B}(C) = \left\{ f:\ \rho^B_{\beta,p}(f) \le L,\ \|f\|_\infty \ge C\left(\varepsilon^2 \ln\frac{1}{\varepsilon}\right)^{\frac{\alpha}{2\alpha+1}} \right\},$$

$$\Lambda^0_{\varepsilon,B}(C) = \left\{ f:\ \rho^B_{\beta,p}(f) \le L,\ f(t_0) \ge C\varepsilon^{\frac{2\alpha}{2\alpha+1}} \right\} \ ,$$

where $\alpha = \beta - 1/p$, the constant $g^*(0)$ should be defined as the value of the problem

$$\begin{aligned} &\max g(0),\ \textit{subject to} \\ &\|g\|_2 \le 1, \\ &\rho^B_{\beta,p}(g) \le 1 \ , \end{aligned} \tag{68}$$

and the function g^*_A should be regarded as a solution of (15) with $\rho = \rho^B_{\beta,p}$.

Explicit solutions of (68) and (15) with $\rho = \rho^B_{\beta,p}$ are not known, therefore in the case of Besov classes of alternatives we can only get results on the existence of asymptotically minimax tests.

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