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Nonlinear stochastic wave and heat equations

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Abstract. We study nonlinear wave and heat equations on $\mathbb{R}^d$ driven by a spatially homogeneous Wiener process. For the wave equation we consider the cases of $d = 1, 2, 3$. The heat equation is considered on an arbitrary $\mathbb{R}^d$-space. We give necessary and sufficient conditions for the existence of a function-valued solution in terms of the covariance kernel of the noise.

0. Introduction

The paper is concerned with the following stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u &= \Delta u + f(u) + b(u)\dot{\mathscr{W}}, \\ u(0,x) &= u_0(x),\ \frac{\partial}{\partial t} u(0,x) = v_0(x),\ x \in \mathbb{R}^d \end{cases} \tag{0.1}$$

and heat equation

$$\frac{\partial}{\partial t} u = \Delta u + f(u) + b(u)\dot{\mathscr{W}}, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^d \ . \tag{0.2}$$

In (0.1) and (0.2), u_0 and v_0 are given functions, $f, b : \mathbb{R} \to \mathbb{R}$, and $\mathscr{W}$ is a spatially homogeneous Wiener process defined on a filtered probability space $\mathfrak{U} = (\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$. For the wave equation we consider the cases of $d = 1, 2, 3$. The heat equation is considered on an arbitrary $\mathbb{R}^d$-space.

Process $\mathscr{W}$ takes values in the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, and has the covariance of the following form

$$\mathbb{E}\,\langle \mathscr{W}(t), \psi\rangle\langle \mathscr{W}(s), \varphi\rangle = t \wedge s\ \langle \Gamma, \psi * \varphi_{(s)}\rangle, \qquad \psi, \varphi \in \mathscr{S}(\mathbb{R}^d)\ ,$$

where $\varphi_{(s)}(x) = \overline{\varphi(-x)}$, $x \in \mathbb{R}^d$, and Γ is a positive-definite tempered distribution. Then Γ has to be the Fourier transform of a positive symmetric tempered measure μ on $\mathbb{R}^d$. We call Γ and μ the *space correlation* and *spectral measure* of $\mathscr{W}$. In Section

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1 we gather basic facts concerning a spatially homogeneous Wiener process, and the stochastic integral with respect to it.

In the paper we are looking for $L^2(\mathbb{R}^d, \mathrm{e}^{-|x|}\mathrm{d}x)$-valued solutions to (0.1) and (0.2). We note that all results of the paper hold true if the exponential weight is replaced by $(1+|x|^2)^{-\rho}$, where $\rho > d/2$.

The problems of existence and uniqueness of solutions to (0.1) and (0.2) have been the subject of numerous investigations with the main aim of finding verifiable conditions in terms of f, b and Γ or μ. For bibliographic comments on the parabolic equation (0.2) we refer to [19]. Necessary and sufficient conditions for the existence of function-valued solutions of linear equations of the form (0.1) and (0.2) have been obtained in [11] and [12]. These papers were inspired by Dalang and Frangos [3] which proved the existence of a local in time solution to the nonlinear equation (0.1) for dimension $d = 2$. Dalang and Frangos assumed that the functions f and b are Lipschitz, and that the space correlation Γ is a non-negative function, continuous outside 0 and satisfies the integrability condition

$$\int_{\{|y|\le 1\}} \log(|y|^{-1})\Gamma(y)\mathrm{d}y < \infty \ .$$

The results of [3] were improved by Millet and Sanz-Solé [15], who showed, still for dimension 2, that under the above conditions the solution to (0.1) is global and that the integrability condition is, in a sense, also necessary for the existence. The methods applied in [3] and [15] rely on the integration theory with respect to martingale measures developed by Walsh [20]. This theory was originally thought not to be applicable for the wave equation in dimension $d = 3$ and higher because the integrand is a distribution valued process. In fact the referee of the present paper has indicated that Dalang [2] has recently extended the theory of stochastic integration with respect to martingale measures to a class of distribution-valued processes, and applied this to 3 dimensional wave equations.

The aim of the present paper is twofold. First we strengthen the results on nonlinear stochastic wave equations from [3] and [15], to cover noise processes with space correlations Γ which can be generalized functions, and we treat also equations in 3 space dimension. Secondly we complement the results of [19] dealing with parabolic equation (0.2). We relax the condition, imposed in [19], that the spectral measure μ is absolutely continuous with respect to Lebegues measure. Our approach is based on the general integration theory and harmonic analysis as developed in [19]. In both cases, hyperbolic and parabolic, our existence results are necessary and sufficient.

Before formulating our results we introduce some notation and definitions. Let $\vartheta \in \mathscr{S}(\mathbb{R}^d)$ be a strictly positive even function such that $\vartheta(x) = \mathrm{e}^{-|x|}$ for $|x| \ge 1$. Clearly, the spaces $L^2(\mathbb{R}^d, \mathrm{e}^{-|x|}\mathrm{d}x)$ and $L^2_\vartheta = L^2(\mathbb{R}^d, \vartheta(x)\mathrm{d}x)$ are isomorphic. Let us denote by H^1_ϑ the weighted Sobolev space being the completion of $\mathscr{S}(\mathbb{R}^d)$ with respect to the norm

$$|\psi|_{H^1_\vartheta} = \left(\int_{\mathbb{R}^d} \left\{|\psi(x)|^2 + |\nabla\psi(x)|^2\right\}\vartheta(x)\mathrm{d}x\right)^{1/2} \ .$$

Let G be the fundamental solution to the Cauchy problem for $\frac{\partial^2}{\partial t^2}u - \Delta u = 0$, see Section 2.

Definition 0.1. Let $u_0 \in H^1_\vartheta$ and $v_0 \in L^2_\vartheta$, and let $T > 0$. By an L^2_ϑ*-valued solution to* (0.1) *on a time interval* $[0, T]$ we understand an L^2_ϑ-valued measurable $(\mathfrak{F}_t)$-adapted process u such that:

$$\sup_{0\le t\le T} \mathbb{E}\,|u(t)|^2_{L^2_\vartheta} < \infty \ , \tag{0.3}$$

the stochastic integrals

$$\int_0^t G(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big), \qquad t \in [0, T]$$

are well defined in L^2_ϑ, and for $t \in [0, T]$,

$$\begin{aligned} u(t) = \tfrac{\partial}{\partial t}G(t) * u_0 + G(t) * v_0 + \textstyle\int_0^t G(t-s) * f(u(s))\mathrm{d}s \\ + \textstyle\int_0^t G(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big) \ . \end{aligned} \tag{0.4}$$

We say that an $(\mathfrak{F}_t)$-adapted measurable process $u : \Omega \times [0,\infty) \to L^2_\vartheta$ is an L^2_ϑ*-valued global solution to* (0.1) iff it is a solution on any finite time interval, that is, it satisfies (0.3) and (0.4) for an arbitrary $T > 0$.

Let $P(t)(x) = (4\pi t)^{-d/2}\mathrm{e}^{-|x|^2/(4t)}$ be the fundamental solution to the Cauchy problem for the heat equation $\frac{\partial}{\partial t}u - \Delta u = 0$.

Definition 0.2. Let $u_0 \in L^2_\vartheta$, and let $T > 0$. By an L^2_ϑ*-valued solution to* (0.2) *on* $[0, T]$ we understand an L^2_ϑ-valued measurable $(\mathfrak{F}_t)$-adapted process u such that: (0.3) holds, the stochastic integrals

$$\int_0^t P(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big), \qquad t \in [0, T]$$

are well defined in L^2_ϑ, and for $t \in [0, T]$,

$$u(t) = P(t) * u_0 + \int_0^t P(t-s) * f(u(s))\mathrm{d}s + \int_0^t P(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big) \ . \tag{0.5}$$

We say that an $(\mathfrak{F}_t)$-adapted measurable process $u : \Omega \times [0,\infty) \to L^2_\vartheta$ is an L^2_ϑ*-valued global solution to* (0.2) iff it is a solution on any finite time interval.

Definitions 0.1 and 0.2 explicitly require that the stochastic integrals in (0.4) and (0.5) are well-defined processes with values in L^2_ϑ. The integrands act on the noise process as the composition of the multiplication by $b(u)$ with the convolution operators with kernels G and P respectively. In the next section we give a precise meaning of the stochastic integral with respect to $\mathscr{W}$.

Let λ denote Lebesgue measure on $\mathbb{R}^d$. In our exposition the following hypotheses (H) and (G) play an essential role.

(H) There is a $\kappa \geq 0$ such that $\Gamma + \kappa\lambda$ is a non-negative measure.

$$
\text{(G)} \quad \begin{cases} \text{The condition (H) holds true, and} \\ \int_{\{|y|\leq 1\}} \log(|y|^{-1})\Gamma(\mathrm{d}y) < \infty \quad \text{if } d = 2, \\ \int_{\{|y|\leq 1\}} |y|^{-d+2}\Gamma(\mathrm{d}y) < \infty \quad \text{if } d > 2. \\ \text{In dimension } d = 1, \text{(G) is identical with (H).} \end{cases}
$$

Note that (G) is stronger than (H), and its formulation depends on the dimension d of the underlying space $\mathbb{R}^d$.

Remark 0.1. The hypothesis (H) can be equivalently stated in terms of the spectral measure μ as a requirement that there exists a κ such that the measure $\mu + \kappa\delta_0$ is a positive-definite distribution.

Remark 0.2. If $\Gamma = \mathscr{F}(\mu)$ is an $\overline{\mathbb{R}}$-valued function bounded from below, then (H) holds. If $\Gamma = \mathscr{F}(\mu)$ and μ is a finite non-negative measure then (G) holds. For more examples we refer the reader to [12] and [19].

The main results of the present paper are the following existence theorems. The first one deals with the wave equation.

Theorem 0.1. *Assume that* $d \leq 3$. *Let* f *and* b *be Lipschitz continuous functions.*
(i) If (G) *holds then for arbitrary* $u_0 \in H^1_\vartheta$ *and* $v_0 \in L^2_\vartheta$ *there exists a unique global* L^2_ϑ*-valued solution to* (0.1).
(ii) Assume that (H) *holds and that there is a constant* $c > 0$ *such that* $|b(x)| \geq c$ *for every* $x \in \mathbb{R}$. *If for some* $u_0 \in H^1_\vartheta$, $v_0 \in L^2_\vartheta$, *and* $T > 0$ *there exists an* L^2_ϑ*-valued solution to* (0.1) *on* $[0, T]$, *then* (G) *holds.*

The second result is concerned with the heat equation.

Theorem 0.2. *Let* $d \in \mathbb{N}$, *and let* f *and* b *be Lipschitz continuous functions.*
(i) If (G) *holds then for every* $u_0 \in L^2_\vartheta$ *there exists a unique global* L^2_ϑ*-valued solution to* (0.2).
(ii) Assume that (H) *holds and there is a constant* $c > 0$ *such that* $|b(x)| \geq c$ *for every* $x \in \mathbb{R}$. *If for certain* $u_0 \in L^2_\vartheta$ *and* $T > 0$ *there exists an* L^2_ϑ*-valued solution to* (0.2) *on* $[0, T]$, *then* (G) *holds.*

The existence of a solution to the parabolic problem (0.2) was obtained in [19], see also [1], under the following condition.

(A) The spectral measure μ of $\mathscr{W}$ is either finite or absolutely continuous with respect to Lebesgue measure and its density $\gamma = \mathrm{d}\mu/\mathrm{d}x$ belongs to $L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$ satisfying $(1 - 1/p)d/2 < 1$.

Clearly, (A) implies that

$$
\int_{\mathbb{R}^d} \frac{1}{1 + |x|^2} \mathrm{d}\mu(x) < \infty \ . \tag{0.6}
$$

We will show, see Proposition 3.1, that if (H) holds then (0.6) and (G) are equivalent.

Theorem 0.2 provides conditions for the existence and uniqueness of an L^2_ϑ solution satisfying the regularity condition (0.3). In fact, see [18], and [19], one can show that the unique solution u to the heat equation has continuous trajectories in L^2_ϑ and satisfies

$$\mathbb{E} \sup_{t\in[0,T]} |u(t)|^p_{L^2_\vartheta} < \infty \quad \text{for all } T > 0 \text{ and } p \geq 1 \ .$$

The paper is organized as follows. In Section 1 we introduce some basic notation and recall the construction of the stochastic integral with respect to a spatially homogeneous Wiener process. Next section is devoted to solutions on the weighted space L^2_ϑ of the linear deterministic wave equation. Then we establish some analytical lemmas needed to estimate stochastic integrals with respect to $\mathscr{W}$. Sections 4 and 5 are devoted to the proofs of Theorems 0.1 and 0.2, respectively.

1. Stochastic integration

Let $\mathscr{S}(\mathbb{R}^d;\mathbb{C})$ and $\mathscr{S}(\mathbb{R}^d)$ denote the spaces of all infinitely differentiable rapidly decreasing complex and real functions on $\mathbb{R}^d$. Let $\mathscr{S}'(\mathbb{R}^d;\mathbb{C})$ and $\mathscr{S}'(\mathbb{R}^d)$ be the spaces of complex and real tempered distributions on $\mathbb{R}^d$. The value of a distribution ξ on a test function ψ is denoted by $\langle \xi, \psi\rangle$. We use also the same notation for the products on $\mathbb{R}^d$ and $L^2(\mathbb{R}^d;\mathbb{C})$.

For $\psi \in \mathscr{S}(\mathbb{R}^d;\mathbb{C})$ we set $\psi_{(s)}(x) = \overline{\psi(-x)}$, $x \in \mathbb{R}^d$. Denote by $\mathscr{S}_{(s)}(\mathbb{R}^d)$ the space of all $\psi \in \mathscr{S}(\mathbb{R}^d;\mathbb{C})$ such that $\psi = \psi_{(s)}$, and by $\mathscr{S}'_{(s)}(\mathbb{R}^d)$ the space of all $\xi \in \mathscr{S}'(\mathbb{R}^d;\mathbb{C})$ such that $\langle \xi, \psi\rangle = \langle \xi, \psi_{(s)}\rangle$ for every $\psi \in \mathscr{S}(\mathbb{R}^d;\mathbb{C})$.

In the paper we denote by $\mathscr{F}$ the Fourier transform on $\mathscr{S}(\mathbb{R}^d;\mathbb{C})$, that is

$$\mathscr{F}\psi(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi i\langle x,y\rangle}\psi(y)\mathrm{d}y \ .$$

Recall that the inverse Fourier transform $\mathscr{F}^{-1}$ is given by the formula

$$\mathscr{F}^{-1}\psi(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i\langle x,y\rangle}\psi(y)\mathrm{d}y \ .$$

Let $\xi \in \mathscr{S}'(\mathbb{R}^d;\mathbb{C})$. We define $\mathscr{F}\xi$ putting $\langle \mathscr{F}\xi, \psi\rangle = \langle \xi, \mathscr{F}^{-1}\psi\rangle$ for $\psi \in \mathscr{S}(\mathbb{R}^d;\mathbb{C})$. Note that $\mathscr{F}$ transforms $\mathscr{S}'(\mathbb{R}^d)$ into $\mathscr{S}'_{(s)}(\mathbb{R}^d)$.

Let $\mathfrak{U} = (\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. In the paper we assume that $\mathscr{W}$ is a spatially homogeneous Wiener process on $\mathfrak{U}$, that is an $\mathscr{S}'(\mathbb{R}^d)$-valued Wiener process satisfying the following conditions:

(A.1) For each $\psi \in \mathscr{S}(\mathbb{R}^d)$, $\{\langle \mathscr{W}(t), \psi\rangle\}_{t\in[0,\infty)}$ is a real-valued $(\mathfrak{F}_t)$-adapted Wiener process.

(A.2) There exists a $\Gamma \in \mathscr{S}'(\mathbb{R}^d)$ such that for all $\psi, \varphi \in \mathscr{S}(\mathbb{R}^d)$ one has

$$Q(\psi,\varphi) \overset{\text{def}}{=} \mathbb{E}\,\langle \mathscr{W}(1), \psi\rangle\langle \mathscr{W}(1), \varphi\rangle = \langle \Gamma, \psi * \varphi_{(s)}\rangle \ .$$

We recall, see [19], that a process satisfying (A.1) satisfies (A.2) iff the laws $\mathscr{L}(\mathscr{W}(t))$, $t \geq 0$ are invariant with respect to all translations $\tau'_h : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, where $\tau_h : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$, $\tau_h\psi(\cdot) = \psi(\cdot + h)$ for $\psi \in \mathscr{S}(\mathbb{R}^d)$. We call Q appearing in (A.2) the *covariance form* of $\mathscr{W}$. Since Γ is a positive-definite distribution there exists a positive symmetric tempered measure μ on $\mathbb{R}^d$ such that $\Gamma = \mathscr{F}(\mu)$, see [9, Theorem 6, p. 169]. Recall, see Section 0, that Γ appearing in (A.2) is the space correlation and $\mu = \mathscr{F}^{-1}\Gamma$ is the spectral measure of $\mathscr{W}$.

If the spectral measure μ is finite, then Γ is a continuous positive-definite real-valued function on $\mathbb{R}^d$, and for an arbitrary $t \geq 0$, $\mathscr{W}(t,\cdot)$ is a stationary random field such that for all $x, y \in \mathbb{R}^d$,

$$\mathbb{E}\,\mathscr{W}(t,x)\mathscr{W}(t,y) = t\,\Gamma(x-y) \ .$$

The crucial role for stochastic integration with respect to $\mathscr{W}$ is played by the Hilbert space $\mathscr{H}_{\mathscr{W}} \subset \mathscr{S}'(\mathbb{R}^d)$ consisting of all distributions ξ for which there exists a constant C such that

$$|(\xi,\psi)| \leq C\sqrt{(\Gamma, \psi * \psi_{(s)})}, \quad \psi \in \mathscr{S}(\mathbb{R}^d) \ .$$

The norm in $\mathscr{H}_{\mathscr{W}}$ is given by the formula

$$|\xi|_{\mathscr{H}_{\mathscr{W}}} = \sup_{\psi\in\mathscr{S}(\mathbb{R}^d)} \frac{|(\xi,\psi)|}{\sqrt{(\Gamma, \psi * \psi_{(s)})}} \ .$$

The space $\mathscr{H}_{\mathscr{W}}$ is called the *reproducing kernel Hilbert space* of $\mathscr{W}$, see e.g. [4], [10], or [19].

Let H be a Hilbert space and let $L(\mathscr{S}'(\mathbb{R}^d), H)$ be the space of linear continuous operators from $\mathscr{S}'(\mathbb{R}^d)$ into H. A mapping Ψ from $[0,\infty)\times\Omega$ into $L(\mathscr{S}'(\mathbb{R}^d), H)$ is called *simple* if it takes only a finite number of values and there exists a finite sequence $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = \infty$ such that

$$\Psi(t,\omega) = \Psi(t_k,\omega), \quad t \in [t_k, t_{k+1}), \quad \omega\in\Omega, \quad k = 0,\ldots,m \ ,$$

with $\mathfrak{F}_{t_k}$-measurable random variables $\Psi(t_k)$. For simple processes the stochastic integral

$$\int_0^t \Psi(s)\mathrm{d}\mathscr{W}(s), \quad t \geq 0$$

is defined in the usual way

$$\int_0^t \Psi(s)\mathrm{d}\mathscr{W}(s) = \sum_{k=0}^m \Psi(t_k)[\mathscr{W}(t\wedge t_{k+1}) - \mathscr{W}(t\wedge t_k)] \ .$$

It is an H-valued martingale for which

$$\mathbb{E}\int_0^t \Psi(s)\mathrm{d}\mathscr{W}(s) = 0, \quad t \geq 0 \ .$$

Moreover, if $L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, H)$ denotes the space of Hilbert–Schmidt operators from $\mathscr{H}_{\mathscr{W}}$ into H then, see e.g. [4], [10], [19],

$$\mathbb{E}\left|\int_0^t \Psi(s)\mathrm{d}\mathscr{W}(s)\right|_H^2 = \mathbb{E}\int_0^t ||\Psi(s)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},H)}\mathrm{d}s \quad \text{for } t \geq 0 \ . \tag{1.1}$$

The formula (1.1) is fundamental for stochastic integration. It underlines importance of the kernel space $\mathscr{H}_{\mathscr{W}}$. It allows to extend the definition of the stochastic integral, by an approximation procedure, to all measurable $(\mathfrak{F}_t)$-adapted $L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, H)$-valued processes Ψ such that

$$\mathbb{E}\int_0^t ||\Psi(s)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},H)}\mathrm{d}s < \infty \quad \text{for } t \geq 0 \ . \tag{1.2}$$

In this more general case the identity (1.1) is still valid.

Finally the condition (1.2) can be relaxed to

$$\mathbb{P}\left(\int_0^t ||\Psi(s)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},H)}\mathrm{d}s < \infty\right) = 1, \quad t \geq 0 \ . \tag{1.3}$$

If (1.3) holds the stochastic integral can be defined by the standard localization procedure.

We will need a characterization of the space $\mathscr{H}_{\mathscr{W}}$ from [19, Proposition 1.2]. In the proposition below $L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu)$ denotes the subspace of $L^2(\mathbb{R}^d, \mathrm{d}\mu; \mathbb{C})$ consisting of all functions u such that $u_{(\mathrm{s})} = u$.

Proposition 1.1. *A distribution ξ belongs to $\mathscr{H}_{\mathscr{W}}$ if and only if $\xi = \mathscr{F}(u\mu)$ for a certain $u \in L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu)$. Moreover, if $\xi = \mathscr{F}(u\mu)$ and $\eta = \mathscr{F}(v\mu)$, then*

$$\langle \xi, \eta\rangle_{\mathscr{H}_{\mathscr{W}}} = \langle u, v\rangle_{L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu)} \ .$$

Recall that G and P denote the fundamental solutions to the Cauchy problems for $\frac{\partial^2}{\partial t^2}u - \Delta u = 0$ and $\frac{\partial}{\partial t}u - \Delta u = 0$, respectively. Let us denote by $\mathscr{H}^0_{\mathscr{W}}$ the dense subspace of $\mathscr{H}_{\mathscr{W}}$ consisting of all $\eta = \mathscr{F}(\psi\mu)$, where $\psi \in \mathscr{S}_{(\mathrm{s})}(\mathbb{R}^d)$. Note that

$$\int_{\mathbb{R}^d} |\psi(x)|\mu(\mathrm{d}x) < \infty \quad \text{for } \psi \in \mathscr{S}_{(\mathrm{s})}(\mathbb{R}^d) \ .$$

Thus $\mathscr{H}^0_{\mathscr{W}} \subset C_{\mathrm{b}}(\mathbb{R}^d)$. For $u \in L^2_\vartheta$, $t > 0$ and $\eta \in \mathscr{H}^0_{\mathscr{W}}$ we write

$$\mathscr{K}(t, u)\eta = G(t) * (u\eta) \tag{1.4}$$

and

$$\mathscr{P}(t, u)\eta = P(t) * (u\eta) \ . \tag{1.5}$$

We will show, see Lemmas 3.3 and 5.2, that if (G) holds then for all $t > 0$ and u, $\mathscr{K}(t, u)$ and $\mathscr{P}(t, u)$ have (unique) extensions to Hilbert–Schmidt operators from $\mathscr{H}_{\mathscr{W}}$ into L^2_ϑ. Moreover,

$$||\mathscr{K}(t, u)||_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)} \leq C_1(t)|u|_{L^2_\vartheta} \ ,$$
$$||\mathscr{P}(t, u)||_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)} \leq C_2(t)|u|_{L^2_\vartheta} \ ,$$

where $C_1(t)$ and $C_2(t)$ can be chosen such that

$$\int_0^T C_i^2(t)\mathrm{d}t < \infty \quad \text{for } T > 0 \text{ and } i = 1, 2 \ .$$

Thus, $\mathscr{K}(t,\cdot)$ and $\mathscr{P}(t,\cdot)$ can be uniquely extended to linear bounded operators from L^2_ϑ into $L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)$. Now, let u be an L^2_ϑ-valued measurable $(\mathfrak{F}_t)$-adapted process satisfying (0.3). Then for any fixed $t > 0$, the operator-valued processes $\mathscr{K}(t-s, b(u(s)))$ and $\mathscr{P}(t-s, b(u(s)))$, $s \in (0,t)$ are adapted and satisfy (1.2). We define the stochastic integrals in (0.4) and (0.5) in the following way

$$\int_0^t G(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big) \stackrel{\mathrm{def}}{=} \int_0^t \mathscr{K}(t-s, b(u(s)))\mathrm{d}\mathscr{W}(s) \ ,$$
$$\int_0^t P(t-s) * \big(b(u(s))\mathrm{d}\mathscr{W}(s)\big) \stackrel{\mathrm{def}}{=} \int_0^t \mathscr{P}(t-s, b(u(s)))\mathrm{d}\mathscr{W}(s) \ .$$

In the formulae above $b(u(s))$ denotes the random field

$$b(u(s))(\omega, x) = b(u(s,x,\omega)), x \in \mathbb{R}^d, \omega \in \Omega.$$

2. Linear deterministic hyperbolic equation on $\mathbb{R}^d$

Consider the linear wave equation on $\mathbb{R}^d$,

$$\frac{\partial^2}{\partial t^2}u(t) = \Delta u(t), \quad t > 0, \quad u(0) = u_0, \quad \frac{\partial}{\partial t}u(0) = v_0 \ , \tag{2.1}$$

were $u_0, v_0 \in \mathscr{S}'(\mathbb{R}^d; \mathbb{C})$. Passing to Fourier transforms we arrive at the problem

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathscr{F}u(t)(x) = -4\pi^2|x|^2\mathscr{F}u(t)(x), \quad t > 0, \quad \mathscr{F}u(0) = \mathscr{F}u_0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}u(0) = \mathscr{F}v_0 \ .$$

Hence, by direct computation we get

$$\mathscr{F}u(t)(x) = \cos(2\pi|x|t)\mathscr{F}u_0(x) + \frac{\sin(2\pi|x|t)}{2\pi|x|}\mathscr{F}v_0(x) \ .$$

In the formula above, we multiply distributions u_0 and v_0 by functions $\cos(2\pi|x|t)$ and $(2\pi|x|)^{-1}\sin(2\pi|x|t)$ of x-variable. These products are well defined because the functions are infinitely differentiable, with all derivatives of a polynomial growth. Note that $\hat{u}(t)$, $t \geq 0$ is an $\mathscr{S}'(\mathbb{R}^d; \mathbb{C})$-valued mapping infinitely differentiable in t.

Let G be the solution to (2.1) with $G(0) \equiv 0$ and $\frac{\partial}{\partial t}G(0)$ being equal to the Dirac δ_0-distribution. Then

$$\mathscr{F}\Big(\frac{\partial}{\partial t}G(t)\Big)(x) = \cos(2\pi|x|t) \quad \text{and} \quad \mathscr{F}G(t)(x) = \frac{\sin(2\pi|x|t)}{2\pi|x|} \ .$$

Consequently, the unique solution to (2.1) is given by

$$u(t) = \frac{\partial}{\partial t} G(t) * u_0 + G(t) * v_0 \ .$$

where $*$ denotes the convolution operator defined by $\xi * \eta = \mathscr{F}^{-1}(\mathscr{F}\xi\mathscr{F}\eta)$ for tempered distributions ξ and η such that their product $\mathscr{F}\xi\mathscr{F}\eta$ is well defined.

We call G the *fundamental solution* to the wave equation $\frac{\partial^2}{\partial t^2}u - \Delta u = 0$. Explicit formulae for $G(t)$, $t \geq 0$ are well known, see e.g. [16, pp. 279–280]. Namely, if $d = 1$, then $G(t)$, $t > 0$ are functions of x-variable, and

$$G(t)(x) = \frac{1}{2}\chi_{\{|x|<t\}} \ .$$

If $d = 2k + 1$ for some integer $k \geq 1$, then

$$G(t) = \frac{1}{2(2\pi)^k}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\frac{\sigma_t^d}{t} \ ,$$

where σ_t^d is the surface measure on the sphere in $\mathbb{R}^d$ with center at 0 and radius t. If $d = 2k$, then

$$G(t)(x) = \frac{1}{(2\pi)^k}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\frac{1}{(t^2 - |x|^2)_+^{1/2}}\chi_{\{|x|<t\}} \ .$$

Recall that in any dimension $G(0) \equiv 0$. Clearly, if $d = 1, 2, 3$, then each $G(t)$, $t \geq 0$ is a finite non-negative measure with the support contained in the closed ball $\overline{B(0,t)}$ in $\mathbb{R}^d$ with center at 0 and radius t.

Recall that the weighted spaces L^2_ϑ and H^1_ϑ were introduced in the first section. Note that there is a constant C_ϑ such that

$$\vartheta(x - z) \leq C_\vartheta e^t \vartheta(x) \quad \text{for } t \geq 0, x \in \mathbb{R}^d, z \in \overline{B(0,t)} \ . \tag{2.2}$$

Lemma 2.1. *Let* $d = 1, 2, 3$, *and let* C_ϑ *satisfy* (2.2). *Then:*

$$G(t) * \vartheta(x) = \int_{\mathbb{R}^d} \vartheta(x - y)G(t)(\mathrm{d}y) \leq C_\vartheta t e^t \vartheta(x), \quad t \geq 0, \ x \in \mathbb{R}^d \ . \tag{2.3}$$

Moreover, for all $t \geq 0$ *and* $\psi \in \mathscr{S}(\mathbb{R}^d)$, $G(t) * \psi \in L^2_\vartheta$ *and* $\frac{\partial}{\partial t}G(t) * \psi \in H^1_\vartheta$, *and*

$$|G(t) * \psi|_{L^2_\vartheta} \leq C_\vartheta^{\frac{1}{2}} t e^{\frac{t}{2}} |\psi|_{L^2_\vartheta} \ , \tag{2.4}$$

$$|\frac{\partial}{\partial t}G(t) * \psi|_{L^2_\vartheta} \leq C_\vartheta^{\frac{1}{2}} e^{\frac{t}{2}}\big(1 + t\big) |\psi|^2_{H^1_\vartheta} \ . \tag{2.5}$$

Proof. By direct calculation we obtain

$$\int_{\mathbb{R}^d} G(t)(\mathrm{d}z) = t \quad \text{for } t \geq 0 \text{ and } d = 1, 2, 3 \ . \tag{2.6}$$

Hence

$$G(t) * \vartheta(x) = \int_{\overline{B(0,t)}} \vartheta(x-z)G(t)(\mathrm{d}z) \leq C_\vartheta t \mathrm{e}^t \vartheta(x) \ ,$$

which gives (2.3).

We now prove (2.4). Since $G(0) * \psi \equiv 0$ we can assume that $t > 0$. By (2.6) and the Schwartz inequality we get

$$\begin{aligned}\int_{\mathbb{R}^d} |G(t) * \psi(x)|^2 \vartheta(x)\mathrm{d}x &= \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \psi(x-z)G(t)(\mathrm{d}z)\Big|^2 \vartheta(x)\mathrm{d}x \\ &\leq t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(x-z)|^2 G(t)(\mathrm{d}z)\vartheta(x)\mathrm{d}x \\ &\leq t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(y)|^2 \vartheta(y+z)G(t)(\mathrm{d}z)\mathrm{d}y\end{aligned}$$

since $G(t)(\mathrm{d}z) = G(t)(-\mathrm{d}z)$ and $\vartheta(y-z) = \vartheta(z-y)$, this is

$$\leq t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(y)|^2 \vartheta(y-z)G(t)(\mathrm{d}z)\mathrm{d}y$$

and from (2.3) we obtain

$$\leq C_\vartheta t^2 \mathrm{e}^t \int_{\mathbb{R}^d} |\psi(y)|^2 \vartheta(y)\mathrm{d}y = C_\vartheta t^2 \mathrm{e}^t |\psi|^2_{L^2_\vartheta} \ ,$$

which proves (2.4).

We now prove (2.5). To this end note that if $d = 1$, then we have

$$\begin{aligned}\Big| \frac{\partial}{\partial t} G(t) * \psi(x) \Big| &= \frac{1}{2} \Big| \frac{\partial}{\partial t} \int_{\{|y|<t\}} \psi(x-y)\mathrm{d}y \Big| = \frac{1}{2} \Big| \frac{\partial}{\partial t}\, t \int_{\{|z|<1\}} \psi(x-tz)\mathrm{d}z \Big| \\ &\leq \frac{1}{t} G(t) * |\psi|(x) + G(t) * |\psi'|(x) \ .\end{aligned}$$

Now, for $d = 2$ we have

$$\begin{aligned}\Big| \frac{\partial}{\partial t} G(t) * \psi(x) \Big| &= \frac{1}{2\pi} \Big| \frac{\partial}{\partial t} \int_{\{|y|<t\}} \frac{\psi(x-y)}{(t^2 - |y|^2)^{1/2}} \mathrm{d}y \Big| \\ &= \frac{1}{2\pi} \Big| \frac{\partial}{\partial t}\, t \int_{\{|z|<1\}} \frac{\psi(x-tz)}{(1-|z|^2)^{1/2}} \mathrm{d}z \Big| \\ &\leq \frac{1}{t} G(t) * |\psi|(x) + G(t) * |\nabla\psi|(x) \ .\end{aligned}$$

Finally, for $d = 3$ we have

$$\Big| \frac{\partial}{\partial t} G(t) * \psi(x) \Big| = \frac{1}{4\pi} \Big| \frac{\partial}{\partial t}\, t \int_{\{|z|=1\}} \psi(x - tz)\sigma_1^3(\mathrm{d}z) \Big|$$
$$\leq \frac{1}{t} G(t) * |\psi|(x) + G(t) * |\nabla\psi|(x) \ .$$

Summing up, for $d = 1, 2, 3$, $t > 0$, $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$|\frac{\partial}{\partial t} G(t) * \psi(x)| \leq \frac{1}{t} G(t) * |\psi|(x) + G(t) * |\nabla\psi|(x) \ .$$

Hence, by (2.4) we have

$$|\frac{\partial}{\partial t} G(t) * \psi|_{L^2_\vartheta} \leq C_\vartheta^{\frac{1}{2}} \mathrm{e}^{\frac{t}{2}} |\psi|_{L^2_\vartheta} + C_\vartheta^{\frac{1}{2}} t \mathrm{e}^{\frac{t}{2}} |\nabla\psi|_{L^2_\vartheta} \ ,$$

which is our claim. □

As a direct consequence of the lemma we have the following corollary.

Corollary 2.1. *For any $t \geq 0$ there are unique operators $\mathscr{G}(t) \in L(L^2_\vartheta, L^2_\vartheta)$ and $\dot{\mathscr{G}}(t) \in L(H^1_\vartheta, L^2_\vartheta)$ such that for every $\psi \in \mathscr{S}(\mathbb{R}^d)$, $\mathscr{G}(t)\psi = G(t) * \psi$ and $\dot{\mathscr{G}}(t) = \frac{\partial}{\partial t} G(t) * \psi$. Moreover, there is a constant C such that*

$$||\mathscr{G}(t)||_{L(L^2_\vartheta, L^2_\vartheta)} + ||\dot{\mathscr{G}}(t)||_{L(H^1_\vartheta, L^2_\vartheta)} \leq C(t+1)\mathrm{e}^t \quad \text{for every } t \geq 0 \ .$$

3. Main estimates in the hyperbolic case

Recall that $\mathscr{K}$ is given by (1.4). Note that by Corollary 2.1 we have

$$|\mathscr{K}(t,u)\eta|_{L^2_\vartheta} \leq C(t+1)\mathrm{e}^t |u\eta|_{L^2_\vartheta} \quad \text{for all } t \geq 0, u \in L^2_\vartheta, \eta \in \mathscr{H}^0_{\mathscr{W}} \ .$$

Clearly,

$$\mathscr{K}(t,u)\eta(x) = \int_{\mathbb{R}^d} u(x-y)\eta(x-y)G(t)(\mathrm{d}y), \quad u \in L^2_\vartheta,\ \eta \in \mathscr{H}^0_{\mathscr{W}},\ x \in \mathbb{R}^d \ .$$

The aim of this section is to show that for any $t > 0$, $\mathscr{K}(t,\cdot)$ has an extension to a bounded linear operator from L^2_ϑ into the space of Hilbert–Schmidt operators $L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)$, provided that the integrability condition (0.6) is fulfilled.

The following result has been proved, under slightly less general conditions, in [12].

Proposition 3.1. *If hypothesis (H) is satisfied, then conditions (G) and (0.6) are equivalent.*

Proof. Define $\Gamma_\kappa = \mathscr{F}(\mu + \kappa\delta_0)$. By Theorem 2 of [12], Γ_κ satisfies (G) if and only if (0.6) holds with μ being replaced by $\mu + \kappa\delta_0$. Since the measure δ_0 obviously satisfies (0.6) and functions 1, $\log(|\cdot|^{-1})$, and $|\cdot|^{-1}$ are locally integrable in dimensions 1, 2 and 3 respectively, the result follows. □

Lemma 3.1. *Let $d \le 3$. Let $u \in C_{\mathrm{b}}(\mathbb{R}^d)$, $t > 0$, and let $\{f_k\} \subset \mathscr{H}^0_{\mathscr{W}}$ be an orthonormal basis of $\mathscr{H}_{\mathscr{W}}$. Then*

$$\sum_k |\mathscr{K}(t,u) f_k|^2_{L^2_\vartheta} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mu(\mathrm{d}y)\vartheta(x)\mathrm{d}x \ . \tag{3.1}$$

Proof. Let $x \in \mathbb{R}^d$ and $t \ge 0$. Then as $G(t)(x-\cdot)u$ is a finite measure, we have $\mathscr{F}\big(G(t)(x-\cdot)u\big) \in C_{\mathrm{b}}(\mathbb{R}^d)$. Thus the right hand side of (3.1) is well defined. Let $\{e_k\} \subset \mathscr{S}_{(\mathrm{s})}(\mathbb{R}^d)$ be an orthonormal basis of $L^2_{(\mathrm{s})}(\mu)$ such that $f_k = \mathscr{F}(e_k\mu)$. Then

$$\begin{aligned} \sum_k |\mathscr{K}(t,u) f_k|^2_{L^2_\vartheta} &= \sum_k |G(t) * \big(u\mathscr{F}(e_k\mu)\big)|^2_{L^2_\vartheta} \\ &= \sum_k \int_{\mathbb{R}^d} |\langle \mathscr{F}(G(t)(x-\cdot)u), e_k\mu\rangle|^2 \vartheta(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mu(\mathrm{d}y)\vartheta(x)\mathrm{d}x \ , \end{aligned}$$

which is the desired conclusion. □

The following lemma is essentially contained in [12].

Lemma 3.2. *Let $d = 1, 2, 3$. The following conditions are equivalent:*

(i) *$\mathscr{K}(s,1)$, $s \ge 0$ are Hilbert–Schmidt operators acting from $\mathscr{H}_{\mathscr{W}}$ into L^2_ϑ, and*

$$\int_0^t ||\mathscr{K}(s,1)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)} \mathrm{d}s < \infty$$

for every, or equivalently for a certain $t > 0$.

(ii) *The spectral measure μ of $\mathscr{W}$ satisfies the integral condition* (0.6).

Proof. Let $\{f_k\} \subset \mathscr{H}^0_{\mathscr{W}}$ be an orthonormal basis of $\mathscr{H}_{\mathscr{W}}$. Then, by Lemma 3.1, we have

$$\begin{aligned} \sum_k |\mathscr{K}(s,1) f_k|^2_{L^2_\vartheta} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(s)(x-\cdot)\big)(y)|^2 \mu(\mathrm{d}y)\vartheta(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\sin^2(2\pi|y|s)}{4\pi^2|y|^2} \mu(\mathrm{d}y)\vartheta(x)\mathrm{d}x \\ &= \left(\int_{\mathbb{R}^d} \vartheta(x)\mathrm{d}x\right) \int_{\mathbb{R}^d} \frac{\sin^2(2\pi|y|s)}{4\pi^2|y|^2} \mu(\mathrm{d}y) \ . \end{aligned} \tag{3.2}$$

Since $\int_{\mathbb{R}^d} \vartheta(x)\mathrm{d}x < \infty$, the desired conclusion follows from the following elementary estimates

$$\frac{\sin^2(2\pi|y|s)}{4\pi^2|y|^2} \le \frac{c(s)}{1+|y|^2}, \quad y \in \mathbb{R}^d$$

and

$$\begin{aligned}\frac{c_1(t)}{1+|y|^2} &\le \int_0^t \frac{\sin^2(2\pi|y|s)}{4\pi^2|y|^2}\mathrm{d}s \\ &= \frac{t}{8\pi^2|y|^2}\left\{1-\frac{\sin(4\pi|y|t)}{4\pi|y|t}\right\} \le \frac{c_2(t)}{1+|y|^2}, \quad y\in\mathbb{R}^d\end{aligned}$$

with properly chosen $c(s), c_1(t), c_2(t) \in (0,\infty)$. □

Since there is a constant C such that $(\sin^2 r)/r^2 \le C/(1+r^2)$ for every $r\in\mathbb{R}$, we have the following direct consequence of (3.2).

Corollary 3.1. *There is a constant C such that it μ satisfies* (0.6)*, then*

$$||\mathscr{K}(t,1)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}};L^2_\vartheta)} \le Ct^2\Big(\int_{\mathbb{R}^d}\vartheta(x)\mathrm{d}x\Big)\int_{\mathbb{R}^d}\frac{\mu(\mathrm{d}y)}{1+|y|^2}, \quad t\ge 0\ .$$

To go further we need the following reformulation of the hypothesis (H).

Proposition 3.2. *Let μ be the spectral measure of $\mathscr{W}$, and let δ_0 be the Dirac distribution. Condition (H) holds if and only if there exists $\kappa\ge 0$ such that for $N\in\mathbb{N}$, the Fourier transforms of the measures*

$$\mu_{N,\kappa}(\mathrm{d}y) = \mathrm{e}^{-\frac{|y|^2}{N}}\big(\mu(\mathrm{d}y)+\kappa\delta_0(\mathrm{d}y)\big)\ ,$$

are non-negative functions.

Proof. Note that the sequence $\{\mu_{N,\kappa}\}$ converges in $\mathscr{S}'(\mathbb{R}^d)$, as $N\to\infty$, to $\mu+\kappa\delta_0$. Therefore the sequence of non-negative measures $\Gamma_{N,\kappa}=\mathscr{F}(\mu_{N,\kappa})$ converges to a non-negative measure $\Gamma_\kappa=\mathscr{F}(\mu+\kappa\delta_0)$. Consequently $\Gamma=\Gamma_\kappa-\kappa\lambda$, as required. If $\Gamma=\Gamma_\kappa-\kappa\lambda$ for some tempered non-negative measure Γ_κ and $\kappa\ge 0$, then $\Gamma_\kappa=\Gamma+\kappa\lambda$ and $\Gamma_\kappa=\mathscr{F}(\mu+\kappa\delta_0)$. Moreover,

$$\mathscr{F}(\mu_{N,\kappa}) = (\pi N)^{d/2}\mathrm{e}^{-N\pi^2|\cdot|^2} * \Gamma_\kappa\ ,$$

and the distributions $\mathscr{F}(\mu_{N,\kappa})$ are non-negative functions, as required. □

Lemma 3.3. *Let $d\le 3$. (i) If condition (G) is satisfied, then for all $t>0$ and $u\in L^2_\vartheta$, $\mathscr{K}(t,u)$ belongs to $L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)$. Moreover, there is a constant C such that*

$$||\mathscr{K}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)} \le Ct^2\mathrm{e}^t|u|^2_{L^2_\vartheta} \quad \textit{for all } t\ge 0 \textit{ and } u\in L^2_\vartheta\ .$$

(ii) Let (H) hold. There exists a constant $C>0$ such that if $u\in L^2_\vartheta$ satisfies $u(x)\ge 1$ for almost all $x\in\mathbb{R}^d$, then

$$\sum_k|\mathscr{K}(t,1)f_k|^2_{L^2_\vartheta} \le \sum_k|\mathscr{K}(t,u)f_k|^2_{L^2_\vartheta} + Ct^2\mathrm{e}^t|u|^2_{L^2_\vartheta}, \quad t\ge 0\ ,$$

for an arbitrary orthonormal basis $\{f_k\}\subset\mathscr{H}^0_{\mathscr{W}}$ of $\mathscr{H}_{\mathscr{W}}$.

Proof of (i). From (H) and Proposition 3.2, there is a $\kappa \geq 0$ such that $\Gamma_{N,\kappa}(x) \geq 0$ for all $N \in \mathbb{N}$ and $x \in \mathbb{R}^d$. First we prove the desired estimate under the additional assumption that $\kappa = 0$, that is

$$\Gamma_{N,0}(x) = \mathscr{F}(\mu_{N,0})(x) \geq 0 \quad \text{for all } N \in \mathbb{N} \text{ and } x \in \mathbb{R}^d \ .$$

Let $u \in C_{\mathrm{b}}(\mathbb{R}^d)$. We have

$$\begin{aligned} &|\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-2\pi i\langle z-z',y\rangle} u(z)u(z')G(t)\big(\mathrm{d}(x-z)\big)G(t)\big(\mathrm{d}(x-z')\big) \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-2\pi i\langle \eta-\eta',y\rangle} u(x-\eta)u(x-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta') \ . \end{aligned}$$

Since $\mu_{N,0}$, $N \in \mathbb{N}$, and $G(t)(x-\cdot)$, $t \geq 0$, $x \in \mathbb{R}^d$ are finite positive measures, and u is a bounded function we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathrm{e}^{-2\pi i\langle \eta-\eta',y\rangle} u(x-\eta)u(x-\eta')\right| G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\mu_{N,0}(\mathrm{d}y) < \infty \ .$$

Thus, by Fubini's theorem, for any N we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mu_{N,0}(\mathrm{d}y) \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')u(x-\eta)u(x-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta') \ . \end{aligned} \tag{3.3}$$

Let $\{f_k\} \subset \mathscr{K}^0_{\mathscr{W}}$ be an orthonormal basis of $\mathscr{K}_{\mathscr{W}}$. Then, by Lemma 3.1 and (3.3) we have

$$\begin{aligned} \sum_k |\mathscr{K}(t,u)f_k|^2_{L^2_\vartheta} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mu(\mathrm{d}y) \\ &= \lim_{N\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mathrm{e}^{-\frac{|y|^2}{N}}\mu(\mathrm{d}y) \\ &= \lim_{N\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \mu_{N,0}(\mathrm{d}y) \\ &= \lim_{N\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')u(x-\eta)u(x-\eta') \\ &\qquad G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\vartheta(x)\mathrm{d}x \ . \end{aligned} \tag{3.4}$$

Since for each $t > 0$, $\operatorname{supp} G(t) \subset \overline{B(0,t)}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')u(x-\eta)u(x-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\vartheta(x)\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^d}\int_{\overline{B(0,t)}}\int_{\overline{B(0,t)}} \Gamma_{N,0}(\eta-\eta')u(x-\eta)u(x-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\vartheta(x)\mathrm{d}x \ . \end{aligned}$$

Using now (2.2) we get for all $t \geq 0$ and $\eta, \eta' \in \overline{B(0,t)}$ the following estimate

$$\Big| \int_{\mathbb{R}^d} u(x-\eta)u(x-\eta')\vartheta(x)\mathrm{d}x \Big| \\ \leq \Big(\int_{\mathbb{R}^d} |u(x-\eta)|^2 \vartheta(x)\mathrm{d}x \Big)^{1/2} \Big(\int_{\mathbb{R}^d} |u(x-\eta')|^2 \vartheta(x)\mathrm{d}x \Big)^{1/2} \leq C_\vartheta \mathrm{e}^t |u|^2_{L^2_\vartheta} \ .$$

Hence, as $\Gamma_{N,0} \geq 0$, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')u(x-\eta)u(x-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\vartheta(x)\mathrm{d}x \\ \leq C_\vartheta \mathrm{e}^t |u|^2_{L^2_\vartheta} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta').$$

Taking in (3.4), $u = 1$ we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta') \\ = \Big(\int_{\mathbb{R}^d} \vartheta(x)\mathrm{d}x \Big)^{-1} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,0}(\eta-\eta')G(t)(\mathrm{d}\eta)G(t)(\mathrm{d}\eta')\vartheta(x)\mathrm{d}x \\ \underset{N\to\infty}{\longrightarrow} \Big(\int_{\mathbb{R}^d} \vartheta(x)\mathrm{d}x \Big)^{-1} \sum_k |\mathscr{K}(t,1)f_k|^2_{L^2_\vartheta} \ .$$

To summarize, for all $u \in C_{\mathrm{b}}(\mathbb{R}^d)$ and $t \geq 0$ we have the following estimate

$$\sum_k |\mathscr{K}(t,u)f_k|^2_{L^2_\vartheta} \leq C_\vartheta \mathrm{e}^t |u|^2_{L^2_\vartheta} \Big(\int_{\mathbb{R}^d} \vartheta(x)\mathrm{d}x \Big)^{-1} \sum_k |\mathscr{K}(t,1)f_k|^2_{L^2_\vartheta} \ .$$

Using now Corollary 3.1 we obtain the required estimate in (i). Let us assume now that (H) is satisfied with a given fixed $\kappa_0 > 0$. Then $\nu = \mu + \kappa_0\delta_0$ satisfies (H) with $\kappa = 0$. Let us denote by $\mathscr{H}_\mathscr{V}$ the kernel of the spatially homogeneous Wiener process $\mathscr{V}$ with the spectral measure ν. Then, since ν satisfies (0.6), we can find a constant C such that

$$||\mathscr{K}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V}, L^2_\vartheta)} \leq Ct^2\mathrm{e}^t|u|^2_{L^2_\vartheta} \quad \text{for all } t \geq 0, u \in C_{\mathrm{b}}(\mathbb{R}^d) \ . \tag{3.5}$$

Now, as

$$\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)\big)(y)|^2\mu(\mathrm{d}y) \leq \int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)\big)(y)|^2\nu(\mathrm{d}y) \ , \tag{3.6}$$

Lemma 3.1 yields that for arbitrary $u \in C_{\mathrm{b}}(\mathbb{R}^d)$, $t \geq 0$ and an orthonormal basis $\{f_k\} \subset \mathscr{H}^0_\mathscr{W}$ of $\mathscr{H}_\mathscr{W}$ one has

$$\sum_k |\mathscr{K}(t,u)f_k|^2_{L^2_\vartheta} \leq ||\mathscr{K}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V}, L^2_\vartheta)} \ .$$

Thus, by (3.5), $\mathscr{K}(t,u) \in L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_{\vartheta})$ and

$$||\mathscr{K}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_{\vartheta})} \le Ct^2 \mathrm{e}^t |u|^2_{L^2_{\vartheta}} \tag{3.7}$$

for all $t \ge 0$ and $u \in C_{\mathrm{b}}(\mathbb{R}^d)$. Since $C_{\mathrm{b}}(\mathbb{R}^d)$ is dense in L^2_{ϑ}, and $\mathscr{K}(t,u)$ is linear with respect to u, we have (3.7) for all $t \ge 0$ and $u \in L^2_{\vartheta}$, which completes the proof of (i). □

Proof of (ii) . Let (H) be satisfied with a given $\kappa_0 \ge 0$. Let $\nu = \mu + \kappa_0 \delta_0$, and let $\mathscr{V}$ be the homogeneous Wiener process with the spectral measure ν. Let $\mathscr{H}_{\mathscr{V}}$ denote the kernel of $\mathscr{V}$, and let $\mathscr{H}^0_{\mathscr{V}}$ be the subspace of $\mathscr{H}_{\mathscr{V}}$ consisting of all $\mathscr{F}(\eta\nu)$, where $\eta \in \mathscr{S}_{(\mathrm{s})}(\mathbb{R}^d)$, and let $\{g_k\} \subset \mathscr{H}^0_{\mathscr{V}}$ be an arbitrary orthonormal basis of $\mathscr{H}_{\mathscr{V}}$. Let $u \in L^2_{\vartheta}$, satisfies $u(x) \ge 1$ for $x \in \mathbb{R}^d$. Then applying (3.4) for ν and $\{g_k\}$ we obtain

$$\sum_k |\mathscr{K}(t,1)g_k|^2_{L^2_{\vartheta}} \le \sum_k |\mathscr{K}(t,u)g_k|^2_{L^2_{\vartheta}} \quad \text{for } t \ge 0 \ . \tag{3.8}$$

By Lemma 3.1 and (3.6) we have

$$\sum_k |\mathscr{K}(t,1)f_k|^2_{L^2_{\vartheta}} \le \sum_k |\mathscr{K}(t,1)g_k|^2_{L^2_{\vartheta}} \quad \text{for } t \ge 0 \tag{3.9}$$

and

$$\sum_k |\mathscr{K}(t,u)g_k|^2_{L^2_{\vartheta}} = \sum_k |\mathscr{K}(t,u)f_k|^2_{L^2_{\vartheta}} + \kappa_0 I(t,u) \quad \text{for } t \ge 0 \ , \tag{3.10}$$

where

$$I(t,u) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(G(t)(x-\cdot)u\big)(y)|^2 \delta_0(\mathrm{d}y)\vartheta(x)\mathrm{d}x \ .$$

Let $\mathscr{B}$ be the Wiener process with the spectral measure δ_0, and let $\mathscr{H}_{\mathscr{B}}$ be its kernel. Obviously, δ_0 satisfies (H) and (0.6). Thus, by Lemmas 3.1 and 3.3(i), $\mathscr{K}(t,u) \in L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{B}}, L^2_{\vartheta})$ for $t \ge 0$, and there is a constant $C_1 > 0$ such that

$$I(t,u) = ||\mathscr{K}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{B}}, L^2_{\vartheta})} \le C_1 t^2 \mathrm{e}^t \, |u|^2_{L^2_{\vartheta}} \quad \text{for } t \ge 0 \ . \tag{3.11}$$

Combining (3.8) to (3.11) we obtain the estimate

$$\sum_k |\mathscr{K}(t,1)f_k|^2_{L^2_{\vartheta}} \le \sum_k |\mathscr{K}(t,u)f_k|^2_{L^2_{\vartheta}} + \kappa_0 C_1 t^2 \mathrm{e}^t |u|^2_{L^2_{\vartheta}}, \quad t \ge 0 \ ,$$

which gives the desired conclusion. □

4. Proof of Theorem 0.1

Proof of (i). Having shown the estimates from the previous section we are able to prove the theorem in a rather short way. We use the Banach fixed point theorem in the space $\mathcal{X}_T$ of all L^2_ϑ-valued $(\mathfrak{F}_t)$-adapted processes z such that

$$\sup_{t\in[0,T]} \mathbb{E}\,|z(t)|^2_{L^2_\vartheta} < \infty \ .$$

Let $T > 0$ be an arbitrary but fixed. For $\rho \geq 0$ we set

$$||z||_\rho \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \mathrm{e}^{-\rho t}\big(\mathbb{E}\,|z(t)|^2_{L^2_\vartheta}\big)^{1/2}, \qquad z \in \mathcal{X}_T \ .$$

Note that the norms $||\cdot||_\rho$, $\rho \geq 0$ are equivalent, and $(\mathcal{X}_T, ||\cdot||_\rho)$ is a Banach space for an arbitrary ρ.

Let $u_0 \in H^1_\vartheta$ and $v_0 \in L^2_\vartheta$. Note that by Corollary 2.1,

$$\frac{\partial}{\partial t} G(t) * u_0 \stackrel{\text{def}}{=} \dot{\mathcal{G}}(t)u_0 \quad \text{and} \quad G(t) * v_0 \stackrel{\text{def}}{=} \mathcal{G}(t)v_0, \qquad t \geq 0 \ ,$$

satisfy

$$\sup_{0\leq t\leq T} \left\{|\dot{\mathcal{G}}(t)u_0|_{L^2_\vartheta} + |\mathcal{G}(t)v_0|_{L^2_\vartheta}\right\} < \infty \quad \text{for any } T \in (0,\infty) \ .$$

Let

$$y(t) = \dot{\mathcal{G}}(t)u_0 + \mathcal{G}(t)v_0 \qquad \text{for } t \geq 0 \ .$$

Let $z \in \mathcal{X}_T$. Then by $f(z)$ and $b(z)$ we denote the random fields $f(z)(t,x) = f(z(t,x))$ and $b(z)(t,x) = b(z(t,x))$, $x \in \mathbb{R}^d$, $t \in [0,T]$. For $t \in [0,T]$ we set $f(z(t))(x) = f(z(t,x))$ and $b(z(t))(x) = b(z(t,x))$. Note that $f(z), b(z) \in \mathcal{X}_T$ for any $z \in \mathcal{X}_T$. For $z \in \mathcal{X}_T$ we set

$$\begin{aligned} I_1(z)(t) &= y(t) + \int_0^t G(t-s) * f(z(s))\mathrm{d}s \\ &\stackrel{\text{def}}{=} y(t) + \int_0^t \mathcal{G}(t-s) f(z(s))\mathrm{d}s \ , \\ I_2(z)(t) &= \int_0^t G(t-s) * \big(b(z(s))\mathrm{d}\mathcal{W}(s)\big) \\ &\stackrel{\text{def}}{=} \int_0^t \mathcal{K}(t-s, b(z(s)))\mathrm{d}\mathcal{W}(s) \ . \end{aligned}$$

Then it is easy to show that I_1 maps $\mathcal{X}_T$ into $\mathcal{X}_T$, and that there is a $\rho_1 \geq 0$ such that

$$||I_1(z) - I_1(\tilde{z})||_\rho \leq \frac{1}{4}||z - \tilde{z}||_\rho \quad \text{for all } z, \tilde{z} \in \mathcal{X}_T, \rho \geq \rho_1 \ .$$

The same holds true for the stochastic integral mapping I_2. For, note that by (1.1) and Lemma 3.3(i) we have

$$\begin{aligned}\mathbb{E}\,|I_2(z)(t)|^2_{L^2_\vartheta} &= \mathbb{E}\int_0^t ||\mathscr{K}(t-s,b(z(s)))||^2_{L_{(HS)}(\mathscr{H}_\mathscr{W},L^2_\vartheta)}\mathrm{d}s\\ &\le Ct^3\mathrm{e}^t \sup_{s\in[0,t]}\mathbb{E}\,|b(z(s))|^2_{L^2_\vartheta}\ ,\end{aligned}$$

where C is independent of z and t. Since $b(z)\in\mathscr{X}_T$, we have $I_2(z)\in\mathscr{X}_T$. Now, let L be the Lipschitz constant for b. Then for arbitrary $\psi,\tilde\psi\in L^2_\vartheta$ we have

$$|b(\psi)-b(\tilde\psi)|_{L^2_\vartheta}\le L|\psi-\tilde\psi|_{L^2_\vartheta}\ ,$$

and consequently, by Lemma 3.3(i), for all $\rho\ge 0$ and $z,\tilde z\in\mathscr{X}_T$, we have

$$\begin{aligned}||I_2(z)-I_2(\tilde z)||^2_\rho &= \sup_{t\in[0,T]}\mathrm{e}^{-2\rho t}\,\mathbb{E}\,|I_2(z)(t)-I_2(\tilde z)(t)|^2_{L^2_\vartheta}\\
&= \sup_{t\in[0,T]}\mathrm{e}^{-2\rho t}\,\mathbb{E}\int_0^t ||\mathscr{K}\left(t-s,b(z(s))-b(\tilde z(s))\right)||^2_{L_{(HS)}(\mathscr{H}_\mathscr{W},L^2_\vartheta)}\mathrm{d}s\\
&\le \sup_{t\in[0,T]}\mathrm{e}^{-2\rho t}\,C\int_0^t (t-s)^2\mathrm{e}^{t-s}\mathbb{E}\,|b(z(s))-b(\tilde z(s))|^2_{L^2_\vartheta}\mathrm{d}s\\
&\le CT^2\mathrm{e}^T L^2\sup_{t\in[0,T]}\mathrm{e}^{-2\rho t}\int_0^t \mathbb{E}\,|z(s)-\tilde z(s)|^2_{L^2_\vartheta}\mathrm{d}s\\
&\le CT^2\mathrm{e}^T L^2\sup_{t\in[0,T]}\int_0^t \mathrm{e}^{-2\rho(t-s)}\mathrm{e}^{-2\rho s}\,\mathbb{E}\,|z(s)-\tilde z(s)|^2_{L^2_\vartheta}\mathrm{d}s\\
&\le CT^2\mathrm{e}^T L^2||z-\tilde z||^2_\rho\sup_{0\le t\le T}\int_0^t \mathrm{e}^{-2\rho(t-s)}\mathrm{d}s\\
&\le CT^2\mathrm{e}^T L^2(2\rho)^{-1}||z-\tilde z||^2_\rho\ ,\end{aligned}$$

which gives the desired conclusion. Thus we can find $\rho\ge 0$ such that $I=I_1+I_2$ is a contraction from $\mathscr{X}_T$ into $\mathscr{X}_T$. The Banach fixed point theorem yields that there is a unique solution $u\in\mathscr{X}_T$ to the equation

$$u=y+I_1(u)+I_2(u),$$

which completes the proof of the existence of a solution to (0.1). □

Proof of (ii) . Let u be a solution to (0.1) on a time interval $[0,T]$, where $T>0$. Then processes

$$\dot{\mathscr{G}}(t)u_0,\quad \mathscr{G}(t)v_0,\quad \int_0^t\mathscr{G}(t-s)f(u(s))\mathrm{d}s,\quad t\in[0,T]\ ,$$

belong to $\mathscr{X}_T$. Thus the process $\int_0^t\mathscr{K}(t-s,b(u(s)))\mathrm{d}\mathscr{W}(s)$, $t\in[0,T]$, also belongs to $\mathscr{X}_T$. Thus

$$\int_0^t\mathbb{E}\,||\mathscr{K}(t-s,b(u(s)))||^2_{L_{(HS)}(\mathscr{H}_\mathscr{W},L^2_\vartheta)}\mathrm{d}s<\infty\quad\text{for every } t\in[0,T]\ .$$

Note that as b is continuous, either $b(x) \geq c > 0$ for every x or $b(x) \leq -c < 0$ for all x. Thus, by Lemma 3.3(ii),

$$\int_0^t \mathbb{E}\,||\mathscr{K}(t-s,1)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)}\mathrm{d}s < \infty \quad \text{for } t \in [0,T] \ ,$$

and we conclude by Lemma 3.2 and Proposition 3.1. □

5. The case of the stochastic heat equation

We pass now to the nonlinear stochastic heat equation (0.2). The proof of Theorem 0.2 follows the same pattern as that of Theorem 0.1, given in the previous sections. We therefore restrict our attention to three basic technical results formulated as Lemmas 5.1 to 5.3 below. Recall that

$$P(t)(x) = (4\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{4t}} \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^d \ ,$$

and $P(0) = \delta_0$ is the fundamental solution to the Cauchy problem for the parabolic equation $\frac{\partial}{\partial t}u - \Delta u = 0$. Let

$$S(t)u(x) = \int_{\mathbb{R}^d} P(t)(x-y)u(y)\mathrm{d}y, \quad u \in \mathscr{S}(\mathbb{R}^d),\ t \geq 0 \ ,$$

be the heat semigroup. Then, see [19, Lemma 3.1], or [7], we have the following fact.

Lemma 5.1. *The semigroup S has a unique extension to a holomorphic semigroup on L^2_ϑ. In particular, denoting this extension also by S, we have the estimate*

$$\forall T > 0\ \exists C_T : \quad |S(t)u|_{L^2_\vartheta} \leq C_T|u|_{L^2_\vartheta} \quad \textit{for all } t \in [0,T],\, u \in L^2_\vartheta \ .$$

Recall that $\mathscr{P}$ is given by (1.5). Clearly,

$$\mathscr{P}(t,u)\eta = S(t)(u\eta) = P(t) * (u\eta) \quad \text{for } u \in L^2_\vartheta,\, \eta \in \mathscr{H}^0_{\mathscr{W}} \ .$$

We have the following version of Lemma 3.3.

Lemma 5.2. *Let condition (H) formulated in Section 0 be satisfied. Then for all $t > 0$ and $u \in C_{\mathrm{b}}(\mathbb{R}^d)$, $\mathscr{P}(t,u) \in L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)$. Moreover, there exists a constant C such that*

$$||\mathscr{P}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)} \leq C\mathrm{e}^t|u|^2_{L^2_\vartheta}\int_{\mathbb{R}^d}\mathrm{e}^{-8\pi^2 t|y|^2}\mu(\mathrm{d}y) \tag{5.1}$$

for all $t > 0$ and $u \in L^2_\vartheta$, and

$$||\mathscr{P}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_{\mathscr{W}},L^2_\vartheta)} + C\mathrm{e}^t|u|^2_{L^2_\vartheta} \geq \left(\int_{\mathbb{R}^d}\vartheta(x)\mathrm{d}x\right)\int_{\mathbb{R}^d}\mathrm{e}^{-4\pi^2 t|y|^2}\mu(\mathrm{d}y) \tag{5.2}$$

for all $t > 0$ and $u \in L^2_\vartheta$ satisfying $u(x) \geq 1$, $x \in \mathbb{R}^d$.

Proof. Let κ_0 be such that $\mu_{N,\kappa}(x) \geq 0$ for all $N \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Let $\nu = \mu + \kappa_0\delta_0$, and let $\mathscr{V}$ and $\mathscr{B}$ be homogeneous Wiener processes with the spectral measures ν and $\kappa_0\delta_0$, respectively. Let $\{f_k\} \subset \mathscr{K}^0_{\mathscr{W}}$, and $\{g_k\}$ be orthonormal bases of $\mathscr{H}_{\mathscr{W}}$ and $\mathscr{H}_{\mathscr{V}}$.

Note first that in the proofs of (3.1), (3.3) and (3.4) we did not use the form of the fundamental solution. Thus we have

$$\sum_k |\mathscr{P}(t,u)g_k|^2_{L^2_\vartheta} = \lim_{N\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,\kappa_0}(\eta-\eta')u(x-\eta)u(x-\eta') \times P(t)(\eta)P(t)(\eta')\vartheta(x)\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}\eta' \ . \tag{5.3}$$

Now for all $\eta, \eta' \in \mathbb{R}^d$ we have the following estimate

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} u(x-\eta)u(x-\eta')\vartheta(x)\mathrm{d}x\Big| \\ &\leq \Big(\int_{\mathbb{R}^d}|u(x-\eta)|^2\vartheta(x)\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^d}|u(x-\eta')|^2\vartheta(x)\mathrm{d}x\Big)^{1/2} \leq C_1\,\mathrm{e}^{\frac{|\eta|+|\eta'|}{2}}|u|^2_{L^2_\vartheta} \ .\end{aligned}$$

Hence, as $\Gamma_{N,\kappa_0} \geq 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_{N,\kappa_0}(\eta-\eta')u(x-\eta)u(x-\eta')P(t)(\eta)P(t)(\eta')\vartheta(x)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}\eta' \\ &\leq C_1|u|^2_{L^2_\vartheta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_{N,\kappa_0}(\eta-\eta')\,\mathrm{e}^{\frac{|\eta|+|\eta'|}{2}}P(t)(\eta)P(t)(\eta')\mathrm{d}\eta\,\mathrm{d}\eta' \ .\end{aligned}$$

Now note that

$$\begin{aligned}\mathrm{e}^{\frac{|\eta|+|\eta'|}{2}}P(t)(\eta)P(t)(\eta') &= (4\pi t)^{-d}\mathrm{e}^{\frac{|\eta|+|\eta'|}{2}-\frac{|\eta|^2+|\eta'|^2}{4t}} \\ &= (4\pi t)^{-d}\mathrm{e}^{\frac{4t|\eta|-|\eta|^2+4t|\eta'|-|\eta'|^2}{8t}}\ \mathrm{e}^{-\frac{|\eta|^2}{8t}-\frac{|\eta'|^2}{8t}} \\ &\leq 2^d\mathrm{e}^t\,(8\pi t)^{-d}\mathrm{e}^{-\frac{|\eta|^2}{8t}-\frac{|\eta'|^2}{8t}} \leq 2^d\mathrm{e}^t\ P(2t)(\eta)P(2t)(\eta') \ .\end{aligned}$$

Thus we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_{N,\kappa_0}(\eta-\eta')u(x-\eta)u(x-\eta')P(t)(\eta)P(t)(\eta')\vartheta(x)\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}\eta' \\ &\leq C_2\mathrm{e}^t\,|u|^2_{L^2_\vartheta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_{N,\kappa_0}(\eta-\eta')P(2t)(\eta)P(2t)(\eta')\vartheta(x)\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}\eta' \ ,\end{aligned}$$

where $C_2 = 2^dC_1\big(\int_{\mathbb{R}^d}\vartheta(x)\mathrm{d}x\big)^{-1}$. Letting $N \to \infty$ we obtain

$$\sum_k |\mathscr{P}(t,u)g_k|^2_{L^2_\vartheta} \leq C_2\mathrm{e}^t|u|^2_{L^2_\vartheta}\sum_k|\mathscr{P}(2t,1)g_k|^2_{L^2_\vartheta} \quad \text{for } u \in C_\mathrm{b}(\mathbb{R}^d),\ t \geq 0 \ . \tag{5.4}$$

Now

$$\sum_k |\mathscr{P}(t,1)g_k|^2_{L^2_\vartheta} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(P(t)(x-\cdot)\big)(y)|^2\nu(\mathrm{d}y)\vartheta(x)\mathrm{d}x$$
$$= \left(\int_{\mathbb{R}^d}\vartheta(x)\mathrm{d}x\right)\int_{\mathbb{R}^d} \mathrm{e}^{-4\pi^2 t|y|^2}\nu(\mathrm{d}y) \ . \tag{5.5}$$

Note that from (5.4) and (5.5) we have $\mathscr{P}(t,u) \in L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V}, L^2_\vartheta)$. Since, by Lemma 3.1,

$$\sum_k |\mathscr{P}(t,u)f_k|^2_{L^2_\vartheta} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(P(t)(x-\cdot)\big)(y)|^2\mu(\mathrm{d}y)\vartheta(x)\mathrm{d}x$$
$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(P(t)(x-\cdot)\big)(y)|^2\big(\mu(\mathrm{d}y)+\kappa_0\delta_0(\mathrm{d}y)\big)\vartheta(x)\mathrm{d}x$$
$$\le ||\mathscr{P}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V},L^2_\vartheta)} \ ,$$

we have $\mathscr{P}(t,u) \in L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{W}, L^2_\vartheta)$ and (5.1) holds true for all $t>0$ and $u \in L^2_\vartheta$. We now show (5.2). Clearly, it is enough to show this for $u \in C_\mathrm{b}(\mathbb{R}^d)$. Let $u \in C_\mathrm{b}(\mathbb{R}^d)$ satisfy $u(x) \ge 1$ for every $x \in \mathbb{R}^d$. Then by Lemma 3.1 and (5.3) we have

$$||\mathscr{P}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{W},L^2_\vartheta)} = ||\mathscr{P}(t,u)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V},L^2_\vartheta)} - \kappa_0 I(t)$$
$$\ge ||\mathscr{P}(t,1)||^2_{L_{(\mathrm{HS})}(\mathscr{H}_\mathscr{V},L^2_\vartheta)} - \kappa_0 I(t) \ , \tag{5.6}$$

where

$$I(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{F}\big(P(t)(x-\cdot)u\big)(y)|^2\delta_0(\mathrm{d}y)\vartheta(x)\mathrm{d}x$$
$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P(t)(x-z)|u(z)|^2\vartheta(x)\mathrm{d}z\,\mathrm{d}x$$
$$\le C_3\left(\int_{\mathbb{R}^d}(4\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|r|^2}{4t}+|r|}\mathrm{d}r\right)|u|^2_{L^2_\vartheta} \le C_4\mathrm{e}^t|u|^2_{L^2_\vartheta} \ .$$

Combining this with (5.5) and (5.6) we obtain (5.2), which completes the proof. □

Note that

$$\int_0^t\int_{\mathbb{R}^d}\mathrm{e}^{-4\pi^2 s|y|^2}\mu(\mathrm{d}y)\mathrm{d}s = \int_{\mathbb{R}^d}\frac{1-\mathrm{e}^{-4\pi^2 t|y|^2}}{4\pi^2|y|^2}\mu(\mathrm{d}y) \ .$$

Thus there are continuous functions $c, \tilde{c} : (0,\infty) \to (0,\infty)$ such that

$$c(t)\int_{\mathbb{R}^d}\frac{\mu(\mathrm{d}y)}{1+|y|^2} \le \int_0^t\int_{\mathbb{R}^d}\mathrm{e}^{-4\pi^2 s|y|^2}\mu(\mathrm{d}y)\mathrm{d}s$$
$$\le \tilde{c}(t)\int_{\mathbb{R}^d}\frac{\mu(\mathrm{d}y)}{1+|y|^2} \quad \text{for all } t>0 \ . \tag{5.7}$$

Combining (5.7) with Lemma 5.2 we get the following result.

Lemma 5.3. *Let condition* (H) *be satisfied. Then there are continuous function* $C_1, C_2, C_3 : (0,\infty) \to (0,\infty)$ *such that*

$$\int_0^t ||\mathscr{P}(s,u)||^2_{L_{(HS)}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)} ds \le C_1(t)|u|^2_{L^2_\vartheta} \int_{\mathbb{R}^d} \frac{\mu(dy)}{1+|y|^2}$$

for all $t > 0$ *and* $u \in L^2_\vartheta$, *and*

$$\int_0^t ||\mathscr{P}(s,u)||^2_{L_{(HS)}(\mathscr{H}_{\mathscr{W}}, L^2_\vartheta)} ds + C_2(t)|u|^2_{L^2_\vartheta} \ge C_3(t) \int_{\mathbb{R}^d} \frac{\mu(dy)}{1+|y|^2}$$

for all $t > 0$ *and* $u \in L^2_\vartheta$ *satisfying* $u(x) \ge 1$, $x \in \mathbb{R}^d$.

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