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The log-Sobolev inequality for weakly coupled lattice fields

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Abstract. We consider a ferromagnetic spin system with unbounded interactions on the d-dimensional integer lattice ($d \geq 1$). Under mild assumptions on the one-body interactions (so that arbitrarily deep double wells are allowed), we prove that if the coupling constants are small enough, then the finite volume Gibbs states satisfy the log-Sobolev inequality uniformly in the volume and the boundary condition.

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1. Introduction

In this paper we address a question of understanding the ergodic property of unbounded lattice spin systems. We will consider a random field on $\mathbf{Z}^d$ described by the formal Hamiltonian

$$H(\sigma) = -\frac{1}{2} \sum_{x,y \in \mathbf{Z}^d} J_{x,y} \sigma_x \sigma_y + \sum_{x \in \mathbf{Z}^d} (U(\sigma_x) - h_x \sigma_x) \quad , \qquad (1.1)$$

where $\sigma_x \in \mathbf{R}$ is the spin at the site $x \in \mathbf{Z}^d$, $J_{x,y}$ are finite range, ferromagnetic coupling constants (cf. (1.18)–(1.21) below), $h_x \in \mathbf{R}$ and $U(s)$ is a function which diverges to $+\infty$ faster than any constant multiple of s^2 as $|s| \nearrow \infty$.

Literatures on the equilibrium statistical mechanics for models of this kind are vast and their phase structure are fairly well understood in some cases (See [FSS76], [COPP78] and [BH82] for example and references therein). On the other hand, from the view point of probability theory, it seems very natural to be interested in the time evolution of the model, or more specifically, the associated stochastic dynamics called Glauber dynamics. Glauber dynamics is a natural model to describe the way a given configuration relaxes to equilibrium as time goes by. Therefore, its ergodic properties are very interesting subject to work on. The study of such dynamical theory for unbounded spin systems seems to be in a much more primitive stage as compared with that for models with compact spin spaces, but it begins to capture the attentions of many probabilists in recent years.

A key to rapid progresses in the study of dynamical lattice spin systems with compact spin spaces was the log-Sobolev inequality (See [SZ92], [LY93], [MO94] and references therein). Although the proof of the log-Sobolev inequality for unbounded spin systems are much harder, it has become increasingly feasible in view of some successful examples. The simplest case is when one can apply the Bakry-Emery Γ_2 criterion ([BE85]) of log-Sobolev inequality. The Γ_2 criterion is applicable to continuous spin systems when the Hamiltonian of the model is a strictly convex function. This is in particular the case with the Hamiltonian (1.1) when $\inf U''$ is positive (To be precise, $\inf U'' > -\lambda(\mathbf{J})$ is enough. cf. (1.21) below). In [Z96], the log-Sobolev inequality is extended beyond the applicability region of the Γ_2-criterion. In fact, the results in [Z96] are applicable to (1.1) even when U'' can take large negative values and we see that the log-Sobolev inequality is true in at least the following two cases;

(a) $d = 1$ (cf. [Z96, Theorem 4.1]),

(b) $d \geq 2$, $J_{x,y}$ are large enough and U is a small perturbation from a strictly convex function in the sense of the sup-norm (cf. paragraphs following [Z96, Proposition 5.2]).

In this article, we mainly consider the restriction of the lattice field described by the Hamiltonian (1.1) to a finite set Λ in the lattice by imposing a boundary condition $\omega \in \mathbf{R}^{\Lambda^c}$. We prove that, if $J_{x,y}$'s ($x \neq y$) are small enough, then the log-Sobolev inequality holds uniformly in Λ and ω. This implies in particular that the (unique) infinite volume tempared DLR-state also satisfies the log-Sobolev inequality. In view of the corresponding phase structure (See Theorem 1.1 below), our assumption seems to be one of the most natural example to be investigated. Heuristically speaking, the proof of log-Sobolev inequality under our assumption is possible for the following reason. As is well known, the log-Sobolev constant is invariant under independent product of underlying measure space ([G94, Theorem 2.3]) and, since the interaction is weak under our assumption, spins on each site in the lattice behave almost independently.

The proof is divided into three steps:

First of all, we prepare a theorem which says that if a certain mixing condition for the Gibbs measure is satisfied, then the spectral gap is uniformly positive in the volume Λ and the boundary condition ω (Theorem 2.2). Here, we will use the method of [LY93] as the basic strategy, as well as ideas from [Z96].

We next prove that the mixing condition mentioned above implies the log-Sobolev inequality which is uniform in Λ and ω (Theorem 2.1). We will do this with the help of the bound on spectral gap obtained in Theorem 2.2. Here, the basic strategy is again the one in [LY93].

Finally, we show that the mixing condition referred to above holds assuming that the coupling constants are small enough (Theorem 2.4). This is carried out by using a formula for the Vassershtein distance ([COPP78]) and a "constructive criterion" (Proposition 2.3), which is very similar to the famous condition for the phase uniqueness presented in [DSh85].

We begin by introducing the standard setup of the model.

The lattice. We will work on the d-dimensional integer lattice $\mathbf{Z}^d = \{x = (x^i)_{i=1}^d : x^i \in \mathbf{Z}\}$ on which we consider the l_∞-metric; $d(x_1, x_2) = \max_{1 \le i \le d} \times |x_1^i - x_2^i|$ $(x_1, x_2 \in \mathbf{Z}^d)$. For a set $\Lambda \subset \mathbf{Z}^d$, diam$\Lambda$ and $|\Lambda|$ stand respectively for its diameter and the cardinality. We write $\Lambda \subset\subset \mathbf{Z}^d$ when $1 \le |\Lambda| < \infty$. The distance between two subsets Λ_1 and Λ_2 of $\mathbf{Z}^d$ will be denoted by $d(\Lambda_1, \Lambda_2)$. For $R \ge 1$, the R-boundary of a set Λ is defined by

$$\partial_R \Lambda = \{x \notin \Lambda; d(x, \Lambda) \le R\} \quad . \tag{1.2}$$

The value of R will eventually be chosen as the range $R(\mathbf{J})$ of the interaction we consider (See (1.18) below). We say $\Lambda \subset\subset \mathbf{Z}^d$ is a *generalized box* with size $(n_1, \ldots, n_d)$, if it can be decomposed as follows;

$$\Lambda = \overset{\circ}{\Lambda} \cup \delta\Lambda \quad , \tag{1.3}$$

where

$$\overset{\circ}{\Lambda} = \{x \in \mathbf{Z}^d;\ v^i \le x^i < v^i + n_i,\ i = 1, \ldots, d\} \text{ for some } (v^i)_{i=1}^d \in \mathbf{Z}^d ,$$

$$\delta\Lambda \subset \left\{ x \notin \overset{\circ}{\Lambda};\ \sum_{i=1}^d |x^i - y^i| = 1 \text{ for some } y \in \overset{\circ}{\Lambda} \right\} \quad .$$

Thus a generalized box is a box with (or without) "dust" on its faces. We call $\min_{1 \le i \le d} n_i$ in the above definition the *minimum side-length* of Λ. We set

$$\mathscr{A} = \{\Lambda;\ \Lambda \subset\subset \mathbf{Z}^d\} \quad , \tag{1.4}$$

$$\mathscr{B}(n_0) = \text{all generalized boxes with the minimum side-length at least } n_0 \quad , \tag{1.5}$$

where $n_0 \ge 1$.

The configuration spaces. The configuration spaces are defined as follows;

$$\mathbf{R}^\Lambda = \{\sigma = (\sigma_x)_{x\in\Lambda};\ \sigma_x \in \mathbf{R}\}, \quad \Lambda \subset \mathbf{Z}^d,$$

$$\Omega = \mathbf{R}^{\mathbf{Z}^d},$$

$$\mathscr{S} = \bigcap_{n\geq 1}\left\{\sigma \in \Omega;\ \sup_{x\in\mathbf{Z}^d}(1+d(x,0))^n|\sigma_x| < \infty\right\},$$

$$\mathscr{S}' = \bigcup_{n\geq 1}\left\{\sigma \in \Omega;\ \sup_{x\in\mathbf{Z}^d}(1+d(x,0))^{-n}|\sigma_x| < \infty\right\} \ .$$

The functions of the configuration. Function spaces $\mathscr{C}$ and $\mathscr{C}_\Lambda$ ($\Lambda \subset \mathbf{Z}^d$) on the configuration space Ω are introduced as follows;

$$\mathscr{C} = \{f : \Omega \longrightarrow \mathbf{R} \mid f \text{ satisfies the properties } \textbf{(C1)} \text{ and } \textbf{(C2)} \text{ below}\} \ . \tag{1.6}$$

(**C1**) There is $\Lambda \subset\subset \mathbf{Z}^d$ such that f depends only on $(\sigma_x)_{x\in\Lambda}$ and is of C^∞ with respect to these variables.

(**C2**) There are positive constants $B_{1.7}$ and $C_{1.7}$ such that

$$|f(\sigma)| + |\nabla_\Lambda f(\sigma)| \ \leq \ B_{1.7}\exp\left(C_{1.7}\sum_{x\in\Lambda}|\sigma_x|\right) \tag{1.7}$$

for any $\sigma \in \Omega$, where

$$|\nabla_\Lambda f(\sigma)|^2 = \sum_{x\in\Lambda}\left|\frac{\partial}{\partial\sigma_x}f(\sigma)\right|^2 \ . \tag{1.8}$$

For $f \in \mathscr{C}$, we denote by S_f the minimal set among those Λ's which satisfy the property referred to in **(C1)** above. We define

$$\mathscr{C}_\Lambda = \left\{f \in \mathscr{C};\ S_f \subset \Lambda\right\}, \quad \Lambda \subset \mathbf{Z}^d \ . \tag{1.9}$$

The Hamiltonian. We introduce a function $U : \mathbf{R} \to \mathbf{R}$ which satisfies **(U1)** and **(U2)** below.

(U1) For any $m > 0$, there exist $V, W \in C^\infty(\mathbf{R} \to \mathbf{R})$ and $C_{1.12} \in (0, \infty)$ such that

$$U(s) = V(s) + W(s) \quad \text{for all } s \in \mathbf{R}, \tag{1.10}$$

$$\inf_s V''(s) \ \geq \ m \ , \tag{1.11}$$

$$W(s) = 0 \quad \text{for } |s| \ \geq \ C_{1.12} \tag{1.12}$$

$$\|W\|_\infty \ \leq \ C_{1.12} \ \ , \tag{1.13}$$

where $\|W\|_\infty = \sup_s |W(s)|$.

(U2)

$$pq \frac{\partial^2 \mathscr{U}}{\partial p \partial q}(p, q) \ \geq \ 0 \quad \text{for all } (p, q) \in \mathbf{R}^2 \ \ , \tag{1.14}$$

where

$$\mathscr{U}(p, q) = U\left(\frac{q+p}{\sqrt{2}}\right) + U\left(\frac{q-p}{\sqrt{2}}\right) \ \ . \tag{1.15}$$

A typical example of U is given by the following polynomial;

$$U(s) = \sum_{\nu=1}^{N} a_{2\nu} s^{2\nu} \tag{1.16}$$

where $N \ \geq \ 2$, $a_2 \in \mathbf{R}$, $a_4 \ \geq \ 0, \ldots, a_{2(N-1)} \ \geq \ 0$ and $a_{2N} > 0$. Since a_2 can be large negative value, U in (1.16) may have arbitrarily deep double wells.

For $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$, we define a function $H^{\Lambda,\omega} : \Omega \to \mathbf{R}$, by;

$$H^{\Lambda,\omega}(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{x,y}\sigma_x\sigma_y + \sum_{x\in\Lambda} \left(U(\sigma_x) - h_x\sigma_x - \sum_{y\notin\Lambda} J_{x,y}\sigma_x\omega_y \right) \ . \tag{1.17}$$

Here, $\mathbf{J} = (J_{x,y} \in \mathbf{R}; \ x, y \in \mathbf{Z}^d)$, $\mathbf{h} = (h_z \in \mathbf{R}; \ z \in \mathbf{Z}^d)$ are such that

$$R(\mathbf{J}) \stackrel{\text{def.}}{=} \sup \left\{ d(x, y) \ ; \ J_{x,y} \neq 0 \right\} < \infty \ \ , \tag{1.18}$$

$$\|\mathbf{J}\| \stackrel{\text{def.}}{=} \sup_x \sum_y |J_{x,y}| < \infty \ \ , \tag{1.19}$$

$$J_{x,y} \ = \ J_{y,x} \ \geq \ 0 \quad \text{if } x \neq y \ \ , \tag{1.20}$$

$$\lambda(\mathbf{J}) \stackrel{\text{def.}}{=} \inf_x \left\{ -\frac{1}{2} \sum_y J_{xy} \right\} > 0 \ \ , \tag{1.21}$$

Note that we have from (1.20) and (1.21) that for any $\Lambda \subset\subset \mathbf{Z}^d$ and $\sigma \in \mathbf{R}^\Lambda$

$$-\frac{1}{2} \sum_{x,y\in\Lambda} J_{x,y}\sigma_x\sigma_y \ \geq \ \lambda(\mathbf{J}) \sum_{x\in\Lambda} |\sigma_x|^2 \ \ . \tag{1.22}$$

Remark 1.1. If a matrix $\mathbf{J}$ satisfies (1.18)–(1.20), we may also assume (1.21) without changing the model. In fact, define $\hat{J}_{x,y} = J_{x,y} - (\|\mathbf{J}\| + 1)\delta_{x,y}$ and $\hat{U}(s) = U(s) - \frac{\|\mathbf{J}\|+1}{2}s^2$. Then $\hat{J}_{x,y}$ satisfies (1.18)–(1.21) and $\hat{U}$ satisfies (1.10)–(1.14) and the replacement of $(J_{x,y}, U)$ by $(\hat{J}_{x,y}, \hat{U})$ does not change the Hamiltonian (1.17).

The local specifications and the DLR-state. For a topological space X, we let $\mathscr{M}_1(X)$ denote the set of Borel probability measures on X. For $\Lambda \subset\subset \mathbf{Z}^d$ and a boundary condition $\omega \in \Omega$, we define $E^{\Lambda,\omega} \in \mathscr{M}_1(\mathbf{R}^\Lambda)$ by;

$$E^{\Lambda,\omega}(d\sigma_\Lambda) = \frac{\exp(-H^{\Lambda,\omega}(\sigma))}{Z^{\Lambda,\omega}} \prod_{x\in\Lambda} d\sigma_x \tag{1.23}$$

where $Z^{\Lambda,\omega}$ is the normalizing constant. $E^{\Lambda,\omega}$ is called the *finite volume Gibbs state* and the family $\left\{E^{\Lambda,\omega} \mid \Lambda \subset\subset \mathbf{Z}^d,\ \omega \in \mathscr{S}\right\}$ is called the *local specification*. The following exponential integrability estimate is true; there is $C_{1.24} = C_{1.24}(\mathbf{J}, \mathbf{h}, U) \in (0, \infty)$ such that

$$E^{\Lambda,\omega} \exp(\lambda|\sigma_x|) \leq \exp\left(C_{1.24}\left(1 + \lambda^2 + \lambda \sum_{y\in\partial_R\Lambda} |\omega_y|\right)\right), \quad \text{for } \lambda > 0\ , \tag{1.24}$$

whenever $x \in \Lambda \subset\subset \mathbf{Z}^d$, $\omega \in \Omega$ and $\lambda > 0$. This follows from the arguments in Proposition III.1 and Theorem III.2 in [BH82]. The bound (1.24) implies that a function in the class $\mathscr{C}$ and its first derivatives have moments of all order with respect to the measure $E^{\Lambda,\omega}$.

For $\nu \in \mathscr{M}_1(\Omega)$, we define a new measure $\nu E^\Lambda \in \mathscr{M}_1(\Omega)$ by;

$$\nu E^\Lambda f = \int \nu(d\omega) \int E^{\Lambda,\omega}(d\sigma) f(\sigma_\Lambda \cdot \omega_{\Lambda^c})\ , \tag{1.25}$$

where $\sigma_\Lambda \cdot \omega_{\Lambda^c}$ denotes the following configuration;

$$(\sigma_\Lambda \cdot \omega_{\Lambda^c})_x = \begin{cases} \sigma_x & \text{if } x \in \Lambda, \\ \omega_x & \text{if } x \notin \Lambda\ . \end{cases}$$

It is a common practice to regard the measure $E^{\Lambda,\omega}$, which was originally defined as a measure on $\mathbf{R}^\Lambda$, as a measure on the full configuration space Ω by identifying it with $\delta_\omega E^\Lambda$, where δ_w is the Dirac measure concentrated on ω. With this in mind, we introduce an integral operator $E^\Lambda : \mathscr{C} \longrightarrow \mathscr{C}$ by;

$$E^\Lambda f(\sigma) = E^{\Lambda,\sigma}(f)\ . \tag{1.26}$$

We now define two subsets $\mathscr{G}$ and $\mathscr{G}_{\mathrm{t}}$ of $\mathscr{M}_1(\Omega)$ as follows;

$$\mathscr{G} = \left\{ \nu \in \mathscr{M}_1(\Omega);\ \ \nu E^{\Lambda} = \nu \text{ for any } \Lambda \subset\subset \mathbf{Z}^d \right\}, \tag{1.27}$$

$$\mathscr{G}_{\mathrm{t}} = \mathscr{G} \cap \mathscr{M}_{1,\mathrm{t}}(\Omega), \tag{1.28}$$

where

$$\mathscr{M}_{1,\mathrm{t}}(\Omega) = \left\{ \nu \in \mathscr{M}_1(\Omega);\ \ (\nu(|\sigma_x|))_{x \in \mathbf{Z}^d} \in \mathscr{S}' \right\}. \tag{1.29}$$

A measure in $\mathscr{G}$ and $\mathscr{G}_{\mathrm{t}}$ is called respectively, the *DLR-state* and the *tempared DLR-state*. It is known that the tempared DLR-state is unique if $J_{x,y}$ $(x \neq y)$ are small enough.

Theorem 1.1. ([COPP78, DSh85]) *There exists* $\beta = \beta(U) \in (0, \infty)$ *such that* $\mathscr{G}_{\mathrm{t}}$ *is a singleton if* $\sup_x \sum_{y : y \neq x} J_{x,y} \ \leq\ \beta$.

The inverse spectral gap and log-Sobolev constant. We define the *inverse spectral gap* $\gamma_{\mathrm{SG}}(\Lambda)$ as the smallest γ for which the following inequality is true for all $f \in \mathscr{C}$ and $\omega \in \Omega$;

$$E^{\Lambda,\omega}(f; f) \ \leq\ \gamma E^{\Lambda,\omega}(|\nabla_\Lambda f|^2)\ . \tag{1.30}$$

Here and in what follows, the following common notaiton for the covariance of functions f and g with respect to a probability measure m is used;

$$m\,(f; g) = m(fg) - m(f) \cdot m(g)\ . \tag{1.31}$$

We define the *log-Sobolev constant* $\gamma_{\mathrm{LS}}(\Lambda)$ as the smallest γ for which the following inequality is true for all $f \in \mathscr{C}$ and $\omega \in \Omega$;

$$E^{\Lambda,\omega}\left(f^2 \log \frac{f^2}{E^{\Lambda,\omega}(f^2)} \right) \ \leq\ \gamma E^{\Lambda,\omega}(|\nabla_\Lambda f|^2)\ . \tag{1.32}$$

It is well known that $2\gamma_{\mathrm{SG}} \ \leq\ \gamma_{\mathrm{LS}}$ (cf. [DS89, Corollary 6.1.17]).

Measures $\mathscr{E}_{\Lambda,q}$, $\mathscr{E}_{\Lambda,+}$ *and* $\mathscr{E}_{\Lambda,q}^{\Lambda\setminus\Delta,\bar{p}}$. We now introduce some new measures on the configuration space, which plays important roles in this article. In fact, a mixing condition (2.1) we will assume to derive the log-Sobolev inequality will be described in terms of these new measures rather than the original local specification defined by (1.23). For $\Lambda \subset\subset \mathbf{Z}^d$, $\omega \in \mathbf{R}^{\Lambda^{\mathrm{c}}}$ and $q \in \mathbf{R}^{\Lambda}$, we define a measure $\mathscr{E}_{\Lambda,q} \in \mathscr{M}_1(\mathbf{R}^{\Lambda})$ by

$$\begin{aligned} \mathscr{E}_{\Lambda,q}(dp) &= E^{\Lambda,\omega} \otimes E^{\Lambda,\omega}\left((\sigma^1, \sigma^2);\ \ \frac{\sigma^1 - \sigma^2}{\sqrt{2}} \in dp \,\middle|\, \frac{\sigma^1 + \sigma^2}{\sqrt{2}} = q \right) \\ &= \frac{\exp(-\mathscr{H}_{\Lambda,q}(p))}{\mathscr{Z}_{\Lambda,q}} \prod_{x \in \Lambda} dp_x\ \ , \end{aligned} \tag{1.33}$$

where

$$\mathscr{H}_{\Lambda,q}(p) = -\frac{1}{2}\sum_{x,y\in\Lambda} J_{x,y}p_x p_y + \sum_{x\in\Lambda}\mathscr{U}(p_x, q_x) \tag{1.34}$$

(Recall that we have defined the function $\mathscr{U}$ by (1.15)). It is also convenient to introduce the following measure;

$$\mathscr{E}_{\Lambda,+}(dq) = E^{\Lambda,\omega}\otimes E^{\Lambda,\omega}\left\{(\sigma^1,\sigma^2);\ \frac{\sigma^1+\sigma^2}{\sqrt{2}}\in dq\right\} \ . \tag{1.35}$$

The use of these measures can be demonstrated in the following expression of the spin-spin correlation function;

$$\begin{aligned} E^{\Lambda,\omega}(\sigma_x;\sigma_y) &= \tfrac{1}{2}\int E^{\Lambda,\omega}\otimes E^{\Lambda,\omega}(d\sigma^1 d\sigma^2)\left(\sigma_x^1-\sigma_x^2\right)\left(\sigma_y^1-\sigma_y^2\right) \\ &= \int \mathscr{E}_{\Lambda,+}(dq)\mathscr{E}_{\Lambda,q}\left(p_x p_y\right) \ . \end{aligned} \tag{1.36}$$

As can be seen from the above expression, if the correlation function $\mathscr{E}_{\Lambda,q}\left(p_x p_y\right)$ decays in $d(x,y)$ uniformly in q-variable, so does $E^{\Lambda,\omega}(\sigma_x;\sigma_y)$, uniformly in ω and $\mathbf{h}$. This idea will be used repeatedly in proofs in Section 3 (cf. Lemma 3.3 and Lemma 3.4). Furthermore, we define $\mathscr{E}^{W,\bar{p}}_{\Lambda,q}\in\mathscr{M}_1(\mathbf{R}^W)$ for $W\subset\Lambda$ and $\bar{p}\in\mathbf{R}^\Lambda$ by

$$\begin{aligned} \mathscr{E}^{W,\bar{p}}_{\Lambda,q}(dp_W) &= \mathscr{E}_{\Lambda,q}(dp_\Lambda \mid p\equiv\bar{p} \text{ on } \Lambda\backslash W) \\ &= \frac{\exp(-\mathscr{H}^{W,\bar{p}}_{\Lambda,q}(p))}{\mathscr{Z}^{W,\bar{p}}_{\Lambda,q}}\prod_{x\in W}dp_x \ , \end{aligned} \tag{1.37}$$

where

$$\mathscr{H}^{W,\bar{p}}_{\Lambda,q}(p) = \mathscr{H}_{W,q}(p) - \sum_{x\in W,y\in\Lambda\backslash W} J_{x,y}p_x\bar{p}_y \ . \tag{1.38}$$

The Vassershtein distance. For a metric space (X,ρ), which is separable and complete (hence is a Polish space), we define

$$\mathscr{M}_{1,\rho}(X) = \{\mu\in\mathscr{M}_1(X);\ \ \rho(x,\cdot)\in L_1(\mu) \text{ for some } x\in X\} \ . \tag{1.39}$$

We introduce the Vassershtein distance $\mathscr{R}_\rho$ on $\mathscr{M}_{1,\rho}(X)$ as follows;

$$\mathscr{R}_\rho(\mu_1,\mu_2) = \inf\left\{\int_{X^2}\mu(dx_1dx_2)\rho(x_1,x_2);\ \mu\in\mathscr{K}(\mu_1,\mu_2)\right\} \ , \tag{1.40}$$

where

$$\mathscr{K}(\mu_1,\mu_2)=\left\{\mu\in\mathscr{M}_1(X^2);\ \mu(dx_1\times X)=\mu_1,\ \mu(X\times dx_2)=\mu_2\right\}\ . \tag{1.41}$$

The case of $X=\mathbf{R}^\Lambda$ and $\rho(\sigma^1,\sigma^2)=\sum_{x\in\Lambda}|\sigma_x^1-\sigma_x^2|$ is especially relevant to us. The Vassershtein distance in this case is denoted by $\mathscr{R}_\Lambda$; for $\mu_i\in\mathscr{M}_{1,\rho}(\mathbf{R}^\Lambda)$ $(i=1,2)$,

$$\mathscr{R}_\Lambda(\mu_1,\mu_2)=\inf\left\{\int_{\mathbf{R}^\Lambda\times\mathbf{R}^\Lambda}\mu(d\sigma^1d\sigma^2)\sum_{x\in\Lambda}|\sigma_x^1-\sigma_x^2|;\ \mu\in\mathscr{K}(\mu_1,\mu_2)\right\}\ . \tag{1.42}$$

2. Results

We have the following results for the lattice field described by the Hamiltonian (1.17). Recall that we have defined the measure $\mathscr{E}_{\Lambda,q}^{W,\bar{p}}$ by (1.37).

Theorem 2.1. *Let $\mathscr{F}$ be either $\mathscr{A}$ or $\mathscr{B}(n_0)$ for arbitrarily fixed $n_0>0$ (cf.* (1.4) *and* (1.5)*). Suppose that the following mixing condition holds; there exist positive constants $B_{2.1}$ and $C_{2.1}$ such that*

$$\sup_{q\in\mathbf{R}^\Lambda}\left|\mathscr{E}_{\Lambda,q}^{W,\bar{p}^1}(p_z)-\mathscr{E}_{\Lambda,q}^{W,\bar{p}^2}(p_z)\right| \le B_{2.1}\left(1+\sum_{w\in\Lambda\cap\partial_R W}(|\bar{p}_w^1|+|\bar{p}_w^2|)\right)\exp\left(-\frac{d(y,z)}{C_{2.1}}\right) \tag{2.1}$$

whenever $\Lambda\in\mathscr{F}$, $W\subset\Lambda$, $y\in\Lambda\backslash W$ and $\bar{p}^i\in\mathbf{R}^\Lambda$ $(i=1,2)$ differs only at y. Then the log-Sobolev constants (cf. (1.32)*) are uniformly bounded in the sense that;*

$$\sup\{\gamma_{\mathrm{LS}}(\Lambda)\mid\Lambda\in\mathscr{F}\}\le C_{2.2}<\infty\ , \tag{2.2}$$

where the constant $C_{2.2}$ depends only on d, U, $\mathbf{J}$, $B_{2.1}$ and $C_{2.1}$. Therefore, the unique element μ in $\mathscr{G}_{\mathrm{t}}$ (cf. Remark 2.1) satisfies the log-Sobolev inequality;

$$\mu\left(f^2\log\frac{f^2}{\mu(f^2)}\right)\le C_{2.2}\mu(|\nabla_\Lambda f|^2) \tag{2.3}$$

for any $\Lambda\subset\subset\mathbf{Z}^d$ and $f\in\mathscr{C}_\Lambda$.

Remark 2.1. The mixing condition (2.1) implies the uniqueness of the tempared DLR-state. In fact, as we will see in Lemma 3.3 below, (2.1) implies the following;

$$|E^{\Lambda,\omega}(\sigma_x;\sigma_y)| \le C_{2.4}\exp\left(-\frac{d(x,y)}{C_{2.4}}\right) \ , \tag{2.4}$$

for any $\Lambda \in \mathscr{F}$, $\omega \in \mathbf{R}^{\Lambda^c}$ and $x, y \in \Lambda$, where $C_{2.4} = C_{2.4}(U, \mathbf{J}) \in (0, \infty)$. Furthermore, it is not difficult to prove by standard arguments that (2.4) implies the uniqueness of the tempared DLR-state (For example, Theorem IV.3, Proposition IV.4 and Proposition V.1 in [BH82] can be used to show that "pure phases" $\mu_\pm$ coincide.).

Remark 2.2. We will see that the mixing condition (2.1) holds if $\sup_x \sum_{y:y\neq x} \times J_{x,y}$ is small enough (cf. Theorem 2.4). It is also known that if $d = 1$, then (2.1) holds regardless of the value of $\sup_x \sum_{y:y\neq x} J_{x,y}$. This can be seen from [Z96, Lemma 4.5].

Remark 2.3. As we experience in models with compact spin spaces, it may well be the case that for some $\mathbf{J}$, U and $\mathbf{h}$, a mixing condition like (2.1) does not hold for *all* Λ, but its restriction to "nice" Λ's (typically to cubes or to "fat" enough boxes) does. This is why we introduced the class $\mathscr{B}(n_0)$.

Remark 2.4. Theorem 2.1 above is strongly motivated by [Z96, Theorem 5.1]. There, the potential function U is assumed only to satisfy (1.10)–(1.13) for *some* $m > 0$. [Z96, Theorem 5.1] says that if there is $C_{2.4} = C_{2.4}(U, \mathbf{J}, \mathbf{h}) \in (0, \infty)$ such that (2.4) holds for large enough cubes $\Lambda \subset \mathbf{Z}^d$, $\omega \in \mathbf{R}^{\Lambda^c}$ and $x, y \in \Lambda$, then there is $\gamma \in (0, \infty)$ for which the log-Sobolev inequality (1.32) holds for $\omega \equiv 0$ and for large enough cubes $\Lambda \subset \mathbf{Z}^d$. Therefore, as compared with [Z96, Theorem 5.1], Theorem 2.1 in this paper says more or less that a stronger results follows from stronger assumptions.

Remark 2.5. Let us briefly remark that the uniform bound (2.2) of the log-Sobolev constants implies "exponential convergence to the equibrium" of the associated stochastic dynamics (cf. [Y98] [Z96, Theorem 3.1]).

We also present the following weaker result:

Theorem 2.2. *Suppose that the same mixing condition as in Theorem 2.2 holds. Then the inverse spectral gaps* (*cf.* (1.30)) *are uniformly bounded in the sense that*;

$$\sup\{\gamma_{\mathrm{SG}}(\Lambda) \mid \Lambda \in \mathscr{F}\} \le C_{2.5} < \infty \ , \tag{2.5}$$

where the constant $C_{2.5}$ depends only on d, U, $\mathbf{J}$, $B_{2.1}$ and $C_{2.1}$. Therefore, the unique element μ in $\mathscr{G}_{\mathrm{t}}$ (cf. Remark 2.1) satisfies

$$\mu(f;f) \le C_{2.5}\mu(|\nabla_\Lambda f|^2) \tag{2.6}$$

for any $\Lambda \subset\subset \mathbf{Z}^d$ and $f \in \mathscr{C}_\Lambda$.

This statement about the spectral gap follows easily from Theorem 2.1, since the inverse spectral gap (cf. (1.30)) is always bounded from above by a constant multiple of the log-Sobolev constant. In this paper, however, we prove Theorem 2.2 in advance of Theorem 2.1 and use it as a step to show Theorem 2.1.

We provide the following “constructive criterion” of the mixing condition (2.1). Recall that for $\Lambda \subset\subset \mathbf{Z}^d$, we have defined the Vassershtein distance $\mathscr{R}_\Lambda$ by (1.42).

Proposition 2.3. *Let $\Lambda \subset\subset \mathbf{Z}^d$ be arbitrary. Suppose that there exist $V \subset\subset \mathbf{Z}^d$, $\varepsilon_{2.8} \in (0,1)$ and a matrix $\mathbf{K} = (K_{x,y} \ge 0 : x, y \in \mathbf{Z}^d)$ such that*

$$K_{x,y} = 0 \quad \text{if } y - x \notin V \cup \partial_R V \ , \tag{2.7}$$

$$\|\mathbf{K}\| \stackrel{\text{def.}}{=} \sup_y \sum_x K_{x,y} \le \varepsilon_{2.8}|V| \ , \tag{2.8}$$

$$\mathscr{R}_{W\cap(x+V)}\big(\mathscr{E}^{W\cap(x+V),\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W\cap(x+V),\bar{p}^2}_{\Lambda,q}\big) \le \sum_y K_{x,y}|\bar{p}^1_y - \bar{p}^2_y| \tag{2.9}$$

for all $W \subset \Lambda$, $x \in \mathbf{Z}^d$ and $\bar{p}^i \in \mathbf{R}^\Lambda (i = 1, 2)$. Then there exist constants $B_{2.1}$ and $C_{2.1}$ which depend only on d, R, V and $\varepsilon_{2.8}$ such that (2.1) *holds whenever $W \subset \Lambda$, $y \in \Lambda \backslash W$ and $\bar{p}^i \in \mathbf{R}^\Lambda$ $(i = 1, 2)$ differs only at y.*

Remark 2.6. In Proposition 2.3, we did not assume that V and $\varepsilon_{2.8}$ are independent of the choice of Λ. However, in application of the proposition we have in mind (cf. Theorem 2.2, Theorem 2.1), it is important to find V and $\varepsilon_{2.8}$ independently of the choice of Λ, since we need (2.1) with constants $B_{2.1}$ and $C_{2.1}$ not depending on Λ.

Remark 2.7. Conditions (2.7)–(2.9) in Proposition 2.3 are reminiscient of [DSh85, Theorem 2.1]. Note however we have put stronger assumptions to get stronger conclusion than that of above mentioned result.

By combining Theorem 2.1 and Proposition 2.3, we can prove the following main result in this article:

Theorem 2.4. *There is $\beta \in (0, \infty)$ such that if $\sup_x \sum_{y:y\neq x} J_{x,y} \le \beta$, then the following hold.*

(a) *There exists $\varepsilon_{2.8} < 1$ which depends only on $d, U, \mathbf{J}$ such that conditions* (2.7)–(2.9) *with $V = \{0\}$ and with some $\mathbf{K} = (K_{x,y} \ge 0 : x, y \in \mathbf{Z}^d)$ are satisfied for all $\Lambda \subset\subset \mathbf{Z}^d$, $W \subset \Lambda$, $x \in \mathbf{Z}^d$ and $\bar{p}^i \in \mathbf{R}^\Lambda$ $(i = 1, 2)$.*

(b) *There exist constants $B_{2.1}$ and $C_{2.1}$ which depend only on d, U, $\mathbf{J}$ such that* (2.1) *holds whenever $\Lambda \subset\subset \mathbf{Z}^d$, $W \subset \Lambda$, $y \in \Lambda \backslash W$ and $\bar{p}^i \in \mathbf{R}^\Lambda$ $(i = 1, 2)$ differs only at y.*

(c) *The uniform bound on the log-Sobolev constants* (2.2) *with $\mathscr{F} = \mathscr{A}$ holds with the constant $C_{2.2}$ depending only on d, U, $\mathbf{J}$ and thus, the inequality* (2.3) *for the unique $\mu \in \mathscr{G}_t$ holds.*

Remark 2.8. Part(b) of Theorem 2.4 implies that if $\sup_x \sum_{y:y\neq x} J_{x,y}$ is small enough, then mixing condition (2.1), and thus (2.4) holds (cf. Lemma 3.3). In particular, we see that [Z96, Theorem 5.1] applies if $\sup_x \sum_{y:y\neq x} J_{x,y}$ is small enough.

Remark 2.9. The assumptions **(U1)** and **(U2)** for the one body interaction U are mild enough to include examples like (1.16) which are often discussed in physical litratures. However, from the mathematical point of view, these assumptions are not minimal to prove the part(c) of Theorem 2.4 with. Based on the result of B. Helffer [He97], T. Bodineau and B. Helffer improved the proof of part(c) of Theorem 2.4 in a very recent paper [BH98] and they succeeded in reducing the assumptions for U to, more or less, the minimal ones. In fact, they only assume (1.10)–(1.13) for *some* $m > 0$ and, (1.14) is not required.

3. Lemmas

In this section, we prove a couple of lemmas which will be used later. Here, we will use many ideas from [Z96, Section 4].

Lemma 3.1. *For any $\lambda > 0$, there exists $C_{3.1}(\lambda) = C_{3.1}(\lambda, U, \mathbf{J}) \in (0, \infty)$ such that*

$$\mathscr{E}^{W,\bar{p}}_{\Lambda,q} \exp\big(\lambda |p_x|^2\big) \le \exp\left(C_{3.1}(\lambda) \left(1 + \sum_{y \in \Lambda \cap \partial_R W} |\bar{p}_y|^2 \right) \right) , \tag{3.1}$$

whenever $W \subset \Lambda \subset\subset \mathbf{Z}^d$, $q \in \mathbf{R}^\Lambda$ and $\bar{p} \in \mathbf{R}^\Lambda$.

Proof. To prove (3.1), we begin by proving that

$$\mathscr{E}^{x,\bar{p}}_{\Lambda,q} \exp\big(\lambda |p_x|^2\big) \le \mathscr{E}^{x,\bar{p}}_{\Lambda,q} \exp\big(\lambda |p_x|^2\big) \big|_{q \equiv 0} \tag{3.2}$$

Since the left-hand-side of (3.2) depends on q only through q_x, it is enough to prove that

$$\frac{\partial}{\partial q_x} \mathscr{E}^{x,\bar{p}}_{\Lambda,q} \exp\big(\lambda |p_x|^2\big) \begin{cases} \le 0 & \text{if } q_x \ge 0, \\ \ge 0 & \text{if } q_x \le 0 \ . \end{cases} \tag{3.3}$$

We have that

$$\begin{aligned}\tfrac{\partial}{\partial q_x}\mathscr{E}^{x,\bar p}_{\Lambda,q}\exp\bigl(\lambda|p_x|^2\bigr) &= -\mathscr{E}^{x,\bar p}_{\Lambda,q}\left(\tfrac{\partial\mathscr{U}}{\partial q_x}(p_x,q_x);\exp\bigl(\lambda|p_x|^2\bigr)\right)\\ &= -\tfrac{1}{2}\int\mathscr{E}^{x,\bar p}_{\Lambda,q}(dp^1_x)\int\mathscr{E}^{x,\bar p}_{\Lambda,q}(dp^2_x)\\ &\quad\cdot\left(\tfrac{\partial\mathscr{U}}{\partial q_x}\left(p^1_x,q_x\right)-\tfrac{\partial\mathscr{U}}{\partial q_x}\left(p^2_x,q_x\right)\right)\\ &\quad\cdot\left(\exp\bigl(\lambda|p^1_x|^2\bigr)-\exp\bigl(\lambda|p^2_x|^2\bigr)\right)\ . \end{aligned}\tag{3.4}$$

Suppose now that $q_x \geq 0$, then $|p_x| \mapsto \frac{\partial\mathscr{U}}{\partial q_x}$ is non-decreasing by (1.14). This implies that the integrand in (3.4) is non-negative for all (p^1_x, p^2_x). Similarly, the integrand in (3.4) is non-positive for all (p^1_x, p^2_x), if $q_x \leq 0$. Therefore, we have proved (3.3).

We next find a constant $C_{3.5}(\lambda) \in (0,\infty)$ and a sequence $\mathbf{r} = (r_x \geq 0) \in \mathscr{S}$, which depend only on λ, $\mathbf{J}$ and U such that

$$\sum_x r_x < 1,$$

$$\mathscr{E}^{x,\bar p}_{\Lambda,q}\exp\bigl(\lambda|p_x|^2\bigr)\big|_{q\equiv 0} \leq \exp\left(C_{3.5}(\lambda)+\lambda\sum_y r_{x-y}|\bar p_y|^2\right)\tag{3.5}$$

whenever $x \in \Lambda \subset\subset \mathbf{Z}^d$ and $\bar p \in \mathbf{R}^\Lambda$. This can be done in the same way as in the proof of [BH82, Theorem III.2], where similar estimate for the integral of $\exp(\lambda|p_x|)$ is obtained. Here, we have to consider the integral of $\exp\bigl(\lambda|p_x|^2\bigr)$. However, it is not diffucult to generalize the computations in [BH82, Lemmas III.5 and III.6] to cover our case (by taking $m > 0$ in (1.10)–(1.13) large enough, depending on λ). Once (3.2) and (3.5) are established, one can proceed as in the proof of [BH82, Proposition III.1] to obtain (3.1). □

Lemma 3.2. *For any $n = 1, 2, \ldots$, there exists $C_{3.6}(n) = C_{3.6}(U, \mathbf{J}, n) \in (0,\infty)$ such that*

$$E^{\Lambda,\omega}\left(f^2\log\frac{f^2}{E^{\Lambda,\omega}(f^2)}\right) \leq C_{3.6}(|\Delta|)E^{\Lambda,\omega}(|\nabla_\Lambda f|^2)\tag{3.6}$$

whenever $\Delta, \Lambda \subset\subset \mathbf{Z}^d$ and $f \in \mathscr{C}$ satisfy $S_f \cap \Lambda \subset \Delta \subset \Lambda$.

To prove Lemma 3.2, we need another lemma.

Lemma 3.3. *Suppose that the mixing condition* (2.1) *holds. Then,*

$$|E^{\Lambda,\omega}(\sigma_x;\sigma_y)| \leq C_{3.7}\exp\left(-\frac{d(x,y)}{C_{2.1}}\right)\ ,\tag{3.7}$$

for any $\Lambda \in \mathscr{F}$, $\omega \in \mathbf{R}^{\Lambda^c}$ *and* $x, y \in \Lambda$, *where* $C_{3.7} = C_{3.7}(U, \mathbf{J}, B_{2.1}) \in (0, \infty)$. *In general* (*without the mixing condition* (2.1)), *the following is true*;

$$\sup\{|E^{\Lambda,\omega}(\sigma_x; \sigma_x)|;\ \Lambda \subset\subset \mathbf{Z}^d,\ x \in \Lambda,\ \mathbf{h} \in \mathbf{R}^{\Lambda},\ \omega \in \mathbf{R}^{\Lambda^c}\} \le C_{3.8} < \infty\,, \tag{3.8}$$

where $C_{3.8} = C_{3.8}(U, \mathbf{J}) \in (0, \infty)$.

Proof of Lemma 3.3. To prove the Lemma, recall that we have defined measures $\mathscr{E}_{\Lambda,q}$, $\mathscr{E}_{\Lambda,+}$ and $\mathscr{E}^{\Lambda\setminus\Delta}_{\Lambda,q}$ respectively by (1.33), (1.35) and (1.37). Note that

$$\begin{aligned} E^{\Lambda}(\sigma_x; \sigma_y) &= \tfrac{1}{2}\int E^{\Lambda} \otimes E^{\Lambda}(d\sigma^1 d\sigma^2)(\sigma^1_x - \sigma^2_x)(\sigma^1_y - \sigma^2_y) \\ &= \int \mathscr{E}_{\Lambda,+}(dq)\int \mathscr{E}_{\Lambda,q}(dp)\, p_x p_y \\ &= \int \mathscr{E}_{\Lambda,+}(dq)\int \mathscr{E}_{\Lambda,q}(dp)\, p_x \mathscr{E}^{\Lambda\setminus x,p}_{\Lambda,q}(p_y) \end{aligned} \tag{3.9}$$

and that we have from (2.1) that

$$|\mathscr{E}^{\Lambda\setminus x,p}_{\Lambda,q}(p_y)| \ \le\ 2B_{2.1}\,(1 + |p_x|)\exp\left(-\frac{d(x, y)}{C_{2.1}}\right)\ .$$

Plugging this into (3.9), we have

$$\begin{aligned} |E^{\Lambda,\omega}(\sigma_x; \sigma_y)| \ \le\ 2B_{2.1}\exp\left(-\frac{d(x, y)}{C_{2.1}}\right)&\int \mathscr{E}_{\Lambda,+}(dq) \\ \int \mathscr{E}_{\Lambda,q}(dp)|p_x|\,(1 + |p_x|)\ &. \end{aligned} \tag{3.10}$$

By (3.1), the integral in the right-hand-side of (3) is bounded by some constant which depends only on $\mathbf{J}$ and U. Therefore we get (3.7). The same argument as above, but without the use of (2.1), proves (3.8). □

Proof of Lemma 3.2. Let us set $E^{\Lambda,\omega}_{\Delta}(d\sigma_\Delta) = E^{\Lambda,\omega}(\mathbf{R}^{\Lambda\setminus\Delta} \times d\sigma_\Delta)$. Since

$$\begin{aligned} H^{\Lambda,\omega}(\sigma) = -\frac{1}{2}\sum_{x,y\in\Delta} J_{x,y}\sigma_x\sigma_y + \sum_{x\in\Delta}(U(\sigma_x) - h_x\sigma_x) - \sum_{x\in\Delta}\sum_{y\notin\Lambda} J_{x,y}\sigma_x\omega_y \\ -\frac{1}{2}\sum_{x,y\in\Lambda\setminus\Delta} J_{x,y}\sigma_x\sigma_y + \sum_{x\in\Lambda\setminus\Delta}(U(\sigma_x) - h_x\sigma_x) \\ - \sum_{x\in\Lambda\setminus\Delta}\sum_{y\notin\Lambda} J_{x,y}\sigma_x\omega_y - \sum_{x\in\Lambda\setminus\Delta}\sum_{y\in\Delta} J_{x,y}\sigma_x\sigma_y\ , \end{aligned} \tag{3.11}$$

the measure $E^{\Lambda,\omega}_{\Delta}(d\sigma_\Delta)$ has the following expression;

$$E^{\Lambda,\omega}_{\Delta}(d\sigma_\Delta) = \frac{\exp\left(-\sum_{x\in\Delta}\left(U(\sigma_x) - h^{\Lambda,\omega}_x\sigma_x\right)\right) Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}}\,\nu^{\Delta}_{\mathbf{J}}(d\sigma_\Delta)}{\int \nu^{\Delta}_{\mathbf{J}}(d\sigma_\Delta)\exp(-\sum_{x\in\Delta}(U(\sigma_x) - h^{\Lambda,\omega}_x\sigma_x)) Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}}} \quad , \tag{3.12}$$

where

$$\begin{aligned} h^{\Lambda,\omega}_x &= h_x + \sum_{y\notin\Lambda} J_{x,y}\omega_y, \\ \nu^{\Delta}_{\mathbf{J}}(d\sigma_\Delta) &= \exp\Big(\tfrac{1}{2}\sum_{x,y\in\Delta} J_{x,y}\sigma_x\sigma_y\Big)\prod_{x\in\Delta} d\sigma_x, \\ Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}} &= \int \nu^{\Lambda\setminus\Delta}_{\mathbf{J}}(d\sigma_{\Lambda\setminus\Delta})\exp\Big(-\sum_{x\in\Lambda\setminus\Delta}(U(\sigma_x) - h_x\sigma_x) \\ &\qquad + \sum_{x\in\Lambda\setminus\Delta}\sum_{y\in\Delta\cup\Lambda^c} J_{x,y}\sigma_x(\sigma_\Delta\cdot\omega_{\Lambda^c})_y\Big) \ . \end{aligned}$$

We would like to deform the above measure into another measure $\bar{E}^{\Lambda,\omega}_{\Delta}(d\sigma_\Delta)$ to which we can apply the Bakry-Emery Γ_2-criterion of the log-Sobolev inequality ([BE85, Corollaire 2]). To this end, we decompose U into V and W in a manner described in (1.10)–(1.13), where the parameter $m > 0$ is specified later, depending on $|\Delta|$. We then have by (1.13) that

$$\exp(-2C_{1.12}) \le \frac{dE^{\Lambda,\omega}_{\Delta}}{d\bar{E}^{\Lambda,\omega}_{\Delta}}(\sigma_\Delta) \le \exp(2C_{1.12}) \quad , \tag{3.13}$$

where

$$\bar{E}^{\Lambda,\omega}_{\Delta}(d\sigma_\Delta) = \frac{\exp\left(-\sum_{x\in\Delta}\left(V(\sigma_x) - h^{\Lambda,\omega}_x\sigma_x\right)\right) Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}}\,\nu^{\Delta}_{\mathbf{J}}(d\sigma_\Delta)}{\int \nu^{\Delta}_{\mathbf{J}}(d\sigma_\Delta)\exp(-\sum_{x\in\Delta}(V(\sigma_x) - h^{\Lambda,\omega}_x\sigma_x)) Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}}} \quad , \tag{3.14}$$

Let us next prove that

$$\sum_{x,y\in\Delta}\left|\xi_x\xi_y\frac{\partial^2}{\partial\sigma_x\partial\sigma_y}\log Z^{\Lambda\setminus\Delta,\sigma_\Delta}_{\mathbf{J}}\right| \le C_{3.8}|\Delta|\|\mathbf{J}\|^2\sum_{x\in\Delta}|\xi_x|^2 \quad , \tag{3.15}$$

for any $(\xi_x)_{x\in\Delta}\in\mathbf{R}^\Delta$. In fact,

$$\begin{aligned}
&\sum_{x,y\in\Delta}\left|\xi_x\xi_y\frac{\partial^2}{\partial\sigma_x\partial\sigma_y}\log Z_{\mathbf{J}}^{\Lambda\setminus\Delta,\sigma_\Delta}\right| \\
&\quad = \sum_{x,y\in\Delta}\left|\xi_x\xi_y\sum_{z,w\in\Lambda\setminus\Delta}J_{x,z}J_{y,w}E^{\Lambda\setminus\Delta,\sigma_\Delta\cdot\omega_{\Lambda^c}}(\sigma_z;\sigma_w)\right| \\
&\quad \le C_{3.8}\|\mathbf{J}\|^2\left(\sum_{x\in\Delta}|\xi_x|\right)^2 \\
&\quad \le C_{3.8}\|\mathbf{J}\|^2|\Delta|\sum_{x\in\Delta}|\xi_x|^2 \ .
\end{aligned}$$

At this point, we take $m = C_{3.8}\|\mathbf{J}\|^2|\Delta| + 1$ to have that

$$\sum_{x,y\in\Delta}\xi_x\xi_y\frac{\partial^2}{\partial\sigma_x\partial\sigma_y}\left(\sum_{x\in\Delta}\left(V(\sigma_x)-h_x^{\Lambda,\omega}\sigma_x\right)-\log Z_{\mathbf{J}}^{\Lambda\setminus\Delta,\sigma_\Delta}\right) \ge \sum_{x\in\Delta}|\xi_x|^2. \tag{3.16}$$

This implies that Γ_2 in the sense of [BE85], computed with respect to the measure $\bar{E}^{\Lambda,\omega}$ is bounded from below by 1 and thus that we have

$$\bar{E}_\Delta^{\Lambda,\omega}\left(f^2\log\frac{f^2}{\bar{E}_\Delta^{\Lambda,\omega}(f^2)}\right) \le 2\bar{E}_\Delta^{\Lambda,\omega}(|\nabla_\Lambda f|^2) \ . \tag{3.17}$$

whenever $f\in\mathscr{C}$ and $\Lambda\subset\subset\mathbf{Z}^d$ satisfy $S_f\cap\Lambda\subset\Delta\subset\Lambda$. By (3.13), (3.17) and a standard comparison argument ([HS87, Lemma 5.1]), we get (3.6) with

$$C_{3.6} = 2\exp(4C_{1.12}) \ . \tag{3.18}$$

□

The following lemma is technically the most important step in our proof of Theorem 2.2 and Theorem 2.1;

Lemma 3.4. *Suppose that mixing condition* (2.1) *is true. Then, for any* $f\in\mathscr{C}$, $\Lambda\in\mathscr{F}$, $x\in\Lambda$ *and* $\Delta\subset\Lambda$ *such that* $S_f\cap\Lambda\subset\Delta\subset\Lambda$,

$$\left|E^\Lambda(f;\sigma_x)\right| \le C_{3.19}|\Delta|^2\exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right)E^\Lambda(f;f)^{1/2}, \tag{3.19}$$

$$\begin{aligned}
\left|E^\Lambda(f^2;\sigma_x)\right| &\le C_{3.19}|\Delta|^2\exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right)E^\Lambda(f^2)^{1/2} \\
&\quad\times\left(E^\Lambda(f;f)^{1/2}+E^\Lambda\left(f^2\log\frac{f^2}{E^\Lambda(f^2)}\right)^{1/2}\right) \ ,
\end{aligned} \tag{3.20}$$

where $C_{3.19} = C_{3.19}(B_{2.1}, C_{3.1}(1)) \in (0, \infty)$. *Furthermore, for* $y \notin \Lambda$,

$$\left|\nabla_y E^\Lambda(f)\right| \le \left|E^\Lambda\left(\nabla_y f\right)\right| + \|\mathbf{J}\| \sup_{\substack{x\in\Lambda \\ d(x,y)\le R}} \left|E^\Lambda(f;\sigma_x)\right| \tag{3.21}$$

$$\le \left|E^\Lambda\left(\nabla_y f\right)\right| + C_{3.19}\|\mathbf{J}\||\Delta|^2 \exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right) E^\Lambda(f;f)^{1/2} \quad , \tag{3.22}$$

$$|\nabla_y \sqrt{E^\Lambda(f^2)}| \le E^\Lambda\left(|\nabla_y f|^2\right)^{1/2}$$
$$+ \tfrac{1}{2} E^\Lambda(f^2)^{-1/2}\|\mathbf{J}\| \sup_{\substack{x\in\Lambda \\ d(x,y)\le R}} \left|E^\Lambda(f^2;\sigma_x)\right| \tag{3.23}$$
$$\le E^\Lambda\left(|\nabla_y f|^2\right)^{1/2} + C_{3.19}\|\mathbf{J}\||\Delta|^2 \exp\left(-\frac{d(y,\Delta)}{C_{2.1}}\right)$$
$$\cdot\left(E^\Lambda(f;f)^{1/2} + E^\Lambda\left(f^2 \log\frac{f^2}{E^\Lambda(f^2)}\right)^{1/2}\right) \quad . \tag{3.24}$$

Remark 3.1. We will use (3.19), (3.21) and (3.22) to prove Theorem 2.2. On the other hand, (3.20), (3.23) and (3.24) will be used in the proof of Theorem 2.1, where the term $E^\Lambda(f;f)$ on the right-hand-side of (3.20) and (3.24) will eventually be bounded by $C_{2.5}E^\Lambda\left(|\nabla_\Lambda f|^2\right)$ by using Theorem 2.2.

Remark 3.2. As will become clear from the proof, if we do not assume any mixing condition, we have (3.19)–(3.24) without the factor $\exp(-d(y,\Delta)/C_{2.1})$.

Proof of Lemma 3.4. Let us begin by proving (3.21) and (3.23). In fact, it is easy to see that

$$\left|\nabla_y E^\Lambda(f)\right| = \left|E^\Lambda\left(\nabla_y f\right) + \sum_{x\in\Lambda} J_{x,y} E^\Lambda(f;\sigma_x)\right|$$
$$\le \left|E^\Lambda\left(\nabla_y f\right)\right| + \|\mathbf{J}\| \sup_{\substack{x\in\Lambda \\ d(x,y)\le R}} \left|E^\Lambda(f;\sigma_x)\right|$$

and that

$$\left|\nabla_y \sqrt{E^\Lambda(f^2)}\right| = \left|\tfrac{1}{2} E^\Lambda(f^2)^{-1/2} \nabla_y E^\Lambda(f^2)\right|$$
$$= \left|\frac{1}{2} E^\Lambda(f^2)^{-1/2}\left(2E^\Lambda\left(f\nabla_y f\right) + \sum_{x\in\Lambda} J_{x,y} E^\Lambda(f^2;\sigma_x)\right)\right|$$

$$
\begin{aligned}
&\le \tfrac{1}{2}E^{\Lambda}(f^2)^{-1/2}\Bigg(2E^{\Lambda}\left(f^2\right)^{1/2}E^{\Lambda}\left(|\nabla_y f|^2\right)^{1/2} \\
&\quad +\|\mathbf{J}\| \sup_{\substack{x\in\Lambda \\ d(x,y)\le R}} \left|E^{\Lambda}(f^2;\sigma_x)\right|\Bigg) \\
&= E^{\Lambda}\left(|\nabla_y f|^2\right)^{1/2} \\
&\quad +\tfrac{1}{2}E^{\Lambda}(f^2)^{-1/2}\|\mathbf{J}\| \sup_{\substack{x\in\Lambda \\ d(x,y)\le R}} \left|E^{\Lambda}(f^2;\sigma_x)\right|.
\end{aligned}
$$

By (3.21) and (3.23), the proof of (3.22) and (3.24) comes down to that of (3.19) and (3.20).

To prove (3.19), recall that we have defined mesures $\mathscr{E}_{\Lambda,q}$, $\mathscr{E}^{\Lambda\setminus\Delta}_{\Lambda,q}$ and $\mathscr{E}_{\Lambda,+}$ respectively by (1.33) and (1.37) and (1.35). The correlation $E^{\Lambda}(f;\sigma_x)$ can be expressed in terms of these measures as follows;

$$
\begin{aligned}
E^{\Lambda}(f;\sigma_x) &= \tfrac{1}{2}\int E^{\Lambda}\otimes E^{\Lambda}(d\sigma^1 d\sigma^2)\left(\sigma_x^1-\sigma_x^2\right)\left(f(\sigma^1)-f(\sigma^2)\right) \\
&= \tfrac{1}{\sqrt{2}}\int \mathscr{E}_{\Lambda,+}(dq)\int \mathscr{E}_{\Lambda,q}(dp)\left(f\left(\frac{q+p}{\sqrt{2}}\right)-f\left(\frac{q-p}{\sqrt{2}}\right)\right)p_x \\
&= \tfrac{1}{\sqrt{2}}\int \mathscr{E}_{\Lambda,+}(dq)\int \mathscr{E}_{\Lambda,q}(dp) \\
&\quad \cdot\left(f\left(\frac{q+p}{\sqrt{2}}\right)-f\left(\frac{q-p}{\sqrt{2}}\right)\right)\mathscr{E}^{\Lambda\setminus\Delta,p}_{\Lambda,q}(p_x)
\end{aligned}
\tag{3.25}
$$

On the other hand, we have from (2.1) that

$$
\left|\mathscr{E}^{\Lambda\setminus\Delta,p}_{\Lambda,q}(p_x)\right| \le 2B_{2.1}|\Delta|\left(1+\sum_{z\in\Delta}|p_z|\right)\exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right)
$$

Plugging this into (3.25), we see that

$$
|E^{\Lambda}(f;\sigma_x)| \le \sqrt{2}B_{2.1}|\Delta|\exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right)\left(I_1(f)+\sum_{z\in\Delta}I_2(f,z)\right), \tag{3.26}
$$

where

$$I_1(f) = \int \mathscr{E}_{\Lambda,+}(dq) \int \mathscr{E}_{\Lambda,q}(dp) \left| f\left(\frac{q+p}{\sqrt{2}}\right) - f\left(\frac{q-p}{\sqrt{2}}\right) \right|$$
$$= \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \left| f(\sigma^1) - f(\sigma^2) \right| \ , \tag{3.27}$$

$$I_2(f,z) = \int \mathscr{E}_{\Lambda,+}(dq) \int \mathscr{E}_{\Lambda,q}(dp) \left| f\left(\frac{q+p}{\sqrt{2}}\right) - f\left(\frac{q-p}{\sqrt{2}}\right) \right| |p_z|$$
$$= \tfrac{1}{\sqrt{2}} \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \left| \left(f(\sigma^1) - f(\sigma^2)\right)\left(\sigma_z^1 - \sigma_z^2\right) \right| \tag{3.28}$$

We first observe that

$$I_2(f,z) \ \le \ \tfrac{1}{\sqrt{2}} I_3(f)^{1/2} I_4(f,z)^{1/2}, \tag{3.29}$$

where

$$I_3(f) \ = \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \left| f(\sigma^1) - f(\sigma^2) \right|^2 \ ,$$

$$I_4(f,z) \ = \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \left| \sigma_z^1 - \sigma_z^2 \right|^2 ,$$

The integrals $I_1(f)$ and $I_3(f)$ can be estimated as follows;

$$I_1(f) \ \le \ I_3(f)^{1/2} \ \le \ 2(E^\Lambda \left| f - E^\Lambda f \right|^2)^{1/2}$$
$$= \ 2E^\Lambda(f;f)^{1/2} \tag{3.30}$$

On the other hand, it follows from (3.1) and Jensen inequality that

$$I_4(f,z) \ \le \ C_{3.31} \tag{3.31}$$

for some $C_{3.31} = C_{3.31}(C_{3.1}(1)) \in (0,\infty)$. Putting (3.29), (3.30) and (3.31) together, we obtain

$$I_1(f) + \sum_{z\in\Delta} I_2(f,z) \ \le \ I_1(f) + \tfrac{1}{\sqrt{2}} I_3(f)^{1/2} \sum_{z\in\Delta} I_4(f,z)^{1/2}$$
$$\le \ 2E^\Lambda(f;f)^{1/2}(1 + C_{3.31}^{1/2}|\Delta|)$$

which, in conjunction with (3.26), implies (3.19).

The proof of (3.20) is similar to that of (3.19). We see from (3.26) that

$$|E^\Lambda(f^2;\sigma_x)| \le \sqrt{2} B_{2.1} |\Delta| \exp\left(-\frac{d(x,\Delta)}{C_{2.1}}\right) \left(I_1(f^2) + \sum_{z\in\Delta} I_2(f^2,z)\right) \ . \tag{3.32}$$

We first observe that

$$\begin{aligned} I_1(f^2) &= \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \\ &\qquad \left| f(\sigma^1)^2 - E^\Lambda(f)^2 - \left(f(\sigma^2)^2 - E^\Lambda(f)^2\right)\right| \\ &\leq 2E^\Lambda(f;f)^{1/2} E^\Lambda(f^2)^{1/2}. \end{aligned} \tag{3.33}$$

Next, we have by Schwartz inequality that

$$I_2(f^2, z) \leq \frac{1}{\sqrt{2}} I_3(f)^{1/2} I_5(f,z)^{1/2} \ , \tag{3.34}$$

where

$$I_5(f,z) = \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) | \left(f(\sigma^1) + f(\sigma^2)\right)^2 \left(\sigma_z^1 - \sigma_z^2\right)^2 | \ .$$

Let us note that $ab \leq \exp(a) + b\log b$ for $a, b \geq 0$ and that

$$\begin{aligned} &\int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \exp(\left|\sigma_z^1 - \sigma_z^2\right|^2) \\ &\qquad = 2\int \mathscr{E}_{\Lambda,+}(dq) \int \mathscr{E}_{\Lambda,q}(dp) \exp\left(|p_z|^2\right) \\ &\qquad \leq C_{3.35} \end{aligned} \tag{3.35}$$

by (3.1), where $C_{3.35} = C_{3.35}(C_{3.1}(1))$. We then have that

$$\begin{aligned} I_5(f,z) &\leq 2\int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) | f(\sigma^1)^2 \left(\sigma_z^1 - \sigma_z^2\right)^2 | \\ &= 2E^\Lambda(f^2) \int E^\Lambda \otimes E^\Lambda(d\sigma^1 d\sigma^2) \left| \left(\sigma_z^1 - \sigma_z^2\right)^2 \frac{f(\sigma^1)^2}{E^\Lambda(f^2)} \right| \\ &\leq 2C_{3.35} E^\Lambda(f^2) + 2E^\Lambda\left(f^2 \log \frac{f^2}{E^\Lambda(f^2)}\right) \ . \end{aligned} \tag{3.36}$$

Putting (3.30), (3.33), (3.34) and (3.36) together, we obtain

$$\begin{aligned} &I_1(f^2) + \sum_{z\in\Delta} I_2(f^2, z) \\ &\quad \leq I_1(f^2) + \frac{1}{\sqrt{2}} I_3(f)^{1/2} \sum_{z\in\Delta} I_5(f,z)^{1/2} \\ &\quad \leq E^\Lambda(f;f) + \frac{2}{\sqrt{2}} E^\Lambda(f;f)^{1/2} \\ &\qquad \cdot \sqrt{2}\,|\Delta| \left(\sqrt{C_{3.35}} E^\Lambda(f^2)^{1/2} + E^\Lambda\left(f^2 \log \frac{f^2}{E^\Lambda(f^2)}\right)^{1/2} \right) \end{aligned}$$

$$\leq C_{3.37}\,|\Delta|\,E^{\Lambda}(f^2)^{1/2}\left(E^{\Lambda}(f;\,f)^{1/2}+E^{\Lambda}\left(f^2\log\frac{f^2}{E^{\Lambda}(f^2)}\right)^{1/2}\right) \tag{3.37}$$

which, in conjunction with (3.32), implies (3.20). □

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by using lemmas presented in the previous section. We first consider the case $\mathscr{F} = \mathscr{A}$. What we want to prove is equivalent to that

$$\sup_{n\geq 1}\gamma_n < \infty \ , \tag{4.1}$$

where

$$\gamma_n = \sup\{\gamma_{\mathrm{SG}}(\Lambda)\,|\,|\Lambda| \leq n\}\,. \tag{4.2}$$

To prove (4.1), it is enough to find some constants $C_{4.3}$ and $N_{4.3}$ which depend only on d, R, U, $\mathbf{J}$, $B_{2.1}$ and $C_{2.1}$ such that

$$\gamma_{2n} \leq \tfrac{4}{5}\gamma_n + C_{4.3}, \quad \text{for } n \geq N_{4.3} \ . \tag{4.3}$$

To this end, we take arbitrary $\Lambda \subset\subset \mathbf{Z}^d$ with $|\Lambda| \leq 2n$ and $0 < f \in \mathscr{C}_{\Lambda}$. We then choose $\Lambda_0 \subset \Lambda$ such that $\max\{|\Lambda_0|,\ |\Lambda\backslash\Lambda_0|\} \leq n$ and define

$$\Lambda_j = \Lambda_0 \cup \{x_1,\ldots,x_j\}, \quad j = 1,2,\ldots,m \ , \tag{4.4}$$

$$f_j = E^{\Lambda_j} f \ , \tag{4.5}$$

$$\Lambda_{j,k} = \{x \in \Lambda_j;\ d(x,x_{j+1}) < (k/2)\ \}, \quad k = 0,1,2,\ldots \ , \tag{4.6}$$

$$f_{j,k} = f, \quad \text{if } k \geq 0 \text{ and } \Lambda_{j,k} = \phi \ ,$$

$$f_{j,k} = E^{\Lambda_{j,k}} f, \quad \text{if } k \geq 1 \text{ and } \Lambda_{j,k} \neq \phi \ , \tag{4.7}$$

where $\{x_j\}_{j=1}^m$ is an enumeration of $\Lambda\backslash\Lambda_0$. We will prove (4.3) after a series of lemmas.

Lemma 4.1.

$$E^{\Lambda}(f;\, f) \;\le\; \gamma_n E^{\Lambda}\big(|\nabla_{\Lambda_0} f|^2\big) + C_{4.8} E^{\Lambda}\left(|\nabla_{\Lambda\setminus\Lambda_0} f|^2\right) + C_{4.8} \sum_{j=0}^{m-1} E^{\Lambda} \mathscr{Q}_j(f) \ , \tag{4.8}$$

where $C_{4.8} = C_{4.8}(C_{3.6}(1), \|\mathbf{J}\|) \in (0, \infty)$ *and*

$$\mathscr{Q}_j(f) = \sup\left\{E^{\Lambda_j}(f; \sigma_x)^2;\ x \in \Lambda_j,\ d(x, x_{j+1}) \;\le\; R\right\} \ . \tag{4.9}$$

Proof. We first divide the left-hand-side of (4.8) into two terms;

$$E^{\Lambda}(f;\, f) = E^{\Lambda}\left(f^2 - f_0^2\right) + E^{\Lambda}\left(f_0^2 - f_m^2\right) \ . \tag{4.10}$$

The first term on the right-hand-side can be estimated as follows;

$$\begin{aligned} E^{\Lambda}\left(f^2 - f_0^2\right) \;&=\; E^{\Lambda} E^{\Lambda_0}\left(f^2 - f_0^2\right) \\ &\le\; \gamma_n E^{\Lambda} E^{\Lambda_0}\left(|\nabla_{\Lambda_0} f|^2\right) \\ &=\; \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda_0} f|^2\right) \ . \end{aligned} \tag{4.11}$$

As for the second term, we have

$$\begin{aligned} E^{\Lambda}\left(f_0^2 - f_m^2\right) &= \sum_{j=0}^{m-1} E^{\Lambda}\left(f_j^2 - f_{j+1}^2\right) \\ &= \sum_{j=0}^{m-1} E^{\Lambda} E^{\Lambda_{j+1}}\left(f_j^2 - f_{j+1}^2\right) \end{aligned} \tag{4.12}$$

Note that $S_{f_j} \cap \Lambda_{j+1} = \left\{x_{j+1}\right\}$ and hence by (3.6) and (3.21) that

$$\begin{aligned} E^{\Lambda_{j+1}}\left(f_j^2 - f_{j+1}^2\right) &\le C_{3.6}(1) E^{\Lambda_{j+1}}\left(|\nabla_{x_{j+1}} f_j|^2\right) \\ &\le C_{4.13} E^{\Lambda_{j+1}}\left(|\nabla_{x_{j+1}} f|^2\right) + C_{4.13} E^{\Lambda_{j+1}} \mathscr{Q}_j(f) \ , \end{aligned} \tag{4.13}$$

where $C_{4.13} = C_{4.13}(C_{3.6}(1), \|\mathbf{J}\|)$. It follows from (4.12) and (4.13) that

$$E^{\Lambda}\left(f_0^2 - f_m^2\right) \le C_{4.13} E^{\Lambda}\left(|\nabla_{\Lambda\setminus\Lambda_0} f|^2\right) + C_{4.13} \sum_{j=0}^{m-1} E^{\Lambda} \mathscr{Q}_j(f) \ . \tag{4.14}$$

By (4.10), (4.11) and (4.14), we conclude (4.8). □

Lemma 4.2.

$$\mathcal{Q}_j(f) \;\leq\; C_{4.15} \sum_{k=0}^{\infty} \exp\left(-\frac{k}{2C_{2.1}}\right) E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}(f;\, f)^{1/2}\right)^2, \qquad (4.15)$$

where $C_{4.15} = C_{4.15}(d, R, C_{2.1}, C_{3.19}) \in (0, \infty)$.

Proof. Suppose that $x \in \Lambda_j$ and $d(x, x_{j+1}) \;\leq\; R$. We then have that

$$\begin{aligned} E^{\Lambda_j}(f;\sigma_x)^2 \;&=\; E^{\Lambda_j}\left((f - f_j)\sigma_x\right)^2 \\ &=\; \left(\sum_{k=0}^{\infty} E^{\Lambda_j}\left((f_{j,k} - f_{j,k+1})\sigma_x\right)\right)^2 \\ &=\; \left(\sum_{k=0}^{\infty} E^{\Lambda_j} E^{\Lambda_{j,k+1}}(f_{j,k};\sigma_x)\right)^2 \\ &\leq\; C_{4.16} \sum_{k=0}^{\infty} (k+1)^2 \left(E^{\Lambda_j} E^{\Lambda_{j,k+1}}(f_{j,k};\sigma_x)\right)^2 \quad (4.16) \end{aligned}$$

Since $\Lambda_{j,k+1} \cap S_{f_{j,k}} \;\subset\; \Lambda_{j,k+1} \backslash \Lambda_{j,k}$ and $d(x, \Lambda_{j,k+1} \backslash \Lambda_{j,k}) \;\geq\; \frac{k}{2} - R$, it follows from (3.19) that

$$\left|E^{\Lambda_{j,k+1}}(f_{j,k};\sigma_x)\right| \leq C_{4.17}(k+1)^{2d} \exp\left(-\frac{k}{2C_{2.1}}\right) E^{\Lambda_{j,k+1}}(f_{j,k};\, f_{j,k})^{1/2} \qquad (4.17)$$

where $C_{4.17} = C_{4.17}(R, C_{3.19})$. We see from Jensen inequality that

$$\begin{aligned} E^{\Lambda_{j,k+1}}(f_{j,k};\, f_{j,k}) \;&=\; E^{\Lambda_{j,k+1}}\left(f_{j,k}^2 - f_{j,k+1}^2\right) \\ &\leq\; E^{\Lambda_{j,k+1}}\left(f^2 - f_{j,k+1}^2\right) \\ &=\; E^{\Lambda_{j,k+1}}(f;\, f) \qquad (4.18) \end{aligned}$$

Putting (4.17) and (4.18) together, we have that

$$\begin{aligned} &E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}(f_{j,k};\sigma_x)\right)^2 \\ &\quad\leq C_{4.17}^2 (k+1)^{4d} \exp\left(-\frac{k}{C_{2.1}}\right) E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}(f_{j,k};\, f_{j,k})^{1/2}\right)^2 \\ &\quad\leq C_{4.17}^2 (k+1)^{4d} \exp\left(-\frac{k}{C_{2.1}}\right) E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}(f;\, f)^{1/2}\right)^2 \end{aligned}$$

Plugging this into (4.16), we arrive at (4.15). □

Remark 4.1. This remark will become relevant when we turn to the proof of Theorem 2.2 for the case $\mathscr{F} = \mathscr{B}(n_0)$. From Remark *3.2* and the proof of Lemma 4.1 we presented above, we see that (4.15) without the factor $\exp(-(k.2C_{2.1}))$ in the right-hand-side summation is true when we do not assume any mixing condition. On the other hand, as will be seen from the way (4.15) is used later (cf. (4.23) in the proof of Lemma 4.3), the factor $\exp(-(k/2C_{2.1}))$ in the right-hand-side summation of (4.15) are used only for sufficiently large k's. It is thus sufficient for us to require (3.19)–(3.22) to be valid only for $\Lambda = \Lambda_{j,k}$ with sufficiently large k's.

Lemma 4.3. *For* $k_0 = 1, 2, \ldots, \lfloor n^{1/d} \rfloor - 1$,

$$\begin{aligned} E^{\Lambda}(f; f) \le\ & \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda_0} f|^2\right) + \gamma_n C_{4.19} \exp\left(-\frac{k_0}{3C_{2.1}}\right) E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) \\ & + D_{4.19} E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) + C_{4.19} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) E^{\Lambda}(f; f) \end{aligned} \tag{4.19}$$

where $C_{4.19} = C_{4.19}(d, U, \mathbf{J}, B_{2.1}, C_{2.1})$ *and* $D_{4.19} = D_{4.19}(k_0, d, U, \mathbf{J}, B_{2.1}, C_{2.1})$.

Proof. We see from (4.15) that

$$\sum_{j=0}^{m-1} E^{\Lambda} \mathscr{Q}_j(f) \le C_{4.15} \sum_{k=0}^{\infty} \exp\left(-\frac{k}{2C_{2.1}}\right) \sum_{j=0}^{m-1} E^{\Lambda} E^{\Lambda_{j,k+1}}(f; f) \ . \tag{4.20}$$

Let us first note that $E^{\Lambda} E^{\Lambda_{j,k+1}}(f; f)$ has the following two upper bounds;

$$\gamma_{(k+1)^d} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right), \quad E^{\Lambda}(f; f) \ . \tag{4.21}$$

The first bound in (4.21) come from (4.2) and the Markov property. In fact,

$$\begin{aligned} E^{\Lambda} E^{\Lambda_{j,k+1}}(f; f) &\le \gamma_{(k+1)^d} E^{\Lambda} E^{\Lambda_{j,k+1}}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right) \\ &= \gamma_{(k+1)^d} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right) \ . \end{aligned}$$

The second bound $E^{\Lambda}(f; f)$ in (4.21) can be seen as follows;

$$\begin{aligned} E^{\Lambda} E^{\Lambda_{j,k+1}}(f; f) &= E^{\Lambda} E^{\Lambda_{j,k+1}}(f^2) - E^{\Lambda}(f_{j,k+1}^2) \\ &\le E^{\Lambda}(f^2) - E^{\Lambda}(f)^2 \end{aligned}$$

where we have used Jensen inequality and Markov property.

We will divide the summation in k on the right-hand-side of (4.20) in three parts as follows;

$$\sum_{k=0}^{\infty} = \sum_{k=0}^{k_0-1} + \sum_{k=k_0}^{k_1-1} + \sum_{k=k_1}^{\infty} \ , \tag{4.22}$$

where $k_1 = \lfloor n^{1/d} \rfloor - 1$. We use the first bound in (4.21) to estimate the first and the second summations in (4.22) as follows;

$$\begin{aligned} \sum_{k=0}^{k_0-1} &\leq \sum_{k=0}^{k_0-1} \exp\left(-\frac{k}{2C_{2.1}}\right) \sum_{j=0}^{m-1} \gamma_{(k+1)^d} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right) \\ &\leq \sum_{x\in\Lambda} E^{\Lambda}\left(|\nabla_x f|^2\right) \sum_{k=0}^{k_0-1} \exp\left(-\frac{k}{2C_{2.1}}\right) \gamma_{(k+1)^d} \sum_{j:\Lambda_{j,k+1}\ni x} 1 \\ &\leq C_{4.23}(k_0) E^{\Lambda}\left(|\nabla_\Lambda f|^2\right) \ . \end{aligned} \tag{4.23}$$

$$\begin{aligned} \sum_{k=k_0}^{k_1-1} &\leq \sum_{k=k_0}^{k_1-1} \exp\left(-\frac{k}{2C_{2.1}}\right) \sum_{j=0}^{m-1} \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right) \\ &\leq \gamma_n C_{4.24} \exp\left(-\frac{k_0}{3C_{2.1}}\right) E^{\Lambda}\left(|\nabla_\Lambda f|^2\right) \ . \end{aligned} \tag{4.24}$$

Note that $\gamma_{(k+1)^d} \leq C_{3.6}(k_0^d)$ for $k \leq k_0 - 1$ by Lemma 3.2 and hence that we can make $C_{4.23}(k_0)$ depend only on $k_0, d, U, \mathbf{J}, B_{2.1}$ and $C_{2.1}$. To estimate the second summation in (4.22), we make use of the second bound in (4.21);

$$\begin{aligned} \sum_{k=k_1}^{\infty} &\leq \sum_{k=k_1}^{\infty} \exp\left(-\frac{k}{2C_{2.1}}\right) \sum_{j=0}^{m-1} E^{\Lambda}\left(f;\, f\right) \\ &\leq C_{4.25} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) E^{\Lambda}\left(f;\, f\right) \end{aligned} \tag{4.25}$$

Now, (4.19) can be seen from (4.8), (4.20) and (4.23)–(4.25) . □

Proof of (4.3). With Lemma 4.3 in hand, (4.3) can be proved as follows. By exchanging the role of Λ_0 and $\Lambda\backslash\Lambda_0$, we have that for $k_0 \leq \lfloor n^{1/d} \rfloor - 1$,

$$\begin{aligned} E^{\Lambda}(f;\, f) \leq \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda\backslash\Lambda_0} f|^2\right) + \gamma_n C_{4.19} \exp\left(-\frac{k_0}{3C_{2.1}}\right) E^{\Lambda}\left(|\nabla_\Lambda f|^2\right) \\ + D_{4.19} E^{\Lambda}\left(|\nabla_\Lambda f|^2\right) s + C_{4.19} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) E^{\Lambda}(f;\, f) \ , \end{aligned} \tag{4.26}$$

and hence by averaging (4.19) and (4.26) that

$$E^{\Lambda}(f;f) \le \frac{\gamma_n}{2} E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) + \gamma_n C_{4.19} \exp\left(-\frac{k_0}{3C_{2.1}}\right) E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) + D_{4.19} E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) + C_{4.19} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) E^{\Lambda}(f;f) \ . \tag{4.27}$$

Since our choice of Λ and $0 < f \in \mathscr{C}_{\Lambda}$ was arbitrary as long as $|\Lambda| \le 2n$, we see from (4.27) that

$$\gamma_{2n} \le \frac{\left(\frac{1}{2} + C_{4.19} \exp\left(-\frac{k_0}{3C_{2.1}}\right)\right)\gamma_n + D_{4.19}}{1 - C_{4.19} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right)} \ . \tag{4.28}$$

At this point, we choose k_0 such that $\frac{1}{2} + C_{4.19} \exp\left(-(k_0/3C_{2.1})\right) < \frac{3}{5}$. We then have (4.3) with $C_{4.3} = \frac{4}{3} D_{4.19}$, whenever $1 - C_{4.19} \exp\left(-(n^{1/d}/3C_{2.1})\right) \ge \frac{3}{4}$. □

Proof of Theorem 2.2. for $\mathscr{F} = \mathscr{B}(n_0)$: We modify the proof for the case $\mathscr{F} = \mathscr{A}$ as follows. The goal is equivalent to that;

$$\sup_{n_0 \le n_i < \infty} \gamma(n_1, \dots, n_d) < \infty \ , \tag{4.29}$$

where

$$\gamma(n_1, \dots, n_d) = \sup\left\{\gamma_{\mathrm{SG}}(\Lambda); \begin{array}{l} \Lambda \text{ is a generalized box with size } (m_1, \dots, m_d), \\ n_0 \le m_i \le n_i, \quad 1 \le i \le d \end{array}\right\} \tag{4.30}$$

To prove (4.29), it is enough to find some constants $C_{4.31}$ and $N_{4.31}$ which depend only on d, R, U, $\mathbf{J}$, n_0, $B_{2.1}$ and $C_{2.1}$ such that

$$\gamma(2n_1, n_2, \dots, n_d) \le \tfrac{4}{5}\gamma(n_1, n_2, \dots, n_d) + C_{4.31} \ , \tag{4.31}$$

for $n_i \ge N_{4.31}$. To this end, we take arbitrary $\Lambda \in \mathscr{B}(n_0)$ with the size at most $(2n_1, n_2, \dots, n_d)$ and $0 < f \in \mathscr{C}_{\Lambda}$. We then decompose Λ into $\Lambda_0 \in \mathscr{B}(n_0)$ and $\Lambda \backslash \Lambda_0 \in \mathscr{B}(n_0)$ such that both of them are of the size at most $(n_1, n_2, \dots, n_d)$. We can now choose an enumeration $\{x_j\}_{j=1}^m$ of $\Lambda \backslash \Lambda_0$ so that $\Lambda_j \in \mathscr{B}(n_0)$ for all $j = 1, \dots, m$, where Λ_j is defined by

(4.4). Note also that $\Lambda_{j,k} \in \mathscr{B}(n_0)$ for $k > 2n_0$, where $\Lambda_{j,k}$ is defined by (4.6). We thus see that the proof of (4.3) for the case $\mathscr{F} = \mathscr{A}$ works almost without change (Recall that we need to apply (3.19)–(3.22) to the set $\Lambda_{j,k}$ with $k > 2n_0$, cf. Remark 4.1. □

5. Proof of Theorem 2.1

In this section, we prove Theorem 2.1 by using Theorem 2.2 as well as lemmas presented in Section 3. The basic strategy is the same as that in the proof of Theorem 2.2. We first consider the case $\mathscr{F} = \mathscr{A}$. What we want to prove is equivalent to that

$$\sup_{n \geq 1} \gamma_n < \infty \ , \tag{5.1}$$

where

$$\gamma_n = \sup \{\gamma_{\mathrm{LS}}(\Lambda) \mid |\Lambda| \leq n\} \ . \tag{5.2}$$

To prove (5.1), it is enough to find some constants $C_{5.3}$ and $N_{5.3}$ which depend only on d, R, U, $\mathbf{J}$, $B_{2.1}$ and $C_{2.1}$ such that

$$\gamma_{2n} \leq \tfrac{4}{5}\gamma_n + C_{5.3}, \quad \text{for} \quad n \geq N_{5.3} \ . \tag{5.3}$$

To this end, we take arbitrary $\Lambda \subset\subset \mathbf{Z}^d$ with $|\Lambda| \leq 2n$ and $0 < f \in \mathscr{C}_\Lambda$. We then choose $\Lambda_0 \subset \Lambda$ such that $\max\{|\Lambda_0|,\ |\Lambda\backslash\Lambda_0|\} \leq n$ and define

$$\Lambda_j = \Lambda_0 \cup \left\{x_1, \ldots, x_j\right\}, \quad j = 1, 2, \ldots, m \ , \tag{5.4}$$

$$f_j = \sqrt{E^{\Lambda_j}(f^2)} \ , \tag{5.5}$$

$$\Lambda_{j,k} = \{x \in \Lambda_j;\ d(x, x_{j+1}) < (k/2)\}, \quad k = 0, 1, 2, \ldots \ , \tag{5.6}$$

$$f_{j,k} = f, \quad \text{if } k \geq 0 \text{ and } \Lambda_{j,k} = \phi \ ,$$

$$f_{j,k} = \sqrt{E^{\Lambda_{j,k}}(f^2)}, \quad \text{if } k \geq 1 \text{ and } \Lambda_{j,k} \neq \phi \ , \tag{5.7}$$

where $\{x_j\}_{j=1}^m$ is an enumeration of $\Lambda\backslash\Lambda_0$. We will prove (5.3) after a series of lemmas.

Lemma 5.1.

$$E^\Lambda\left(f^2 \log \frac{f^2}{E^\Lambda\left(f^2\right)}\right) \leq \gamma_n E^\Lambda\left(|\nabla_{\Lambda_0} f|^2\right) + C_{5.8} E^\Lambda\left(|\nabla_{\Lambda\backslash\Lambda_0} f|^2\right) + C_{5.8} \sum_{j=0}^{m-1} E^\Lambda\left(\frac{\mathscr{R}_j(f)}{f_j^2}\right) \ , \tag{5.8}$$

where $C_{5.8} = C_{5.8}(C_{3.6}(1), \|\mathbf{J}\|) \in (0, \infty)$ *and*

$$\mathscr{R}_j(f) = \sup\left\{E^{\Lambda_j}(f^2; \sigma_x)^2;\ x \in \Lambda_j,\ d(x, x_{j+1}) \le R\right\} \ . \tag{5.9}$$

Proof. We first divide the left-hand-side of (5.8) into two terms;

$$E^{\Lambda}\left(f^2 \log \frac{f^2}{E^{\Lambda}\left(f^2\right)}\right) = E^{\Lambda}\left(f^2 \log \frac{f^2}{f_0^2}\right) + E^{\Lambda}\left(f^2 \log \frac{f_0^2}{f_m^2}\right) \ . \tag{5.10}$$

The first term on the right-hand-side can be estimated as follows;

$$\begin{aligned} E^{\Lambda}\left(f^2 \log \frac{f^2}{f_0^2}\right) &= E^{\Lambda} E^{\Lambda_0}\left(f^2 \log \frac{f^2}{f_0^2}\right) \\ &\le \gamma_n E^{\Lambda} E^{\Lambda_0}\left(|\nabla_{\Lambda_0} f|^2\right) \\ &= \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda_0} f|^2\right) \ . \end{aligned} \tag{5.11}$$

As for the second term, we have

$$\begin{aligned} E^{\Lambda}\left(f^2 \log \frac{f_0^2}{f_m^2}\right) &= \sum_{j=0}^{m-1} E^{\Lambda}\left(f_j^2 \log f_j^2 - f_{j+1}^2 \log f_{j+1}^2\right) \\ &= \sum_{j=0}^{m-1} E^{\Lambda}\left(f_j^2 \log f_j^2 - E^{\Lambda_{j+1}}(f_j^2) \log f_{j+1}^2\right) \\ &= \sum_{j=0}^{m-1} E^{\Lambda} E^{\Lambda_{j+1}}\left(f_j^2 \log \frac{f_j^2}{f_{j+1}^2}\right) \ . \end{aligned} \tag{5.12}$$

Note that $S_{f_j} \cap \Lambda_{j+1} = \left\{x_{j+1}\right\}$ and hence by (3.6) and (3.23) that

$$\begin{aligned} E^{\Lambda_{j+1}}\left(f_j^2 \log \frac{f_j^2}{f_{j+1}^2}\right) &\le C_{3.6}(1) E^{\Lambda_{j+1}}\left(|\nabla_{x_{j+1}} f_j|^2\right) \\ &\le C_{5.13} E^{\Lambda_{j+1}}\left(|\nabla_{x_{j+1}} f|^2\right) + C_{5.13} \frac{\mathscr{R}_j(f)}{f_j^2}, \end{aligned} \tag{5.13}$$

where $C_{5.13} = C_{5.13}(C_{3.6}(1), \|\mathbf{J}\|) \in (0, \infty)$. It follows from (5.12) and (5.13) that

$$E^{\Lambda}\left(f^2 \log \frac{f_0^2}{f_m^2}\right) \le C_{5.13} E^{\Lambda}\left(|\nabla_{\Lambda \setminus \Lambda_0} f|^2\right) + C_{5.13} \sum_{j=0}^{m-1} E^{\Lambda}\left(\frac{\mathscr{R}_j(f)}{f_j^2}\right) \ . \tag{5.14}$$

By (5.10), (5.11) and (5.14), we conclude (5.8). □

Lemma 5.2.

$$\frac{\mathscr{R}_j(f)}{f_j^2} \le C_{5.15} \sum_{k=0}^{\infty} \exp\left(-\frac{k}{2C_{2.1}}\right) E^{\Lambda_j}\left(|\nabla_{\Lambda_{j,k+1}} f|^2 + f^2 \log\frac{f^2}{f_{j,k}^2} + f^2 \log\frac{f^2}{f_{j,k+1}^2}\right) , \tag{5.15}$$

where $C_{5.15} = C_{5.15}(d, R, C_{2.1}, C_{2.5}, C_{3.19}) \in (0, \infty)$.

Proof. Suppose that $x \in \Lambda_j$ and $d(x, x_{j+1}) \le R$. We then have that

$$\begin{aligned} E^{\Lambda_j}(f^2; \sigma_x)^2 &= E^{\Lambda_j}\left(\left(f^2 - f_j^2\right)\sigma_x\right)^2 \\ &= \left(\sum_{k=0}^{\infty} E^{\Lambda_j}\left(\left(f_{j,k}^2 - f_{j,k+1}^2\right)\sigma_x\right)\right)^2 \\ &= \left(\sum_{k=0}^{\infty} E^{\Lambda_j} E^{\Lambda_{j,k+1}}\left(f_{j,k}^2; \sigma_x\right)\right)^2 \\ &\le C_{5.16} \sum_{k=0}^{\infty} (k+1)^2 E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}\left(f_{j,k}^2; \sigma_x\right)\right)^2 \end{aligned} \tag{5.16}$$

Since $\Lambda_{j,k+1} \cap S_{f_{j,k}} \subset \Lambda_{j,k+1} \backslash \Lambda_{j,k}$ and $d(x, \Lambda_{j,k+1} \backslash \Lambda_{j,k}) \ge \frac{k}{2} - R$, it follows from (3.20) that

$$\left|E^{\Lambda_{j,k+1}}\left(f_{j,k}^2; \sigma_x\right)\right| \le C_{5.17}(k+1)^{2d} \exp\left(-\frac{k}{2C_{2.1}}\right) f_{j,k+1}\left(I_1^{1/2} + I_2^{1/2}\right) , \tag{5.17}$$

where $C_{5.17} = C_{5.17}(R, C_{3.19})$,

$$I_1 = E^{\Lambda_{j,k+1}}\left(f_{j,k}; f_{j,k}\right) \quad \text{and} \quad I_2 = E^{\Lambda_{j,k+1}}\left(f_{j,k}^2 \log\frac{f_{j,k}^2}{f_{j,k+1}^2}\right) .$$

I_1 can be estimated as follows;

$$I_1 \le C_{2.5} E^{\Lambda_{j,k+1}}\left(|\nabla_{\Lambda_{j,k+1}} f_{j,k}|^2\right) \tag{5.18}$$

$$\le 2C_{2.5} E^{\Lambda_{j,k+1}}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right) + C_{5.19}(k+1)^{5d} \left(E^{\Lambda_{j,k+1}}(f; f) + E^{\Lambda_{j,k+1}}\left(f^2 \log\frac{f^2}{f_{j,k}^2}\right)\right) \tag{5.19}$$

$$\leq C_{5.20}(k+1)^{5d}\Bigg(E^{\Lambda_{j,k+1}}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right)$$
$$+E^{\Lambda_{j,k+1}}\left(f^2\log\frac{f^2}{f_{j,k}^2}\right)\Bigg)\ . \qquad (5.20)$$

Here, we have used (2.5) in both (5.18) and (5.20), whereas (5.19) is an application of (3.24). On the other hand, we see from Jensen inequality that

$$I_2 \leq E^{\Lambda_{j,k+1}}\left(f^2\log\frac{f^2}{f_{j,k+1}^2}\right)\ . \qquad (5.21)$$

Putting (5.17), (5.20) and (5.21) together, we have that

$$E^{\Lambda_j}\left(E^{\Lambda_{j,k+1}}(f_{j,k}^2;\sigma_x)\right)^2$$
$$\leq C_{5.17}^2(k+1)^{4d}\exp\left(-\frac{k}{C_{2.1}}\right)E^{\Lambda_j}\left(f_{j,k+1}\left(I_1^{1/2}+I_2^{1/2}\right)\right)^2$$
$$\leq 2C_{5.17}^2(k+1)^{4d}\exp\left(-\frac{k}{C_{2.1}}\right)f_j^2E^{\Lambda_j}(I_1+I_2)$$
$$\leq C_{5.22}(k+1)^{9d}\exp\left(-\frac{k}{C_{2.1}}\right)$$
$$\cdot f_j^2E^{\Lambda_j}\left(|\nabla_{\Lambda_{j,k+1}} f|^2+f^2\log\frac{f^2}{f_{j,k}^2}+f^2\log\frac{f^2}{f_{j,k+1}^2}\right)\ . \qquad (5.22)$$

Plugging this into (5.16), we arrive at the following bound;

$$\frac{E^{\Lambda_j}(f^2;\sigma_x)^2}{f_j^2} \leq C_{5.23}\sum_{k=0}^{\infty}(k+1)^{9d+2}\exp\left(-\frac{2k}{C_{2.1}}\right)$$
$$\times E^{\Lambda_j}\left(|\nabla_{\Lambda_{j,k+1}} f|^2+f^2\log\frac{f^2}{f_{j,k}^2}+f^2\log\frac{f^2}{f_{j,k+1}^2}\right)\ , \qquad (5.23)$$

which proves (5.15). □

Remark 5.1. This remark will become relevant when we turn to the proof of Theorem 2.1 for the case $\mathscr{F}=\mathscr{B}(n_0)$. From Remark 3.2 and the proof of Lemma 5.1 we presented above, we see that (5.15) without the factor $\exp\left(-\frac{k}{C_{2.1}}\right)$ in the right-hand-side summation is true when we do not assume any mixing condition. On the other hand, as will be seen from the way (5.15) is used later (cf. (5.28) in the proof of Lemma 5.3), the factor $\exp\left(-\frac{k}{C_{2.1}}\right)$ in the right-hand-side summation of (5.15) are used only for sufficiently large k's. It is thus sufficient for us to require (3.19)–(3.22) to be valid only for $\Lambda = \Lambda_{j,k}$ with sufficiently large k's.

Lemma 5.3. *For* $k_0 = 1, 2, \ldots, \lfloor n^{1/d} \rfloor - 1$,

$$
\begin{aligned}
&E^{\Lambda}\left(f^2 \log \frac{f^2}{E^{\Lambda}(f^2)}\right) \\
&\quad \le \gamma_n E^{\Lambda}\left(|\nabla_{\Lambda_0} f|^2\right) + \gamma_n C_{5.24} \exp\left(-\frac{k_0}{3C_{2.1}}\right) E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) \\
&\qquad + D_{5.24} E^{\Lambda}\left(|\nabla_{\Lambda} f|^2\right) \\
&\qquad + C_{5.24} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) E^{\Lambda}\left(f^2 \log \frac{f^2}{E^{\Lambda}(f^2)}\right) \qquad (5.24)
\end{aligned}
$$

where $C_{5.24} = C_{5.24}(d, U, \mathbf{J}, B_{2.1}, C_{2.1})$ *and* $D_{5.24} = D_{5.24}(k_0, d, U, \mathbf{J}, B_{2.1}, C_{2.1})$.

Proof. We see from (5.15) that

$$
\begin{aligned}
\sum_{j=0}^{m-1} E^{\Lambda}\left(\frac{\mathscr{R}_j(f)}{f_j^2}\right) &\le C_{5.15} \sum_{k=0}^{\infty} \exp\left(-\frac{k}{2C_{2.1}}\right) \\
&\quad \cdot \sum_{j=0}^{m-1} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2 + f^2 \log \frac{f^2}{f_{j,k}^2} + f^2 \log \frac{f^2}{f_{j,k+1}^2}\right) \ . \qquad (5.25)
\end{aligned}
$$

Let us first note that $E^{\Lambda}(f^2 \log(f^2/f_{j,k}^2) + f^2 \log(f^2/f_{j,k+1}^2))$ has the following two upper bounds;

$$
2\gamma_{(k+1)^d} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k+1}} f|^2\right), \quad 2E^{\Lambda}\left(f^2 \log \frac{f^2}{E^{\Lambda}(f^2)}\right) \ . \qquad (5.26)
$$

The first bound in (5.26) comes from (5.2) and the Markov property. For example,

$$
\begin{aligned}
E^{\Lambda}\left(f^2 \log \frac{f^2}{f_{j,k}^2}\right) &\le E^{\Lambda} E^{\Lambda_{j,k}}\left(f^2 \log \frac{f^2}{f_{j,k}^2}\right) \\
&\le \gamma_{k^d} E^{\Lambda} E^{\Lambda_{j,k}}\left(|\nabla_{\Lambda_{j,k}} f|^2\right) \\
&= \gamma_{k^d} E^{\Lambda}\left(|\nabla_{\Lambda_{j,k}} f|^2\right) \ .
\end{aligned}
$$

The second bound $E^\Lambda(f^2\log(f^2/E^\Lambda(f^2)))$ in (5.26) can be seen as follows;

$$\begin{aligned} E^\Lambda\left(f^2\log\frac{f^2}{f_{j,k}^2}\right) &= E^\Lambda\left(f^2\log(f^2)\right) - E^\Lambda\left(f^2\log(f_{j,k}^2)\right) \\ &= E^\Lambda\left(f^2\log(f^2)\right) - E^\Lambda\left(f_{j,k}^2\log(f_{j,k}^2)\right) \\ &\le E^\Lambda\left(f^2\log(f^2)\right) - E^\Lambda(f^2)\log E^\Lambda(f^2)\ , \end{aligned}$$

where we have used Jensen inequality and Markov property in the last line.

We will divide the summation in k on the right-hand-side of (**??**) in three parts as follows;

$$\sum_{k=0}^{\infty} = \sum_{k=0}^{k_0-1} + \sum_{k=k_0}^{k_1-1} + \sum_{k=k_1}^{\infty}\ , \tag{5.27}$$

where $k_1 = \lfloor n^{1/d}\rfloor - 1$. We use the first bound in (5.26) to estimate the first and the second summations in (5.27) as follows;

$$\begin{aligned} \sum_{k=0}^{k_0-1} &\le \sum_{k=0}^{k_0-1}\exp\left(-\frac{k}{2C_{2.1}}\right)\sum_{j=0}^{m-1}\left(1+2\gamma_{(k+1)^d}\right)E^\Lambda\left(|\nabla_{\Lambda_{j,k+1}}f|^2\right) \\ &\le \sum_{x\in\Lambda}E^\Lambda\left(|\nabla_x f|^2\right)\sum_{k=0}^{k_0-1}\exp\left(-\frac{k}{2C_{2.1}}\right)\left(1+2\gamma_{(k+1)^d}\right)\sum_{j:\Lambda_{j,k+1}\ni x}1 \\ &\le C_{5.28}(k_0)E^\Lambda\left(|\nabla_\Lambda f|^2\right)\ . \end{aligned} \tag{5.28}$$

$$\begin{aligned} \sum_{k=k_0}^{k_1-1} &\le \sum_{k=k_0}^{k_1-1}\exp\left(-\frac{k}{2C_{2.1}}\right)\sum_{j=0}^{m-1}(1+2\gamma_n)\,E^\Lambda\left(|\nabla_{\Lambda_{j,k+1}}f|^2\right) \\ &\le (1+2\gamma_n)\,C_{5.29}\exp\left(-\frac{k_0}{3C_{2.1}}\right)E^\Lambda\left(|\nabla_\Lambda f|^2\right)\ . \end{aligned} \tag{5.29}$$

Note that $\gamma_{(k+1)^d} \le C_{3.6}(k_0^d)$ for $k \le k_0-1$ by Lemma 3.2 and hence that we can make $C_{5.28}(k_0)$ depend only on $k_0, d, U, \mathbf{J}, B_{2.1}$ and $C_{2.1}$. To estimate the second summation in (5.27), we make use of the second bound in (5.26);

$$\begin{aligned} \sum_{k=k_1}^{\infty} \le &\sum_{k=k_1}^{\infty}\exp\left(-\frac{k}{2C_{2.1}}\right) \\ &\times\sum_{j=0}^{m-1}\left(E^\Lambda\left(|\nabla_{\Lambda_{j,k+1}}f|^2\right)+2E^\Lambda\left(f^2\log\frac{f^2}{E^\Lambda(f^2)}\right)\right) \end{aligned}$$

$$\leq\ C_{5.30} E^{\Lambda} \left(|\nabla_{\Lambda} f|^2\right) + C_{5.30} \exp\left(-\frac{n^{1/d}}{3C_{2.1}}\right) \times E^{\Lambda}\left(f^2 \log \frac{f^2}{E^{\Lambda}(f^2)}\right) \tag{5.30}$$

Now, (5.24) can be seen from (5.8), (**??**) and (5.28)–(5.30). □

Proof of (5.3). With Lemma 5.3 in hand, (5.3) can be proved in the same way as we derived (4.3) from Lemma 4.3. □

6. Proof of Proposition 2.3 and Theorem 2.4

In this section, we prove Proposition 2.3 and Theorem 2.4. The proof of Proposition 2.3 is based on the two lemmas presented below.

Lemma 6.1. *For $W \subset \Lambda$ and for $\bar{p}^i \in \mathbf{R}^{\Lambda}$ $(i = 1, 2)$, there exists a measure $\mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \in \mathscr{K}(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q})$ such that*

$$\sum_{z \in W \cap (x+V)} f_z^{W,\bar{p}^1,\bar{p}^2} \leq \sum_{z} K_{x,z} f_z^{W,\bar{p}^1,\bar{p}^2} \ , \tag{6.1}$$

where

$$f_z^{W,\bar{p}^1,\bar{p}^2} = \int \mathscr{E}^{W,\bar{p}^1,\bar{p}^2}(dp^1 dp^2)|p_z^1 - p_z^2| \ . \tag{6.2}$$

Proof. Let us take $\mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \in \mathscr{K}(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q})$ which attains the Vassershtein distance of $\mathscr{E}^{W,\bar{p}^i}_{\Lambda,q}$ $(i = 1, 2)$, i.e.,

$$\sum_{z \in W} f_z^{W,\bar{p}^1,\bar{p}^2} = \mathscr{R}_W(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}) \ . \tag{6.3}$$

The existence of such measure is guaranteed by the compactness of the set $\mathscr{K}(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q})$ and the fact that the map $\mu \mapsto \int \mu(dp^1 dp^2)|p_z^1 - p_z^2|$ from $\mathscr{M}_1(\mathbf{R}^W \times \mathbf{R}^W)$ to $[0, \infty)$ is lower semi-continuous.

We next take a measure

$$\hat{\mathscr{E}}^x(\cdot|\hat{p}^1, \hat{p}^2) \in \mathscr{K}(\mathscr{E}^{W\cap(x+V),\hat{p}^1}_{\Lambda,q}, \mathscr{E}^{W\cap(x+V),\hat{p}^2}_{\Lambda,q})$$

in such a way that it attains the Vassershtein distance of $\mathscr{E}^{W\cap(x+V),\hat{p}^i}_{\Lambda,q}$ $(i = 1, 2)$, i.e.,

$$\mathscr{R}_{W\cap(x+V)}(\mathscr{E}^{W\cap(x+V),\hat{p}^1}_{\Lambda,q}, \mathscr{E}^{W\cap(x+V),\hat{p}^2}_{\Lambda,q}) = \int \hat{\mathscr{E}}^x\left(dp^1 dp^2|\hat{p}^1, \hat{p}^2\right) \sum_{z \in W\cap(x+V)} |p_z^1 - p_z^2| \ , \tag{6.4}$$

and that the map $(\hat{p}^1, \hat{p}^2) \mapsto \hat{\mathscr{E}}^x(\cdot|\hat{p}^1, \hat{p}^2)$ from $\mathbf{R}^\Lambda \times \mathbf{R}^\Lambda$ to $\mathscr{M}_1(\mathbf{R}^{W\cap(x+V)} \times \mathbf{R}^{W\cap(x+V)})$ is measurable. The possibility of such measurable selection can be shown as an apllication of [SV79, Theorem 12.1.10]. (Use also Lemma 12.1.7 in that book to check that the set of minimizers of the Vassershtein distance of $\mathscr{E}^{W\cap(x+V),\hat{p}^i}_{\Lambda,q}$ $(i = 1, 2)$ is measurable as a set-valued function of $(\hat{p}^1, \hat{p}^2)$).

We now define a measure $\widetilde{\mathscr{E}}^{x,\bar{p}^1,\bar{p}^2} \in \mathscr{M}_1(\mathbf{R}^W \times \mathbf{R}^W)$ by

$$\widetilde{\mathscr{E}}^{x,\bar{p}^1,\bar{p}^2}(A \times B) = \int_A \mathscr{E}^{W,\bar{p}^1,\bar{p}^2}(d\hat{p}^1 d\hat{p}^2) \int_B \hat{\mathscr{E}}^x(dp^1 dp^2|\hat{p}^1, \hat{p}^2) \ ,$$

where $A \subset \mathbf{R}^{W\setminus(x+V)} \times \mathbf{R}^{W\setminus(x+V)}$ and $B \subset \mathbf{R}^{(x+V)\cap W} \times \mathbf{R}^{(x+V)\cap W}$. It follows from the above definition that

$$\widetilde{\mathscr{E}}^{x,\bar{p}^1,\bar{p}^2} \in \mathscr{K}\left(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}\right), \tag{6.5}$$

$$\widetilde{\mathscr{E}}^{x,\bar{p}^1,\bar{p}^2} = \mathscr{E}^{x,\bar{p}^1,\bar{p}^2} \quad \text{on } \mathbf{R}^{W\setminus(x+V)} \times \mathbf{R}^{W\setminus(x+V)}. \tag{6.6}$$

To see (6.1), it is sufficient to prove that

$$\sum_{z\in W\cap(x+V)} f_z^{W,\bar{p}^1,\bar{p}^2} \le \sum_{z\in W\cap(x+V)} \tilde{f}_z^{x,\bar{p}^1,\bar{p}^2} \ , \tag{6.7}$$

$$\sum_{z\in W\cap(x+V)} \tilde{f}_z^{x,\bar{p}^1,\bar{p}^2} \le \sum_z K_{x,z} f_z^{W,\bar{p}^1,\bar{p}^2} \ , \tag{6.8}$$

where

$$\tilde{f}_z^{x,\bar{p}^1,\bar{p}^2} = \int \widetilde{\mathscr{E}}^{x,\bar{p}^1,\bar{p}^2}(dp^1 dp^2)|p_z^1 - p_z^2| \ . \tag{6.9}$$

The first inequality (6.7) can be seen as follows. Since (6.6) implies that $f_z^{x,\bar{p}^1,\bar{p}^2} = \tilde{f}_z^{x,\bar{p}^1,\bar{p}^2}$ for $z \notin W \cap (x + V)$, we have from this, (6.3) and (6.5) that

$$\begin{aligned}
\sum_{z\in W\cap(x+V)} (f_z^{W,\bar{p}^1,\bar{p}^2} - \tilde{f}_z^{x,\bar{p}^1,\bar{p}^2}) &= \sum_{z\in W} (f_z^{W,\bar{p}^1,\bar{p}^2} - \tilde{f}_z^{x,\bar{p}^1,\bar{p}^2}) \\
&\le \sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} - \mathscr{R}_W(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}) \\
&= 0.
\end{aligned}$$

To prove the second inequality (6.8), we will use (6.4) and (2.8) as follows;

$$\sum_{z\in W\cap(x+V)} \widetilde{f}_z^{x,\bar p^1,\bar p^2} = \int \mathscr{E}^{W,\bar p^1,\bar p^2}(d\hat p^1 d\hat p^2)\int \hat{\mathscr{E}}^{x,\bar p^1,\bar p^2}(dp^1dp^2|\hat p^1,\hat p^2) \times \sum_{z\in W\cap(x+V)} |p_z^1 - p_z^2|$$
$$\le \int \mathscr{E}^{W,\bar p^1,\bar p^2}(d\hat p^1 d\hat p^2)\sum_z K_{x,z}|\hat p_z^1 - \hat p_z^2|$$
$$= \sum_z K_{x,z} f_z^{W,\bar p^1,\bar p^2}.$$

This completes the proof of Lemma 6.1. □

Lemma 6.2. *For any* $A \subset W \subset \Lambda$, $L \ge 1$ *and* $\bar p^i \in \mathbf{R}^\Lambda$ $(i = 1, 2)$ *with* $\bar p^1 \equiv \bar p^2$ *off* y

$$\sum_{z\in W} f_z^{W,\bar p^1,\bar p^2}\exp\left(-\frac{d(z,A)}{C_{6.10}}\right) \le B_{6.10}\sum_{\substack{z\in W\\ d(z,y)\le L+D_{6.10}}} f_z^{W,\bar p^1,\bar p^2}\exp\left(-\frac{d(z,A)}{C_{6.10}}\right) + B_{6.10}\sum_{z;d(z,y)>L} K_{z,y}|\bar p_y^1 - \bar p_y^2| \times \exp\left(-\frac{d(z,A)}{C_{6.10}}\right), \qquad (6.10)$$

where $f_z^{W,\bar p^1,\bar p^2}$ *is defined by* (6.2), $D_{6.10} = \mathrm{diam}(V \cup \partial_R V)$, $B_{6.10} = B_{6.10}(R, V, \varepsilon_{2.8})$ *and* $C_{6.10} = C_{6.10}(R, V, \varepsilon_{2.8})$. *In addition, by* (2.7), *the second term on the right-hand-side of* (6.10) *is zero when* $L \ge D_{6.10}$.

Proof. We set $e_x = \exp(-(d(x,A)/C_{6.10}))$, where $C_{6.10} = C_{6.10}(R, V, \varepsilon_{2.8})$ is choosen so large that

$$C_{6.11} \overset{\text{def.}}{=} \exp\left(-\frac{D_{6.10}}{C_{6.10}}\right) - \varepsilon_{2.8}\exp\left(\frac{D_{6.10}}{C_{6.10}}\right) > 0 . \qquad (6.11)$$

We then define $l_z^0 = \sum_{x:x+V\ni z} e_x$, $l_z^1 = \sum_x e_x K_{x,z}$, $l_z = l_z^0 - l_z^1$ and $r_z = \sum_{\substack{x:x+V\ni z\\ d(x,y)\le L}} e_x$. Let us first prove that

$$\sum_{z\in W} f_z^{W,\bar p^1,\bar p^2} l_z \le \sum_{z\in W} f_z^{W,\bar p^1,\bar p^2} r_z + \sum_{z:d(z,y)\ge L} e_z K_{z,y}|\bar p_y^1 - \bar p_y^2| . \qquad (6.12)$$

We have by (6.1) that

$$\sum_{x:d(x,y)>L} e_x \sum_{z\in W\cap(x+V)} f_z^{W,\bar{p}^1,\bar{p}^2}$$
$$\leq \sum_{x:d(x,y)>L} e_x \sum_{z\in W} K_{x,z} f_z^{W,\bar{p}^1,\bar{p}^2} + \sum_{x:d(x,y)>L} e_x K_{x,y} |\bar{p}_y^1 - \bar{p}_y^2| \ . \quad (6.13)$$

Since

$$\sum_{x:d(x,y)>L} e_x \sum_{z\in W\cap(x+V)} f_z^{W,\bar{p}^1,\bar{p}^2} = \sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} \sum_{\substack{x:d(x,y)>L \\ x+V\ni z}} e_x$$
$$= \sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} \left(l_z^0 - r_z\right) \ ,$$
$$\sum_{x:d(x,y)>L} e_x \sum_{z\in W} K_{x,z} f_z^{W,\bar{p}^1,\bar{p}^2} \leq \sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} l_z^1 \ ,$$

it follows from (6.13) that

$$\sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} \left(l_z^0 - r_z\right) \leq \sum_{z\in W} f_z^{W,\bar{p}^1,\bar{p}^2} l_z^1 + \sum_{x:d(x,y)>L} e_x K_{x,y} |\bar{p}_y^1 - \bar{p}_y^2| \ ,$$

which is equivalent to (6.12).

Let us next prove that

$$r_z \leq C_{6.14} e_z \ , \quad (6.14)$$
$$r_z = 0 \quad \text{if } d(z,y) > L + D_{6.10} \ , \quad (6.15)$$
$$l_z \geq C_{6.16} e_z \ , \quad (6.16)$$

where $C_{6.14}, C_{6.16} \in (0,\infty)$ depend only on R, V and $\varepsilon_{2.8}$. To verify (6.14) and (6.15), note first that an easy to prove fact that

$$\exp\left(-\frac{D_{6.10}}{C_{6.10}}\right) \leq \frac{e_x}{e_z} \leq \exp\left(\frac{D_{6.10}}{C_{6.10}}\right) \quad \text{if } d(x,z) \leq D_{6.10} \ . \quad (6.17)$$

We thus see that

$$r_z \leq \exp\left(\frac{D_{6.10}}{C_{6.10}}\right) e_z \sum_{\substack{x:x+V\ni z \\ d(x,y)\leq L}} 1 \ ,$$

which proves (6.14) and (6.15).

On the other hand, it follows from (6.17) and (2.8) that

$$\begin{aligned} l_z \ &\geq \ e_z \left\{ \exp\left(-\frac{D_{6.10}}{C_{6.10}}\right) \sum_{x:x+V\ni z} 1 - \exp\left(\frac{D_{6.10}}{C_{6.10}}\right) \sum_{x\in\mathbb{Z}^d} K_{x,z} \right\} \\ &\geq \ e_z \left\{ \exp\left(-\frac{D_{6.10}}{C_{6.10}}\right) |V| - \exp\left(\frac{D_{6.10}}{C_{6.10}}\right) \varepsilon_{2.8}|V| \right\} \\ &= \ C_{6.11}|V|e_z \ , \end{aligned} \tag{6.18}$$

which proves (6.16).

By plugging (6.14), (6.15) and (6.16) into (6.12), we obtain (6.10). □

Proof of Proposition 2.3. Suppose that $A \subset W \subset \Lambda$. We denote by $(\mathscr{E}^{W,\bar{p}}_{\Lambda,q})_A$ the restriction of $\mathscr{E}^{W,\bar{p}}_{\Lambda,q}$ to $\mathbf{R}^A$. Let $\mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \in \mathscr{K}(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q})$ be the measure we have found in Lemma 6.1. Note that the restriction of $\mathscr{E}^{W,\bar{p}^1,\bar{p}^2}$ to $\mathbf{R}^A \times \mathbf{R}^A$ is an element of $\mathscr{K}((\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q})_A, (\mathscr{E}^{W,\bar{p}^2}_{\Lambda,q})_A)$. We thus have that

$$\begin{aligned} &\left| \sum_{z\in A} \left(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}(p_z) - \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}(p_z) \right) \right| \\ &\quad \leq \mathscr{R}_A \left(\left(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q} \right)_A, \left(\mathscr{E}^{W,\bar{p}^2}_{\Lambda,q} \right)_A \right) \\ &\quad \leq \sum_{z\in A} \mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \left(|p^1_z - p^2_z| \right) \\ &\quad \leq \sum_{z\in W} \mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \left(|p^1_z - p^2_z| \right) \exp\left(-\frac{d(z,A)}{C_{6.10}} \right) \\ &\quad \leq B_{6.10} \sum_{\substack{z\in W \\ d(z,y)\leq 2D_{6.10}}} \mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \left(|p^1_z - p^2_z| \right) \exp\left(-\frac{d(z,A)}{C_{6.10}} \right) \ . \end{aligned} \tag{6.19}$$

Here, in passage to the last line, we have used (6.10) with $L = D_{6.10}$ (and thus without the second term on the right-hand-side of (6.10)). To proceed from (6.19), note that we have $d(z, A) \geq d(y, A) - 2D_{6.10}$ in the exponential in (6.19) and that by (3.1),

$$\begin{aligned} \mathscr{E}^{W,\bar{p}^1,\bar{p}^2} \left(|p^1_z - p^2_z| \right) \ &\leq \ \mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}(|p_z|) + \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}(|p_z|) \\ &\leq \ C_{3.1}(4 + \sum_{w\in\Lambda\cap\partial_R W} \left(|\bar{p}^1_w| + |\bar{p}^2_w| \right)) \ . \end{aligned} \tag{6.20}$$

Plugging these into (6.19), we conclude that

$$\left| \sum_{z \in A} \left(\mathscr{E}^{W,\bar{p}^1}_{\Lambda,q}(p_z) - \mathscr{E}^{W,\bar{p}^2}_{\Lambda,q}(p_z) \right) \right| \leq C_{6.21} \left(1 + \sum_{y \in \Lambda \cap \partial_R W} \left(|\bar{p}^1_y| + |\bar{p}^2_y| \right) \right. \\ \left. \times \exp\left(-\frac{d(y, A)}{C_{6.10}} \right) \right) . \tag{6.21}$$

The mixing condition (2.1) can be obtained as a special case of $A = \{z\}$. □

Proof of Theorem 2.4 By Theorem 2.1 and Proposition 2.3, it is sufficient for us to prove part **(a)** of the theorem. It can be seen from the same computation as in the proof of [COPP78, Theorem 2.3] that

$$\mathscr{R}_x \left(\mathscr{E}^{x,\bar{p}^1}_{\Lambda,q}, \mathscr{E}^{x,\bar{p}^2}_{\Lambda,q} \right) \leq \sum_y K_{x,y} |\bar{p}^1_y - \bar{p}^2_y| \tag{6.22}$$

for all $x \in W$ and $\bar{p}^i \in \mathbf{R}^\Lambda$ $(i = 1, 2)$, where

$$K_{x,y} = \begin{cases} 0, & \text{if } x = y, \\ J_{x,y} \sup_{\bar{p} \in \mathbf{R}^\Lambda} \mathscr{E}^{x,\bar{p}}_{\Lambda,q}(p_x;\, p_x), & \text{if } x \neq y \ . \end{cases} \tag{6.23}$$

If we set $V = \{0\}$ and define $\mathbf{K} = (K_{x,y} \geq 0 : x, y \in \mathbf{Z}^d)$ by (6.23), then we have (2.7) and (2.9). To see that (2.8) is satisfied if $\sup_x \sum_{y:y \neq x} J_{x,y}$ is small enough, it is sufficient to prove that

$$\mathscr{E}^{x,\bar{p}}_{\Lambda,q}(p_x;\, p_x) \leq C_{6.24}, \tag{6.24}$$

where $C_{6.24} = C_{6.24}(U) \in (0, \infty)$. In fact, (6.22) and (6.24) imply that

$$\sup_y \sum_x K_{x,y} \leq C_{6.24} \sup_x \sum_{y:y \neq x} J_{x,y}$$

and therefore that (2.8) is true if $\sup_x \sum_{y:y \neq x} J_{x,y} < \min\left\{1, C_{6.24}^{-1}\right\}$.

The proof of (6.24) can be given as an application of the log-Sobolev inequality to the measure $\mathscr{E}^{x,\bar{p}}_{\Lambda,q}$ as follows. We begin by decomposing U into V and W as in (1.10)–(1.13), where the parameter $m > 0$ is arbitrary. We then have by (1.38) that

$$\begin{aligned} \mathscr{H}^{x,\bar{p}}_{\Lambda,q}(p_x) &= \mathscr{U}(p_x, q_x) - p_x \sum_{y \in \Lambda \setminus x} J_{x,y} \bar{p}_y \\ &= \mathscr{V}(p_x, q_x) + \mathscr{W}(p_x, q_x) \ , \end{aligned} \tag{6.25}$$

where $\mathscr{U}(p_x, q_x) = U((q_x + p_x)\sqrt{2}) + U((q_x - p_x)/\sqrt{2})$, $\mathscr{V}(p_x, q_x) = V((q_x + p_x)/\sqrt{2}) + V((q_x - p_x)/\sqrt{2}) - p_x \sum_{y \in \Lambda \setminus x} J_{x,y} \bar{p}_y$, and $\mathscr{W}(p_x, q_x) = W((q_x + p_x)/\sqrt{2}) + W((q_x - p_x)/\sqrt{2})$. Since $(\partial^2/\partial p_x^2)$

$\mathscr{V}(p_x, q_x) \geq m$ and $|\mathscr{W}(p_x, q_x)| \leq 2\|W\|_\infty$, we see from the Bakry-Emery criterion together with a comparison argument ([HS87, Lemma 5.1], cf. proof of Lemma 3.2) that

$$\mathscr{E}^{x,\bar{p}}_{\Lambda,q}\left(f^2 \log \frac{f^2}{\mathscr{E}^{x,\bar{p}}_{\Lambda,q}(f^2)}\right) \leq \gamma \mathscr{E}^{x,\bar{p}}_{\Lambda,q}\left(\left|\frac{\partial f}{\partial p_x}\right|^2\right) \tag{6.26}$$

for all $f \in \mathscr{C}_{\{x\}}$ with $\gamma = 2\exp(16\|W\|_\infty)/m$. It is well known that (6.26) implies that

$$\mathscr{E}^{x,\bar{p}}_{\Lambda,q}(f; f) \leq \frac{\gamma}{2} \mathscr{E}^{x,\bar{p}}_{\Lambda,q}\left(\left|\frac{\partial f}{\partial p_x}\right|^2\right) \tag{6.27}$$

for all $f \in \mathscr{C}_{\{x\}}$ (See [DS89, Corollary 6.1.17]). Putting $f(p_x) = p_x$ in (6.27), we get (6.24) with $C_{6.24} = \frac{\gamma}{2}$. □

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